



Theory article

Predator extinction and total extinction in a non-autonomous predator–prey system with Allee effect and feedback control

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Abstract: Lai et al. [1] recently studied a non-autonomous predator–prey system with an additive Allee effect, prey refuge, antipredator behavior, and indirect feedback control. Their Theorem 5.1 gives a sufficient condition for predator extinction and prey persistence. We show that its hypotheses involve unknown solution components and therefore cannot be checked *a priori*; additionally, we identify a sign error in the proof. Then, we replace that result with conditions stated entirely in terms of the prescribed coefficients and parameters. The first condition yields predator extinction while the prey remains uniformly persistent. Its main ingredient is an optimization bound for the predator's net benefit from the prey, which avoids the sign issue in the original argument. Additionally, we give three sufficient mechanisms for total extinction: strong intraspecific competition, uniformly bounded intrinsic prey growth combined with strong competition, and a quantitative integral condition involving the prey dissipativity bound. These conditions are not intended to be necessary, but they are directly checkable. For the autonomous model, we derive an explicit critical refuge level for predator invasion and discuss invasion numbers in autonomous and periodic settings. Numerical examples illustrate both the autonomous and non-autonomous cases.

Keywords: predator–prey system; Allee effect; non-autonomous system; extinction; persistence; feedback control; anti-predator behavior

1. Introduction

A central question in mathematical ecology is whether a population persists or eventually dies out. The conditions separating these outcomes also matter in conservation biology and ecosystem management.

The Allee effect, first systematically described by Dennis [2], models reduced per-capita growth at a low density. Here, the additive formulation contains the term $-n/(x + h)$ in the prey equation. For

constant coefficients, the low-density growth rate is $a - n/h$: the cases $a > n/h$ and $a < n/h$ correspond to weak and strong Allee effects, respectively. When the coefficients vary with time, $a(t)$ may cross this threshold, so the effective regime can change with the environment. We directly account for that variation rather than imposing a fixed weak/strong classification.

Allee effects arise in several related settings. They may act directly on the predator [3], emerge in the prey through predation [4], or interact with non-lethal predator-induced fear [5]. Recent work has also considered eco-epidemiological models in which Allee thresholds interact with behavioral feedback and memory effects [6]. Additive Allee effects, prey refuge, and antipredator behaviors have likewise been studied together in a range of predator–prey models; for example, see [7–14].

A related but distinct modeling paradigm is the “fear effect”, namely the reduction of prey reproduction due to perceived predation risk, typically modeled through a fear function multiplying the prey birth rate. We follow the formulation of [1], where an anti-predator behavior enters as a direct interaction term $-\eta(t)xy$, representing active defence by the prey. Within the fear-effect literature, Pal et al. [15] explored how fear carryover effects combined with an Allee effect in the predator can drive diffusion-driven pattern formation, Umrao and Srivastava [16] showed that fear in prey together with hunting cooperation in predators can create multistability, and Tripathi et al. [17] investigated how high levels of anti-predator behaviors can increase the extinction risk when fear-induced Allee effects are present. Three other recent papers are also directly relevant: Qi et al. [18] examined the stability, bifurcation, and chaos of a stage-structured predator–prey model under fear-induced delay; Song et al. [19] analyzed a predator–prey model with fear effect and group defense; and Du et al. [20] established the persistence of a plankton allelopathy model in a time–space periodic environment.

Non-autonomous predator–prey systems, in which coefficients vary with time, are a more realistic setting for seasonally or environmentally forced ecosystems, and they complicate the analysis considerably: standard equilibrium and eigenvalue techniques no longer directly apply. Wu et al. [21] established persistence and extinction criteria for a non-autonomous Holling-type II system with prey refuge by means of comparison theorems and differential inequalities. Tiwari et al. [22] showed that non-autonomous forcing can reverse the stabilizing or destabilizing tendencies observed in autonomous counterparts. In the Leslie–Gower modeling tradition, Zhu et al. [23] investigated stability and bifurcation phenomena induced by the Allee effect, while Liu et al. [24] showed that hunting cooperation among predators can enlarge the Allee-effect-driven extinction region. Rakshit and Raghunathan [25] demonstrated that strong Allee effects can induce regime shifts in the classical Rosenzweig–MacArthur model.

Feedback control is another important regulatory mechanism in population dynamics [26]. It describes a closed-loop response in which a change in one component of an ecosystem affects other components, and the induced response then acts back on the original component. Feedback control has been studied in single-species models with the Allee effect [27], competitive systems [28], and predator–prey systems [29, 30]. Recent work by Yang et al. [31] examined anti-predation strategies in models combining prey refuge and group defense, while Jia and Wang [32] analyzed the stability of non-autonomous reaction-diffusion food-chain systems with feedback control and time-varying delays.

In a recent paper, Lai et al. [1] proposed and analyzed the following non-autonomous predator–prey system combining additive Allee effect, prey refuge, anti-predator behavior, and feedback control:

$$\frac{dx}{dt} = x \left[a(t) - b_1(t)x - \frac{n}{x+h} - \frac{c(t)(1-m)y}{b(t) + (1-m)x} - n_1(t)u_1 \right], \quad (1.1a)$$

$$\frac{dy}{dt} = y \left[-d(t) - b_2(t)y + \frac{c(t)\beta(t)(1-m)x}{b(t) + (1-m)x} - \eta(t)x - n_2(t)u_2 \right], \quad (1.1b)$$

$$\frac{du_1}{dt} = -f_1(t)u_1 + g_1(t)x, \quad (1.1c)$$

$$\frac{du_2}{dt} = -f_2(t)u_2 + g_2(t)y. \quad (1.1d)$$

Here, $x(t)$ and $y(t)$ denote the prey and predator densities, respectively, while $u_1(t)$ and $u_2(t)$ are indirect feedback control variables. The coefficient functions

$$a(t), b_1(t), b_2(t), c(t), d(t), b(t), \beta(t), \eta(t), f_1(t), f_2(t), g_1(t), g_2(t), n_1(t), n_2(t)$$

are assumed to be nonnegative, continuous, and bounded on $[0, +\infty)$, with additional positive lower bounds specified in Assumption 2.1 below. The constant parameters satisfy the following:

$$n > 0, \quad h > 0, \quad m \in (0, 1).$$

The term $n/(x+h)$ represents the additive Allee effect; the functional response

$$\frac{c(t)(1-m)x}{b(t) + (1-m)x}$$

incorporates prey refuge, since a fraction m of the prey is inaccessible, and $-\eta(t)x$ in the predator per-capita growth rate models anti-predator behavior by the prey.

Lai et al. [1] stated, among other results, a theorem asserting predator extinction and prey persistence under hypotheses $(\mathfrak{H}_1)$ – $(\mathfrak{H}_3)$. A close examination shows that this result has two serious difficulties: first, the nontrivial hypotheses involve unknown solution components such as the feedback trajectories and population densities, and therefore are not *a priori* verifiable; and second, the proof contains a sign error when a lower bound for $x(t)$ is substituted into a term whose coefficient has no fixed sign. These issues are examined in detail in Section 3.

The aim of this paper is to give a logically clean and fully verifiable extinction theory for System (1.1). Our main contributions are as follows:

- (i) We identify the flaws in the original Theorem 5.1 in [1], thereby showing that its hypotheses are not verifiable in advance and that the proof contains a sign error.
- (ii) We establish a verifiable sufficient condition, solely expressed in terms of prescribed coefficient functions and parameters, for extinction of the predator y together with uniform persistence of the prey x .
- (iii) We prove three sufficient conditions for total extinction of both species, corresponding to distinct structural mechanisms: strong intraspecific competition, uniformly bounded intrinsic prey growth with sufficiently strong competition, and an integral condition incorporating the dissipativity bound for the prey population.
- (iv) We include numerical examples in both autonomous and non-autonomous settings.
- (v) We characterize the effect of the prey refuge quantitatively: the predator's invasion fitness strictly decreases with the refuge parameter m , and we determine the critical refuge level m_c in a closed form.

- (vi) We introduce invasion reproduction numbers for the autonomous and periodic cases, and we show that failure of Condition (4.2) is necessary for predator persistence.

These conditions describe different mechanisms, and an intermediate parameter range remains where none of them applies.

The rest of the paper is organized as follows: Section 2 sets up the notation, proves dissipativity and positivity, recalls the comparison lemmas, and states the algebraic optimization result behind the predator-extinction criterion; Section 3 revisits the original Theorem 5.1 in [1]; Section 4 contains the main results; Section 5 presents numerical illustrations, including the effect of the prey refuge; and Section 6 closes with a discussion of invasion thresholds, reproduction numbers, and open problems.

2. Preliminaries and notation

For a bounded function $r : [0, +\infty) \rightarrow \mathbb{R}$, define the following:

$$r^M := \sup_{t \geq 0} r(t), \quad r^L := \inf_{t \geq 0} r(t). \quad (2.1)$$

Throughout the paper, we consider the following positive initial conditions:

$$x(0) > 0, \quad y(0) > 0, \quad u_1(0) > 0, \quad u_2(0) > 0.$$

For convenience, we summarize the principal notation.

Table 1. Principal notation.

Symbol	Meaning
r^M, r^L	$\sup_{t \geq 0} r(t), \inf_{t \geq 0} r(t)$ for bounded r
M_1, M_3	dissipativity bounds for $x(t)$ and $u_1(t)$, defined in (2.4)
M_2, M_4	dissipativity bounds for $y(t)$ and $u_2(t)$, defined in (2.4)
$\Phi(t)$	maximal net predator benefit defined in (2.5)
λ_i, δ_i	positive constants in integral conditions

Assumption 2.1. All coefficient functions appearing in (1.1) are nonnegative, continuous, and bounded on $[0, +\infty)$. Moreover,

$$b^L > 0, \quad b_1^L > 0, \quad b_2^L > 0, \quad f_1^L > 0, \quad f_2^L > 0, \quad d^L > 0, \quad g_1^L > 0. \quad (2.2)$$

The conditions $b^L > 0$ and $b_1^L > 0$ ensure that the functional response and the prey self-limitation are well behaved, $b_2^L > 0$ guarantees self-limitation of the predator, $f_1^L, f_2^L > 0$ ensure exponential decay of feedback variables in the absence of input, $g_1^L > 0$ ensures a uniformly positive input to u_1 when the prey persists, and $d^L > 0$ provides a baseline predator mortality.

Lemma 2.2 (Dissipativity). *Under Assumption 2.1, every positive solution of (1.1) is ultimately bounded. More precisely,*

$$\begin{aligned} \limsup_{t \rightarrow +\infty} x(t) &\leq M_1, & \limsup_{t \rightarrow +\infty} y(t) &\leq M_2, \\ \limsup_{t \rightarrow +\infty} u_1(t) &\leq M_3, & \limsup_{t \rightarrow +\infty} u_2(t) &\leq M_4, \end{aligned} \quad (2.3)$$

where one may take

$$M_1 = \frac{a^M}{b_1^L}, \quad M_2 = \frac{(c^M \beta^M - d^L)_+}{b_2^L}, \quad M_3 = \frac{g_1^M M_1}{f_1^L}, \quad M_4 = \frac{g_2^M M_2}{f_2^L}, \quad (2.4)$$

with $r_+ = \max\{r, 0\}$.

