



Research article

Robust numerical techniques for solving nonlinear ordinary and fractional partial differential problems

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Abstract: This manuscript presents novel spectral techniques based on shifted orthonormal normalized Jacobi polynomials (SONJPs) for solving two types of fractional differential equations. The first method uses the tau technique to solve the nonlinear boundary value problem (NBVP), and the second method uses the collocation approach to solve the time-fractional generalized Fisher problem (TFGFP) that has importance in mathematical biology, population dynamics, and economic modeling. We derive new relations that form the foundation for the proposed numerical techniques, thereby reducing the problem and its governing conditions to a nonlinear algebraic system that can be solved efficiently using Newton's method or another suitable iterative technique. We present numerical results with comparisons to illustrate the suggested algorithm's high accuracy and applicability.

Keywords: Jacobi polynomials; spectral methods; nonlinear boundary value problems; time-fractional generalized Fisher equation

1. Introduction

Because fractional differential equations (FDEs) can capture memory effects and genetic features in complex systems, they have found enormous utility in many scientific and industrial fields. Their superior representation of anomalous diffusion and viscoelastic behavior compared to more conventional models makes them useful in the physical sciences. Some applications of FDEs in different disciplines can be found in [1–3].

Because most FDEs are not analytically solvable, numerical algorithms are necessary. Many numerical algorithms have been developed in this regard. For example, the authors of [4] applied the Adomian decomposition method to solve some FDEs. In [5], the homotopy method was applied to solve two-dimensional cable equations. The authors of [6] adopted an adaptive step-size approach to solve FDEs. A finite difference method was employed in [7] to treat FDEs. Another finite difference approach was used in [8] to solve certain nonlinear FDEs. Other FDEs with multiorder terms were studied in [9] using a certain operational matrix of integration. In [10], a predictor–corrector approach was applied to solve certain types of FDEs. Operational matrix methods were used in [11, 12] to treat different types of FDEs. A spatial sixth-order method was employed in [13] to solve FDEs. A transform iterative scheme was followed in [14] to solve nonlinear FDEs. Recent polynomial-based spectral algorithms include shifted Pell–Lucas techniques for fractional and classical Rosenau–Hyman models [15], shifted Pell polynomial schemes for third-order Rosenau–Hyman and Gilson–Pickering equations [16], and a shifted Dickson polynomial method for the time-fractional Huxley equation [17].

Numerous fractional and classical differential models have been numerically treated using Jacobi polynomials and their modified forms. The shifted Jacobi polynomials (SJPs) were used by the authors of [18] to develop an effective numerical scheme for solving tempered fractional quadratic integro-differential equations. A hybrid framework combining classical methods and SJPs was developed in [19] to approximate solutions of two-dimensional time-fractional diffusion–wave equations. In [20], a computational method based on SJPs was suggested to deal with general classes of FDEs. In [21], Jacobi polynomial-based spectral techniques were further applied for the numerical solution of multidimensional stochastic multiorder time-fractional diffusion–wave problems. In [22], Romanovski–Jacobi polynomials were employed to analyze fractional mobile–immobile advection–dispersion models involving the Atangana–Baleanu–Caputo derivative. In [23], coupled systems of Hadamard fractional Klein–Gordon–Schrödinger equations were solved using Jacobi polynomials. In [24], spectral formulations for the Benjamin–Bona–Mahony–Burgers model using transformed generalized Jacobi polynomials were presented. Jacobi polynomials were used to solve nonlinear fractional variational problems [25] and to solve laminar nanofluid flow models using Jacobi wavelets [26]. In [27], multiterm time-fractional diffusion and diffusion–wave equations were solved using Galerkin schemes. Using spectral collocation techniques, space-fractional Navier–Stokes equations were treated in [28].

Spectral methods are a highly accurate family of numerical approaches that use global basis functions to approximate solutions to differential, integral, and integro-differential equations. The principal idea of the spectral method is to express the proposed approximate solution as a combination of selected special functions, such as special polynomials. The problem, subject to its governing constraints, is transformed into an algebraic system of equations using weighted-residual formulations. There are three strategies for spectral methods: the collocation, tau, and Galerkin methods. Every approach has its own characteristics. The type of the investigated equation and its governing conditions determine the appropriate method to apply. The collocation method can be applied to various types of DEs; see, for example, [29]. The tau and Galerkin methods are based on the application of two sets of basis functions. In the tau method, there are no constraints on choosing these sets; see, for example, [30], whereas the two sets coincide when applying the Galerkin method; see, for example, [31].

The operational matrix of derivatives (OMDs) is a useful tool for efficiently solving various DEs using spectral methods. The advantage of using such matrices is that the investigated equation and its governing conditions are converted into an algebraic system that can be numerically solved. Many

authors have applied this approach to treating DEs of various types. For example, the authors in [32] utilized OMDs based on Bernoulli wavelets for treating some delay FDEs. In [33], OMDs of SJPs were constructed and employed to solve some ordinary differential equations and FDEs. Vieta Lucas OMDs were employed in [34] to treat certain models. Certain OMDs in the Caputo sense were established in [35], accompanied by some applications. A family of shifted Chebyshev polynomials was also employed to construct a numerical method for a gas-dynamics equation involving the Caputo time-fractional derivative [36]. Other FDEs were handled in [37] using certain OMDs.

The main contributions of the current paper may be listed as follows:

- An efficient spectral tau method based on shifted orthonormal normalized Jacobi polynomials (SONJPs) is developed for two types of FDEs.
- An SONJP-based collocation method is constructed for solving the time-fractional generalized Fisher problem (TFGFP).
- Suitable operational matrices are derived and employed within both the tau and collocation methods to transform the considered fractional differential equations into systems of algebraic equations.

The novelty of the proposed approach can be summarized as follows:

- For the tau approach, a new employment of the linearization formula for SONJPs is introduced. This formula plays an essential role in explicitly computing the inner products involving products of shifted Jacobi polynomials that are required to construct the resulting algebraic system.
- The orthonormality of the SONJP basis further simplifies the tau formulation by eliminating many cross terms in the resulting inner products and consequently leads to more compact algebraic expressions.

This manuscript is arranged as follows. Section 2 presents the theoretical background on fractional calculus and defines the Jacobi polynomial basis adopted in the analysis. In Section 3, a tau method is constructed to solve the nonlinear boundary value problem (NBVP). A collocation formulation for the TFGFP is examined in Section 4. Numerical tests are reported in Section 5, and concluding remarks are summarized in Section 6.

2. Some fundamentals

This section recalls the fractional-calculus ingredients needed below and collects the principal Jacobi-polynomial relations used in constructing the proposed schemes.

2.1. Fractional derivative in the Caputo sense

Definition 1. The Caputo fractional derivative $D_s^\mu g(s)$ is given by [38] as follows:

$$D_s^\mu g(s) = \frac{1}{\Gamma(r-\mu)} \int_0^s (s-t)^{r-\mu-1} g^{(r)}(t) dt, \quad \mu > 0, \quad s > 0, \quad r-1 < \mu \leq r, \quad r \in \mathbb{N}. \quad (2.1)$$

When $r-1 < \mu \leq r$ with $r \in \mathbb{N}$, the operator D_s^μ satisfies the identities

$$D_s^\mu C = 0, \quad C \text{ is a constant}, \quad (2.2)$$

$$D_s^\mu s^r = \begin{cases} 0, & \text{if } r \in \mathbb{N}_0 \text{ and } r < \lceil \mu \rceil, \\ \frac{r!}{\Gamma(r-\mu+1)} s^{r-\mu}, & \text{if } r \in \mathbb{N}_0 \text{ and } r \geq \lceil \mu \rceil, \end{cases} \quad (2.3)$$

where $\mathbb{N} = \{1, 2, \dots\}$, and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, and $\lceil \mu \rceil$ is the ceiling function.

