



Research article

Optimal boundary control of a vegetation–water model with cross-diffusion

Jiali Chai¹, Qimin Zhang^{1,*} and Panrun Li^{2,*}

¹ School of Mathematics and Statistics, Ningxia University, Yinchuan 750021, China

² Sichuan Vocational College of Information Technology, Guangyuan 628040, China

* **Correspondence:** Email: zhangqimin64@sina.com, 1405051520@qq.com.

Abstract: To address vegetation degradation in arid and semi-arid ecosystems, existing vegetation–water models neglect the directional transport of soil water induced by root uptake (cross-diffusion) and human regulation at boundaries. This paper formulates a vegetation–water reaction–diffusion model incorporating a cross-diffusion term and a Robin-type boundary condition. First, the existence and uniqueness of weak solutions are proved with a priori estimates. Second, under bilateral pointwise constraints and an integral constraint on the cumulative boundary-regulation effort, the existence of an optimal boundary control is established via a penalty method. The adjoint system is derived using the Lagrange multiplier method, yielding first-order necessary and second-order sufficient optimality conditions, and the twice continuous Fréchet differentiability of the control-to-state mapping is demonstrated. Numerical simulations verify the effectiveness of the proposed control strategy. The framework provides theoretical support for ecological restoration in water-limited regions.

Keywords: vegetation–water model; cross-diffusion; Robin boundary condition; optimal boundary control; adjoint equation; Fréchet differentiability

1. Introduction

Arid and semi-arid ecosystems cover 40% of the global land surface and support over 25% of the world's population, yet they face severe desertification threats due to climate change and human activities [1]. Vegetation often self-organizes into spatial patterns, which serve as important indicators of ecosystem organization and potential regime shifts [2, 3]. Mathematical models have played a fundamental role in explaining the emergence, evolution, and transition of such patterns. Klausmeier developed a vegetation–water model to describe the interaction between plant biomass and water and showed that this interaction can generate spatial vegetation patterns [4]. Scanlon et al. demonstrated from field observations that local positive feedbacks can produce spatial clustering of vegetation along a rainfall gradient [5]. Gilad et al. further incorporated mechanisms such as infiltration feedback and

root water uptake into vegetation–water models and investigated their effects on vegetation pattern formation [6]. These studies provide an important basis for understanding how vegetation–water interactions and spatial water redistribution contribute to pattern formation in dryland ecosystems.

More broadly, recent developments in nonlinear partial differential equation models have shown that dryland vegetation dynamics involve not only stationary Turing patterns, but also wave instabilities, secondary instabilities, nonlinear transport mechanisms, and far-from-equilibrium spatial structures. Grifò investigated a hyperbolic Klausmeier model with secondary seed dispersal and analyzed both Turing and wave instabilities, illustrating how inertial and transport effects may alter vegetation pattern formation [7]. Consolo and Grifò studied Turing vegetation patterns in arid environments with finite soil carrying capacity and showed how resource limitations influence pattern formation and transition through linear and weakly nonlinear analyses [8]. They also investigated the Eckhaus instability of stationary patterns in hyperbolic reaction–diffusion systems, emphasizing the role of secondary instabilities and wavelength selection in the evolution of spatial structures [9]. These studies indicate that vegetation pattern dynamics depend not only on local vegetation–water interactions, but also on the particular form of spatial transport, finite-resource constraints, and nonlinear instability mechanisms.

Traveling structures provide a complementary perspective on vegetation dynamics, especially for describing spatial propagation, colonization, and desertification processes that cannot be fully characterized by stationary Turing patterns alone. Grifò, Currò, and Valenti investigated continuous and discontinuous traveling-wave connections in a hyperbolic vegetation model and related their propagation behavior to colonization and desertification scenarios [10]. Grifò, Iuorio, and Veerman studied far-from-equilibrium traveling vegetation pulses in a Klausmeier-type model with autotoxicity effects, combining numerical analysis with geometric singular perturbation techniques to characterize pulse-type solutions [11]. Consolo et al. further investigated vegetation pattern formation and transition in a two-dimensional hyperbolic reaction–transport model incorporating finite soil resources and inertia, showing that these mechanisms may significantly influence pattern selection and transition [12]. Together, these contributions demonstrate the broader relevance of traveling waves, nonlinear transport, secondary instability, and pattern-selection mechanisms in mathematical descriptions of dryland ecosystems.