Proof. From (1.1a), since the Allee, predation, and feedback terms are nonpositive in the prey equation,

$$\frac{dx}{dt} \leq x(a^M - b_1^L x).$$

The comparison principle for the logistic equation gives the following:

$$\limsup_{t \rightarrow +\infty} x(t) \leq \frac{a^M}{b_1^L} = M_1.$$

For the predator, using

$$0 \leq \frac{(1-m)x}{b(t) + (1-m)x} \leq 1$$

and dropping the nonpositive terms $-\eta(t)x$ and $-n_2(t)u_2$, we have

$$\frac{dy}{dt} \leq y[-d^L - b_2^L y + c^M \beta^M].$$

Hence,

$$\limsup_{t \rightarrow +\infty} y(t) \leq \frac{(c^M \beta^M - d^L)_+}{b_2^L} = M_2.$$

If $c^M \beta^M < d^L$, then $y'(t) \leq -d^L y(t) + c^M \beta^M y(t) < 0$, which gives exponential decay of $y(t)$. If $c^M \beta^M = d^L$, then $y'(t) \leq -b_2^L y(t)^2$, whence $y(t) \leq (y(0)^{-1} + b_2^L t)^{-1} \rightarrow 0$. Thus, $M_2 = 0$ is admissible in these degenerate cases.

For the feedback variables, let $\varepsilon > 0$ be arbitrary. From $\limsup x(t) \leq M_1$ and $\limsup y(t) \leq M_2$, there exists $T > 0$ such that

$$x(t) \leq M_1 + \varepsilon, \quad y(t) \leq M_2 + \varepsilon, \quad t \geq T.$$

Then, for $t \geq T$,

$$u_1'(t) = -f_1(t)u_1(t) + g_1(t)x(t) \leq -f_1^L u_1(t) + g_1^M(M_1 + \varepsilon).$$

By comparison,

$$\limsup_{t \rightarrow +\infty} u_1(t) \leq \frac{g_1^M(M_1 + \varepsilon)}{f_1^L}.$$

Letting $\varepsilon \rightarrow 0$ yields the following:

$$\limsup_{t \rightarrow +\infty} u_1(t) \leq \frac{g_1^M M_1}{f_1^L} = M_3.$$

The same argument applied to (1.1d) gives the following:

$$\limsup_{t \rightarrow +\infty} u_2(t) \leq \frac{g_2^M M_2}{f_2^L} = M_4.$$

□

Lemma 2.3 (Positivity and global existence). *Under Assumption 2.1, for every positive initial condition, System (1.1) admits a unique solution on $[0, +\infty)$. Moreover,*

$$x(t) > 0, \quad y(t) > 0, \quad u_1(t) > 0, \quad u_2(t) > 0, \quad t \geq 0.$$

Proof. Since $b(t) \geq b^L > 0$, $m \in (0, 1)$, and $h > 0$, the denominators $b(t) + (1 - m)x$ and $x + h$ are positive in a neighborhood of the nonnegative orthant. The vector field is continuous in t and locally Lipschitz in (x, y, u_1, u_2) on the nonnegative orthant. Hence, a unique local solution exists.

The equations for x and y have the following form:

$$x'(t) = x(t)P(t, x(t), y(t), u_1(t)), \quad y'(t) = y(t)Q(t, x(t), y(t), u_2(t)).$$

Therefore, as long as the solution exists,

$$x(t) = x(0) \exp\left(\int_0^t P(s, x(s), y(s), u_1(s)) ds\right) > 0,$$

and similarly $y(t) > 0$.

For the feedback variables, if $u_1(t) = 0$, then

$$u_1'(t) = g_1(t)x(t) \geq 0,$$

and if $u_2(t) = 0$, then

$$u_2'(t) = g_2(t)y(t) \geq 0.$$

Thus, the nonnegative orthant is positively invariant. Since the initial values are strictly positive and the inputs $g_1(t)x(t)$ and $g_2(t)y(t)$ are nonnegative, $u_1(t)$ and $u_2(t)$ remain nonnegative; strict positivity follows from the variation-of-constants formula.

Finally, Lemma 2.2 gives ultimate boundedness of all components, which excludes finite-time blow-up. Hence, the local solution extends to $[0, +\infty)$. $\square$

We will use two standard comparison criteria for scalar non-autonomous differential inequalities. Both follow from the comparison principle together with a uniform sign condition on the averaged growth rate; for example, see Teng and Chen [33].

Lemma 2.4 (Extinction criterion). *Let $z : [0, +\infty) \rightarrow [0, +\infty)$ be absolutely continuous and satisfy*

$$z'(t) \leq \alpha(t)z(t)$$

for all sufficiently large t , where $\alpha : [0, +\infty) \rightarrow \mathbb{R}$ is bounded. If there exist constants $\lambda > 0$ and $\delta > 0$ such that

$$\limsup_{t \rightarrow +\infty} \int_t^{t+\lambda} \alpha(s) ds < -\delta,$$

then

$$\lim_{t \rightarrow +\infty} z(t) = 0.$$

Lemma 2.5 (Persistence criterion). *Let $z : [0, +\infty) \rightarrow [0, +\infty)$ be absolutely continuous with $z(0) > 0$ and satisfy*

$$z'(t) \geq z(t)(\alpha(t) - \beta z(t))$$

for all sufficiently large t , where $\beta > 0$ is constant and $\alpha : [0, +\infty) \rightarrow \mathbb{R}$ is bounded. If there exist constants $\lambda > 0$ and $\mu > 0$ such that

$$\liminf_{t \rightarrow +\infty} \int_t^{t+\lambda} \alpha(s) ds > \mu,$$

then there exists $\eta_0 > 0$, only depending only on the bounds of α , β , λ , and μ , such that

$$\liminf_{t \rightarrow +\infty} z(t) \geq \eta_0.$$

The next algebraic lemma is the key ingredient in the verifiable predator extinction criterion. It provides a uniform upper bound for the net benefit that the predator can obtain from the prey population, by optimizing the combined functional-response and anti-predator expression over all possible prey densities.

Lemma 2.6. *For each fixed $t \geq 0$, define the following:*

$$\phi_t(\xi) := \frac{c(t)\beta(t)(1-m)\xi}{b(t) + (1-m)\xi} - \eta(t)\xi, \quad \xi \geq 0.$$

Then,

$$\Phi(t) := \sup_{\xi \geq 0} \phi_t(\xi) = \begin{cases} 0, & \eta(t) \geq \frac{c(t)\beta(t)(1-m)}{b(t)}, \\ \left(\sqrt{c(t)\beta(t)} - \sqrt{\frac{\eta(t)b(t)}{1-m}} \right)^2, & \eta(t) < \frac{c(t)\beta(t)(1-m)}{b(t)}. \end{cases} \quad (2.5)$$

Consequently,

$$\frac{c(t)\beta(t)(1-m)x}{b(t) + (1-m)x} - \eta(t)x \leq \Phi(t) \quad (2.6)$$

for all $t \geq 0$ and $x \geq 0$.

Proof. Fix $t \geq 0$ and suppress the dependence on t . Set

$$A = c\beta, \quad B = \frac{b}{1-m} > 0.$$

Then,

$$\phi_t(\xi) = \frac{A\xi}{B + \xi} - \eta\xi.$$

If $\eta \geq A/B$, then

$$\phi'(\xi) = \frac{AB}{(B + \xi)^2} - \eta \leq \frac{A}{B} - \eta \leq 0$$

for all $\xi \geq 0$. Hence, ϕ is nonincreasing and

$$\sup_{\xi \geq 0} \phi(\xi) = \phi(0) = 0.$$

If $0 < \eta < A/B$, then ϕ has a unique critical point

$$\xi^* = \sqrt{\frac{AB}{\eta}} - B > 0.$$

At this point,

$$\begin{aligned}\phi(\xi^*) &= \frac{A\xi^*}{B + \xi^*} - \eta\xi^* \\ &= A \left(1 - \frac{B}{B + \xi^*} \right) - \eta \left(\sqrt{\frac{AB}{\eta}} - B \right) \\ &= A - 2\sqrt{A\eta B} + \eta B = (\sqrt{A} - \sqrt{\eta B})^2.\end{aligned}$$

Substituting $A = c\beta$ and $B = b/(1-m)$ gives the second branch in (2.5). The case $\eta = 0$ is also covered by the second branch, since then $\phi(\xi)$ is increasing and tends to $A = c\beta$ as $\xi \rightarrow \infty$. $\square$

Remark 2.7. The quantity $\Phi(t)$ is the maximal possible net benefit that the predator can extract from the prey population at time t , after accounting for an anti-predator behavior. It is optimized over all possible prey densities $x \geq 0$ and only depends on the known functions $c(t)$, $\beta(t)$, $b(t)$, and $\eta(t)$ and the parameter m . Importantly, it does not require the actual prey trajectory $x(t)$. Since $\phi_t(0) = 0$, one always has $\Phi(t) \geq 0$.

For later use, we record three further properties of $\Phi(t)$.

- Explicit maximizer. If $0 < \eta(t) < c(t)\beta(t)(1-m)/b(t)$, then the supremum in (2.5) is attained at the unique positive prey density

$$\xi_*(t) = \sqrt{\frac{c(t)\beta(t)b(t)}{\eta(t)(1-m)}} - \frac{b(t)}{1-m} > 0,$$

while if $\eta(t) \geq c(t)\beta(t)(1-m)/b(t)$, then the supremum is attained at $\xi_* = 0$. Thus, $\Phi(t) = \phi_t(\xi_*(t))$ is precisely the explicit maximal predator net-growth function at time t , and $\xi_*(t)$ is the prey density most favorable to the predator. For $\eta(t) = 0$ and $c(t)\beta(t) > 0$, the supremum $c(t)\beta(t)$ is approached as $\xi \rightarrow +\infty$ but is not attained, so in this limiting case, no maximizing density exists.

- Monotonicity in the refuge parameter. For each fixed t , the quantity $\Phi(t)$ is nonincreasing in m : the first branch in (2.5) is identically zero, and in the second branch $\sqrt{\eta(t)b(t)/(1-m)}$ increases with m , so that

$$\frac{\partial \Phi}{\partial m} = -\frac{1}{1-m} \left(\sqrt{c(t)\beta(t)} - \sqrt{\frac{\eta(t)b(t)}{1-m}} \right) \sqrt{\frac{\eta(t)b(t)}{1-m}} \leq 0.$$

Consequently, enlarging the refuge never increases the maximal predator benefit, and condition (4.2) is monotone in m : if it holds for a given refuge level m_0 , then it holds for every $m \geq m_0$.

- Refuge as a rescaling of the half-saturation constant. The net benefit $\phi_t(\xi)$ only depends on m through the combination $b(t)/(1-m)$: increasing the refuge fraction m is exactly equivalent to increasing the half-saturation constant $b(t)$ by the factor $1/(1-m)$. Therefore, the refuge acts by weakening the saturation of the functional response, while the anti-predator term $-\eta(t)\xi$ acts linearly and independently; the two mechanisms interact multiplicatively through the single effective parameter $b(t)/(1-m)$ entering (2.5).

3. Critique of the original Theorem 5.1

Theorem 5.1 in [1] does not provide a fully verifiable extinction criterion, and its proof also contains a sign error. Hypothesis ($\mathfrak{H}_1$) is a regularity assumption on the coefficient functions and is not the source of the difficulty. The issues lie in ($\mathfrak{H}_2$) and ($\mathfrak{H}_3$).

3.1. The non-verifiable hypotheses

Hypothesis ($\mathfrak{H}_3$) of Theorem 5.1 in [1], transcribed into the notation of the present paper, reads as follows:

$$\limsup_{t \rightarrow +\infty} \int_t^{t+\lambda_2} \left[-d(s) + \frac{c(s)\beta(s)(1-m)}{b(s)} y(s) - \eta(s)x(s) - n_2(s)u_{20}(s) \right] ds \leq 0, \quad (3.1)$$

where $u_{20}(s)$ denotes a reference solution used in the original paper. Thus, the hypothesis involves three unknown solution components: $y(s)$, $x(s)$, and $u_{20}(s)$.