2.2. An overview of SONJPs

For real parameters $\zeta, \rho > -1$, the standard JPs $P_\ell^{(\zeta, \rho)}(x)$ defined on the reference interval $x \in [-1, 1]$ are mapped onto the unit interval $z \in [0, 1]$ using transformation $x = 2z - 1$.

The shifted normalized Jacobi polynomials on $[0, 1]$ are defined by

$$\phi_\ell^{(\zeta, \rho)}(2z - 1) = \frac{\ell! \Gamma(\zeta + 1)}{\Gamma(\zeta + \ell + 1)} P_\ell^{(\zeta, \rho)}(2z - 1), \quad (2.4)$$

where $P_\ell^{(\zeta, \rho)}(2z - 1)$ denotes the classical SJPs.

Define the SONJPs as

$$\psi_\ell^{(\zeta, \rho)}(z) = \sqrt{h_\ell^{(\zeta, \rho)}} \phi_\ell^{(\zeta, \rho)}(2z - 1), \quad (2.5)$$

where

$$h_\ell^{(\zeta, \rho)} = \frac{(\zeta + \rho + 2\ell + 1) \Gamma(\zeta + \ell + 1) \Gamma(\zeta + \rho + \ell + 1)}{\ell! \Gamma(\zeta + 1)^2 \Gamma(\rho + \ell + 1)}. \quad (2.6)$$

The orthogonality property of the SONJPs is

$$\int_0^1 \psi_n^{(\zeta, \rho)}(z) \psi_m^{(\zeta, \rho)}(z) \omega^{(\zeta, \rho)}(z) dz = \delta_{n,m}, \quad (2.7)$$

where

$$\omega^{(\zeta, \rho)}(z) = (1 - z)^\zeta z^\rho,$$

and $\delta_{n,m}$ represents the Kronecker delta function

$$\delta_{n,m} = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m. \end{cases} \quad (2.8)$$

The series form of $\psi_\ell^{(\zeta, \rho)}(z)$ takes the form

$$\psi_\ell^{(\zeta, \rho)}(z) = \sum_{s=0}^{\ell} \theta_{s,\ell}^{\zeta, \rho} z^s, \quad (2.9)$$

where the coefficients $\theta_{s,\ell}^{\zeta, \rho}$ are given by

$$\theta_{s,\ell}^{\zeta, \rho} = \frac{(-1)^{\ell+s} (\zeta + \rho + 2\ell + 1) \Gamma(\zeta + \rho + \ell + s + 1)}{\sqrt{h_\ell^{(\zeta, \rho)}} s! \Gamma(\zeta + 1) (\ell - s)! \Gamma(\rho + s + 1)}. \quad (2.10)$$

Moreover, the inversion formula expressing the monomial z^ℓ in terms of the basis $\{\psi_s^{(\zeta, \rho)}(z)\}$ is written as

$$z^\ell = \sum_{s=0}^{\ell} B_{s,\ell} \psi_s^{(\zeta, \rho)}(z), \quad (2.11)$$

where $B_{s,\ell}$ are given by

$$B_{s,\ell} = \frac{(\zeta + \rho + 2s + 1)(\zeta + 1)_s \Gamma(\rho + \ell + 1)(\ell - s + 1)_s \Gamma(\zeta + \rho + s + 1)}{\sqrt{h_s^{(\zeta,\rho)}} s! \Gamma(\rho + s + 1) \Gamma(\zeta + \rho + \ell + s + 2)}. \quad (2.12)$$

Lemma 1. Consider the non-negative integers m and n . The following linearization formula holds [39]:

$$\psi_m^{(\zeta,\zeta+1)}(t) \psi_n^{(\zeta,\zeta+1)}(t) = \sum_{s=|m-n|}^{m+n} \sqrt{\frac{h_m^{(\zeta,\zeta+1)} h_n^{(\zeta,\zeta+1)}}{h_s^{(\zeta,\zeta+1)}}} \chi_{s,\zeta,m,n} \psi_s^{(\zeta,\zeta+1)}(t). \quad (2.13)$$

The coefficients $\chi_{s,\zeta,m,n}$ are given explicitly by

$$\chi_{s,\zeta,m,n} = \frac{2^{2\zeta+1} m! n! \Gamma(\zeta + 1)}{\sqrt{\pi} \Gamma(\zeta + \frac{3}{2}) \Gamma(m + 2\zeta + 2) \Gamma(n + 2\zeta + 2)} \times \begin{cases} \frac{\Gamma(\frac{m+n-s+3}{2} + \zeta) \Gamma(\frac{m-n+s+3}{2} + \zeta) \Gamma(\frac{-m+n+s+3}{2} + \zeta) \Gamma(\frac{m+n+s+4\zeta+4}{2})}{\Gamma(\frac{m+n-s+2}{2}) \Gamma(\frac{m-n+s+2}{2}) \Gamma(\frac{-m+n+s+2}{2}) \Gamma(\frac{m+n+s+3}{2} + \zeta)}, & \text{if } (m+n-s) \text{ is even,} \\ -\frac{\Gamma(\frac{m+n-s}{2} + \zeta + 1) \Gamma(\frac{m-n+s+2}{2} + \zeta) \Gamma(\frac{-m+n+s+2}{2} + \zeta) \Gamma(\frac{m+n+s+4\zeta+5}{2})}{\Gamma(\frac{m+n-s+1}{2}) \Gamma(\frac{m-n+s+1}{2}) \Gamma(\frac{-m+n+s+1}{2}) \Gamma(\frac{m+n+s+4}{2} + \zeta)}, & \text{otherwise.} \end{cases} \quad (2.14)$$

Theorem 1. [40] The r th derivative of $\phi_\ell^{(\zeta,\rho)}(z)$ may be represented as

$$D^r \phi_\ell^{(\zeta,\rho)}(z) = \sum_{i=0}^{\ell-r} \lambda_{i,\ell,r}^{(\zeta,\rho)} \phi_i^{(\zeta,\rho)}(z), \quad (2.15)$$

where

$$\lambda_{i,\ell,r}^{(\zeta,\rho)} = \frac{2^{-r} \ell! \Gamma(\zeta + \rho + i + 1)(\zeta + \rho + \ell + 1)_r (\zeta + i + r + 1)_{-i+\ell-r} (\zeta + \rho + \ell + r + 1)_i}{i! (-i + \ell - r)! \Gamma(\zeta + \rho + 2i + 1)(\zeta + i + 1)_{\ell-i}} \times {}_3F_2 \left(\begin{matrix} i - \ell + r, \zeta + \rho + i + \ell + r + 1, \zeta + i + 1 \\ \zeta + i + r + 1, \zeta + \rho + 2i + 2 \end{matrix} \middle| 1 \right), \quad (2.16)$$

and the generalized hypergeometric function is defined by [41]

$${}_rF_s \left(\begin{matrix} r_1, r_2, \dots, r_r \\ s_1, s_2, \dots, s_s \end{matrix} \middle| z \right) = \sum_{\ell=0}^{\infty} \frac{(r_1)_\ell (r_2)_\ell \dots (r_r)_\ell}{(s_1)_\ell (s_2)_\ell \dots (s_s)_\ell} \frac{z^\ell}{\ell!}. \quad (2.17)$$

