



Research article

Continuous filtering and kernel transformations in series representations of $\zeta(5)$

Sunyoung Bu^{1,2} and Jongkyum Kwon^{3,*}

¹ Department of Mathematics, College of Natural Science, Kwangwoon University, Seoul 01897, South Korea

² Research Institute of Mathematical Sciences, Kwangwoon University, Seoul 01897, South Korea

³ Department of Mathematics Education, Gyeongsang National University, Jinju 52828, South Korea

* **Correspondence:** Email: mathkjk26@gnu.ac.kr.

Abstract: The evaluation of the Riemann zeta function at odd integers remains an important analytical problem, partly because many of its classical series representations converge slowly. In this paper, we develop a continuous filtering framework for analyzing weighted series representations associated with $\zeta(5)$. Rather than relying on closed-form algebraic transformations, the proposed approach introduces analytically designed weighting sequences (filters) within a continuous integral formulation. We show that this construction naturally interprets discrete filtering as a continuous transformation of the underlying integral kernel. Furthermore, the proposed framework reveals a fundamental trade-off between convergence enhancement and preservation of the kernel structure. By establishing a continuous–discrete connection, the framework provides a unified perspective for studying filtered series representations and offers new insight into the analytical structure of zeta-related series.

Keywords: Riemann zeta function; odd zeta values; continuous filter; integral kernel transformation; asymptotic analysis

1. Introduction

The Riemann zeta function, denoted by $\zeta(s)$, occupies a central position in analytic number theory and complex analysis. While its properties have been extensively studied [1–3], the precise nature of its values at positive integers remains one of the most intriguing subjects in modern mathematics. In particular, Euler established that the values of $\zeta(s)$ at even positive integers admit explicit closed-form expressions involving Bernoulli numbers and powers of π . In contrast, the values at odd integers, such as $\zeta(3)$, $\zeta(5)$, and higher, remain far less understood and lack comparable closed-form representations.

Finding special values of the Riemann zeta function, particularly $\zeta(5)$, has been an active topic in both numerical and theoretical research. Following Apéry's celebrated proof of the irrationality of $\zeta(3)$ [4], substantial efforts have been devoted to extending Apéry-type constructions to higher odd zeta values. In particular, Zudilin [5] developed a third-order Apéry-like recursion for $\zeta(5)$, providing a structured framework for generating rapidly convergent rational approximations. Subsequent works have further explored such constructions using symbolic and algorithmic approaches [6–8].

In parallel, various studies [9, 10] have focused on deriving fast convergent series representations for odd zeta values through hypergeometric functions, binomial sums, and harmonic number expansions. Despite these advances, closed-form expressions for $\zeta(5)$ remain unknown, and its irrationality is still an open problem, although partial results are available [11]. More recently, attention has shifted toward structural interpretations based on integral representations and kernel-based formulations, which emphasize the connection between discrete series and continuous operators [12, 13]. In particular, such perspectives naturally relate to generalized integral operators and fractional calculus, where nonlocal operators and memory effects provide a broader analytical framework for interpreting series transformations.

A fundamental difficulty in the numerical evaluation of $\zeta(5)$ arises from the slow algebraic convergence of its defining series

$$\zeta(5) = \sum_{n=1}^{\infty} \frac{1}{n^5}, \quad (1.1)$$

whose truncation error satisfies

$$\sum_{n=N+1}^{\infty} \frac{1}{n^5} = O(N^{-4}). \quad (1.2)$$

This polynomial decay renders direct summation inefficient for high-precision computation. Even when restricting to the odd subsequence

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^5}, \quad (1.3)$$

the convergence rate remains algebraic, suggesting that meaningful acceleration may be achieved by modifying the asymptotic structure of the summand.

To alleviate the computational inefficiencies caused by slowly converging series, a wide range of sequence transformations and extrapolation techniques have been developed in numerical analysis. Methods such as Levin-type transformations and nonlinear sequence accelerators are well known for their remarkable efficiency in purely numerical contexts. However, these approaches primarily focus on improving computational performance and often provide limited analytical insight into the intrinsic structure of the underlying number-theoretic objects.

In the context of number theory, convergence acceleration has traditionally relied on the discovery of highly specialized and structurally rigid hypergeometric identities. While such identities can yield exceptionally fast convergence, they are typically problem-specific, difficult to generalize, and fundamentally based on discrete algebraic manipulations. As a result, they often lack a unified geometric or continuous analytical interpretation.

A related classical perspective is provided by the Mellin–Poisson–modular framework, where discrete series are connected to continuous kernel representations through Mellin transforms and

subsequently transformed using Poisson, modular, or theta symmetries [14]. This framework establishes a broad connection between discrete series, continuous kernels, and rapidly convergent expansions. The approach considered here is complementary to this viewpoint, focusing instead on how prescribed weighting sequences induce transformations of the associated kernels.

In this work, we develop a continuous filtering framework that reinterprets discrete sequence acceleration as transformations of an underlying integral kernel. Within this framework, different filtering strategies arise as specific kernel transformations, providing a unified perspective on convergence acceleration. This viewpoint further explains the emergence of the proportionality constant through the transformed kernel structure and establishes a direct connection between discrete weighted series and continuous integral representations.

It is important to clarify the precise analytical scope of the proposed framework and the role of $\zeta(5)$ within it. The primary objective of this work is not to present a standalone, independent computational algorithm for discovering unknown numerical digits of $\zeta(5)$ or to claim that arbitrary weighted series preserve direct algebraic identities for $\zeta(5)$. Instead, the odd-term series representation of $\zeta(5)$ serves as a well-defined baseline reference for establishing a continuous filtering methodology.

Our main contribution lies in providing a structural and analytical framework that reinterprets discrete sequence acceleration as continuous transformations of an underlying integral kernel. Within this formulation, the normalization constant C_w serves as a theoretical scaling factor that measures the degree of kernel deformation and places different weighted series on a mathematically unified reference scale.

The remainder of this paper is organized as follows. In Section 2, we introduce the filtered series formulation and establish integral kernel representations. Also, by the truncation error analysis, the convergence properties of the proposed filtered series are proven. Section 3 presents the construction of polynomial, exponential, and integral-based filters. Numerical experiments demonstrating the effectiveness of the proposed framework in Section 4. Finally, Section 5 summarizes the proposed filtered series and provides future research directions.