Note that in (3.1), the density $y(s)$ multiplies the positive functional-response coefficient, which is inconsistent with the model: the predator equation (1.1b) contains the functional response $\frac{c\beta(1-m)x}{b+(1-m)x}$, which depends on the prey density $x(t)$, not on $y(t)$. Equation (5.1) in [1], the joint integral form of the prey and predator equations used in the proof, likewise depends on $x(t)$ in the functional-response term. Replacing $y(s)$ by $x(s)$ in (3.1) would correct the apparent typographical error, but would not remove the fundamental problem: the hypothesis would still involve the unknown solution component $x(s)$.

Hypothesis ($\mathfrak{H}_2$) has a similar difficulty because it involves the feedback trajectory $u_{10}(s)$. It is stated in [1] as follows:

$$\liminf_{t \rightarrow +\infty} \int_t^{t+\lambda_1} [a(s) - n(s) - n_1(s)u_{10}(s)] ds > 0.$$

The notation $n(s)$ is ambiguous: Model (1.1) contains the constant Allee parameter n , not a function $n(s)$. There are two possible interpretations:

- If $n(s)$ is intended to denote $n/(x(s) + h)$, then the hypothesis depends on the unknown prey trajectory $x(s)$.
- If the intended argument is based on the low-density estimate $-n/(x+h) \geq -n/h$, then the constant appearing in the hypothesis should be n/h rather than n . In that case, the hypothesis is a verifiable condition on the prescribed data, but it differs from what is written in [1] and no justification for the n/h replacement is given there.

The difficulty is in fact sharper than a matter of notation. The list of coefficient functions in ($\mathfrak{H}_1$) of [1] does not include n or m , so the symbol $n(s)$ appearing in ($\mathfrak{H}_2$) is never defined there. Nevertheless, the persistence criteria of that paper treat both n and the refuge coefficient m as bounded time-varying functions, writing n^M , n^L , and m^M (and the derived quantities C_1 , C_2 , and D_1) in its Theorem 4.2, while Model (1.1) and the numerical examples fix them as constants. Thus, ($\mathfrak{H}_2$), the model, and the auxiliary estimates of the original paper are mutually inconsistent, and ($\mathfrak{H}_2$) cannot be validated from the published statement alone.

A theorem whose assumptions involve the solution itself is not logically meaningless, but it is of a limited predictive value: a useful extinction criterion should be expressible in terms of the prescribed coefficient functions and model parameters.

3.2. A sign error in the proof

Even apart from the non-verifiability issue, the proof of Theorem 5.1 contains a sign error. From the predator equation (1.1b), dropping nonpositive terms and using the crude bound

$$\frac{(1-m)x}{b+(1-m)x} \leq \frac{(1-m)x}{b},$$

the original proof obtains the upper differential inequality

$$\frac{dy}{dt} \leq y \left[-d(t) + \left(\frac{c(t)\beta(t)(1-m)}{b(t)} - \eta(t) \right) x(t) \right], \quad (3.2)$$

which it then integrates to obtain an upper bound for $\log y$. After deriving a positive lower bound for $x(t)$, the proof substitutes that lower bound into the integrand of the resulting upper *integral* inequality, Eq (5.6) of [1]. The validity of this substitution depends on the sign of

$$\Delta(t) := \frac{c(t)\beta(t)(1-m)}{b(t)} - \eta(t).$$

Indeed, the integrand contributed by the functional-response and anti-predator terms is $\Delta(t)x(t)$, which is increasing in $x(t)$ whenever $\Delta(t) > 0$. Then, replacing $x(t)$ by a lower bound produces a lower bound rather than an upper bound, so the substitution does not preserve the upper integral inequality; the same obstruction directly applies to the differential inequality (3.2). If $\Delta(t) \leq 0$, then the substitution is legitimate, but this sign condition is neither assumed in the original hypotheses nor implied by them.

The resulting gap is not a borderline case. In the illustration used in [1] to demonstrate Theorem 5.1, the parameters retained from the preceding numerical experiment give $c(t) = 0.7 + 0.5 \sin t$, $\beta(t) = 0.8 + 0.1 \cos t$, $b(t) = 0.1 + 0.05 \cos t$, $m = 0.5$, and $\eta(t) = 0.15 + 0.1 \sin t$, so that

$$\Delta(t) = \frac{(0.7 + 0.5 \sin t)(0.8 + 0.1 \cos t)(0.5)}{0.1 + 0.05 \cos t} - (0.15 + 0.1 \sin t) > 0$$

for every t (its minimum is approximately 0.73). Hence, the branch on which the substitution is unjustified is precisely the one that is displayed.

Thus, the proof is only valid under an additional sign restriction ensuring that an anti-predator behavior dominates the linearized predator gain at all times. Such a restriction is neither assumed nor generally justified.

3.3. Resolution via direct optimization

The problem arises from the crude linearization

$$\frac{(1-m)x}{b+(1-m)x} \leq \frac{(1-m)x}{b},$$

which discards saturation and creates the sign-sensitive term $\Delta(t)x(t)$. Lemma 2.6 avoids this difficulty by directly optimizing the combined expression

$$\frac{c(t)\beta(t)(1-m)x}{b(t) + (1-m)x} - \eta(t)x$$

over all $x \geq 0$. The resulting bound $\Phi(t)$ is independent of the actual trajectory $x(t)$, requires no sign condition on $\Delta(t)$, and directly leads to the verifiable extinction criterion in Theorem 4.1.

4. Main results

4.1. Predator extinction with prey persistence

Theorem 4.1 (Predator extinction with prey persistence). *Assume Assumption 2.1. Recall that $g_1^L > 0$ is part of Assumption 2.1; the persistence conclusion for u_1 stated below relies on this positive lower bound. Let M_1 and M_3 be given by (2.4). Suppose that the following two conditions hold.*

(H1) *There exist constants $\lambda_1 > 0$ and $\delta_1 > 0$ such that*

$$\liminf_{t \rightarrow +\infty} \int_t^{t+\lambda_1} \left[a(s) - \frac{n}{h} - n_1^M M_3 \right] ds > \delta_1. \quad (4.1)$$

(H2) *There exist constants $\lambda_2 > 0$ and $\delta_2 > 0$ such that*

$$\limsup_{t \rightarrow +\infty} \int_t^{t+\lambda_2} [-d(s) + \Phi(s)] ds < -\delta_2, \quad (4.2)$$

where $\Phi(t)$ is defined by (2.5).

Then, for every positive solution of (1.1),

$$\lim_{t \rightarrow +\infty} y(t) = 0, \quad \lim_{t \rightarrow +\infty} u_2(t) = 0.$$

Moreover, there exists $\eta_0 > 0$, independent of the particular solution, such that

$$\liminf_{t \rightarrow +\infty} x(t) \geq \eta_0, \quad \liminf_{t \rightarrow +\infty} u_1(t) \geq \frac{g_1^L \eta_0}{2f_1^M}.$$

Thus, the predator becomes extinct, while the prey and its feedback control variable are uniformly persistent.

Remark 4.2. Both conditions are a priori verifiable. Condition (4.1) only involves $a(t)$, n , h , n_1^M , and the dissipativity bound M_3 , all of which are computable from the prescribed data. Condition (4.2) only involves $d(t)$ and the explicit function $\Phi(t)$.

The term $n_1^M M_3$ in (4.1) is essential. Since $-n_1(t)u_1(t) \leq 0$, omitting this term would increase the right-hand side, and the resulting inequality would no longer be a valid lower bound for dx/dt . Replacing the feedback term by its worst-case value $-n_1^M M_3$ is conservative, but it yields a fully verifiable criterion. Sharper criteria could be obtained by exploiting the variation-of-constants formula for the u_1 equation.

Proof. We divide the proof into three steps.

Step 1: predator extinction. From the predator equation (1.1b), dropping the nonpositive terms $-b_2(t)y$ and $-n_2(t)u_2$, we obtain the following:

$$\frac{dy}{dt} \leq y \left[-d(t) + \frac{c(t)\beta(t)(1-m)x(t)}{b(t) + (1-m)x(t)} - \eta(t)x(t) \right].$$

By Lemma 2.6,

$$\frac{c(t)\beta(t)(1-m)x(t)}{b(t) + (1-m)x(t)} - \eta(t)x(t) \leq \Phi(t).$$

Hence,

$$\frac{dy}{dt} \leq y[-d(t) + \Phi(t)]. \quad (4.3)$$

The function $-d(t) + \Phi(t)$ is bounded by Assumption 2.1 and the explicit formula (2.5). Therefore, Condition (4.2) and Lemma 2.4 imply the following:

$$\lim_{t \rightarrow +\infty} y(t) = 0.$$

Step 2: decay of $u_2(t)$. Since

$$u_2'(t) = -f_2(t)u_2(t) + g_2(t)y(t) \leq -f_2^L u_2(t) + g_2^M y(t),$$

and $y(t) \rightarrow 0$, a standard comparison argument gives $u_2(t) \rightarrow 0$. Indeed, for any $\varepsilon > 0$, there exists $T > 0$ such that $y(t) \leq \varepsilon$ for $t \geq T$, and hence

$$u_2'(t) \leq -f_2^L u_2(t) + g_2^M \varepsilon, \quad t \geq T.$$

It follows that

$$\limsup_{t \rightarrow +\infty} u_2(t) \leq \frac{g_2^M}{f_2^L} \varepsilon.$$

Letting $\varepsilon \rightarrow 0$ yields the following:

$$\lim_{t \rightarrow +\infty} u_2(t) = 0.$$

Step 3: prey persistence. Since $y(t) \rightarrow 0$, for every $\varepsilon > 0$, there exists $T_1 > 0$ such that

$$y(t) < \varepsilon, \quad t \geq T_1.$$

By Lemma 2.2,

$$\limsup_{t \rightarrow +\infty} u_1(t) \leq M_3.$$

Thus, for every $\rho > 0$, there exists $T_2 > 0$ such that

$$u_1(t) \leq M_3 + \rho, \quad t \geq T_2.$$

Let $T_* = \max\{T_1, T_2\}$.

For $t \geq T_*$, the prey equation gives the following:

$$\begin{aligned} \frac{1}{x} \frac{dx}{dt} &= a(t) - b_1(t)x - \frac{n}{x+h} - \frac{c(t)(1-m)y}{b(t) + (1-m)x} - n_1(t)u_1(t) \\ &\geq a(t) - b_1^M x - \frac{n}{h} - \frac{c^M(1-m)}{b^L} \varepsilon - n_1^M(M_3 + \rho). \end{aligned}$$

Therefore,

$$\frac{dx}{dt} \geq x[\alpha_{\varepsilon,\rho}(t) - b_1^M x],$$

where

$$\alpha_{\varepsilon,\rho}(t) = a(t) - \frac{n}{h} - \frac{c^M(1-m)}{b^L} \varepsilon - n_1^M(M_3 + \rho).$$

By (4.1), for all sufficiently large t ,

$$\begin{aligned} \int_t^{t+\lambda_1} \alpha_{\varepsilon,\rho}(s) ds &= \int_t^{t+\lambda_1} \left[a(s) - \frac{n}{h} - n_1^M M_3 \right] ds \\ &\quad - \lambda_1 \frac{c^M(1-m)}{b^L} \varepsilon - \lambda_1 n_1^M \rho \\ &> \delta_1 - \lambda_1 \frac{c^M(1-m)}{b^L} \varepsilon - \lambda_1 n_1^M \rho. \end{aligned}$$

Choose $\varepsilon > 0$ and $\rho > 0$ sufficiently small so that the right-hand side is larger than $\delta_1/2$. This choice only depends on the coefficient bounds and the margin δ_1 , not on the particular solution. Then, Lemma 2.5 yields a constant $\eta_0 > 0$, independent of the particular solution, such that

$$\liminf_{t \rightarrow +\infty} x(t) \geq \eta_0.$$

Finally, from $\liminf_{t \rightarrow +\infty} x(t) \geq \eta_0 > 0$ and the equation for u_1 , we have

$$u_1'(t) = -f_1(t)u_1(t) + g_1(t)x(t) \geq -f_1^M u_1(t) + \frac{g_1^L \eta_0}{2}$$

for all sufficiently large t , where we used that $\liminf_{t \rightarrow +\infty} x(t) \geq \eta_0$ implies $x(t) \geq \eta_0/2$ for all large t , and that $g_1^L > 0$ by Assumption 2.1. A standard comparison argument applied to this linear differential inequality yields

$$\liminf_{t \rightarrow +\infty} u_1(t) \geq \frac{g_1^L \eta_0}{2f_1^M} > 0,$$

proving uniform persistence of $u_1(t)$. This last step is the only place where $g_1^L > 0$ is used: if $g_1^L = 0$ (for example, $g_1(t) \equiv 0$), then $u_1'(t) \leq -f_1^L u_1(t)$ and $u_1(t) \rightarrow 0$, even though the prey persists, so the persistence conclusion for u_1 genuinely requires the positive lower bound on g_1 .