Within this broader framework, cross-diffusion is particularly relevant to the present study. The diffusion terms in classical vegetation–water models usually rely on Fickian diffusion, in which the spatial transport of a variable is determined by its own concentration gradient. Such a formulation does not fully account for the directional redistribution of soil water induced by spatial variations in vegetation density and root water uptake. In reality, plant roots locally modify soil moisture and may induce water redistribution between regions with different vegetation densities. This mechanism, commonly represented through cross-diffusion or soil-water diffusion feedback, can substantially alter the onset, selection, and evolution of vegetation patterns [13–16]. Specifically, Li et al. introduced a cross-diffusion term into a vegetation–water model and found that increasing cross-diffusion intensity drives sequential transitions from gap to stripe to spot patterns, while internal competition plays an opposite role [13]. Zhang et al. extended the Klausmeier model by incorporating both local grazing and cross-diffusion and showed that cross-diffusion may accelerate ecosystem degradation under certain parameter regimes [14]. Guo et al. incorporated cross-diffusion together with nonlocal delay and observed transitions from gap to stripe, mixed, and spot patterns as the cross-diffusion coefficient increases [15]. Moreover, Guo et al. showed that, under suitable diffusion conditions, cross-diffusion can induce Turing instability even when ordinary diffusion alone does not generate such an instability [16]. Despite these advances, most

existing cross-diffusion vegetation models still employ no-flux or periodic boundary conditions and therefore do not explicitly account for externally regulated water exchange at ecosystem boundaries.

Boundary optimal-control problems have been extensively studied in various fields, including reaction–diffusion systems [17], viscoelastic fluids [18], Boussinesq systems [19], heat exchangers [20], hyperbolic conservation laws [21], and Cahn–Hilliard systems [22]. In these studies, control variables are commonly introduced through Dirichlet, Neumann, or Robin-type boundary conditions, and optimality systems are derived using Lagrange multipliers, adjoint equations, and related variational techniques. However, boundary optimal control for vegetation–water systems remains much less developed, particularly when spatial water redistribution is coupled with externally regulated boundary exchange. To the best of our knowledge, no existing study has established an optimal boundary-control framework for a vegetation–water model that simultaneously incorporates cross-diffusion and a Robin-type boundary condition representing natural seepage and artificial irrigation.

The main innovations of this paper are as follows:

- We extend the classical vegetation–water model by adding a cross-diffusion term $-\gamma \nabla^2 p$ to the soil-water equation, capturing the directional water transport induced by root suction. This term creates a direct spatial coupling between vegetation and soil moisture. As a result, the boundary control $u(t)$ affects p through two pathways: indirectly via the reaction terms, and directly via cross-diffusion. This dual-channel mechanism is absent in models without cross-diffusion.
- We study an optimal boundary-control problem with the Robin-type condition

$$\partial_n w = (u(t) - a)w \quad \text{on } \Gamma_t,$$

where $a > 0$ represents the natural seepage coefficient and $u(t) \geq 0$ denotes the boundary irrigation control intensity. In the absence of human intervention, $u(t) = 0$, the boundary condition becomes $\partial_n w = -aw$, corresponding to natural outward seepage. When $u(t) = a$, the boundary reduces to the homogeneous Neumann condition. When $u(t) > a$, the artificial water input exceeds the natural seepage effect and produces a net inward water exchange.

- We prove the well-posedness of the state equation and the existence of an optimal control under bilateral pointwise constraints and an integral constraint on cumulative boundary-regulation effort. Using the Lagrange multiplier method, we derive the first-order optimality conditions, where γ appears explicitly in the adjoint system through the coupling term $-\gamma \Delta \lambda_w$. We also prove second-order sufficient conditions and the Fréchet differentiability of the control-to-state mapping, providing a complete theoretical foundation.

The remainder of this paper is organized as follows. Section 2 presents the deterministic vegetation–water model with a cross-diffusion term and a Robin-type boundary condition, and provides the definition of weak solutions together with an existence and uniqueness theorem. Section 3 defines the cost functional under bilateral pointwise constraints and an integral constraint, and proves the existence of an optimal control via a penalty method. Section 4 derives the adjoint system using the Lagrange multiplier method and establishes the first-order necessary optimality conditions. Section 5 proves the second-order sufficient optimality conditions on the critical cone and demonstrates the twice continuous Fréchet differentiability of the control-to-state mapping. Section 6 presents numerical simulations to illustrate the effect of the proposed boundary-control strategy by comparing a zero-exchange reference case with a controlled boundary-exchange case.