Corollary 1. The second-order derivative of the basis function $\psi_\ell^{(\zeta,\zeta+1)}(z)$ admits the expansion

$$\frac{d^2 \psi_\ell^{(\zeta,\zeta+1)}(z)}{dz^2} = \sum_{s=0}^{\ell-2} \mu_{s,\ell}^{(\zeta)} \psi_s^{(\zeta,\zeta+1)}(z), \quad (2.18)$$

where the coefficients $\mu_{s,\ell}^{(\zeta)}$ are given by

$$\mu_{s,\ell}^{(\zeta)} = \frac{\ell! \sqrt{h_\ell^{(\zeta,\zeta+1)}} \Gamma(2\zeta + s + 2) (2\zeta + \ell + 2)_2 (\zeta + s + 3)_{-s+\ell-2} (2\zeta + \ell + 4)_s}{s! \sqrt{h_s^{(\zeta,\zeta+1)}} (-s + \ell - 2)! \Gamma(2\zeta + 2s + 2) (\zeta + s + 1)_{\ell-s}} \times {}_3F_2 \left(\begin{matrix} \zeta + s + 1, s - \ell + 2, 2\zeta + s + \ell + 4 \\ \zeta + s + 3, 2\zeta + 2s + 3 \end{matrix} \middle| 1 \right). \quad (2.19)$$

Proof. The proof of Corollary 1 follows from the definition of $\psi_i^{(\zeta,\zeta+1)}(z)$ together with Theorem 1. $\square$

Theorem 2. [42] Let $\phi(z)$ be the shifted Jacobi polynomial vector

$$\phi(z) = \left[P_{\ell,0}^{(\zeta,\rho)} \left(\frac{2z}{\ell} - 1 \right), P_{\ell,1}^{(\zeta,\rho)} \left(\frac{2z}{\ell} - 1 \right), \dots, P_{\ell,\mathcal{L}}^{(\zeta,\rho)} \left(\frac{2z}{\ell} - 1 \right) \right]^T,$$

and suppose $\gamma > 0$. Then,

$$D_z^\gamma \phi(z) \simeq \mathbf{D}_\gamma^{(\zeta,\rho,\ell)} \phi(z), \quad (2.20)$$

where $\mathbf{D}_\gamma^{(\zeta,\rho,\ell)}$ is the $(\mathcal{L} + 1) \times (\mathcal{L} + 1)$ operational matrix of derivatives of order γ in the Caputo sense and is defined as follows:

$$\mathbf{D}_\gamma^{(\zeta,\rho,\ell)} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ \Lambda_\gamma(\lceil \gamma \rceil, 0) & \Lambda_\gamma(\lceil \gamma \rceil, 1) & \Lambda_\gamma(\lceil \gamma \rceil, 2) & \dots & \Lambda_\gamma(\lceil \gamma \rceil, \mathcal{L}) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \Lambda_\gamma(i, 0) & \Lambda_\gamma(i, 1) & \Lambda_\gamma(i, 2) & \dots & \Lambda_\gamma(i, \mathcal{L}) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \Lambda_\gamma(\mathcal{L}, 0) & \Lambda_\gamma(\mathcal{L}, 1) & \Lambda_\gamma(\mathcal{L}, 2) & \dots & \Lambda_\gamma(\mathcal{L}, \mathcal{L}) \end{pmatrix}. \quad (2.21)$$

Here,

$$\Lambda_\gamma(i, j) = \sum_{k=\lceil \gamma \rceil}^i \Omega_{ijk},$$

and Ω_{ijk} is given by

$$\Omega_{ijk} = \frac{(-1)^{i-k} \ell^{\zeta+\rho-\gamma+1} \Gamma(j + \rho + 1) \Gamma(i + \rho + 1) \Gamma(i + k + \zeta + \rho + 1)}{h_j \Gamma(j + \zeta + \rho + 1) \Gamma(k + \rho + 1) \Gamma(i + \zeta + \rho + 1) \Gamma(k - \gamma + 1) (i - k)!} \times \sum_{l=0}^j \frac{(-1)^{j-l} \Gamma(j + l + \zeta + \rho + 1) \Gamma(\zeta + 1) \Gamma(l + k + \rho - \gamma + 1)}{\Gamma(l + \rho + 1) \Gamma(l + k + \zeta + \rho - \gamma + 2) (j - l)! l!}. \quad (2.22)$$

Note that in $\mathbf{D}_\gamma^{(\zeta,\rho,\ell)}$, the first $\lceil \gamma \rceil$ rows are all zero.

Remark 1. Let $\psi(z)$ be the vector of $\psi_j^{(\zeta, \zeta+1)}(z)$, defined by

$$\psi(z) = \left[\psi_0^{(\zeta, \zeta+1)}(z), \psi_1^{(\zeta, \zeta+1)}(z), \dots, \psi_{\mathcal{L}}^{(\zeta, \zeta+1)}(z) \right]^T.$$

Taking $\ell = 1$ and $\rho = \zeta + 1$ in Theorem 2, the relation between $\psi(z)$ and $\phi(z)$ follows from Eqs (2.5) and (2.4) as

$$\psi(z) = X \phi(z), \quad (2.23)$$

where

$$X = \begin{pmatrix} \frac{\sqrt{h_0^{(\zeta, \zeta+1)}} 0! \Gamma(\zeta + 1)}{\Gamma(\zeta + 1)} & 0 & \dots & 0 \\ 0 & \frac{\sqrt{h_1^{(\zeta, \zeta+1)}} 1! \Gamma(\zeta + 1)}{\Gamma(\zeta + 2)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{\sqrt{h_{\mathcal{L}}^{(\zeta, \zeta+1)}} \mathcal{L}! \Gamma(\zeta + 1)}{\Gamma(\zeta + \mathcal{L} + 1)} \end{pmatrix}. \quad (2.24)$$

Corollary 2. Let $\alpha > 0$, and let $\psi(t)$ be the vector defined in Remark 1. Then,

$$D_t^\alpha \psi(t) \simeq D_\alpha^{(\zeta, \zeta+1, 1)} \psi(t). \quad (2.25)$$

Proof. The proof follows directly by setting $\ell = 1$ and $\rho = \zeta + 1$ in Theorem 2 and then applying Remark 1. $\square$

Remark 2. Consider the vector $\psi(\eta)$ defined in Remark 1. Then, Corollary 1 can be written in matrix form as

$$\frac{d^2 \psi(\eta)}{d\eta^2} = H \psi(\eta), \quad (2.26)$$

where $H = (\mu_{s,i}^{(\zeta)})$ is the $(\mathcal{L} + 1) \times (\mathcal{L} + 1)$ second-derivative operational matrix.

3. A tau approach for some linear BVPs

A spectral tau formulation based on SONJPs is constructed here for fractional boundary value problems (BVPs), together with the inner products needed for its numerical realization. We consider the fractional BVP [43]

$$D_z^\gamma Y(z) + q(z) Y(z) = g(z), \quad z \in (0, 1), \quad (3.1)$$

subject to

$$Y(0) = a_0, \quad Y(1) = a_1, \quad (3.2)$$

where $\gamma \in (1, 2)$, and $g(z)$ and $q(z)$ are given functions.

For Eq (3.1), the unknown solution is approximated in the SONJP basis $\{\psi_j^{(\zeta, \zeta+1)}(z)\}$ as

$$\mathbf{Y}^{\mathcal{L}}(z) = \sum_{j=0}^{\mathcal{L}} y_j \psi_j^{(\zeta, \zeta+1)}(z) = \mathbf{y}^T \boldsymbol{\psi}(z), \quad (3.3)$$

where $\mathbf{y}^T = [y_0, y_1, y_2, \dots, y_{\mathcal{L}}]$, and $\boldsymbol{\psi}(z)$ is defined in Remark 1.