The proof is complete. $\square$

Remark 4.3 (A necessary condition for predator persistence). *Condition (4.2) is sharp in the following one-sided sense. If the predator persists, that is, $\limsup_{t \rightarrow +\infty} y(t) > 0$; then, necessarily*

$$\limsup_{t \rightarrow +\infty} \int_t^{t+\lambda} [-d(s) + \Phi(s)] ds \geq 0 \quad \text{for every } \lambda > 0.$$

Indeed, if there existed $\lambda > 0$ and $\delta > 0$ with

$$\limsup_{t \rightarrow +\infty} \int_t^{t+\lambda} [-d(s) + \Phi(s)] ds < -\delta,$$

then (4.3) and Lemma 2.4 would force $y(t) \rightarrow 0$. Thus, the failure of Condition (4.2) is a necessary condition for predator persistence. It is not sufficient: the examples of Section 5 show that the predator may become extinct even when (4.2) fails.

4.2. Total extinction

Next, we identify conditions that also force prey extinction. Once $x(t) \rightarrow 0$, the predator must vanish as well: its food source disappears while its baseline mortality remains strictly positive ($d^L > 0$).

At a low prey density, the Allee term satisfies

$$\frac{n}{x+h} \approx \frac{n}{h},$$

and hence, the low-density per-capita prey growth rate is approximately

$$a(t) - \frac{n}{h}.$$

Therefore, a natural extinction condition is

$$\limsup_{t \rightarrow +\infty} \int_t^{t+\lambda} \left[a(s) - \frac{n}{h} \right] ds < -\delta \quad (4.4)$$

for some constants $\lambda > 0$ and $\delta > 0$.

However, Condition (4.4) alone does not guarantee global extinction. Even in an autonomous system, the inequality $a - n/h < 0$ only implies that sufficiently small prey populations decline. It does not exclude the possibility that populations starting from larger densities survive, because the Allee mortality $n/(x+h)$ decreases as x increases. In the non-autonomous setting, temporary favorable periods in $a(t)$ may further allow the prey to reach densities where the Allee effect is weaker. Therefore, additional structural conditions are needed. The following theorem gives three such sufficient mechanisms.

Theorem 4.4 (Total extinction). *Assume Assumption 2.1. Then, for every positive solution of (1.1),*

$$\lim_{t \rightarrow +\infty} x(t) = 0, \quad \lim_{t \rightarrow +\infty} y(t) = 0, \quad \lim_{t \rightarrow +\infty} u_1(t) = 0, \quad \lim_{t \rightarrow +\infty} u_2(t) = 0,$$

provided one of the following conditions holds.

(E1) Strong competition with negative low-density average.

$$b_1^L \geq \frac{n}{h^2},$$

and there exist constants $\lambda > 0$ and $\delta > 0$ such that (4.4) holds.

(E2) Uniformly bounded intrinsic growth with strong competition.

$$d^M \leq \frac{n}{h}, \quad b_1^L > \frac{n}{h^2}.$$

(E3) Quantitative integral condition with prey dissipativity.

$$b_1^L \leq \frac{n}{h^2},$$

and there exist constants $\lambda > 0$ and $\delta > 0$ such that

$$\limsup_{t \rightarrow +\infty} \int_t^{t+\lambda} \left[a(s) - \frac{n}{h} \right] ds < -\lambda \left(\frac{n}{h^2} - b_1^L \right) M_1 - \delta, \quad (4.5)$$

where

$$M_1 = \frac{d^M}{b_1^L}.$$

Remark 4.5. The three conditions in Theorem 4.4 are not intended to be mutually exclusive. They are separate sufficient mechanisms for total extinction, corresponding to distinct ecological scenarios, not a classification of all possible extinction regimes.

Remark 4.6. Condition (E3) does not require $d^M \leq n/h$. Nevertheless, the most ecologically natural regime for this condition is the case where the intrinsic prey growth is not substantially above the Allee threshold. If $a(t)$ is often much larger than n/h , then the prey may be able to escape the low-density Allee deficit, and a global extinction criterion necessarily requires a stronger negative average or stronger self-limitation.

Proof. The proof has two parts. First, we establish $x(t) \rightarrow 0$ under each condition, and then show that $u_1(t)$, $y(t)$, and $u_2(t)$ also tend to zero.

Step 1: extinction of the prey. From (1.1a), since the predation and feedback terms are nonpositive,

$$\frac{dx}{dt} \leq x \left[a(t) - b_1(t)x - \frac{n}{x+h} \right]. \quad (4.6)$$

The function $x \mapsto n/(x+h)$ is convex on $[0, +\infty)$. Hence, its tangent line at $x = 0$ gives the following:

$$\frac{n}{x+h} \geq \frac{n}{h} - \frac{n}{h^2}x, \quad x \geq 0.$$

Substitution into (4.6) yields the following:

$$\frac{dx}{dt} \leq x \left[a(t) - \frac{n}{h} - \left(b_1(t) - \frac{n}{h^2} \right) x \right]. \quad (4.7)$$

Case (E1). If

$$b_1^L \geq \frac{n}{h^2},$$

then

$$b_1(t) - \frac{n}{h^2} \geq 0.$$

Therefore, (4.7) gives the following:

$$\frac{dx}{dt} \leq x \left[a(t) - \frac{n}{h} \right].$$

By (4.4) and Lemma 2.4, we obtain the following:

$$\lim_{t \rightarrow +\infty} x(t) = 0.$$

Case (E2). Assume

$$d^M \leq \frac{n}{h}, \quad b_1^L > \frac{n}{h^2}.$$

Then, for every $t \geq 0$ and $x \geq 0$,

$$a(t) - \frac{n}{x+h} \leq \frac{n}{h} - \frac{n}{x+h} = \frac{nx}{h(x+h)} \leq \frac{n}{h^2}x.$$

Using (4.6), we get the following:

$$\frac{dx}{dt} \leq x \left[\frac{n}{h^2} x - b_1(t)x \right] \leq - \left(b_1^L - \frac{n}{h^2} \right) x^2.$$

Let

$$K = b_1^L - \frac{n}{h^2} > 0.$$

Then,

$$x'(t) \leq -Kx(t)^2.$$

Therefore,

$$x(t) \leq \frac{1}{x(0)^{-1} + Kt} \rightarrow 0, \quad t \rightarrow +\infty.$$

Case (E3). Assume

$$b_1^L \leq \frac{n}{h^2}$$

and (4.5). From (4.7) and $b_1(t) \geq b_1^L$, we obtain the following:

$$\begin{aligned} \frac{dx}{dt} &\leq x \left[a(t) - \frac{n}{h} - \left(b_1^L - \frac{n}{h^2} \right) x \right] \\ &= x \left[a(t) - \frac{n}{h} + \left(\frac{n}{h^2} - b_1^L \right) x \right]. \end{aligned}$$

Here,

$$\frac{n}{h^2} - b_1^L \geq 0.$$

By Lemma 2.2,

$$\limsup_{t \rightarrow +\infty} x(t) \leq M_1.$$

Thus, for every $\varepsilon_0 > 0$, there exists $T_0 > 0$ such that

$$x(t) \leq M_1 + \varepsilon_0, \quad t \geq T_0.$$

Since the coefficient

$$\frac{n}{h^2} - b_1^L$$

is nonnegative, replacing $x(t)$ by the upper bound $M_1 + \varepsilon_0$ preserves the upper differential inequality. Hence, for $t \geq T_0$,

$$\frac{dx}{dt} \leq x \left[a(t) - \frac{n}{h} + \left(\frac{n}{h^2} - b_1^L \right) (M_1 + \varepsilon_0) \right].$$

Define the following:

$$\alpha_{\varepsilon_0}(t) = a(t) - \frac{n}{h} + \left(\frac{n}{h^2} - b_1^L \right) (M_1 + \varepsilon_0).$$

By (4.5), choose $\varepsilon_0 > 0$ sufficiently small so that

$$\lambda \left(\frac{n}{h^2} - b_1^L \right) \varepsilon_0 < \frac{\delta}{2}.$$

Then,

$$\limsup_{t \rightarrow +\infty} \int_t^{t+\lambda} \alpha_{\varepsilon_0}(s) ds < -\frac{\delta}{2}.$$

Lemma 2.4 gives the following:

$$\lim_{t \rightarrow +\infty} x(t) = 0.$$

Step 2: extinction of the remaining variables.

First, by (1.1c),

$$u_1'(t) = -f_1(t)u_1(t) + g_1(t)x(t) \leq -f_1^L u_1(t) + g_1^M x(t).$$

Since $x(t) \rightarrow 0$, for every $\varepsilon > 0$, there exists $T > 0$ such that $x(t) \leq \varepsilon$ for $t \geq T$. Thus,

$$u_1'(t) \leq -f_1^L u_1(t) + g_1^M \varepsilon, \quad t \geq T.$$

By comparison,

$$u_1(t) \leq u_1(T)e^{-f_1^L(t-T)} + \frac{g_1^M \varepsilon}{f_1^L} (1 - e^{-f_1^L(t-T)}), \quad t \geq T.$$

Taking $\limsup_{t \rightarrow +\infty}$ and then letting $\varepsilon \rightarrow 0$ gives the following:

$$\lim_{t \rightarrow +\infty} u_1(t) = 0.$$

Next, from (1.1b), dropping the nonpositive terms $-b_2(t)y$, $-\eta(t)x$, and $-n_2(t)u_2$, we have the following:

$$\frac{dy}{dt} \leq y \left[-d^L + \frac{c^M \beta^M (1-m)}{b^L} x(t) \right].$$

If $c^M \beta^M = 0$, then immediately

$$y'(t) \leq -d^L y(t),$$

and hence $y(t) \rightarrow 0$. If $c^M \beta^M > 0$, then since $x(t) \rightarrow 0$, there exists $T_1 > 0$ such that

$$\frac{c^M \beta^M (1-m)}{b^L} x(t) \leq \frac{d^L}{2}, \quad t \geq T_1.$$

Therefore,

$$y'(t) \leq -\frac{d^L}{2} y(t), \quad t \geq T_1,$$

and hence

$$\lim_{t \rightarrow +\infty} y(t) = 0.$$

Finally, by (1.1d),

$$u_2'(t) = -f_2(t)u_2(t) + g_2(t)y(t) \leq -f_2^L u_2(t) + g_2^M y(t).$$

Since $y(t) \rightarrow 0$, the same comparison argument used for u_1 gives the following:

$$\lim_{t \rightarrow +\infty} u_2(t) = 0.$$

The proof is complete. □

Remark 4.7. Theorems 4.1 and 4.4 provide two sets of contrasting sufficient conditions, rather than a necessary-and-sufficient dichotomy. In the first regime, the prey has enough average growth, after accounting for the worst-case feedback loss, to persist, while the predator cannot overcome its mortality. In the second regime, the prey's low-density deficit, together with additional structural restrictions, forces total collapse. The two regimes are not exhaustive; an intermediate parameter regime remains in which neither theorem applies. Determining the dynamics in that gap, including the possible dependence on initial data, is an open problem.