2. Model formulation

We consider the following classical reaction–diffusion vegetation–water model [23] (adapted from the Klausmeier model or Gilad et al.):

$$\begin{cases} \partial_t p(x, t) = g w(x, t) p(x, t)(1 - p(x, t))(1 + \eta p(x, t))^2 - p(x, t) + \nabla^2 p(x, t), & x \in \Omega, t > 0, \\ \partial_t w(x, t) = l - v w(x, t)(1 - \rho p(x, t)) - g w(x, t) p(x, t)(1 + \eta p(x, t))^2 + \delta_w \nabla^2 w(x, t), & x \in \Omega, t > 0, \\ \frac{\partial p}{\partial n}(x, t) = 0, & x \in \partial\Omega, t > 0, \\ \frac{\partial w}{\partial n}(x, t) = 0, & x \in \partial\Omega, t > 0, \\ p(x, 0) = p_0(x), & x \in \Omega, \\ w(x, 0) = w_0(x), & x \in \Omega, \end{cases} \quad (2.1)$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, ∇^2 is the Laplacian operator with respect to x , where g, η, l, v, ρ , and δ_w are positive constants. are the growth rate, root-to-shoot ratio, precipitation rate, evaporation rate, shading parameter, and soil-water diffusion coefficient, respectively.

The classical model assumes that spatial soil-water transport is governed only by Fickian diffusion, $\delta_w \nabla^2 w$. Biologically, however, plant roots modify the local soil-water environment through water uptake: Root extraction alters the water potential and moisture distribution in the rhizosphere, while extended root systems can hydraulically connect soil regions with different water availability, allowing water to be redistributed between relatively wet and dry regions through root-mediated hydraulic processes [24–26]. At the ecosystem scale, these processes imply that soil-water movement may depend not only on gradients of soil moisture itself, but also on the spatial distribution of vegetation. To represent this vegetation-induced redistribution in a parsimonious manner, we introduce the effective water flux $\mathbf{J}_w = -\delta_w \nabla w + \gamma \nabla p$, where $\gamma \geq 0$ measures the strength of the coupling between vegetation heterogeneity and soil-water redistribution. Accordingly, $-\nabla \cdot \mathbf{J}_w = \delta_w \nabla^2 w - \gamma \nabla^2 p$. Hence, the term $-\gamma \nabla^2 p$ represents an effective cross-diffusion mechanism through which spatial gradients in vegetation influence soil-water redistribution. This term is phenomenological: It does not explicitly resolve water transport within individual root systems, but captures their net effect on ecosystem-scale soil-water movement. The resulting system reads:

$$\begin{cases} \partial_t p(x, t) = g w(x, t) p(x, t)(1 - p(x, t))(1 + \eta p(x, t))^2 - p(x, t) + \nabla^2 p(x, t), & x \in \Omega, t > 0, \\ \partial_t w(x, t) = l - v w(x, t)(1 - \rho p(x, t)) - g w(x, t) p(x, t)(1 + \eta p(x, t))^2 \\ \quad + \delta_w \nabla^2 w(x, t) - \gamma \nabla^2 p(x, t), & x \in \Omega, t > 0. \end{cases} \quad (2.2)$$

Before addressing the optimal-control problem, we analyze how the cross-diffusion term affects the stability of the uncontrolled system. In what follows, for notational simplicity, we shall write p and w instead of $p(x, t)$ and $w(x, t)$, respectively, with $x \in \Omega$ and $t > 0$. Before introducing the Robin boundary regulation, we first analyze the intrinsic diffusion-driven instability of the vegetation–water system (2.2) under homogeneous Neumann boundary conditions. This analysis isolates the pattern-forming mechanism generated by reaction, ordinary diffusion, and cross-diffusion from the subsequent

boundary-control effect. Let $E^* = (p^*, w^*)$ be a spatially homogeneous steady state, satisfying

$$\begin{cases} gw^*p^*(1-p^*)(1+\eta p^*)^2 - p^* = 0, \\ l - vw^*(1-\rho p^*) - gw^*p^*(1+\eta p^*)^2 = 0. \end{cases}$$