Substitution of Eq (3.3) into Eq (3.1) gives the residual

$$R(z) = D_z^{\gamma} \mathbf{Y}^{\mathcal{L}}(z) + q(z) \mathbf{Y}^{\mathcal{L}}(z) - g(z). \quad (3.4)$$

We further represent $q(z)$ by the truncated SONJP expansion

$$q(z) = \sum_{i=0}^{\mathcal{L}} \hat{q}_i \psi_i^{(\zeta, \zeta+1)}(z), \quad (3.5)$$

where

$$\hat{q}_i = \int_0^1 q(z) \psi_i^{(\zeta, \zeta+1)}(z) \omega^{(\zeta, \zeta+1)}(z) dz. \quad (3.6)$$

Consequently, Eq (3.4) takes the form

$$\begin{aligned} R(z) &= \sum_{j=0}^{\mathcal{L}} y_j D_z^{\gamma} \psi_j^{(\zeta, \zeta+1)}(z) \\ &+ \sum_{j=0}^{\mathcal{L}} \sum_{i=0}^{\mathcal{L}} y_j \hat{q}_i \psi_i^{(\zeta, \zeta+1)}(z) \psi_j^{(\zeta, \zeta+1)}(z) - g(z). \end{aligned} \quad (3.7)$$

The tau conditions are then imposed as

$$\langle R(z), \psi_s^{(\zeta, \zeta+1)}(z) \rangle_{\omega^{(\zeta, \zeta+1)}} = 0, \quad s = 0, 1, 2, \dots, \mathcal{L} - 2. \quad (3.8)$$

This leads to

$$\sum_{j=0}^{\mathcal{L}} y_j A_{j,s} + \sum_{j=0}^{\mathcal{L}} \sum_{i=0}^{\mathcal{L}} y_j \hat{q}_i B_{j,i,s} = G_s, \quad s = 0, 1, 2, \dots, \mathcal{L} - 2. \quad (3.9)$$

Equivalently, we can write

$$\begin{aligned} &\sum_{j=0}^{\mathcal{L}} y_j \left\langle D_z^{\gamma} \psi_j^{(\zeta, \zeta+1)}(z), \psi_s^{(\zeta, \zeta+1)}(z) \right\rangle_{\omega^{(\zeta, \zeta+1)}} \\ &+ \sum_{j=0}^{\mathcal{L}} \sum_{i=0}^{\mathcal{L}} y_j \hat{q}_i \left\langle \psi_i^{(\zeta, \zeta+1)}(z) \psi_j^{(\zeta, \zeta+1)}(z), \psi_s^{(\zeta, \zeta+1)}(z) \right\rangle_{\omega^{(\zeta, \zeta+1)}} \\ &= \left\langle g(z), \psi_s^{(\zeta, \zeta+1)}(z) \right\rangle_{\omega^{(\zeta, \zeta+1)}}, \quad s = 0, 1, 2, \dots, \mathcal{L} - 2. \end{aligned} \quad (3.10)$$

The resulting system admits the following matrix representation:

$$\mathbf{y}^T \mathbf{A} + [\mathbf{y}^T \mathbf{C}^0 \mathbf{Q}, \mathbf{y}^T \mathbf{C}^1 \mathbf{Q}, \dots, \mathbf{y}^T \mathbf{C}^{\mathcal{L}-2} \mathbf{Q}] = \mathbf{G}, \quad (3.11)$$

where the involved matrices and vectors are defined below.

$$\mathbf{G} = (g_s)_{1 \times (\mathcal{L}-1)}, \quad g_s = \left\langle g(z), \psi_s^{(\zeta, \zeta+1)}(z) \right\rangle_{\omega^{(\zeta, \zeta+1)}}, \quad (3.12)$$

$$\mathbf{A} = (a_{j,s})_{(\mathcal{L}+1) \times (\mathcal{L}-1)}, \quad a_{j,s} = \left\langle D_z^\gamma \psi_j^{(\zeta, \zeta+1)}(z), \psi_s^{(\zeta, \zeta+1)}(z) \right\rangle_{\omega^{(\zeta, \zeta+1)}}, \quad (3.13)$$

$$\mathbf{C}^s = (c_{j,i}^s)_{(\mathcal{L}+1) \times (\mathcal{L}+1)}, \quad c_{j,i}^s = \left\langle \psi_i^{(\zeta, \zeta+1)}(z) \psi_j^{(\zeta, \zeta+1)}(z), \psi_s^{(\zeta, \zeta+1)}(z) \right\rangle_{\omega^{(\zeta, \zeta+1)}}. \quad (3.14)$$

Moreover, $\mathbf{Q} = [\hat{q}_0, \hat{q}_1, \hat{q}_2, \dots, \hat{q}_{\mathcal{L}}]^T$.

From the boundary conditions (3.2), we obtain

$$\mathbf{y}^T \boldsymbol{\psi}(0) = a_0, \quad \mathbf{y}^T \boldsymbol{\psi}(1) = a_1. \quad (3.15)$$

Finally, Eqs (3.11) and (3.15) yield a system of $(\mathcal{L} + 1)$ linear algebraic equations for the coefficients. Solving this system gives the approximation $\mathbf{Y}^{\mathcal{L}}(z)$ in (3.3).

Remark 3. The inner products

$$\left\langle D_z^\gamma \psi_j^{(\zeta, \zeta+1)}(z), \psi_s^{(\zeta, \zeta+1)}(z) \right\rangle_{\omega^{(\zeta, \zeta+1)}} \quad \text{and} \quad \left\langle \psi_i^{(\zeta, \zeta+1)}(z) \psi_j^{(\zeta, \zeta+1)}(z), \psi_s^{(\zeta, \zeta+1)}(z) \right\rangle_{\omega^{(\zeta, \zeta+1)}}$$

are evaluated in the next theorem.

Theorem 3. The following identities hold:

$$\begin{aligned} & \left\langle D_z^\gamma \psi_j^{(\zeta, \zeta+1)}(z), \psi_s^{(\zeta, \zeta+1)}(z) \right\rangle_{\omega^{(\zeta, \zeta+1)}} \\ &= \sum_{r=2}^j \sum_{m=0}^s \frac{r! \Gamma(\zeta + 1) \theta_{r,j}^{(\zeta, \zeta+1)} \theta_{m,s}^{(\zeta, \zeta+1)} \Gamma(\zeta + m + r - \gamma + 2)}{\Gamma(r - \gamma + 1) \Gamma(2\zeta + m + r - \gamma + 3)}, \end{aligned} \quad (3.16)$$

$$\begin{aligned} & \left\langle \psi_i^{(\zeta, \zeta+1)}(z) \psi_j^{(\zeta, \zeta+1)}(z), \psi_s^{(\zeta, \zeta+1)}(z) \right\rangle_{\omega^{(\zeta, \zeta+1)}} \\ &= \sum_{p=|i-j|}^{i+j} \delta_{p,s} \sqrt{\frac{h_i^{(\zeta, \zeta+1)} h_j^{(\zeta, \zeta+1)}}{h_p^{(\zeta, \zeta+1)}}} \chi_{p,\zeta,i,j}. \end{aligned} \quad (3.17)$$

Proof. To prove the first identity, we apply the fractional derivative operator D_z^γ to the power-series representation of $\psi_j^{(\zeta, \zeta+1)}(z)$ given in (2.9), which yields

$$D_z^\gamma \psi_j^{(\zeta, \zeta+1)}(z) = \sum_{r=2}^j \theta_{r,j}^{(\zeta, \zeta+1)} \frac{r!}{\Gamma(r - \gamma + 1)} z^{r-\gamma}. \quad (3.18)$$