5. Numerical illustrations

We illustrate the criteria using constant coefficients inspired by [1]:

$$\begin{aligned} a \text{ variable, } & b_1 = 0.3, & n = 0.4, & h = 0.5, \\ c = 0.3, & m = 0.5, & b = 0.2, & \beta = 0.6, \\ d = 0.01, & b_2 = 0.1, & \eta \text{ variable,} & \\ n_1 = n_2 = 0.1, & f_1 = f_2 = 0.5, & g_1 = g_2 = 0.3. & \end{aligned} \quad (5.1)$$

Then,

$$\frac{n}{h} = 0.8, \quad \frac{n}{h^2} = 1.6.$$

Unless stated otherwise, all numerical integrations use the following five initial conditions:

$$\begin{aligned} \text{IC-1 : } (x_0, y_0, u_{10}, u_{20}) &= (1.0, 3.5, 2.5, 2.5), \\ \text{IC-2 : } (0.3, 1.8, 0.05, 2.0), \\ \text{IC-3 : } (2.0, 0.3, 2.0, 0.05), \\ \text{IC-4 : } (0.6, 1.2, 0.15, 0.15), \\ \text{IC-5 : } (1.5, 0.1, 0.2, 0.3). \end{aligned} \quad (5.2)$$

In all figures, solid lines represent the prey $x(t)$ and its feedback control $u_1(t)$; dashed lines represent the predator $y(t)$ and its feedback control $u_2(t)$. Different colors distinguish the five initial conditions (or the different parameter choices where indicated). Readers viewing a monochrome version should note that the prey/predator distinction is carried by the solid/dashed line style.

5.1. Verification of the prey persistence condition

For constant coefficients,

$$M_1 = \frac{a}{b_1} = \frac{a}{0.3}, \quad M_3 = \frac{g_1 M_1}{f_1} = \frac{0.3}{0.5} \cdot \frac{a}{0.3} = 2a.$$

Therefore, $n_1 M_3 = 0.1 \cdot 2a = 0.2a$. Condition (4.1) reduces to $a - n/h - n_1 M_3 > 0$, that is, $a - 0.8 - 0.2a > 0$, or $0.8a > 0.8$, (i.e., $a > 1$). Accordingly, we take $a = 1.2$ to illustrate predator extinction with prey persistence, which gives $a - n/h - n_1 M_3 = 1.2 - 0.8 - 0.24 = 0.16 > 0$.

To numerically check the prediction we integrate the autonomous system (1.1) with $a = 1.2$, $\eta = 0.35$, and the remaining parameters as in (5.1) over the time interval $t \in [0, 30]$. Condition (4.2)

is verified for this choice of η in Section 5.2, where the sufficient threshold $\eta_{\text{suf}} \approx 0.263 < 0.35$ is computed.

Figure 1 shows the result. Figure 1(a) displays the time series of the prey $x(t)$ and predator $y(t)$. In all five cases, the prey population approaches a positive steady-state value, while the predator population decays to zero. Figure 1(b) shows the corresponding evolution of the feedback controls $u_1(t)$ and $u_2(t)$. Consistent with the structure of the model, $u_1(t)$ converges to a positive limit, driven by the persistent prey, while $u_2(t)$ tends to zero as the source term $g_2 y$ vanishes with the predator. The simulations agree with the theory and suggest that the observed pattern is not sensitive to these particular initial conditions.

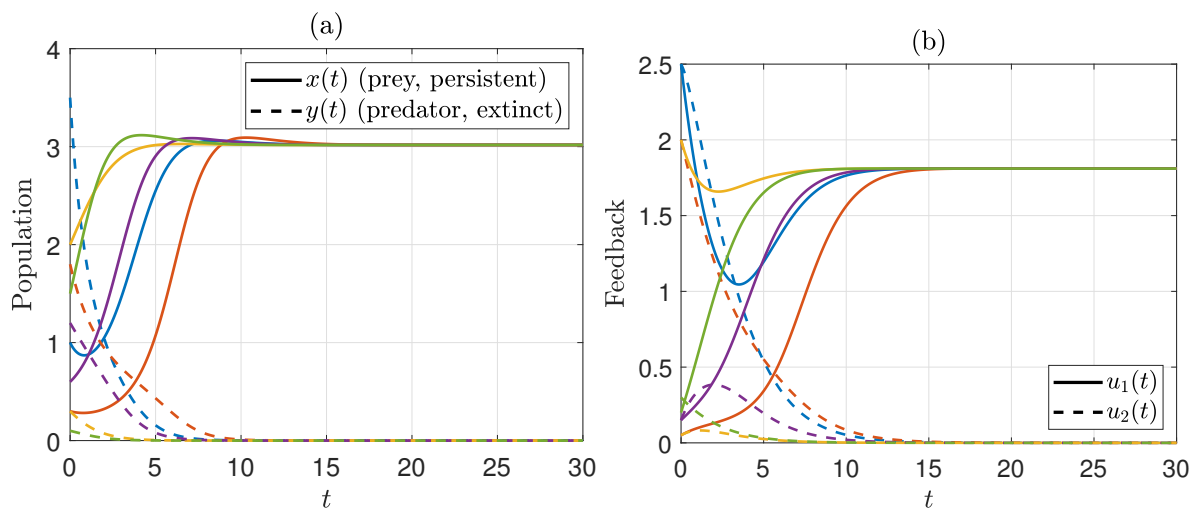


Figure 1. Time series of the autonomous system with $a = 1.2$ and $\eta = 0.35$, showing predator extinction with prey persistence. Five initial conditions (5.2) are shown, integrated over $t \in [0, 30]$. **(a)** Prey $x(t)$ (persistent) and predator $y(t)$ (extinct). **(b)** Feedback controls $u_1(t)$ and $u_2(t)$.

5.2. Predator extinction and invasion thresholds

We turn to the predator-extinction criterion. For the analysis below, we change b_1 from the baseline value 0.3 to $b_1 = 0.8$. At $b_1 = 0.3$ the predator-free prey equilibrium is $x^* \approx 3.0$, so the invasion criterion $\frac{c\beta(1-m)x^*}{b+(1-m)x^*} - \eta x^* - d > 0$ already fails for $\eta \gtrsim 0.05$; hence, the invasion threshold is trivially small (coexistence would be confined to $\eta \lesssim 0.05$), and a comparison with the sufficient threshold $\eta_{\text{suf}} \approx 0.263$ below would be uninformative. Raising b_1 to 0.8 reduces x^* to ≈ 1.106 , yielding a nontrivial $\eta_{\text{act}} \approx 0.111$. Additionally, we record that Condition (4.1) remains satisfied after this change of b_1 : with $M_1 = a/b_1 = 1.5$, one has $M_3 = g_1 M_1 / f_1 = 0.9$, and $a - n/h - n_1^M M_3 = 1.2 - 0.8 - 0.09 = 0.31 > 0$.

For constant coefficients, Condition (4.2) reduces to $-d + \Phi < 0$. With the parameters in (5.1) and b_1 changed to 0.8,

$$\frac{c\beta(1-m)}{b} = \frac{0.3 \times 0.6 \times 0.5}{0.2} = 0.45.$$

Hence, if $\eta \geq 0.45$, then $\Phi = 0$ and $-d + \Phi = -0.01 < 0$, so predator extinction is guaranteed by

Theorem 4.1.

If $\eta < 0.45$, then by (2.5),

$$\Phi = (\sqrt{c\beta} - \sqrt{\eta b/(1-m)})^2 = (\sqrt{0.18} - \sqrt{0.4\eta})^2.$$

The sufficient condition $-d + \Phi < 0$ becomes $(\sqrt{0.18} - \sqrt{0.4\eta})^2 < 0.01$. Taking square roots,

$$|\sqrt{0.18} - \sqrt{0.4\eta}| < 0.1,$$

and since $\sqrt{0.4\eta} < \sqrt{0.18}$ for $\eta < 0.45$, we obtain

$$\sqrt{0.4\eta} > \sqrt{0.18} - 0.1 \approx 0.42426 - 0.1 = 0.32426,$$

hence $0.4\eta > 0.10515$, or

$$\eta > 0.26287 \approx 0.263.$$

We call this cutoff $\eta_{\text{suf}} := 0.263$.

Being a worst-case bound over all possible prey densities (cf. Remark 2.7), this sufficient threshold is conservative. A direct equilibrium analysis gives the *invasion threshold at the predator-free equilibrium*: solving the predator-free equilibrium $0 = a - b_1 x^* - n/(x^* + h) - n_1 g_1 x^*/f_1$ gives $x^* \approx 1.106$, and the invasion criterion $\frac{c\beta(1-m)x^*}{b+(1-m)x^*} - \eta x^* - d > 0$ holds precisely when

$$\eta < \eta_{\text{act}} := \frac{1}{x^*} \left(\frac{c\beta(1-m)x^*}{b+(1-m)x^*} - d \right) \approx 0.111.$$

Throughout, η_{act} denotes this *invasion threshold at the predator-free equilibrium*, that is, the value of η at which the predator-free equilibrium changes the stability with respect to predator introduction. We do not claim as a theorem that extinction occurs for every $\eta > \eta_{\text{act}}$; the numerical bifurcation analysis below (Figure 2) shows that for the present parameter set the computed predator equilibrium is identically zero for all $\eta > \eta_{\text{act}}$, which supports the global interpretation in this example.

Figure 3 shows the three regimes: when $\eta = 0.06$, the predator coexists with the prey at a positive steady state; when $\eta = 0.20$, the predator vanishes despite being outside the reach of the sufficient criterion (4.2); and when $\eta = 0.35$, the extinction predicted by the theorem is confirmed. The prey follows suit: the predator-free equilibrium prey level is higher than the prey level on the computed coexistence branch, because predation reduces prey abundance. As η increases along the coexistence branch, predation weakens, and the computed prey equilibrium monotonically moves toward the predator-free value (from $x^* \approx 0.90$ at $\eta = 0.02$ to $x^* \approx 1.106$ at $\eta = \eta_{\text{act}}$); after the predator equilibrium reaches zero, the prey remains at the predator-free level x^* . This variation is visible in the numerical bifurcation curve in Figure 2.

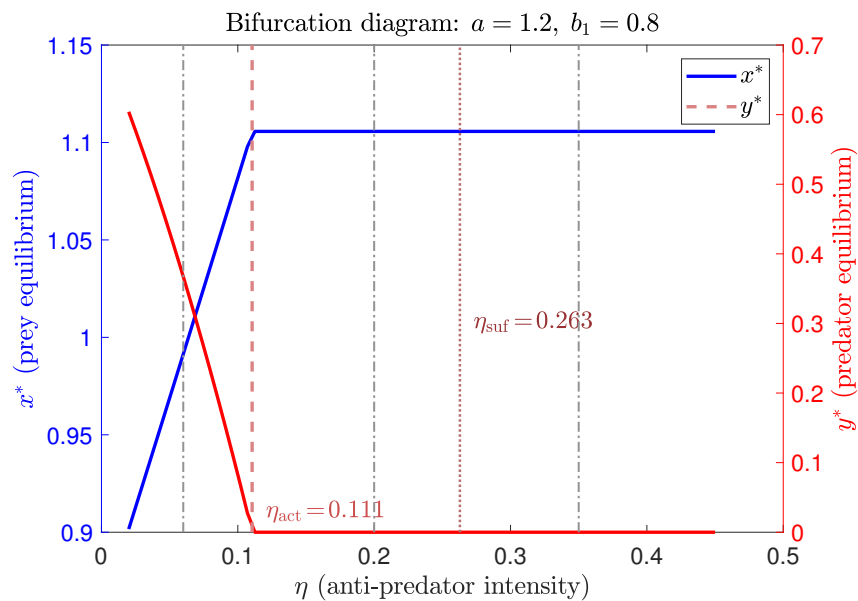


Figure 2. Bifurcation diagram of equilibria versus η for the autonomous system with $a = 1.2$, $b_1 = 0.8$. The blue curve (left axis) is the prey equilibrium x^* ; the red curve (right axis) is the predator equilibrium y^* . The red dashed line marks the invasion threshold at the predator-free equilibrium $\eta_{\text{act}} \approx 0.111$, where y^* reaches zero; the red dotted line marks the cutoff $\eta_{\text{suf}} \approx 0.263$ from (4.2). The gap between them quantifies the conservatism of the worst-case bound Φ . The three grey dash-dotted lines mark the values $\eta = 0.06$, 0.20 , and 0.35 used in Figure 3.