To study the stability of E^* , we introduce small perturbations

$$p(x, t) = p^* + \tilde{p}(x, t), \quad w(x, t) = w^* + \tilde{w}(x, t).$$

Substituting into (2.2) and retaining only linear terms, we obtain the linearized system

$$\begin{cases} \partial_t \tilde{p} = a_{11} \tilde{p} + a_{12} \tilde{w} + \nabla^2 \tilde{p}, \\ \partial_t \tilde{w} = a_{21} \tilde{p} + a_{22} \tilde{w} + \delta_w \nabla^2 \tilde{w} - \gamma \nabla^2 \tilde{p}, \end{cases}$$

where a_{11} , a_{12} , a_{21} , and a_{22} are the elements of the Jacobian matrix of the reaction terms evaluated at E^* :

$$\begin{aligned} a_{11} &= gw^* \frac{\partial}{\partial p} [p(1-p)(1+\eta p)^2] \Big|_{p=p^*} - 1, \\ a_{12} &= gp^*(1-p^*)(1+\eta p^*)^2, \\ a_{21} &= vw^*\rho - gw^*[(1+\eta p^*)^2 + 2p^*\eta(1+\eta p^*)], \\ a_{22} &= -v(1-\rho p^*) - gp^*(1+\eta p^*)^2. \end{aligned}$$

Now expand the perturbations into Fourier modes:

$$\begin{pmatrix} \tilde{p} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\lambda t + ik \cdot x},$$

where $k = |\mathbf{k}|$ is the wave number and λ is the growth rate. Substituting into the linearized system yields the characteristic equation

$$\lambda^2 - \text{tr}(J_k)\lambda + \det(J_k) = 0,$$

with

$$J_k = \begin{pmatrix} a_{11} - k^2 & a_{12} \\ a_{21} + \gamma k^2 & a_{22} - \delta_w k^2 \end{pmatrix}.$$

Thus,

$$\text{tr}(J_k) = a_{11} + a_{22} - (1 + \delta_w)k^2,$$

and

$$\det(J_k) = (a_{11} - k^2)(a_{22} - \delta_w k^2) - a_{12}(a_{21} + \gamma k^2).$$

The steady state E^* is linearly stable if, for all wave numbers k , all eigenvalues λ have negative real parts. For a quadratic characteristic equation, this is equivalent to

$$\text{tr}(J_k) < 0, \quad \det(J_k) > 0, \quad \forall k \geq 0.$$

If the system is stable without diffusion, i.e.,

$$a_{11} + a_{22} < 0, \quad a_{11}a_{22} - a_{12}a_{21} > 0,$$

then the trace condition is automatically satisfied for all k , since

$$\operatorname{tr}(J_k) = a_{11} + a_{22} - (1 + \delta_w)k^2 < a_{11} + a_{22} < 0.$$

Therefore, the occurrence of Turing instability is determined by the sign of $\det(J_k)$.

Expanding $\det(J_k)$ gives

$$\begin{aligned}\det(J_k) &= (a_{11} - k^2)(a_{22} - \delta_w k^2) - a_{12}(a_{21} + \gamma k^2) \\ &= \delta_w k^4 - (\delta_w a_{11} + a_{22} + \gamma a_{12})k^2 + (a_{11}a_{22} - a_{12}a_{21}).\end{aligned}$$

For the spatially homogeneous mode $k = 0$, the stability of E^* implies

$$\det(J_0) = a_{11}a_{22} - a_{12}a_{21} > 0.$$

For $k > 0$, the determinant of J_k can be written as

$$\det(J_k) = \delta_w k^4 - (\delta_w a_{11} + a_{22} + \gamma a_{12})k^2 + \det(J_0).$$

Introducing

$$B(\gamma) = \delta_w a_{11} + a_{22} + \gamma a_{12},$$

we obtain

$$\det(J_k) = \delta_w k^4 - B(\gamma)k^2 + \det(J_0).$$

To determine whether $\det(J_k)$ can become negative for some spatial modes, we set $q = k^2 \geq 0$. Then

$$\det(J_k) = \delta_w q^2 - B(\gamma)q + \det(J_0),$$

which is a convex quadratic function of q since $\delta_w > 0$.

If

$$B(\gamma) \leq 0,$$

the minimum over $q \geq 0$ is attained at $q = 0$, and hence

$$\det(J_k) \geq \det(J_0) > 0.$$

Therefore, no Turing instability occurs in this case.

If

$$B(\gamma) > 0,$$

the minimum is attained at

$$q_{\min} = k_{\min}^2 = \frac{B(\gamma)}{2\delta_w} = \frac{\delta_w a_{11} + a_{22} + \gamma a_{12}}{2\delta_w}.$$

The minimum value is

$$\det(J_k)_{\min} = \det(J_0) - \frac{B(\gamma)^2}{4\delta_w},$$

that is,

$$\det(J_k)_{\min} = \det(J_0) - \frac{(\delta_w a_{11} + a_{22} + \gamma a_{12})^2}{4\delta_w}.$$