2. Filtered series construction

In this section, we formalize the filtering framework and establish its analytical foundation. A main goal is to construct a modified series representation of $\zeta(5)$ whose convergence properties can be systematically controlled through the introduction of a weighting sequence.

2.1. Filtered series formulation

We begin with the classical series representation of the Riemann zeta function (1.1) which corresponds to the Dirichlet lambda function $\lambda(5)$.

Separating odd and even terms yields

$$\zeta(5) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^5} + \sum_{k=1}^{\infty} \frac{1}{(2k)^5} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^5} + \frac{1}{2^5} \zeta(5),$$

which implies

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^5} = \left(1 - \frac{1}{32}\right) \zeta(5) = \frac{31}{32} \zeta(5).$$

This representation serves as the natural baseline for the filtered construction. To systematically accelerate the convergence of the odd-term series representation of $\zeta(5)$, we formalize the concept of a filtered series. Rather than evaluating the series directly, we introduce a weighting sequence that modifies the asymptotic behavior of the summands.

Motivated by accelerating convergence of the sequence, we introduce a weighting sequence, so called filter, $\{w_k\}_{k \geq 0}$ and define the filtered series as follows:

$$S_w = \sum_{k=0}^{\infty} \frac{w_k}{(2k+1)^5}. \quad (2.1)$$

In general, S_w does not coincide with $\zeta(5)$; instead, we obtain a proportional relationship

$$\zeta(5) = C_w S_w, \quad \text{where} \quad C_w = \frac{\zeta(5)}{S_w}. \quad (2.2)$$

The objective is to design filters $\{w_k\}$ such that the series S_w converges rapidly while maintaining a stable proportionality constant C_w .

Remark 2.1. The normalization constant $C_w = \zeta(5)/S_w$ is introduced to place different filtered series on a common limiting scale. Using the corresponding kernel representations, it can equivalently be written as

$$C_w = \frac{\int_0^{\infty} t^4 \frac{1}{2 \sinh(t)} dt}{\int_0^{\infty} t^4 G_w(t) dt}. \quad (2.3)$$

This representation does not, by itself, provide an independent determination of C_w , since the numerator is itself an integral representation associated with $\zeta(5)$. Thus, in the present framework, C_w should be understood primarily as a normalization factor for structural and convergence comparisons.

Remark 2.2. The odd-term series representation can be naturally interpreted in terms of the Dirichlet lambda function,

$$\lambda(s) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^s} = (1 - 2^{-s})\zeta(s).$$

Thus, the baseline representation

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^5} = \frac{31}{32} \zeta(5)$$

can be viewed as a special case of $\lambda(5)$. The filtered series S_w may therefore be interpreted as a weighted deformation of the Dirichlet lambda function.

To provide a deeper analytical interpretation, we will show that the filtered series admits a natural representation in terms of integral kernel transformations. This perspective connects the discrete weighting process with the corresponding transformation of the underlying continuous kernel.

Definition 2.3 (Admissible Filter). A sequence of real numbers $W = \{w_k\}_{k=0}^{\infty}$ is called an admissible filter if it satisfies the following conditions:

- (a) $w_k > 0$ for all $k \geq 0$.
 (b) $\lim_{k \rightarrow \infty} w_k = 0$.

Here, filtering refers to the weighted transformation of the original summands $(2k+1)^{-5}$ into $w_k(2k+1)^{-5}$.

For any admissible filter, the corresponding filtered series S_w is related to the reference value $\zeta(5)$ through $\zeta(5) = C_w S_w$ in (2.2), where C_w serves as a normalization factor.

2.2. Integral kernel transformation

The classical series for $\zeta(5)$ can be expressed via the Mellin transform of the exponential kernel. For the odd terms, we have

$$\frac{1}{(2k+1)^5} = \frac{1}{\Gamma(5)} \int_0^\infty t^4 e^{-(2k+1)t} dt = \frac{1}{24} \int_0^\infty t^4 e^{-(2k+1)t} dt. \quad (2.4)$$

Proposition 2.4. Let $W = \{w_k\}_{k=0}^\infty$ be an admissible filter. The filtered kernel $G_w(t)$ defined by

$$G_w(t) = \sum_{k=0}^\infty w_k e^{-(2k+1)t} \quad (2.5)$$

converges uniformly on any compact subinterval $[\delta, K] \subset (0, \infty)$ for $\delta > 0$. Furthermore, the filtered series S_w satisfies the integral representation

$$S_w = \frac{1}{24} \int_0^\infty t^4 G_w(t) dt. \quad (2.6)$$

Proof. Since W is an admissible filter, the sequence $\{w_k\}$ is bounded by $M = \sup_{k \geq 0} w_k < \infty$. For any $t \geq \delta > 0$, we have $w_k e^{-(2k+1)t} \leq M e^{-(2k+1)\delta}$. By the Weierstrass M -test, the series $\sum_{k=0}^\infty w_k e^{-(2k+1)t}$ converges uniformly on $[\delta, \infty)$ and hence on any compact subinterval $[\delta, K] \subset (0, \infty)$.

To establish (2.6), substitute the Gamma integral identity $\frac{1}{(2k+1)^5} = \frac{1}{24} \int_0^\infty t^4 e^{-(2k+1)t} dt$ into S_w :

$$S_w = \sum_{k=0}^\infty w_k \left(\frac{1}{24} \int_0^\infty t^4 e^{-(2k+1)t} dt \right).$$

Since all terms $f_k(t) = w_k t^4 e^{-(2k+1)t}$ are non-negative and measurable on $(0, \infty)$, Tonelli's theorem allows the interchange of summation and integration

$$\sum_{k=0}^\infty \int_0^\infty w_k t^4 e^{-(2k+1)t} dt = \int_0^\infty t^4 \left(\sum_{k=0}^\infty w_k e^{-(2k+1)t} \right) dt = \int_0^\infty t^4 G_w(t) dt.$$

Moreover, since $w_k \leq M$,

$$\sum_{k=0}^\infty \frac{w_k}{(2k+1)^5} \leq M \sum_{k=0}^\infty \frac{1}{(2k+1)^5} < \infty.$$

Thus, the resulting integral is finite, and (2.6) follows. $\square$

Remark 2.5. The unfiltered series based on odd integers admits the integral representation

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^5} = \frac{1}{24} \int_0^{\infty} t^4 \frac{1}{2 \sinh(t)} dt,$$

where the kernel $\frac{1}{2 \sinh(t)}$ arises from $\sum_{k=0}^{\infty} e^{-(2k+1)t}$.