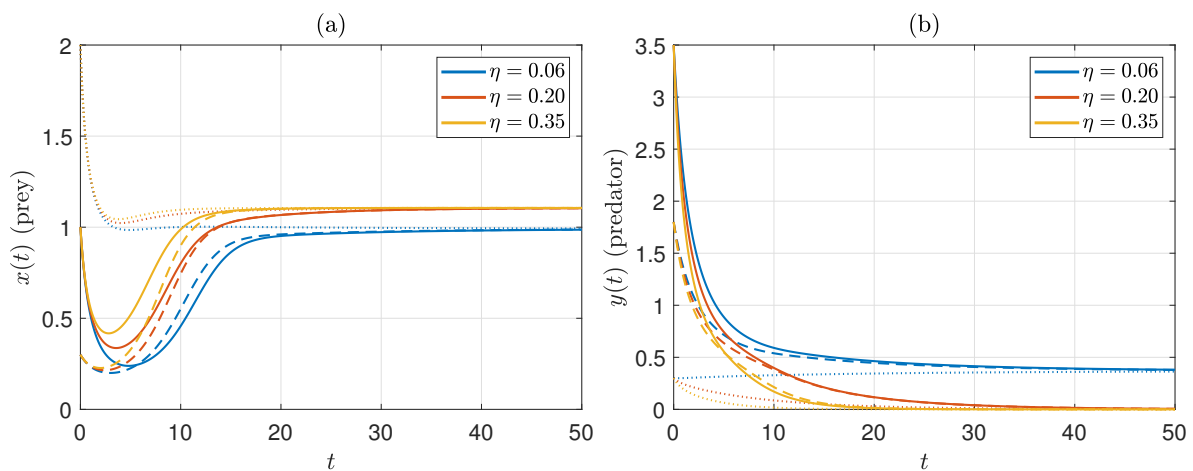


Figure 3. Time series of the autonomous system with $a = 1.2$, $b_1 = 0.8$ and three values of η , integrated over $t \in [0, 50]$ from three distinct initial conditions (solid, dashed, dotted lines). **(a)** Prey $x(t)$. **(b)** Predator $y(t)$. For $\eta = 0.06$ (below the invasion threshold at the predator-free equilibrium η_{act}), the predator persists. For $\eta = 0.20$ (between η_{act} and the sufficient threshold η_{suf}), the predator goes extinct although (4.2) does not apply. For $\eta = 0.35$ (above η_{suf}), extinction is guaranteed by Theorem 4.1.

The full bifurcation diagram is given in Figure 2. As η increases from zero, the coexistence equilibrium (x^*, y^*) moves along a continuous branch: x^* gradually increases toward the predator-free level $x^* \approx 1.106$ while y^* declines toward zero. At $\eta = \eta_{\text{act}} \approx 0.111$, the predator equilibrium hits zero and remains there for all larger η , while x^* stays constant at the predator-free value $x^* \approx 1.106$. Theorem 4.1 guarantees extinction only for $\eta > \eta_{\text{suf}}$; the region $\eta_{\text{act}} < \eta < \eta_{\text{suf}}$ is one where extinction is observed for all tested initial conditions and along the computed equilibrium branch, although it is not captured by the sufficient criterion (4.2). This global interpretation is supported by the numerical bifurcation analysis rather than by a theorem. This gap is a consequence of the optimization in Lemma 2.6—the bound Φ assumes that the prey density can be adversarially chosen, whereas for this parameter set the actual equilibrium prey density $x^* \approx 1.106$ lies substantially *above* the maximizing prey density ξ_* of Remark 2.7, which decreases from $\xi_* \approx 0.41$ at $\eta = \eta_{\text{act}}$ to $\xi_* \approx 0.12$ at $\eta = \eta_{\text{suf}}$; at the realized density $x^* \approx 1.106$, one has $\phi(x^*) \approx 0.132 - 1.106\eta < d$ for every $\eta > \eta_{\text{act}}$, so the realized per-capita net growth $-d + \phi(x^*)$ is negative, which is why extinction is observed throughout the gap even though the worst-case bound Φ remains positive and Condition (4.2) does not apply.

The three values of η used in Figure 3 ($\eta = 0.06, 0.20, 0.35$) are marked on the bifurcation diagram by the grey dash-dotted lines, illustrating the three qualitatively distinct parameter regimes.

Influence of the prey refuge: For the autonomous setting, the influence of the refuge parameter m can be made fully explicit. Since the predator-free prey equilibrium x^* solves

$$0 = a - b_1 x^* - \frac{n}{x^* + h} - \frac{n_1 g_1}{f_1} x^*,$$

it is independent of m . The invasion fitness at the predator-free equilibrium,

$$F(m) := \frac{c\beta(1-m)x^*}{b + (1-m)x^*} - \eta x^* - d,$$

satisfies

$$F'(m) = -\frac{c\beta b x^*}{(b + (1-m)x^*)^2} < 0,$$

so refuge always decreases the predator's invasion fitness. If $F(0) \leq 0$, then the predator cannot invade for any $m \in [0, 1)$ and no critical refuge level exists. If $F(0) > 0$, then F has a unique zero

$$m_c = 1 - \frac{b(\eta x^* + d)}{x^*(c\beta - \eta x^* - d)} \in (0, 1),$$

and the predator can invade (the predator-free equilibrium is unstable with respect to predator introduction) precisely when $m < m_c$. In the example of Figure 2 with $\eta = 0.06$, we have $F(0) \approx 0.0761 > 0$ and

$$m_c \approx 1 - \frac{0.2 \times 0.0764}{1.1057 \times (0.18 - 0.0764)} \approx 0.867,$$

so a refuge protecting about 87% of the prey blocks predator invasion, while for $\eta = 0.35$, we have $F(0) = \frac{c\beta x^*}{b+x^*} - (\eta x^* + d) \approx 0.1524 - (0.3871 + 0.01) \approx -0.2447 < 0$, so no value of m allows invasion. Together with the monotonicity of $\Phi(t)$ in m recorded in Remark 2.7, this shows that the refuge acts in the same direction on the sufficient threshold η_{suf} and on the invasion threshold η_{act} .

5.3. Total extinction

We illustrate two of the three total-extinction criteria in Theorem 4.4. Condition (E1) is omitted for brevity: it requires $b_1 \geq n/h^2$ together with the integral condition (4.4), and its verification follows the same pattern as the cases below. The examples show that (E2) and (E3) are non-vacuous.

Condition (E2). Take $b_1 = 2.0$, $a = 0.75$, $\eta = 0.3$. and Then, $a \leq n/h = 0.8$ and $b_1 = 2.0 > 1.6 = n/h^2$, so Condition (E2) of Theorem 4.4 is satisfied. Figure 4 displays the resulting time series over $t \in [0, 100]$ from the five initial conditions (5.2). Panel (a) shows that both $x(t)$ and $y(t)$ monotonically decay to zero for all five initial conditions shown; panel (b) shows that the feedback controls $u_1(t)$ and $u_2(t)$ likewise vanish. The uniformly bounded intrinsic growth ($a \leq n/h = 0.8$) and strong intraspecific competition ($b_1 > n/h^2 = 1.6$) drive the prey to extinction; then, the predator follows. The conclusion holds for every positive solution by Theorem 4.4.

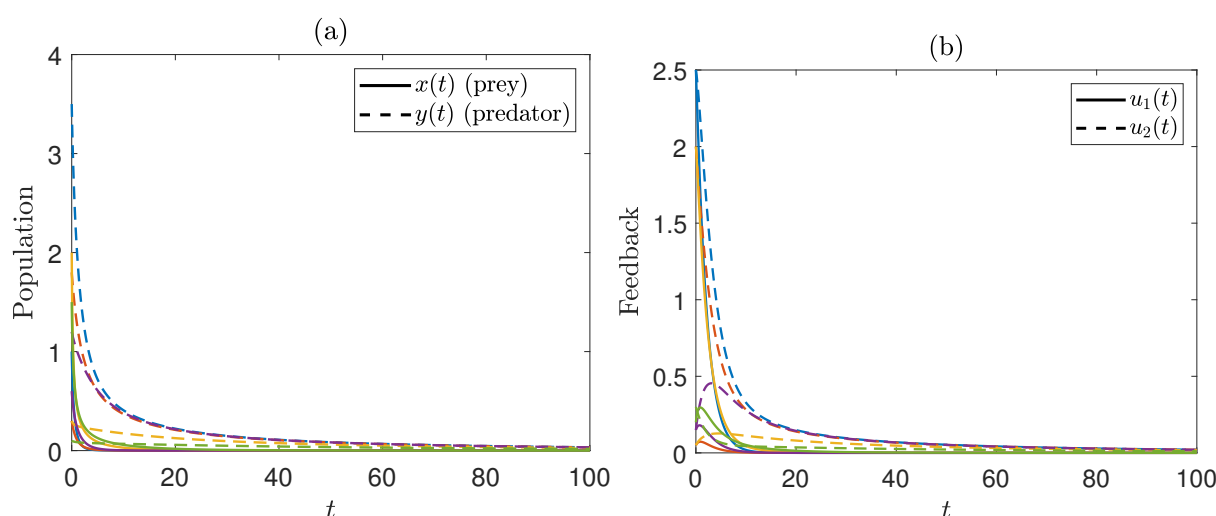


Figure 4. Total extinction under condition (E2): $b_1 = 2.0$, $a = 0.75$, $\eta = 0.3$, and integrated over $t \in [0, 100]$ from five initial conditions (5.2). **(a)** $x(t) \rightarrow 0$ and $y(t) \rightarrow 0$. **(b)** $u_1(t) \rightarrow 0$ and $u_2(t) \rightarrow 0$.

Condition (E3). Take $b_1 = 1.5$, $a = 0.65$, $\eta = 0.3$. and Here, $b_1 = 1.5 < 1.6 = n/h^2$, so (E1) and (E2) do not apply—competition alone is insufficient to guarantee extinction. However, $M_1 = a/b_1 \approx 0.4333$ and

$$\left(\frac{n}{h^2} - b_1\right)M_1 = (1.6 - 1.5) \times 0.4333 \approx 0.04333,$$

while $a - n/h = -0.15$. For every sufficiently large $\lambda > 0$ (in particular $\lambda = 1$; the inequality below holds for all $\lambda > 0.4688$),

$$\int_t^{t+\lambda} \left[a - \frac{n}{h}\right] ds = -\lambda \cdot 0.15 < -\lambda \cdot 0.04333 - 0.05,$$

so (4.5) is satisfied with $\delta = 0.05$. Therefore, Theorem 4.4 guarantees total extinction via the integral mechanism (E3).