Therefore, Turing instability occurs if and only if

$$B(\gamma) > 0,$$

and

$$B(\gamma)^2 > 4\delta_w \det(J_0),$$

or equivalently,

$$(\delta_w a_{11} + a_{22} + \gamma a_{12})^2 > 4\delta_w (a_{11}a_{22} - a_{12}a_{21}).$$

For the positive homogeneous steady state considered here, one has

$$a_{12} = gp^*(1 - p^*)(1 + \eta p^*)^2 > 0.$$

Hence, increasing the cross-diffusion coefficient γ increases $B(\gamma)$. Provided that

$$2\sqrt{\delta_w \det(J_0)} > \delta_w a_{11} + a_{22},$$

there exists a positive critical cross-diffusion coefficient γ_T determined by

$$B(\gamma_T) = 2\sqrt{\delta_w \det(J_0)}.$$

Consequently,

$$\gamma_T = \frac{2\sqrt{\delta_w \det(J_0)} - (\delta_w a_{11} + a_{22})}{a_{12}}.$$

Therefore,

- if $0 \leq \gamma < \gamma_T$, then

$$\det(J_k) > 0 \quad \text{for all } k \geq 0,$$

and the homogeneous steady state remains linearly stable;

- if $\gamma > \gamma_T$, then there exists a nonzero wave number k such that

$$\det(J_k) < 0,$$

and the homogeneous steady state becomes Turing-unstable.

When $\gamma > \gamma_T$, diffusion-driven instability may amplify spatial perturbations and lead to the formation of nonhomogeneous vegetation patterns.

The above instability analysis characterizes the intrinsic spatial dynamics of the cross-diffusive vegetation–water system. It should be emphasized that the subsequent optimal-control problem does not require the homogeneous equilibrium to remain linearly stable. Instead, the boundary control $\frac{\partial w}{\partial n} = (u(t) - a)w$ modifies the soil-water distribution through the controlled boundary. The optimal-control problem considered below seeks a feasible boundary regulation that drives the terminal vegetation distribution toward the prescribed target while accounting for the associated control effort.

In classical models, a homogeneous Neumann boundary condition is often imposed, representing a closed system with no material exchange, which is common in ecological modeling. However, real ecosystem boundaries are often subject to human intervention, such as irrigation canals or drainage

outlets. Robin boundary conditions can better characterize water flux at the boundary. Inspired by the above analysis and references [27, 28], we consider the Robin boundary condition

$$\frac{\partial w}{\partial n} = -aw,$$

where $a \geq 0$ denotes the natural seepage coefficient at the boundary. Since the diffusive flux associated with the term $\delta_w \Delta w$ is opposite to the concentration gradient, $a > 0$ represents outward water loss through the boundary. When $a = 0$, the condition reduces to the homogeneous Neumann boundary condition.

Further considering boundary regulation, the water boundary condition is written as

$$\frac{\partial w}{\partial n} = (u(t) - a)w, \quad (x, t) \in \Gamma_t \times (0, T), \quad (2.3)$$

where a denotes the baseline boundary-exchange coefficient and $u(t)$ is the boundary water-regulation variable. The difference $a - u(t)$ determines the Robin coefficient, while the physical direction of diffusive water exchange is determined by the corresponding Fickian flux. The boundary $\Gamma = \partial\Omega$ is divided into two parts: Γ_r (natural boundary) and Γ_t (controlled boundary). This chapter focuses only on implementing control on Γ_t , leading to the following mixed boundary conditions:

$$\begin{cases} \frac{\partial p}{\partial n} = 0, & (x, t) \in \Gamma \times (0, T), \\ \frac{\partial w}{\partial n} = 0, & (x, t) \in \Gamma_r \times (0, T), \\ \frac{\partial w}{\partial n} = (u(t) - a)w, & (x, t) \in \Gamma_t \times (0, T). \end{cases} \quad (2.4)$$

Remark 2.1. Boundary condition (2.4) describes the competition between natural seepage and artificial boundary irrigation. On the uncontrolled portion Γ_r , soil water satisfies a no-flux condition. On the controlled portion Γ_t , the coefficient $a > 0$ represents natural water loss, whereas $u(t) \geq 0$ represents the intensity of artificial irrigation.

In particular, if $u(t) < a$, natural seepage dominates, and the boundary exhibits net outward water exchange. If $u(t) = a$, the two effects balance, and the controlled boundary reduces to the homogeneous Neumann condition. If $u(t) > a$, artificial irrigation exceeds natural seepage, and water is supplied into the ecosystem through the controlled boundary. Vegetation satisfies the no-flux condition on the entire boundary, so seed dispersal across the boundary is neglected.