Thus, both the original and filtered series are represented through the same odd-kernel framework. Consequently, the proportionality constant C_w can be expressed as

$$C_w = \frac{\int_0^{\infty} t^4 \frac{1}{2 \sinh(t)} dt}{\int_0^{\infty} t^4 G_w(t) dt}, \quad (2.7)$$

which reflects the ratio between the original and filtered kernel representations.

It is also worth noting that the Riemann zeta function admits the classical integral representation

$$\zeta(s) = \frac{1}{2(1-2^{-s})\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{\sinh t} dt, \quad \Re(s) > 1, \quad (2.8)$$

as given in [15, Eq (25.5.8)]. In particular, for $(s=5)$,

$$\zeta(5) = \frac{2}{93} \int_0^{\infty} \frac{t^4}{\sinh t} dt. \quad (2.9)$$

This classical representation provides an alternative integral representation of $\zeta(5)$, complementing the formulation considered in this study.

2.3. Truncation error analysis

In numerical practice, we evaluate the finite truncated series $S_w(N) = \sum_{k=0}^N \frac{w_k}{(2k+1)^5}$. The convergence rate is governed by the remainder term.

Theorem 2.6. The absolute error $\mathcal{E}_N = |\zeta(5) - C_w S_w(N)|$ introduced by truncating the filtered series at index N is directly proportional to the integral of the residual kernel

$$\mathcal{E}_N = C_w \left(\frac{1}{24} \int_0^{\infty} t^4 R_{w,N}(t) dt \right), \quad (2.10)$$

where the residual kernel is $R_{w,N}(t) = \sum_{k=N+1}^{\infty} w_k e^{-(2k+1)t}$.

Proof. By (2.2), we have the exact relation $\zeta(5) = C_w S_w$. Substituting the absolute truncation error yields

$$\mathcal{E}_N = |\zeta(5) - C_w S_w(N)| = |C_w S_w - C_w S_w(N)| = C_w |S_w - S_w(N)|.$$

Then, we consider

$$S_w - S_w(N) = \sum_{k=0}^{\infty} \frac{w_k}{(2k+1)^5} - \sum_{k=0}^N \frac{w_k}{(2k+1)^5} = \sum_{k=N+1}^{\infty} \frac{w_k}{(2k+1)^5}.$$

Applying the integral representation (2.4) for the terms in the tail, we obtain

$$\begin{aligned} S_w - S_w(N) &= \sum_{k=N+1}^{\infty} w_k \left(\frac{1}{24} \int_0^{\infty} t^4 e^{-(2k+1)t} dt \right) \\ &= \frac{1}{24} \int_0^{\infty} t^4 \left(\sum_{k=N+1}^{\infty} w_k e^{-(2k+1)t} \right) dt \\ &= \frac{1}{24} \int_0^{\infty} t^4 R_{w,N}(t) dt, \end{aligned}$$

where $R_{w,N}(t)$ is the residual kernel. This completes the proof. $\square$

This theorem highlights that accelerating convergence heavily depends on choosing a filter sequence w_k that forces $R_{w,N}(t)$ to decay rapidly, while maintaining a moderately sized C_w to avoid amplifying floating-point inaccuracies.

Theorem 2.7. *Let $w(x)$ be a strictly positive, monotonically decreasing, and integrable function on $[0, \infty)$ such that $w(k) = w_k$ for all integers $k \geq 0$. Then, the truncation error $\mathcal{E}_N$ is strictly bounded by the improper integral*

$$\mathcal{E}_N \leq C_w \int_N^{\infty} \frac{w(x)}{(2x+1)^5} dx. \quad (2.11)$$

Proof. From the Theorem 2.5, we have

$$\mathcal{E}_N = \frac{C_w}{24} \int_0^{\infty} t^4 R_{w,N}(t) dt, \quad \text{where} \quad R_{w,N}(t) = \sum_{k=N+1}^{\infty} w_k e^{-(2k+1)t}. \quad (2.12)$$

Since $w(x)$ is monotonically decreasing and $e^{-(2x+1)t}$ is monotonically decreasing with x for $t > 0$, $w(x)e^{-(2x+1)t}$ is also strictly decreasing. Then, we obtain

$$R_{w,N}(t) \leq \int_N^{\infty} w(x) e^{-(2x+1)t} dx.$$

From (2.12), we have

$$\begin{aligned} \mathcal{E}_N &= \frac{C_w}{24} \int_0^{\infty} t^4 R_{w,N}(t) dt \leq \frac{C_w}{24} \int_0^{\infty} t^4 \left(\int_N^{\infty} w(x) e^{-(2x+1)t} dx \right) dt \\ &= \frac{C_w}{24} \int_N^{\infty} w(x) \left(\int_0^{\infty} t^4 e^{-(2x+1)t} dt \right) dx. \end{aligned}$$

Now, we consider the inner integral

$$\int_0^{\infty} t^4 e^{-(2x+1)t} dt = \frac{\Gamma(5)}{(2x+1)^5} = \frac{24}{(2x+1)^5}.$$

This completes the proof. $\square$

Theorem 2.8. Let $\{w_k\}_{k \geq 0}$ be an admissible filter. Suppose there exist constants $A > 0$ and $p > 0$ such that the filter satisfies the asymptotic decay condition $w_k \leq A(k+1)^{-p}$ for all $k \geq N$. Then, the truncation error $\mathcal{E}_N = |\zeta(5) - C_w S_w(N)|$ satisfies the following bound:

$$\mathcal{E}_N \leq \frac{A \cdot C_w}{2^5(p+4)} N^{-(p+4)} + O(N^{-(p+5)}). \quad (2.13)$$