Figure 5 shows the time series over $t \in [0, 100]$ with the same five initial conditions. Compared with the (E2) case, the decay is noticeably slower— $b_1 = 1.5$ is close to the Allee slope $n/h^2 = 1.6$,

so competition-driven suppression is weaker. Nevertheless, the persistent deficit $a - n/h = -0.15$ accumulates over time and eventually forces both populations to zero. This illustrates the qualitatively different mechanisms: (E2) relies on instantaneous competition dominance, while (E3) relies on the integrated net-negative prey growth.

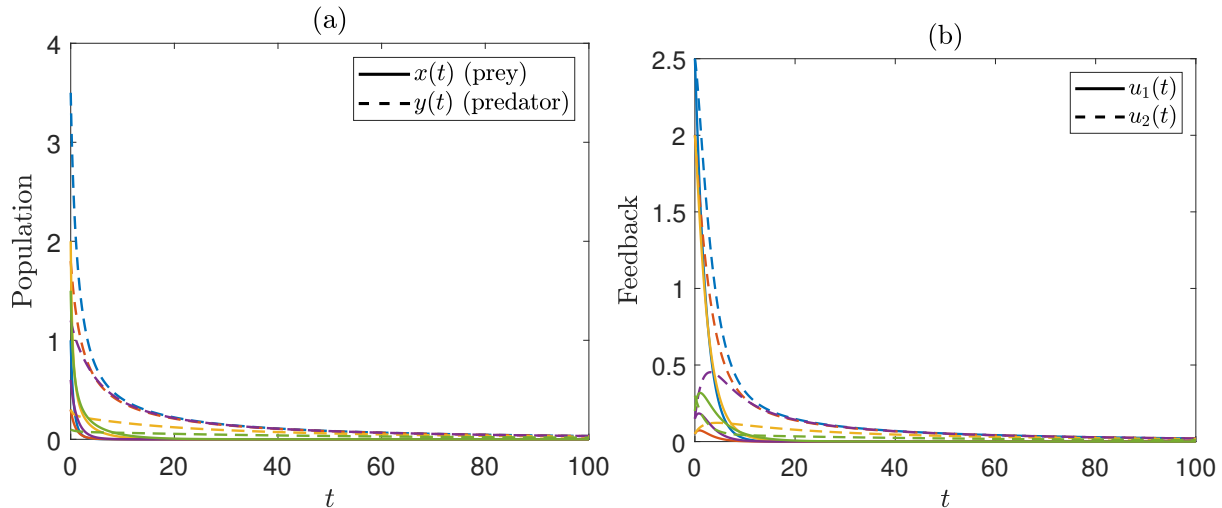


Figure 5. Total extinction under condition (E3): $b_1 = 1.5$, $a = 0.65$, $\eta = 0.3$, and integrated over $t \in [0, 100]$ from five initial conditions (5.2). **(a)** $x(t) \rightarrow 0$ and $y(t) \rightarrow 0$ (slower decay than in Figure 4). **(b)** $u_1(t) \rightarrow 0$ and $u_2(t) \rightarrow 0$.

5.4. Non-autonomous examples

Next, we test the criteria in genuinely non-autonomous settings. Three cases are considered: predator extinction with prey persistence (Section 5.4.1), followed by total extinction through the two distinct mechanisms (E2) and (E3) of Theorem 4.4 (Section 5.4.2).

5.4.1. Predator extinction with prey persistence

Let

$$a(t) = 1.2 + 0.3 \sin t,$$

so that $\frac{1}{2\pi} \int_t^{t+2\pi} a(s) ds = 1.2$ and $a^M = 1.5$. All other parameters are taken from (5.1) with $b_1 = 0.3$ and $\eta = 0.35$.

Condition (4.1). From $M_1 = a^M/b_1 = 1.5/0.3 = 5$, we obtain $M_3 = g_1 M_1/f_1 = \frac{0.3}{0.5} \cdot 5 = 3$, hence $n_1^M M_3 = 0.1 \cdot 3 = 0.3$. Then, for $\lambda_1 = 2\pi$,

$$\begin{aligned} \int_t^{t+2\pi} \left[a(s) - \frac{n}{h} - n_1^M M_3 \right] ds &= \int_t^{t+2\pi} [1.2 + 0.3 \sin s - 0.8 - 0.3] ds \\ &= \int_t^{t+2\pi} [0.1 + 0.3 \sin s] ds = 2\pi \cdot 0.1 > 0, \end{aligned}$$

so (4.1) is satisfied.

Condition (4.2). As computed in Section 5.2, the sufficient threshold for η is 0.263. With $\eta = 0.35 > 0.263$, Condition (4.2) holds. Therefore, Theorem 4.1 guarantees predator extinction with uniform prey persistence.

Figure 6 shows the numerical integration over $t \in [0, 40]$ (approximately six forcing periods) from the five initial conditions (5.2). The effect of the non-autonomous forcing is clearly visible: $x(t)$ settles into a persistent oscillation whose amplitude mirrors the $0.3 \sin t$ modulation of $a(t)$, while $y(t)$ decays to zero from every initial state. Panel (b) shows that $u_1(t)$ tracks the prey with a comparable oscillatory envelope and a positive asymptotic mean, whereas $u_2(t)$ vanishes together with the predator. The five trajectories coincide for a large t in each panel, which is consistent with the theoretical prediction that the persistence–extinction dichotomy is determined by the coefficients.

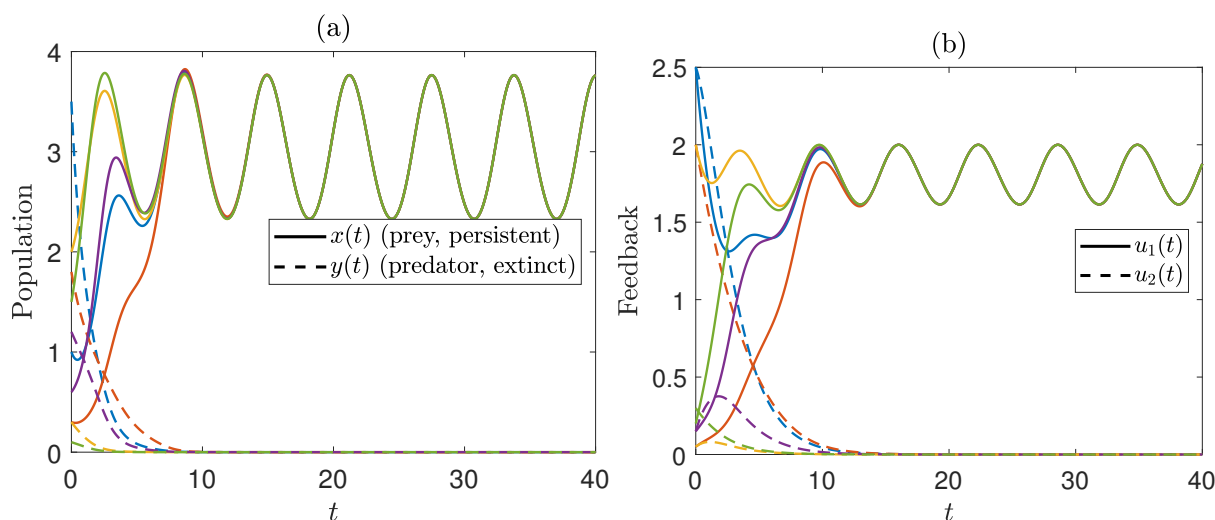


Figure 6. Non-autonomous predator extinction with prey persistence: $a(t) = 1.2 + 0.3 \sin t$, $b_1 = 0.3$, $\eta = 0.35$, integrated over $t \in [0, 40]$ from five initial conditions (5.2). **(a)** Prey $x(t)$ (persistent, with pronounced oscillatory envelope) and predator $y(t) \rightarrow 0$. **(b)** Feedback controls $u_1(t)$ (persistent) and $u_2(t) \rightarrow 0$.

5.4.2. Total extinction

Then, we consider total extinction in a non-autonomous setting. The two examples below correspond to the distinct mechanisms (E2) and (E3) in Theorem 4.4.

Condition (E2). Take

$$a(t) = 0.66 + 0.10 \sin t, \quad b_1 = 2.0,$$

with all other parameters as in (5.1) and $\eta = 0.3$. Then,

$$a^M = 0.76 < \frac{n}{h} = 0.80, \quad b_1 = 2.0 > \frac{n}{h^2} = 1.6.$$

Condition (E2) of Theorem 4.4 is satisfied: $a^M < n/h$ and $b_1^L > n/h^2$. No integral sign condition needs to be checked; the conclusion $x(t) \rightarrow 0$, $y(t) \rightarrow 0$, $u_1(t) \rightarrow 0$, $u_2(t) \rightarrow 0$ holds for every positive solution, regardless of the oscillatory component in $a(t)$.

Figure 7 displays the time series over $t \in [0, 100]$ from the five initial conditions (5.2). The sinusoidal forcing produces visible undulations in $x(t)$ and $y(t)$ during the decay phase, but the strong competition ($b_1 = 2.0 > n/h^2$) eventually drives all four components to zero, as guaranteed by Theorem 4.4.

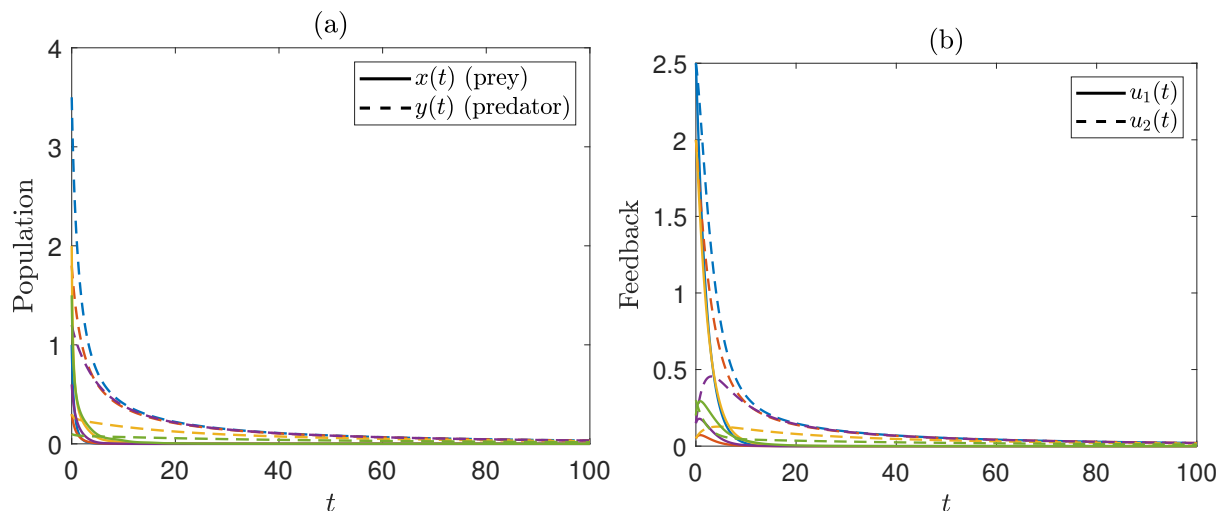


Figure 7. Non-autonomous total extinction under condition (E2): $a(t) = 0.66 + 0.10 \sin t$, $b_1 = 2.0$, $\eta = 0.3$, integrated over $t \in [0, 100]$ from five initial conditions (5.2). **(a)** $x(t) \rightarrow 0$ and $y(t) \rightarrow 0$. **(b)** $u_1(t) \rightarrow 0$ and $u_2(t) \rightarrow 0$.