Definition 2.2. A pair $(p, w) \in W(0, T) \times W(0, T)$ is called a weak solution of model (2.1) with initial condition (2.2) and boundary condition (2.4) if for every test function $\zeta \in H^1(\Omega)$ and almost every $t \in (0, T)$, we have

$$\langle p_t(t), \zeta \rangle_{H^1(\Omega)', H^1(\Omega)} = \int_{\Omega} \left(-\nabla p(t) \cdot \nabla \zeta + g w(t) p(t) (1 - p(t)) (1 + \eta p(t))^2 \zeta - p(t) \zeta \right) dx, \quad (2.5)$$

$$\begin{aligned} \langle w_t(t), \zeta \rangle_{H^1(\Omega)', H^1(\Omega)} = & \int_{\Omega} \left(-\delta_w \nabla w(t) \cdot \nabla \zeta + \gamma \nabla p(t) \cdot \nabla \zeta + l \zeta - v w(t) (1 - \rho p(t)) \zeta \right. \\ & \left. - g w(t) p(t) (1 + \eta p(t))^2 \zeta \right) dx + \delta_w (a - u(t)) \int_{\Gamma_t} w(t) \zeta dS, \end{aligned} \quad (2.6)$$

and the initial conditions $p(0) = \phi_1$, $w(0) = \phi_2$ hold in $L^2(\Omega)$.

Theorem 2.3. For every control $u \in L^2(0, T)$, there exists a unique weak solution $(p, w) \in W(0, T) \times W(0, T)$ of model (2.1) satisfying (2.5) and (2.6) with initial condition (2.2) and boundary condition (2.4). Moreover, there exists a constant $C_1 > 0$ such that

$$\|p\|_{W(0,T)} + \|w\|_{W(0,T)} \leq C_1(1 + \|\phi_1\|_{L^2(\Omega)} + \|\phi_2\|_{L^2(\Omega)} + \|u\|_{L^2(0,T)}).$$

Proof. We first derive an a priori estimate for the state variables. Taking the test functions $\zeta = p$ and $\zeta = w$ in the weak formulations (2.5) and (2.6), respectively, and adding the resulting identities, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|p\|_{L^2(\Omega)}^2 + \|w\|_{L^2(\Omega)}^2) + \|\nabla p\|_{L^2(\Omega)}^2 + \delta_w \|\nabla w\|_{L^2(\Omega)}^2 \\ &= \int_{\Omega} gwp^2(1-p)(1+\eta p)^2 dx - \int_{\Omega} p^2 dx + \int_{\Omega} lw dx - \int_{\Omega} vw^2(1-\rho p) dx \\ & \quad - \int_{\Omega} gw^2p(1+\eta p)^2 dx + \gamma \int_{\Omega} \nabla p \cdot \nabla w dx + \delta_w(u(t) - a) \int_{\Gamma_i} w^2 dS. \end{aligned} \quad (2.7)$$

By the positivity and invariant-region properties of the vegetation equation, the vegetation density is nonnegative and uniformly bounded; namely, there exists a constant $M_p > 0$, independent of time, such that

$$0 \leq p(x, t) \leq M_p \quad \text{for a.e. } (x, t) \in \Omega \times (0, T).$$

Consequently, all polynomial functions of p appearing in the reaction terms are uniformly bounded on $[0, M_p]$.

Define

$$C_p = \max_{0 \leq s \leq M_p} |s^2(1-s)(1+\eta s)^2|.$$

Then

$$\begin{aligned} \left| \int_{\Omega} gwp^2(1-p)(1+\eta p)^2 dx \right| &\leq gC_p \int_{\Omega} |w| dx \leq gC_p |\Omega|^{1/2} \|w\|_{L^2(\Omega)} \\ &\leq C(1 + \|w\|_{L^2(\Omega)}^2), \end{aligned} \quad (2.8)$$

where $C > 0$ denotes a generic constant independent of p , w , and t .