In addition, if the filter satisfies

$$w_k \leq A e^{-\lambda k}$$

for some constants $A > 0$ and $\lambda > 0$, then

$$\mathcal{E}_N = O(N^{-5} e^{-\lambda N}).$$

Proof. From the Theorem 2.5, the truncation error is given by the tail of the filtered series

$$\mathcal{E}_N = C_w \sum_{k=N+1}^{\infty} \frac{w_k}{(2k+1)^5}. \quad (2.14)$$

Applying the given decay condition $w_k \leq A(k+1)^{-p}$, we have

$$\mathcal{E}_N \leq A \cdot C_w \sum_{k=N+1}^{\infty} \frac{1}{(k+1)^p (2k+1)^5}. \quad (2.15)$$

Using the integral test to estimate the tail of the series, we obtain

$$\sum_{k=N+1}^{\infty} \frac{1}{(k+1)^p (2k+1)^5} \leq \int_N^{\infty} \frac{1}{x^p (2x)^5} dx = \frac{1}{2^5} \int_N^{\infty} x^{-(p+5)} dx. \quad (2.16)$$

Evaluating the integral with $p > 0$ yields

$$\frac{1}{2^5} \left[\frac{x^{-(p+4)}}{-(p+4)} \right]_N^{\infty} = \frac{1}{2^5(p+4)} N^{-(p+4)}. \quad (2.17)$$

Multiplying by the constants A and C_w completes the proof for the algebraic case.

For the exponential case, suppose the filter satisfies

$$w_k \leq A e^{-\lambda k}, \quad \lambda > 0.$$

Then, by (2.14),

$$\mathcal{E}_N \leq A C_w \sum_{k=N+1}^{\infty} \frac{e^{-\lambda k}}{(2k+1)^5}. \quad (2.18)$$

Applying the integral test yields

$$\sum_{k=N+1}^{\infty} \frac{e^{-\lambda k}}{(2k+1)^5} \leq \frac{1}{2^5} \int_N^{\infty} x^{-5} e^{-\lambda x} dx. \quad (2.19)$$

Since $x \geq N$ implies $x^{-5} \leq N^{-5}$, we obtain

$$\int_N^\infty x^{-5} e^{-\lambda x} dx \leq N^{-5} \int_N^\infty e^{-\lambda x} dx = \frac{1}{\lambda} N^{-5} e^{-\lambda N}. \quad (2.20)$$

Therefore,

$$\mathcal{E}_N \leq \frac{AC_w}{2^5 \lambda} N^{-5} e^{-\lambda N}, \quad (2.21)$$

which proves that

$$\mathcal{E}_N = O(N^{-5} e^{-\lambda N}).$$

□

For completeness, we record the following Euler–Maclaurin expansion, which will be used as an auxiliary tool in the subsequent analysis.

Theorem 2.9. *Let $w(x)$ be a smooth filter function on $[0, \infty)$ such that $f(x) = \frac{w(x)}{(2x+1)^5}$ satisfies $f^{(r)} \in L^1([N, \infty))$ and $\lim_{x \rightarrow \infty} f^{(r)}(x) = 0$ for all integers $r \geq 0$. Then, for any order $m \geq 1$, the truncation error $\mathcal{E}_N = |\zeta(5) - C_w S_w(N)|$ admits the asymptotic expansion*

$$\mathcal{E}_N = C_w \left| \int_N^\infty \frac{w(x)}{(2x+1)^5} dx - \frac{1}{2} \frac{w(N)}{(2N+1)^5} - \sum_{j=1}^m \frac{B_{2j}}{(2j)!} \frac{d^{2j-1}}{dx^{2j-1}} \left(\frac{w(x)}{(2x+1)^5} \right) \Big|_{x=N} + R_m(N) \right|, \quad (2.22)$$

where B_{2j} are the Bernoulli numbers and the remainder term $R_m(N)$ satisfies the bound

$$|R_m(N)| \leq \frac{|B_{2m}|}{(2m)!} \int_N^\infty |f^{(2m)}(x)| dx = O(f^{(2m-1)}(N)) \quad \text{as } N \rightarrow \infty. \quad (2.23)$$

Proof. Applying the Euler–Maclaurin summation formula of order m to $f(x) = \frac{w(x)}{(2x+1)^5}$ on $[N+1, M]$ yields:

$$\sum_{k=N+1}^M f(k) = \int_N^M f(x) dx + \frac{f(M) - f(N)}{2} + \sum_{j=1}^m \frac{B_{2j}}{(2j)!} (f^{(2j-1)}(M) - f^{(2j-1)}(N)) + R_m(N, M),$$

where $R_m(N, M) = \frac{(-1)^{m+1}}{(2m)!} \int_N^M P_{2m}(x) f^{(2m)}(x) dx$, with $P_{2m}(x)$ denoting the periodic Bernoulli polynomial satisfying $|P_{2m}(x)| \leq |B_{2m}|$. Since $f^{(r)}(x) \rightarrow 0$ as $x \rightarrow \infty$ for all $r \geq 0$, taking $M \rightarrow \infty$ eliminates the upper limit boundary terms. The integrability $f^{(2m)} \in L^1([N, \infty))$ guarantees that $\lim_{M \rightarrow \infty} R_m(N, M) = R_m(N)$ exists and obeys the integral bound. Multiplying by C_w completes the proof. □

3. Specific filter classes and their properties

In this section, based on the filtered series framework established previously, we rigorously analyze the mathematical forms and convergence characteristics of three specific filters : polynomial, exponential, and integral-based filters.

3.1. Polynomial filter

The polynomial filter employs a weighting sequence that decays algebraically, effectively dampening the influence of high-order oscillatory terms while preserving the low-order principal components of the series. Its mathematical form is defined as

$$w_k = \frac{1}{(k+1)^p}, \quad \text{for } p > 0. \quad (3.1)$$

For the numerical test, we apply $p = 2$, yielding an inverse-square decay, $w_k = \frac{1}{(k+1)^2}$. This specific choice of the parameter p is found to prevent the over-smoothing often observed with more aggressive filters such as exponential ones, thereby providing an optimal balance between convergence acceleration and the preservation of the sequence's original asymptotic behavior. Despite its simple mathematical structure, the polynomial filter yields highly robust performance.