Consequently, the inner product can be written as

$$\begin{aligned} \left\langle D_z^\gamma \psi_j^{(\zeta, \zeta+1)}(z), \psi_s^{(\zeta, \zeta+1)}(z) \right\rangle_{\omega^{(\zeta, \zeta+1)}} &= \int_0^1 D_z^\gamma \psi_j^{(\zeta, \zeta+1)}(z) \psi_s^{(\zeta, \zeta+1)}(z) \omega^{(\zeta, \zeta+1)}(z) dz \\ &= \sum_{r=2}^j \theta_{r,j}^{(\zeta, \zeta+1)} \frac{r!}{\Gamma(r - \gamma + 1)} \int_0^1 z^{r-\gamma} \psi_s^{(\zeta, \zeta+1)}(z) \omega^{(\zeta, \zeta+1)}(z) dz \\ &= \sum_{r=2}^j \sum_{m=0}^s \theta_{r,j}^{(\zeta, \zeta+1)} \theta_{m,s}^{(\zeta, \zeta+1)} \frac{r!}{\Gamma(r - \gamma + 1)} \int_0^1 z^{r+m-\gamma} \omega^{(\zeta, \zeta+1)}(z) dz. \end{aligned} \quad (3.19)$$

Using the identity

$$\int_0^1 z^{r+m-\gamma} \omega^{(\zeta, \zeta+1)}(z) dz = \frac{\Gamma(\zeta+1) \Gamma(m+r-\gamma+\zeta+2)}{\Gamma(m+r-\gamma+2\zeta+3)}, \quad (3.20)$$

the first result follows immediately.

The second identity is obtained directly from Lemma 1 together with the orthonormality condition in (2.7) after setting the parameters to $(\zeta, \zeta+1)$. This completes the proof. $\square$

4. A collocation procedure for the time-fractional generalized Fisher equation

This section presents an SONJP-based spectral collocation procedure for the time-fractional generalized Fisher equation. Operational matrices are employed to reduce the problem to a nonlinear algebraic system.

Consider the following time-fractional generalized Fisher problem [43]:

$$D_t^\alpha Y(\eta, t) = Y_{\eta\eta}(\eta, t) + f(Y) + g(\eta, t), \quad 0 < \alpha < 1, \quad (4.1)$$

subject to

$$\begin{aligned} Y(\eta, 0) &= y_0(\eta), \quad 0 < \eta < 1, \\ Y(0, t) &= y_1(t), \quad Y(1, t) = y_2(t), \quad 0 < t < 1, \end{aligned} \quad (4.2)$$

where

$$f(Y) = \mu Y(\eta, t)(1 - Y^\gamma(\eta, t)),$$

$\mu \in \mathbb{R}$, $\gamma \in \mathbb{N}$, and $y_0(\eta)$, $y_1(t)$, $y_2(t)$, and $g(\eta, t)$ are given functions. Introduce the finite-dimensional tensor-product space

$$\Omega^\mathcal{L} = \text{span} \{ \psi_i^{(\zeta, \zeta+1)}(\eta) \psi_j^{(\zeta, \zeta+1)}(t) : 0 \leq i, j \leq \mathcal{L} \}. \quad (4.3)$$

Accordingly, an approximation $Y^\mathcal{L}(\eta, t) \in \Omega^\mathcal{L}$ can be written as

$$Y^\mathcal{L}(\eta, t) = \sum_{i=0}^{\mathcal{L}} \sum_{j=0}^{\mathcal{L}} \hat{y}_{ij} \psi_i^{(\zeta, \zeta+1)}(\eta) \psi_j^{(\zeta, \zeta+1)}(t) = \boldsymbol{\psi}(\eta)^T \hat{\mathbf{y}} \boldsymbol{\psi}(t), \quad (4.4)$$

where

$$\begin{aligned} \boldsymbol{\psi}(\eta) &= [\psi_0^{(\zeta, \zeta+1)}(\eta), \psi_1^{(\zeta, \zeta+1)}(\eta), \dots, \psi_{\mathcal{L}}^{(\zeta, \zeta+1)}(\eta)]^T, \\ \boldsymbol{\psi}(t) &= [\psi_0^{(\zeta, \zeta+1)}(t), \psi_1^{(\zeta, \zeta+1)}(t), \dots, \psi_{\mathcal{L}}^{(\zeta, \zeta+1)}(t)]^T, \end{aligned}$$

and $\hat{\mathbf{y}} = (\hat{y}_{ij})_{0 \leq i, j \leq \mathcal{L}}$ is the unknown $(\mathcal{L}+1) \times (\mathcal{L}+1)$ coefficient matrix.

For Eq (4.1), we introduce the residual

$$R(\eta, t) = D_t^\alpha Y^\mathcal{L}(\eta, t) - Y_{\eta\eta}^\mathcal{L}(\eta, t) - f(Y^\mathcal{L}) - g(\eta, t). \quad (4.5)$$

By using Remark 1, Corollary 2, and the expansion (4.4), the residual (4.5) can be written in matrix form as

$$\begin{aligned} R(\eta, t) &= \boldsymbol{\psi}(\eta)^T \hat{\mathbf{y}} (\mathbf{D}_\alpha^{(\zeta, \zeta+1, 1)} \boldsymbol{\psi}(t)) - (\mathbf{H} \boldsymbol{\psi}(\eta))^T \hat{\mathbf{y}} \boldsymbol{\psi}(t) \\ &\quad - f(\boldsymbol{\psi}(\eta)^T \hat{\mathbf{y}} \boldsymbol{\psi}(t)) - g(\eta, t). \end{aligned} \quad (4.6)$$

To determine the unknown coefficients $\hat{y}_{ij}$, the collocation method is employed by enforcing the residual to vanish at selected nodes (η_r, t_s) :

$$R(\eta_r, t_s) = 0, \quad 1 \leq r \leq \mathcal{L} - 1, \quad 1 \leq s \leq \mathcal{L}. \quad (4.7)$$

In addition, the constraints in (4.2) yield

$$\begin{aligned} \psi(\eta_r)^T \hat{\mathbf{y}} \psi(0) &= y_0(\eta_r), \quad r = 1, 2, \dots, \mathcal{L} + 1, \\ \psi(0)^T \hat{\mathbf{y}} \psi(t_s) &= y_1(t_s), \quad s = 1, 2, \dots, \mathcal{L}, \\ \psi(1)^T \hat{\mathbf{y}} \psi(t_s) &= y_2(t_s), \quad s = 1, 2, \dots, \mathcal{L}. \end{aligned} \quad (4.8)$$

Here, η_r and t_s denote the roots of $\psi_{\mathcal{L}+1}^{(\zeta, \zeta+1)}(\eta)$ and $\psi_{\mathcal{L}+1}^{(\zeta, \zeta+1)}(t)$, respectively.

Finally, Eqs (4.7) and (4.8) provide $(\mathcal{L}+1)^2$ nonlinear algebraic equations for the unknown coefficients $\hat{y}_{ij}$. The resulting system is solved by Newton iteration.

Remark 4. Although high-order spectral differentiation matrices are typically susceptible to ill-conditioning for large $\mathcal{L}$, utilizing the proposed basis functions significantly mitigates condition number growth compared to un-normalized bases. Furthermore, because spectral precision is attained at modest degrees ($\mathcal{L} \leq 14$), the condition numbers of $\mathbf{A}$, $\mathbf{C}^s$, and $\mathbf{D}_\alpha^{(\zeta, \zeta+1, 1)}$ remain well-bounded, ensuring complete solver stability during Newton iterations.