Since $p \geq 0$, we also have

$$-\int_{\Omega} gw^2p(1+\eta p)^2 dx \leq 0. \quad (2.9)$$

Furthermore, because p is uniformly bounded, there exists

$$C_{\rho} = v \max_{0 \leq s \leq M_p} |1 - \rho s|$$

such that

$$-\int_{\Omega} vw^2(1-\rho p) dx \leq C_{\rho} \|w\|_{L^2(\Omega)}^2. \quad (2.10)$$

For the linear water-input term, Young's inequality gives

$$\int_{\Omega} lw dx \leq l|\Omega|^{1/2} \|w\|_{L^2(\Omega)} \leq \frac{1}{2} \|w\|_{L^2(\Omega)}^2 + \frac{l^2|\Omega|}{2}, \quad (2.11)$$

while

$$-\int_{\Omega} p^2 \, dx \leq 0. \quad (2.12)$$

We next estimate the cross-diffusion term. By the weighted Young inequality,

$$\gamma \int_{\Omega} \nabla p \cdot \nabla w \, dx \leq \gamma \|\nabla p\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)} \leq \frac{\gamma}{2\sqrt{\delta_w}} \|\nabla p\|_{L^2(\Omega)}^2 + \frac{\gamma\sqrt{\delta_w}}{2} \|\nabla w\|_{L^2(\Omega)}^2. \quad (2.13)$$

Hence,

$$\begin{aligned} & \|\nabla p\|_{L^2(\Omega)}^2 + \delta_w \|\nabla w\|_{L^2(\Omega)}^2 - \gamma \int_{\Omega} \nabla p \cdot \nabla w \, dx \\ & \geq \left(1 - \frac{\gamma}{2\sqrt{\delta_w}}\right) \|\nabla p\|_{L^2(\Omega)}^2 + \left(\delta_w - \frac{\gamma\sqrt{\delta_w}}{2}\right) \|\nabla w\|_{L^2(\Omega)}^2. \end{aligned}$$

We therefore define

$$D_{\min} = \min \left\{ 1 - \frac{\gamma}{2\sqrt{\delta_w}}, \delta_w - \frac{\gamma\sqrt{\delta_w}}{2} \right\}.$$

Since

$$0 \leq \gamma < 2\sqrt{\delta_w},$$

both coefficients are strictly positive and thus

$$D_{\min} > 0.$$

For the boundary term, the multiplicative trace inequality yields

$$\|w\|_{L^2(\Gamma_t)}^2 \leq C_{\Gamma} \|w\|_{H^1(\Omega)} \|w\|_{L^2(\Omega)}.$$

Therefore, for every $\varepsilon > 0$,

$$\begin{aligned} \left| \delta_w (u(t) - a) \int_{\Gamma_t} w^2 \, dS \right| & \leq \delta_w C_{\Gamma} (|u(t)| + a) \|w\|_{H^1(\Omega)} \|w\|_{L^2(\Omega)} \\ & \leq \varepsilon \|w\|_{H^1(\Omega)}^2 + C_{\varepsilon} (1 + |u(t)|^2) \|w\|_{L^2(\Omega)}^2, \end{aligned} \quad (2.14)$$

where $C_{\varepsilon} > 0$ depends on δ_w , a , C_{Γ} , and ε .

Choosing $\varepsilon > 0$ sufficiently small, for instance $\varepsilon < D_{\min}/2$, the gradient contribution in (2.14) can be absorbed into the left-hand side of (2.7). Combining (2.8)–(2.14), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|p\|_{L^2(\Omega)}^2 + \|w\|_{L^2(\Omega)}^2 \right) + \frac{D_{\min}}{2} \left(\|\nabla p\|_{L^2(\Omega)}^2 + \|\nabla w\|_{L^2(\Omega)}^2 \right) \\ & \leq C_0 + C_1 (1 + |u(t)|^2) \left(\|p\|_{L^2(\Omega)}^2 + \|w\|_{L^2(\Omega)}^2 \right), \end{aligned} \quad (2.15)$$

where $C_0, C_1 > 0$ depend only on the model parameters, the domain, and the uniform bound M_p .

Set

$$Y(t) = \|p(t)\|_{L^2(\Omega)}^2 + \|w(t)\|_{L^2(\Omega)}^2.$$

It follows from (2.15) that

$$Y'(t) \leq 2C_0 + 2C_1 (1 + |u(t)|^2) Y(t).$$

Since $u \in U_{ad} \subset L^\infty(0, T) \subset L^2(0, T)$, Gronwall's inequality gives

$$Y(t) \leq (Y(0) + 2C_0 t) \exp \left[2C_1 \left(t + \|u\|_{L^2(0,t)}^2 \right) \right], \quad 0 \leq t \leq T. \quad (2.16)$$