3.2. Exponential filter

The exponential filter utilizes a weighting sequence characterized by a rapid exponential decay, designed to aggressively suppress high-frequency oscillations and mitigate phenomena, such as the Gibbs effect. Its mathematical form is given by

$$w_k = e^{-\lambda k}, \quad \text{for } \lambda > 0. \quad (3.2)$$

In the numerical experiments, the decay parameter is set to $\lambda = 0.5$. Among the evaluated filter classes, this exponential weighting exerts the most stringent damping effect on higher-order terms. Consequently, it yields an exponentially accelerated convergence rate. However, this rapid error decay comes with an inherent trade-off. The aggressive damping intrinsically introduces a more pronounced distortion to the original integral kernel, potentially leading to the over-smoothing of localized features. This structural deviation is quantitatively captured by the proportionality constant C_w , which reflects the penalty incurred by heavily penalizing the higher-order decay rate components.

3.3. Integral-based filter

The integral-based filter employs a weighting sequence derived from the Beta function. Its mathematical form is defined as

$$w_k = \int_0^1 x^k (1-x)^k, dx = \frac{(k!)^2}{(2k+1)!}. \quad (3.3)$$

Unlike the polynomial and exponential filters, the integral-based filter is determined directly by this integral representation and does not require an additional tuning parameter. The factorial structure of the resulting coefficients naturally induces rapid asymptotic decay, which governs the convergence behavior of the corresponding filtered series. The precise decay rate is established in the following theorem.

Theorem 3.1. *If the filter is given as the integral based filter, then the truncation error $\mathcal{E}_N = |\zeta(5) - C_w S_w(N)|$ converges with a decay rate rate $O(4^{-N} N^{-11/2})$.*

Proof. The result follows from the asymptotic behavior of the Beta-function coefficients obtained using Stirling's formula. The detailed derivation is provided in Appendix. $\square$

3.4. Filter design principle

Synthesizing the analysis above, the optimal design of a convergence-accelerating filter can be understood as a delicate balance between two competing analytical criteria. The first criterion requires that the weighting sequence w_k exhibits sufficiently rapid decay as $k \rightarrow \infty$. Such asymptotic decay plays a crucial role in enhancing the convergence rate of the filtered series S_w , thereby effectively reducing the truncation error associated with a given finite limit N .

On the other hand, this aggressive decay must be carefully controlled to preserve kernel structure. In particular, the induced kernel $G_w(t)$ should maintain a strong structural correspondence with the original generating kernel, ensuring that the proportionality constant remains stable and that no artificial artifacts or excessive smoothing are introduced into the approximation.

In light of this fundamental trade-off, the evaluated filter classes exhibit distinct mechanisms for balancing these competing requirements. The exponential filter emphasizes rapid decay, resulting in strong convergence acceleration but increased kernel distortion. The polynomial filter offers an intermediate regime characterized by algebraic decay. In contrast, the integral-based filter incorporates intrinsic structural information through combinatorial identities, allowing for a more balanced interplay between convergence enhancement and preservation of the underlying analytical structure.

4. Numerical experiments

In this section, we present numerical experiments to evaluate the accuracy and convergence behavior of the proposed filtered representation for the Riemann zeta function at $s = 5$, denoted as $\zeta(5) \approx 1.0369277551$, which is a well-known and well calculated number with sufficient significant digits. We further analyze the role of the filter parameter w and compare the numerical results with the exact values.

4.1. Experimental setup and normalized formulation

To examine the convergence behavior induced by continuous filtering, we consider the truncated filtered series

$$S_w(N) = \sum_{k=0}^N \frac{w_k}{(2k+1)^5}, \quad (4.1)$$

where $\{w_k\}_{k \geq 0}$ denotes a prescribed sequence of filter weights. Its infinite counterpart is given by

$$S_w = \sum_{k=0}^{\infty} \frac{w_k}{(2k+1)^5}. \quad (4.2)$$

Since the application of a nontrivial filter generally changes the limiting value of the original series, we introduce the normalization constant

$$C_w := \frac{\zeta(5)}{S_w}. \quad (4.3)$$

The corresponding normalized filtered quantity is then defined by

$$\zeta_w(N) := C_w S_w(N) = \zeta(5) \frac{S_w(N)}{S_w}. \quad (4.4)$$

By construction, $\zeta_w(N) \rightarrow \zeta(5)$ as $N \rightarrow \infty$. The role of C_w is solely to place the filtered sequences on a common limiting scale, thereby enabling a direct comparison of the truncation behavior generated by different filters. It is not intended to provide an independent computational formula for evaluating $\zeta(5)$. Accordingly, the normalized quantity $\zeta_w(N)$ should be interpreted as a measure of how $S_w(N)$ approaches its limiting value S_w , rather than as a method for computing an unknown value of $\zeta(5)$.

For the numerical experiments, C_w is evaluated using a high-precision reference value of $\zeta(5)$ together with a sufficiently accurate approximation of S_w . Accordingly, the numerical results are intended to assess the normalized convergence behavior of the filtered sequences under this common reference scale.

The absolute truncation error is defined by

$$\mathcal{E}_w(N) = |\zeta(5) - \zeta_w(N)|. \quad (4.5)$$

Using the definition of C_w , this error may equivalently be written as

$$\mathcal{E}_w(N) = |C_w| |S_w - S_w(N)|, \quad (4.6)$$

which shows that the convergence rate of the normalized approximation is governed directly by the tail behavior of the filtered series.

We consider three representative classes of filters. The first is the polynomial filter,

$$w_k = \frac{1}{(k+1)^p}, \quad p > 0, \quad (4.7)$$

which introduces an additional algebraic decay into the summand. The second is the exponential filter,

$$w_k = e^{-\lambda k}, \quad \lambda > 0, \quad (4.8)$$

which produces exponentially decaying weights. The third is the integral-based filter,

$$w_k = \int_0^1 x^k (1-x)^k dx = \frac{(k!)^2}{(2k+1)!}, \quad (4.9)$$

obtained from the Euler Beta-function identity. These three choices provide representative examples of algebraic, exponential, and Beta-function-based decay within the continuous filtering framework.