Remark 5. The truncation-error analysis based on SONJPs was carefully studied and presented in [44].

Remark 6. Although the proposed spectral collocation method shows high empirical accuracy, it is still challenging to establish accurate theoretical stability and error estimates for nonlinear fractional systems. This is due to the nonlocal memory core in Caputo operator and dense algebraic matrices. We refer readers to [45, 46] for recent analysis on stability and convergence in nonlinear Caputo-type time-fractional PDEs.

5. Illustrative examples

To evaluate the efficiency of the proposed approach, we introduce various test problems and compare the findings with those reported in existing literature. After implementing the suggested approach, the problem is transformed into a nonlinear algebraic system of dimensions $(\mathcal{L} + 1)^2$. This system is solved by Newton's method, where the initial guess is chosen as 10^{-i+j} , $i, j = 0, \dots, \mathcal{L}$, and a high working precision of 60 digits is used to ensure convergence to a tolerance of about 10^{-50} . All computations are carried out using the FindRoot function in Mathematica 11 on an HP Z420 Workstation equipped with an Intel(R) Xeon(R) CPU E5-1620 v2 (3.70 GHz), 16 GB of DDR3 RAM, and 512 GB of storage.

Example 1. Consider the following equation:

$$D_t^\gamma Y(t) + q(t) Y(t) = g(t), \quad t \in (0, 1), \quad (5.1)$$

subject to

$$Y(0) = 0, \quad Y(1) = 1, \quad (5.2)$$

where $g(t)$ is selected in accordance with the exact solution given by $Y(t) = t^4$.

The numerical scheme is applied to the problem for two specific forms of the function $q(t)$: $q(t) = 1$ and $q(t) = e^t$.

Case 1: At $q(t) = 1$.

Applying the tau method described in Section 3 for $\mathcal{L} = 4$, $\zeta = 1$, and $\gamma = 1.5$ yields the following system of equations:

$$256 \left(5005 \sqrt{10} y_2 - 2 \left(-6354 \sqrt{35} y_4 + 5915 \sqrt{5} y_3 + 84 \sqrt{3} \right) \right) + 2145 \sqrt{\pi} \left(\sqrt{3} - 28 y_0 \right) = 0, \quad (5.3)$$

$$384 \left(-1044 \sqrt{35} y_4 + 4270 \sqrt{5} y_3 + 455 \sqrt{10} y_2 - 56 \sqrt{3} \right) + 715 \sqrt{\pi} \left(2 \sqrt{3} - 63 y_1 \right) = 0, \quad (5.4)$$

$$256 \left(2975 y_2 - 39865 \sqrt{2} y_3 - 57690 \sqrt{14} y_4 + 84 \sqrt{30} \right) - 2431 \sqrt{\pi} \left(\sqrt{30} - 210 y_2 \right) = 0, \quad (5.5)$$

$$y_0 - 5 \sqrt{5} y_3 + 2 \sqrt{10} y_2 + 3 \sqrt{35} y_4 - 3 y_1 = 0, \quad (5.6)$$

$$2 \sqrt{3} \left(y_0 + 2 y_1 + \sqrt{5} \left(2 y_3 + \sqrt{2} y_2 + \sqrt{7} y_4 \right) \right) = 1, \quad (5.7)$$

which yields $\{y_0 \rightarrow \sqrt{3}/28, y_1 \rightarrow 2/(21 \sqrt{3}), y_2 \rightarrow 1/(7 \sqrt{30}), y_3 \rightarrow 2/(77 \sqrt{15}), y_4 \rightarrow 1/(132 \sqrt{105})\}$, and therefore, $Y^{\mathcal{L}}(t) = t^4$, which is the exact solution.

Case 2: At $q(t) = e^t$.

Table 1 presents the absolute errors (AEs) at various t when $\gamma = 1.5$, and $\mathcal{L} = 4$. Figure 1 shows the exact and approximate solutions and AEs at $\zeta = 0.9$ when $\gamma = 1.8$, and $\mathcal{L} = 4$.

Table 1. The AEs of Example 1 at $\gamma = 1.5$ and $\mathcal{L} = 4$.

t	$\zeta = 0.7$	$\zeta = 1$	$\zeta = 1.4$
0.1	3.38967×10^{-9}	3.61913×10^{-9}	4.05727×10^{-9}
0.2	6.43289×10^{-9}	4.2147×10^{-9}	2.24261×10^{-9}
0.3	1.96107×10^{-8}	1.48785×10^{-8}	1.10587×10^{-8}
0.4	2.92495×10^{-8}	2.23947×10^{-8}	1.70044×10^{-8}
0.5	3.14179×10^{-8}	2.34316×10^{-8}	1.71476×10^{-8}
0.6	2.51473×10^{-8}	1.73029×10^{-8}	1.10103×10^{-8}
0.7	1.24319×10^{-8}	5.96768×10^{-9}	5.68869×10^{-10}
0.8	1.77146×10^{-9}	5.96907×10^{-9}	9.74577×10^{-9}
0.9	9.54293×10^{-9}	1.1257×10^{-8}	1.30486×10^{-8}

Example 2. [43] Consider the following equation:

$$D_t^\alpha Y(\eta, t) = Y_{\eta\eta}(\eta, t) + Y(\eta, t) (1 - Y^2(\eta, t)) + g(\eta, t), \quad 0 < \alpha < 1, \quad (5.8)$$

subject to

$$\begin{aligned} Y(\eta, 0) &= 0, \quad 0 < \eta < 1, \\ Y(0, t) &= Y(1, t) = 0, \quad 0 < t < 1, \end{aligned} \quad (5.9)$$

and $g(\eta, t)$ is selected in accordance with the exact solution given by $Y(\eta, t) = t^2 \sin(2\pi\eta)$.

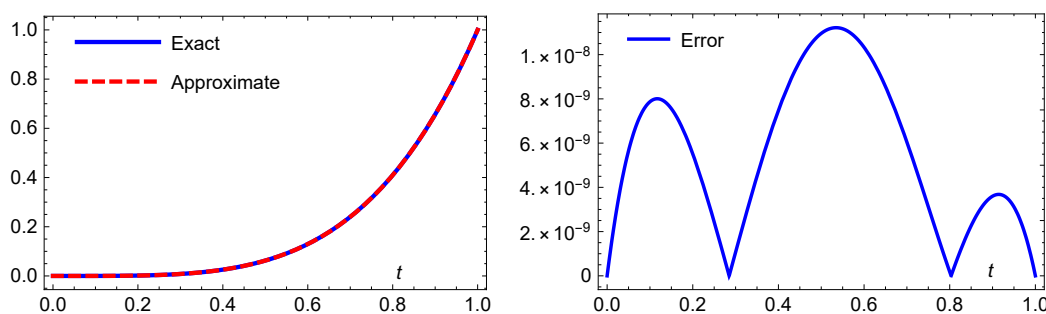


Figure 1. The exact and approximate solutions and AEs of Example 1 at $\zeta = 0.9$ when $\gamma = 1.8$, and $\mathcal{L} = 4$.