Condition (E3). Take

$$a(t) = 0.65 + 0.20 \sin t, \quad b_1 = 1.5,$$

again with other parameters as in (5.1) and $\eta = 0.3$. Here, $b_1 = 1.5 < 1.6 = n/h^2$ and $a^M = 0.85 > n/h = 0.8$, so (E1) and (E2) do not apply: (E1) fails because $b_1^L < n/h^2$ (competition is insufficient), while (E2) fails because $a^M > n/h$ (intrinsic growth exceeds the Allee threshold) and also because $b_1^L < n/h^2$. We have $M_1 = a^M/b_1 = 0.85/1.5 \approx 0.5667$ and

$$\left(\frac{n}{h^2} - b_1\right)M_1 = (1.6 - 1.5) \times 0.5667 \approx 0.05667.$$

For any $\lambda > 0$,

$$\begin{aligned} \int_t^{t+\lambda} \left[a(s) - \frac{n}{h}\right] ds &= \int_t^{t+\lambda} [0.65 + 0.20 \sin s - 0.80] ds \\ &= \int_t^{t+\lambda} [-0.15 + 0.20 \sin s] ds \\ &= -\lambda \cdot 0.15 + 0.20(\cos t - \cos(t + \lambda)). \end{aligned}$$

The oscillatory contribution is bounded by 0.40 in absolute value. Taking $\lambda = 2\pi$ makes it vanish exactly, yielding

$$\int_t^{t+2\pi} \left[a(s) - \frac{n}{h}\right] ds = -2\pi \cdot 0.15 \approx -0.9425 < -2\pi \cdot 0.05667 - 0.05 \approx -0.4061,$$

so (4.5) holds with $\delta = 0.05$. Therefore, Theorem 4.4 guarantees total extinction under (E3) in this non-autonomous setting.

Figure 8 shows the time series over $t \in [0, 150]$ from the five initial conditions (5.2). Because $b_1 = 1.5$ is only slightly below $n/h^2 = 1.6$, the competition-driven suppression is weak and the decay is markedly slower than in the (E2) case. The sinusoidal forcing produces clearly visible oscillations in the transient phase, yet the persistent integrated deficit $\int(a - n/h) < 0$ eventually forces both populations to zero—indicating that (E3) operates through a different mechanism than (E2).

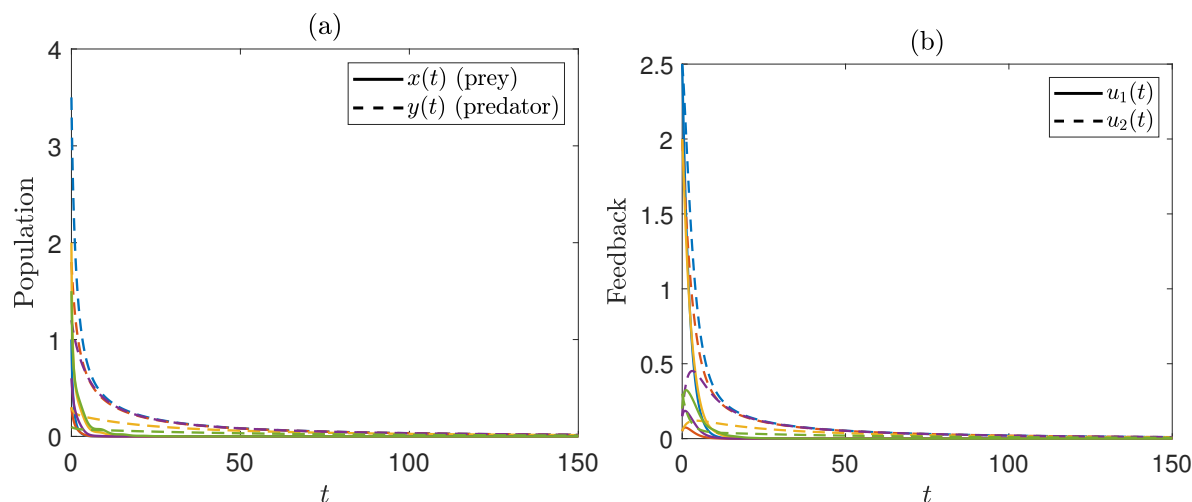


Figure 8. Non-autonomous total extinction under condition (E3): $a(t) = 0.65 + 0.20 \sin t$, $b_1 = 1.5$, $\eta = 0.3$, integrated over $t \in [0, 150]$ from five initial conditions (5.2). **(a)** $x(t) \rightarrow 0$ and $y(t) \rightarrow 0$ (slower decay than in Figure 7). **(b)** $u_1(t) \rightarrow 0$ and $u_2(t) \rightarrow 0$.

6. Discussion and concluding remarks

We developed a revised extinction theory for the non-autonomous predator–prey system (1.1) studied by Lai et al. [1]. The main points are these. First, we replaced the non-verifiable extinction criterion in Theorem 5.1 with an *a priori* condition based only on the prescribed coefficients and parameters. The function $\Phi(t)$ is central to this result: it gives the largest net benefit that the predator can obtain from the prey after an antipredator behavior is included. Then, we proved three separate sufficient conditions for total extinction and illustrated the results in both autonomous and non-autonomous examples. Finally, we quantified the effect of the prey refuge, derived invasion numbers for autonomous and periodic settings, and identified a necessary condition for predator persistence.

6.1. Ecological interpretation

The quantity

$$a(t) - \frac{n}{h}$$

is the low-density net prey growth rate in the absence of predation and feedback control. If its average is sufficiently positive after subtracting the worst-case feedback penalty $n_1^M M_3$, then the prey uniformly persists. If it is sufficiently negative and additional structural conditions prevent high-density escape from the Allee deficit, then the prey becomes extinct.

The predator's fate is governed by the balance between its mortality $d(t)$ and the maximal net benefit $\Phi(t)$. Strong anti-predator behavior, represented by large $\eta(t)$, reduces $\Phi(t)$ and therefore promotes predator extinction. The constant-coefficient example gives a concrete cutoff at $\eta \gtrsim 0.263$: beyond this level of antipredator behavior, the predator cannot persist even though the prey does.

Condition (E2) is particularly simple. It only requires that the maximum intrinsic prey growth rate does not exceed the Allee threshold n/h and that intraspecific competition is strong enough:

$$a^M \leq \frac{n}{h}, \quad b_1^L > \frac{n}{h^2}.$$

Under these assumptions, extinction holds for every positive initial datum and no integral average needs to be checked. Ecologically, this gives a practical warning sign: if the prey's reproductive capacity is uniformly limited relative to the Allee mortality and competition is sufficiently strong, then the whole system collapses.

6.2. Invasion thresholds, reproduction numbers, and refuge effects

This subsection summarizes the implications of Sections 4.1 and 5.2, especially near the borderline between the two regimes of Theorem 4.1.

If the extinction condition fails. By Remark 4.3, a violation of Condition (4.2) is necessary for predator persistence but does not force it. In the autonomous example of Section 5.2, the borderline $\eta = \eta_{\text{act}}$ separates the two outcomes observed numerically: for $\eta < \eta_{\text{act}}$, the predator invades and coexistence is observed, whereas for $\eta > \eta_{\text{act}}$, the computed predator equilibrium vanishes. The gap $\eta_{\text{act}} < \eta < \eta_{\text{suf}}$ is precisely the region where invasion from low predator densities is excluded ($\mathcal{R}_0 < 1$), but global extinction is not yet guaranteed by Theorem 4.1.

Invasion reproduction numbers. A reproduction-number-type threshold can be attached to the predator in the standard way [34, 35]. In the autonomous case, define the following:

$$\mathcal{R}_0 = \frac{c\beta(1-m)x^*}{(b + (1-m)x^*)(\eta x^* + d)},$$

where x^* is the predator-free prey equilibrium. Then, $\mathcal{R}_0 > 1$ exactly when $F(m) > 0$ with F as in Section 5.2, (i.e., exactly when the predator-free equilibrium is unstable with respect to predator introduction); invasion is expected when $\mathcal{R}_0 > 1$ and extinction of the predator when $\mathcal{R}_0 < 1$. In a T -periodic non-autonomous setting, if $\bar{x}(t)$ denotes an attracting T -periodic predator-free solution, then the analogous quantity is

$$\mathcal{R}_0 = \exp \left(\int_0^T \left[-d(s) + \frac{c(s)\beta(s)(1-m)\bar{x}(s)}{b(s) + (1-m)\bar{x}(s)} - \eta(s)\bar{x}(s) \right] ds \right),$$

in the sense of [36]. Indeed, the linearized predator equation $y' = \alpha(t)y$ with a T -periodic coefficient α is asymptotically stable when $\int_0^T \alpha(s) ds < 0$. To pass from Condition (4.2) to $\mathcal{R}_0 < 1$, we use precisely the following averaging argument: with $g(s) := -d(s) + \Phi(s)$ and $F(t) := \int_t^{t+\lambda_2} g(s) ds$, Condition (4.2) yields $F(t) \leq -\delta_2$ for all sufficiently large t ; since g is T -periodic, so is F , and the same bound holds for all $t \geq 0$. Averaging over one period, and $\frac{1}{T} \int_0^T F(t) dt = \lambda_2 \cdot \frac{1}{T} \int_0^T g(s) ds$ (Fubini's theorem), hence $\int_0^T [-d(s) + \Phi(s)] ds < 0$. Since the integrand never exceeds $-d(s) + \Phi(s)$, the sufficient Condition (4.2) implies $\mathcal{R}_0 < 1$: whenever our theorem guarantees extinction, the invasion number is below one.

Scope of the threshold discussion. The conditions in Theorems 4.1 and 4.4 are only sufficient they are not claimed, separately or together, to be necessary and sufficient. Under additional global hypotheses, the invasion-number discussion may distinguish predator extinction from persistence, but it does not provide a classification of total extinction.

The threshold $\mathcal{R}_0 = 1$. At $\mathcal{R}_0 = 1$ the linearized predator equation at the predator-free equilibrium has a zero invasion exponent, so linearization is inconclusive. The strict negative averaged bound in (4.2) does not cover equality; a separate nonlinear or center-manifold analysis is required. For System (1.1), when $\mathcal{R}_0 > 1$, the predator variational equation has positive growth, which gives local instability and rules out attraction of sufficiently nearby strictly positive solutions, but does not by itself establish a global conclusion for every positive initial condition.

6.3. Limitations and open problems

The criteria proved here are only sufficient. They leave an intermediate region in which neither prey persistence with predator extinction nor total extinction is guaranteed. Several questions remain open.

- The threshold obtained from $\Phi(t)$ is conservative because the proof drops the stabilizing terms $-b_2(t)y$ and $-n_2(t)u_2$ in the predator equation. Can one obtain sharper predator extinction conditions by retaining these negative terms?
- The prey persistence condition uses the global dissipativity bound M_3 for the feedback variable $u_1(t)$. Since the u_1 equation is linear, one may use the variation-of-constants formula

$$u_1(t) = u_1(0) \exp\left(-\int_0^t f_1(\tau) d\tau\right) + \int_0^t \exp\left(-\int_s^t f_1(\tau) d\tau\right) g_1(s)x(s) ds$$

to obtain sharper estimates. Such refinements may enlarge the parameter range for which prey persistence can be proved.

- For Condition (E3), the bound M_1 may be crude. It would be useful to replace it by a sharper trajectory-dependent or time-averaged estimate for $x(t)$.
- The intermediate regime in which neither theorem applies remains largely open. It is still unclear whether coexistence, extinction, or initial-condition-dependent outcomes occur in that region.
- Periodic coefficients form a natural special case of System (1.1). In such settings, it may be possible to express extinction or persistence thresholds in terms of Floquet exponents or principal Lyapunov exponents of suitably chosen scalar comparison equations.
- Determine the critical refuge level for non-autonomous or periodic coefficients, where the predator-free solution $\bar{x}(t)$ depends on time and the analogue of m_c is implicitly defined through the invasion integral appearing in Section 6.2.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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