4.2. Convergence behavior and asymptotic analysis

To validate the theoretical formulation presented above, we calculate the asymptotic value of the summation of $C_w S_w(N)$ with a sufficient large value N . For this experiment, we set the parameter p to be 2 for the polynomial filter, and the parameter λ to be 0.5. Based on the almost exact value of $\zeta(5)$ mentioned above, the absolute errors $\mathcal{E}_w(N)$ are plotted the results in Figure 1. The plot exhibits the absolute error $\mathcal{E}_w(N)$ on a semi-logarithmic scale, where the vertical axis represents the degree of precision achieved for a given truncation level N . Also for the comparison, we plot the result obtained from the unfiltered series and compare it with the results from the filtered series.

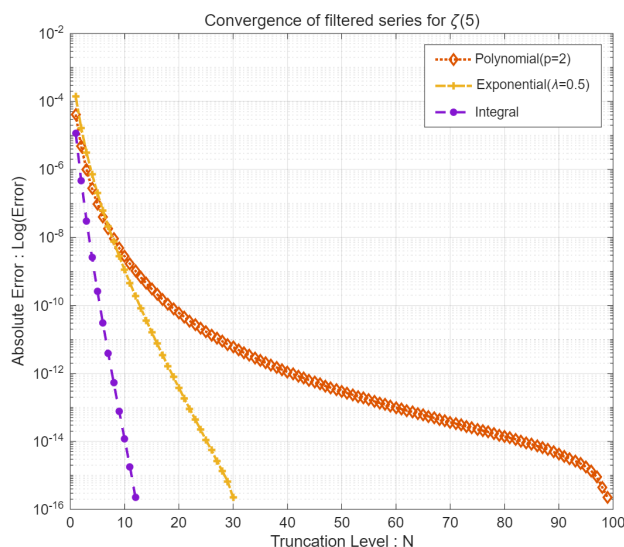


Figure 1. Comparison of the convergence behavior of the three filtered series in terms of the normalized error $\mathcal{E}_w(N)$.

The results show that the proposed filters reach the desired accuracy while the unfiltered series needs more truncation level to reach the desired accuracy. In other words, the results obtained from the proposed filtered converge to the expected value faster, compared with the result obtained from the unfiltered case. Also, we can easily check that the results from the proposed filtered series also generate different aspect in which the summation with the polynomial filter converges to the expected value of $\zeta(5)$ slowly and the summation with the integral filter generate the the ideal results.

Since the polynomial filter with the power $p = 2$ converges slower than other two filters, it is needed to investigate the effect of the power of the polynomial filter to accelerate the convergence rate of the polynomial filter. For this test, we calculate the absolute error by varying the power of the polynomial $p = 2, 4$, and 6 and plot the results in Figure 2.

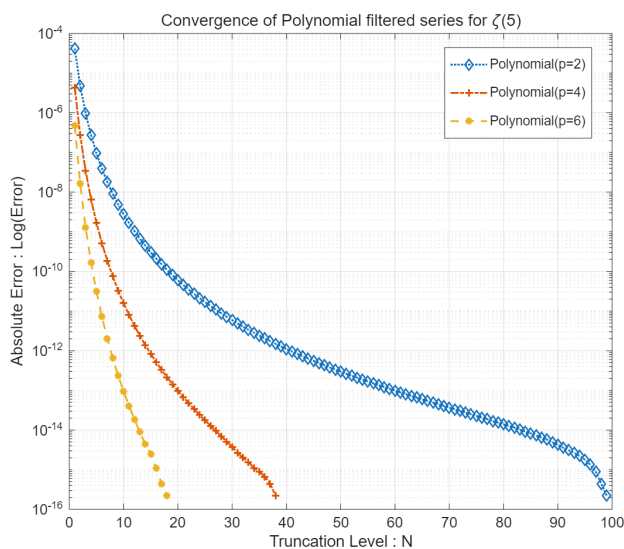


Figure 2. Absolute error $\mathcal{E}_w(N)$ of the polynomial filter for comparing the effect of the polynomial's power.

One can see that the lower the polynomial power is, the slower the convergence rate is. That is, the polynomial filter series can accelerate the convergence rate by increasing the power of the polynomial. Therefore, the result of the Figure 1 cannot explain the performance of the superiority of the series.

Similarly, in order to examine the convergence behavior of the exponential filter series, the parameter λ is systematically varied across the values $\lambda = 0.1, 0.5$, and 1 . The resulting absolute errors are computed for each configuration and plotted in Figure 3 to illustrate the impact of parameter variation on convergence performance.

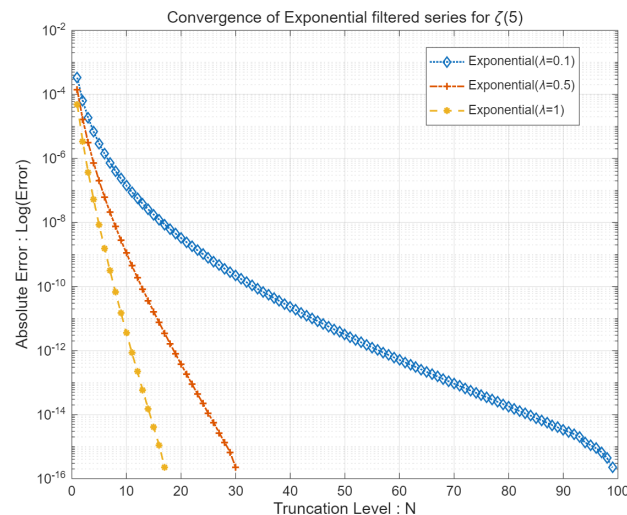


Figure 3. Absolute error $\mathcal{E}_w(N)$ for comparing the effect of the parameter λ of the exponential filter.

The Figure 3 illustrates that the convergence rate of the exponential filter can be systematically regulated through the parameters of the exponential function, thereby providing a flexible mechanism for tuning the convergence behavior to achieve enhanced numerical efficiency.