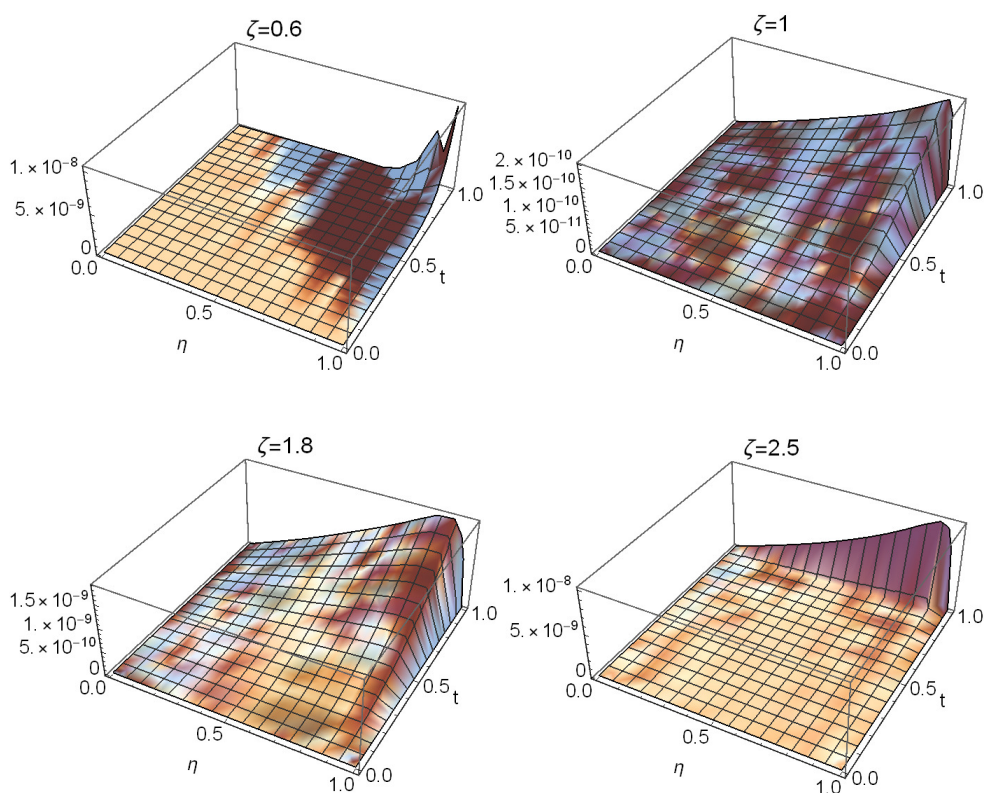


Figure 2. The AEs of Example 2 at different values of ζ for $\alpha = 0.2$ and $\mathcal{L} = 18$. The results show that the AEs remain of order 10^{-10} – 10^{-8} at different values of ζ .

Table 2 reports a comparison of the L_∞ errors and CPU time used with those obtained by the method in [43] using $M = 2001$ and our proposed method at $\mathcal{L} = 18$ when $\alpha = 0.4$, and $\alpha = 0.6$. Table 3 presents the errors at different $\mathcal{L}$ when $\alpha = 0.9$, and $\zeta = 1$. Table 4 presents the L_∞ errors at different values of ζ when $\mathcal{L} = 18$, and $\alpha = 0.5$. Table 5 shows the AEs at $\alpha = 0.5$, $\zeta = 1$, and $\mathcal{L} = 18$. Figure 2 shows the AEs at different ζ when $\alpha = 0.2$, and $\mathcal{L} = 18$. Figure 3 shows the AEs and approximate solution at $\alpha = 0.5$, $\zeta = 1$, and $\mathcal{L} = 18$.

Table 2. Comparison of the L_∞ errors and CPU time used of Example 2.

α	Method in [43] at $M = 2001$			Presented method at $\mathcal{L} = 18$		
	N	Error	CPU time	ζ	Error	CPU time
0.4	1600	1.352997×10^{-5}	39.76	0.5	1.38841×10^{-9}	145.906
	3200	6.728497×10^{-6}	154.14	1	1.95685×10^{-10}	159.163
	6400	3.099865×10^{-6}	638.62	1.5	1.55769×10^{-9}	144.996
0.6	1600	1.352997×10^{-5}	40.19	0.5	1.3816×10^{-9}	145.983
	3200	6.728497×10^{-6}	156.86	1	1.96386×10^{-10}	159.544
	6400	3.099865×10^{-6}	657.40	1.5	1.84562×10^{-9}	145.768

Table 3. The errors of Example 2 at different $\mathcal{L}$, $\alpha = 0.9$, and $\zeta = 1$.

$\mathcal{L}$	6	10	14	18
L_∞ -error	5.44258×10^{-1}	2.30595×10^{-3}	4.61017×10^{-6}	1.96248×10^{-10}

Table 4. The L_∞ errors of Example 2 at $\mathcal{L} = 18$ and $\alpha = 0.5$.

ζ	0.1	0.5	1	1.6	3	6
Error	6.9500×10^{-10}	1.3851×10^{-9}	1.9652×10^{-10}	1.3535×10^{-9}	6.2455×10^{-10}	2.8772×10^{-9}

Table 5. The AEs of Example 2 at $\alpha = 0.5$, $\zeta = 1$, and $\mathcal{L} = 18$.

η	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
0.1	2.373×10^{-13}	5.14023×10^{-13}	8.75394×10^{-13}	1.37643×10^{-12}
0.2	1.1578×10^{-12}	2.4353×10^{-12}	3.96247×10^{-12}	5.88268×10^{-12}
0.3	2.85288×10^{-12}	5.92294×10^{-12}	9.44108×10^{-12}	1.36497×10^{-11}
0.4	5.34597×10^{-12}	1.10202×10^{-11}	1.73586×10^{-11}	2.47104×10^{-11}
0.5	8.61491×10^{-12}	1.76928×10^{-11}	2.76675×10^{-11}	3.90142×10^{-11}
0.6	1.25818×10^{-11}	2.58147×10^{-11}	4.02095×10^{-11}	5.64177×10^{-11}
0.7	1.7096×10^{-11}	3.51382×10^{-11}	5.46787×10^{-11}	7.66539×10^{-11}
0.8	2.19023×10^{-11}	4.52406×10^{-11}	7.05485×10^{-11}	9.92521×10^{-11}
0.9	2.66542×10^{-11}	5.55404×10^{-11}	8.70928×10^{-11}	1.23572×10^{-10}

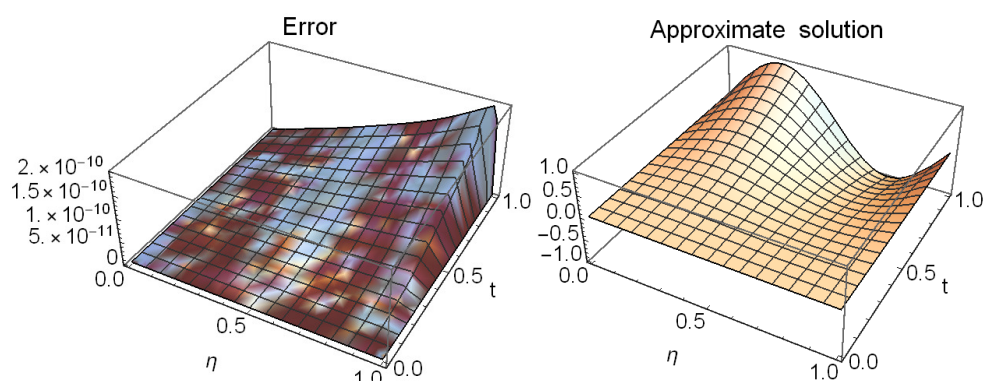


Figure 3. The AEs and approximate solution of Example 2 at $\alpha = 0.5$, $\zeta = 1$, and $\mathcal{L} = 18$.