As seen in Figures 2 and 3, the convergence rate can be controlled by the parameters to represent the characteristics of the given filters. Therefore, attempts to determine the superiority of different filters through direct comparisons may not be particularly informative. Instead, for a meaningful comparison, we select parameter configurations such that the three filters produce similar convergence rates, and subsequently analyze the characteristic convergence behavior of each filter.

For the highly convergence comparison, the absolute value $\mathcal{E}_w(N)$ is evaluated for the three proposed filters under their respective suitable parameter choices—the polynomial filter with $p = 6$, the exponential filters with $\lambda = 1$, and the integral filter. The results are plotted in Figure 4.

As seen in the figure, while the overall convergence rate of the three filters are comparable, subtle differences emerge in their detailed convergence behavior. The polynomial filter initially appears to exhibit the fastest convergence, but it requires more iterations to achieve the desired accuracy. By contrast, the exponential and integral filter have almost linear convergence behavior in semi-log plot. However, integral filter still generates better convergence behavior than the exponential filter with the respective suitable parameter.

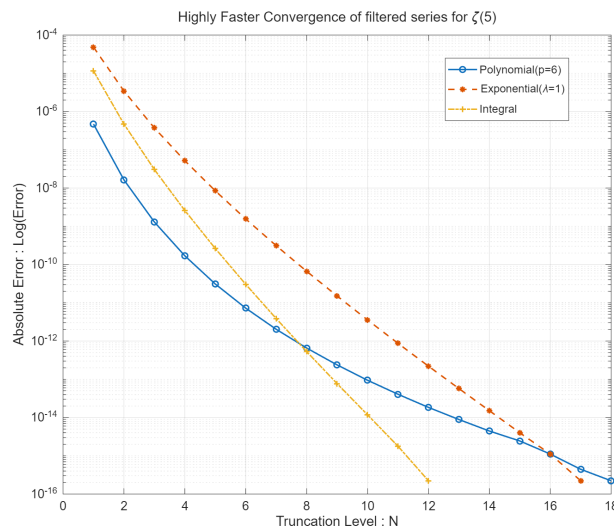


Figure 4. Absolute error $\mathcal{E}_w(N)$ of the three filters with respective suitable parameter choices.

4.3. Discussion

The comparative analysis based on Figures 1–4 is summarized in Table 1. The results show that all three filtering strategies improve the convergence behavior of the original series. Furthermore, the convergence characteristics of each filter can be systematically adjusted through appropriate choices of their associated parameters, providing additional flexibility within the proposed continuous filtering framework.

Table 1. Normalized error $\mathcal{E}_w(N)$ for different filtering strategies and parameter values.

N	Poly. ($p = 2$)	Poly. ($p = 6$)	Exp. ($\lambda = .5$)	Exp. ($\lambda = 1$)	Integral-based
3	9.7664×10^{-7}	1.2928×10^{-9}	3.0980×10^{-6}	3.7291×10^{-7}	3.0438×10^{-8}
10	2.7823×10^{-9}	9.4369×10^{-14}	1.1099×10^{-9}	3.5580×10^{-12}	1.1768×10^{-14}
20	6.0173×10^{-11}	0.0000×10^0	3.7792×10^{-13}	0.0000×10^0	0.0000×10^0
50	2.9687×10^{-13}	0.0000×10^0	0.0000×10^0	0.0000×10^0	0.0000×10^0

Table 1 summarizes the absolute truncation errors obtained with representative parameter choices for each filtering strategy. All three filters substantially reduce the truncation error relative to the unfiltered series, although their convergence characteristics differ considerably.

The exponential filter generally produces the most rapid decay of the truncation error for appropriately chosen values of λ . However, this improvement is accompanied by increased sensitivity to the filter parameter and a greater modification of the underlying kernel, as reflected in the behavior of the normalization constant C_w . In contrast, the polynomial filter exhibits more gradual error reduction while maintaining greater robustness with respect to parameter variation. The integral-based filter provides a balanced compromise, preserving the kernel structure while achieving competitive convergence without requiring explicit parameter tuning.

These observations indicate that the three filtering strategies possess complementary characteristics rather than establishing the universal superiority of any single filter. Consequently, the choice of filter

should depend on the desired balance between convergence enhancement, parameter sensitivity, and preservation of the kernel structure.

The integral-based filter provides a robust and well-balanced alternative. Unlike the polynomial and exponential filters, it does not require parameter tuning while exhibiting consistently stable convergence behavior and maintaining competitive accuracy. Moreover, it preserves the structure of the generating kernel, thereby maintaining the analytical integrity of the underlying formulation. These observations suggest that no single filter is universally superior. Rather, each filtering strategy possesses distinct analytical characteristics, and the choice of filter depends on the desired balance among convergence performance, and parameter sensitivity. Consequently, the proposed framework emphasizes that filter design should be viewed as an analytical trade-off rather than merely a numerical optimization problem.

5. Conclusions

In this paper, we developed a continuous filtering framework for constructing and analyzing weighted series representations associated with the Riemann zeta function, with a primary focus on $\zeta(5)$. Rather than seeking new closed-form identities, the proposed framework introduces analytically designed weighting sequences to generate modified series whose convergence properties can be systematically investigated. By interpreting discrete filtered series through continuous integral representations, we showed that the filtering process can be viewed as a continuous transformation of the underlying kernel. This viewpoint provides a natural explanation for the normalization constant C_w and establishes a unified connection between discrete filtering and continuous kernel-based analysis.

The numerical investigations demonstrate that polynomial, exponential, and integral-based filters exhibit distinct convergence characteristics reflecting their underlying analytical structures. Instead of identifying a universally optimal filter, the proposed framework reveals how different filtering strategies provide complementary convergence behaviors under different design objectives. This continuous filtering perspective offers a flexible analytical framework for studying weighted series representations and suggests a foundation for extending the methodology to other slowly convergent series and related special functions.

This work highlight several promising directions for further refinement and extension of the current methodology. First, the methodology can be naturally extended to evaluate higher odd integers, such as $\zeta(7)$ and $\zeta(9)$, where the suppression of slowly decaying algebraic terms becomes increasingly critical. Furthermore, the design of the optimal weighting sequence w_k can be formalized as a constrained functional optimization problem, mathematically balancing between the rate of convergence and the degree of kernel deformation. Finally, the observed connections between the newly induced kernels and generalized integral operators suggest deeper analytical structures, potentially related to fractional calculus and integral transforms, which merit further theoretical study. A detailed investigation of these directions will be presented in forthcoming work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there are no conflicts of interest.