Example 3. [47] Consider the following equation:

$$D_t^\alpha Y(\eta, t) = Y_{\eta\eta}(\eta, t) + Y(\eta, t)(1 - Y^3(\eta, t)) + g(\eta, t), \quad 0 < \alpha < 1, \quad (5.10)$$

subject to

$$\begin{aligned} Y(\eta, 0) &= e^{2\eta} \eta^2, \quad 0 < \eta < 1, \\ Y(0, t) &= 0, \quad Y(1, t) = e^2(t^2 + 1), \quad 0 < t < 1, \end{aligned} \quad (5.11)$$

where $g(\eta, t)$ is selected in accordance with the exact solution given by $Y(\eta, t) = (t^2 + 1) \eta^2 e^{2\eta}$.

Table 6 presents a comparison of the L_∞ errors and CPU time used between the method in [47] and our proposed method when $\alpha = 0.5$. Also, Table 7 presents a comparison of the L_∞ errors between the method in [47] and our proposed method when $\alpha = 0.95$. Figure 4 shows the AEs at different $\mathcal{L}$ when $\alpha = 0.7$, and $\zeta = 1$. Figure 5 shows the exact solution (contour), approximate solution (contour), and AEs distribution at different $\mathcal{L} = 14$ when $\alpha = 0.3$, and $\zeta = 0.5$.

Table 6. Comparison of the L_∞ errors and CPU time used of Example 3 at $\alpha = 0.5$.

Method in [47] at $N = 10000$			Presented method at $\mathcal{L} = 14$		
M	Error	CPU time	ζ	Error	CPU time
6	1.4174×10^{-4}	3.798	0.4	2.18592×10^{-12}	42.86
12	9.4186×10^{-6}	5.669	1	3.06244×10^{-12}	54.968
24	6.2852×10^{-7}	9.942	1.6	3.78497×10^{-12}	43.393

Table 7. Comparison of the L_∞ errors of Example 3 at $\alpha = 0.95$.

t	Method in [47]	Proposed method at $\mathcal{L} = 14$	
		$\zeta = 2$	$\zeta = 3$
0.5	5.7352×10^{-4}	3.99769×10^{-12}	5.28866×10^{-12}
0.75	1.1253×10^{-6}	4.00657×10^{-12}	5.16343×10^{-12}
1	1.6727×10^{-6}	5.52767×10^{-12}	1.06736×10^{-10}

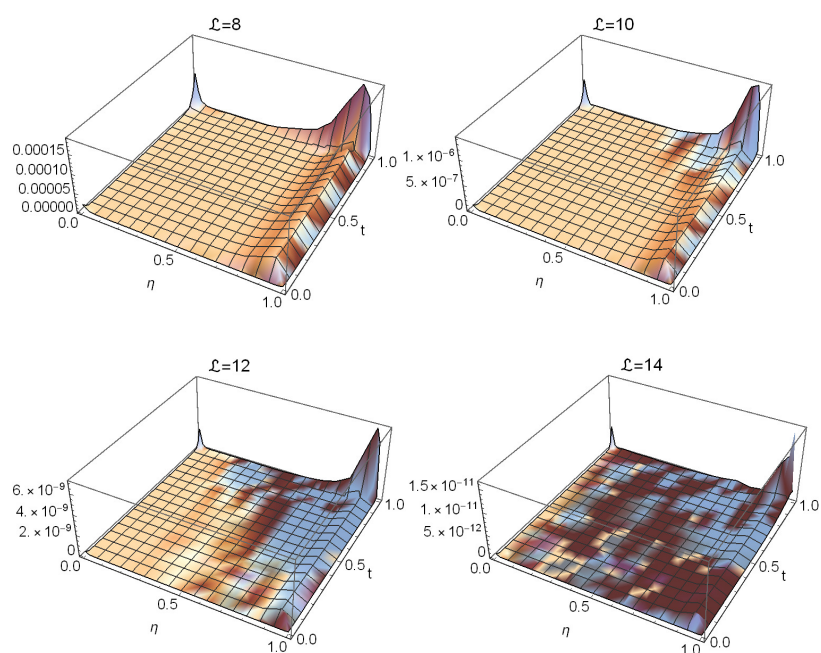


Figure 4. The AEs of Example 3 at different values of $\mathcal{L}$ when $\alpha = 0.7$, and $\zeta = 1$. The results show that the AEs remain of order 10^{-12} – 10^{-4} at different values of $\mathcal{L}$.

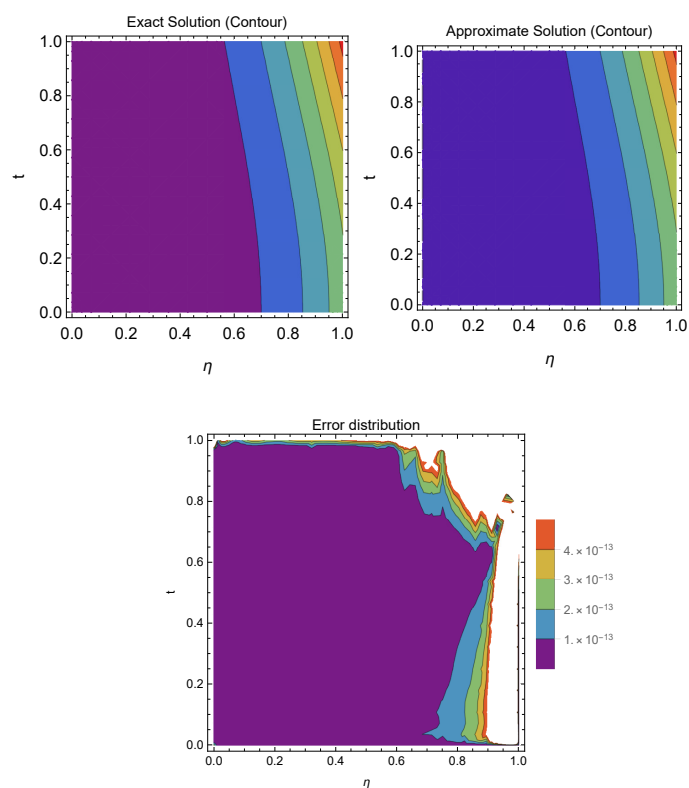


Figure 5. The exact solution (contour), approximate solution (contour), and AE distribution of Example 3 at $\mathcal{L} = 14$, $\alpha = 0.3$, and $\zeta = 0.5$.

Remark 7. *The numerical results indicate that the parameter ζ influences the accuracy of the proposed method. However, no monotonic dependence or universally optimal value of ζ is observed, as its effect is problem-dependent. Nevertheless, high accuracy is maintained over a wide range of ζ .*

Remark 8. *The superior accuracy of the proposed technique over existing methods in [43] and [47] :*

- 1) *Global SONJPs achieve exponential convergence for smooth functions, whereas the method in [43] is restricted to algebraic polynomial rates requiring large grids ($M = 2001$), and the method in [47] depends on localized piecewise polynomial order.*
- 2) *The continuous orthonormality of the SONJP basis ensures well-conditioned operational matrices, attaining higher precision with far fewer degrees of freedom ($\mathcal{L} \leq 14$).*

6. Concluding remarks

This study developed SONJP-based tau and collocation schemes for the numerical treatment of the NBVP and the TFGFP. Integer- and fractional-derivative representations of the SONJPs were incorporated into the required operational matrices, allowing both the equations and their accompanying conditions to be discretized systematically. The resulting nonlinear algebraic systems were then solved numerically. The computational tests demonstrate that the resulting spectral approximations attain high accuracy. In future work, the present techniques may be extended to other fractional orders, systems of FDEs, or higher spatial dimensions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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