References

1. J. M. Borwein, D. M. Bradley, R. E. Crandall, Computational strategies for the Riemann zeta function, *J. Comput. Appl. Math.*, **121** (2000), 247–296. [https://doi.org/10.1016/S0377-0427\(00\)00336-8](https://doi.org/10.1016/S0377-0427(00)00336-8)
2. P. Flajolet, I. Vardi, Zeta function expansions of classical constants, 1996. Available from: <https://algo.inria.fr/flajolet/Publications/FIVa96.pdf>.
3. E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd edition, Oxford University Press, 1986.
4. R. Apéry, Irrationalité de $\zeta(2)$ et $\zeta(3)$, *Astérisque*, **61** (1979), 11–13.
5. V. V. Zudilin, A third-order Apéry-like recursion for $\zeta(5)$, *Math. Notes*, **72** (2002), 733–737. <https://doi.org/10.1023/A:1021473409544>
6. B. Beckermann, G. Labahn, Effective computation of rational approximants and interpolants, *Reliab. Comput.*, **6** (2000), 365–390. <https://doi.org/10.1023/A:1009942122633>
7. A. Soria-Lorente, On Zudilin-like rational approximations to $\zeta(5)$, *Notes Number Theory Discrete Math.*, **24** (2018), 104–116. <https://doi.org/10.7546/nntdm.2018.24.2.104-116>
8. W. V. Zudilin, One of the numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational, *Russ. Math. Surv.*, **56** (2001), 774–776. <https://doi.org/10.1070/RM2001v056n04ABEH000427>
9. T. Amdeberhan, D. Zeilberger, Hypergeometric series acceleration via the WZ method, *Electron. J. Comb.*, **4** (1996), R3. <https://doi.org/10.37236/1318>
10. E. J. Weniger, Nonlinear sequence transformations for the acceleration of convergence and the summation of divergent series, *Comput. Phys. Rep.*, **10** (1989), 189–371. [https://doi.org/10.1016/0167-7977\(89\)90011-7](https://doi.org/10.1016/0167-7977(89)90011-7)
11. W. Zudilin, Arithmetic of linear forms involving odd zeta values, *J. Theor. Nombres Bordeaux*, **16** (2004), 251–291. <https://doi.org/10.5802/jtnb.447>
12. D. Cvijović, J. Klinowski, Integral representations of the Riemann zeta function for odd integer arguments, *J. Comput. Appl. Math.*, **142** (2002), 435–439. [https://doi.org/10.1016/S0377-0427\(02\)00358-8](https://doi.org/10.1016/S0377-0427(02)00358-8)
13. E. A. Galapon, Finite-part integral representation of the Riemann zeta function at odd positive integers and consequent representations, preprint, arXiv:2203.11342.
14. T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Springer-Verlag, 1976. <https://doi.org/10.1007/978-1-4684-9910-0>
15. F. W. J. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark, *The NIST Handbook of Mathematical Functions*, Cambridge University Press, 2010.

Appendix

Detailed Derivation of Theorem 3.1

Proof. From Theorem 2.6, the truncation error of the integral-based filter $w_k = \frac{(k!)^2}{(2k+1)!}$ is given by

$$\mathcal{E}_N = C_w \sum_{k=N+1}^{\infty} \frac{w_k}{(2k+1)^5}. \quad (\text{A.1})$$

By applying Stirling's formula with an explicit error term, $k! = \sqrt{2\pi k} \left(\frac{k}{e}\right)^k (1 + O(k^{-1}))$, the filter coefficients admit the asymptotic expansion as $k \rightarrow \infty$:

$$\begin{aligned} w_k &= \frac{(k!)^2}{(2k+1)!} = \frac{2\pi k \left(\frac{k}{e}\right)^{2k} (1 + O(k^{-1}))}{\sqrt{2\pi(2k+1)} \left(\frac{2k+1}{e}\right)^{2k+1} (1 + O(k^{-1}))} \\ &= \frac{\sqrt{\pi k}}{\sqrt{1 + \frac{1}{2k}}} \cdot \frac{e}{2k+1} \left(\frac{k}{2k+1}\right)^{2k} (1 + O(k^{-1})). \end{aligned} \quad (\text{A.2})$$

Using the identity $\left(\frac{k}{2k+1}\right)^{2k} = 4^{-k} \left(1 + \frac{1}{2k}\right)^{-2k} = 4^{-k} e^{-1} (1 + O(k^{-1}))$, we simplify (A.2) to:

$$w_k = \frac{\sqrt{\pi}}{2} k^{-1/2} 4^{-k} (1 + O(k^{-1})) \quad \text{as } k \rightarrow \infty. \quad (\text{A.3})$$

Substituting (A.3) into the tail sum (A.1) yields:

$$\mathcal{E}_N = C_w \frac{\sqrt{\pi}}{64} \sum_{k=N+1}^{\infty} k^{-\frac{11}{2}} 4^{-k} (1 + O(k^{-1})). \quad (\text{A.4})$$

For $k \geq N+1$, factor out the leading term $(N+1)^{-\frac{11}{2}}$ using discrete integration by parts or tail estimation:

$$\sum_{k=N+1}^{\infty} k^{-\frac{11}{2}} 4^{-k} = (N+1)^{-\frac{11}{2}} \sum_{k=N+1}^{\infty} 4^{-k} (1 + O(N^{-1})) = \frac{1}{3} 4^{-N} N^{-\frac{11}{2}} (1 + O(N^{-1})).$$

Therefore, the truncation error satisfies:

$$\mathcal{E}_N = \frac{C_w \sqrt{\pi}}{192} 4^{-N} N^{-\frac{11}{2}} (1 + O(N^{-1})) = O(4^{-N} N^{-\frac{11}{2}}) \quad \text{as } N \rightarrow \infty, \quad (\text{A.5})$$

which establishes the decay rate bound. $\square$



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>)