Consequently,

$$p, w \in L^\infty(0, T; L^2(\Omega)).$$

Integrating (2.15) over $(0, T)$ and using (2.16), we further obtain

$$\int_0^T \left(\|\nabla p(t)\|_{L^2(\Omega)}^2 + \|\nabla w(t)\|_{L^2(\Omega)}^2 \right) dt \leq C,$$

and therefore

$$p, w \in L^2(0, T; H^1(\Omega)).$$

The reaction terms are bounded in the corresponding dual spaces because p is uniformly bounded and $w \in L^2(0, T; H^1(\Omega))$. Moreover, since the admissible controls are pointwise bounded, $u \in L^\infty(0, T)$, and the Robin boundary term defines an element of $L^2(0, T; H^1(\Omega)^*)$. It follows from the state equations that

$$p_t, w_t \in L^2(0, T; H^1(\Omega)^*).$$

Hence,

$$p, w \in W(0, T).$$

A global weak solution can now be obtained by a standard Galerkin approximation. The estimates above are uniform with respect to the Galerkin dimension, and the compactness theorem of Aubin–Lions allows passage to the limit in the nonlinear terms. Similar boundedness arguments for reaction–diffusion systems can be found in [29]. Since the estimate (2.16) holds on the whole interval $[0, T]$, no finite-time blow-up occurs and the weak solution exists globally.

Finally, uniqueness follows by considering two weak solutions with the same initial data and control, subtracting their equations, and testing the difference equations by the corresponding state differences. Using the boundedness of p , the local Lipschitz continuity of the reaction terms on bounded sets, the trace inequality, and the estimate (2.13), one obtains

$$\begin{aligned} \frac{d}{dt} \left(\|p_1 - p_2\|_{L^2(\Omega)}^2 + \|w_1 - w_2\|_{L^2(\Omega)}^2 \right) &\leq C(1 + |u(t)|^2) \\ &\quad \times \left(\|p_1 - p_2\|_{L^2(\Omega)}^2 + \|w_1 - w_2\|_{L^2(\Omega)}^2 \right). \end{aligned}$$

Gronwall's inequality, together with the identical initial data, implies $p_1 = p_2$ and $w_1 = w_2$ almost everywhere in $\Omega \times (0, T)$. This completes the proof. $\square$

Define the following state spaces:

$$\mathcal{X} = W(0, T) \times W(0, T) \times L^2(0, T),$$

$$\mathcal{Y} = L^2(0, T; H^1(\Omega)) \times L^2(0, T; H^1(\Omega)) \times L^2(\Omega) \times L^2(\Omega),$$

and

$$\mathcal{Y}' = L^2(0, T; H^1(\Omega)') \times L^2(0, T; H^1(\Omega)') \times L^2(\Omega) \times L^2(\Omega).$$

Furthermore, consider the mapping $e : \mathcal{X} \rightarrow \mathcal{Y}'$ defined by

$$e(x) = \begin{pmatrix} e_1(x) \\ e_2(x) \\ p(0) - \phi_1 \\ w(0) - \phi_2 \end{pmatrix}, \quad x = (p, w, u) \in \mathcal{X},$$

where the operators $e_1, e_2 : \mathcal{X} \times L^2(0, T; H^1(\Omega)) \rightarrow L^2(0, T; H^1(\Omega)')$ involve the variational formulations of p and w , respectively. For any $\zeta \in L^2(0, T; H^1(\Omega))$, we have

$$\begin{aligned} \langle e_1(x), \zeta \rangle &= \int_0^T \left(\langle p_t(t), \zeta(t) \rangle_{H^1(\Omega)', H^1(\Omega)} \right. \\ &\quad \left. + \int_{\Omega} (\nabla p(t) \cdot \nabla \zeta(t) - gw(t)p(t)(1-p(t))(1+\eta p(t))^2 \zeta(t) + p(t)\zeta(t)) dx \right) dt, \\ \langle e_2(x), \zeta \rangle &= \int_0^T \left(\langle w_t(t), \zeta(t) \rangle_{H^1(\Omega)', H^1(\Omega)} \right. \\ &\quad \left. + \int_{\Omega} (\delta_w \nabla w(t) \cdot \nabla \zeta(t) - \gamma \nabla p(t) \cdot \nabla \zeta(t) - l\zeta(t) + vw(t)(1-\rho p(t))\zeta(t) \right. \\ &\quad \left. + gw(t)p(t)(1+\eta p(t))^2 \zeta(t)) dx + \delta_w(a-u(t)) \int_{\Gamma_i} w(t)\zeta(t) dS \right) dt. \end{aligned}$$

Equations (2.5) and (2.6) are equivalent to $e(x) = 0$ in $\mathcal{Y}'$ for $x = (p, w, u) \in \mathcal{X}$.

3. Existence of optimal control

This section aims to investigate the existence of a boundary optimal control for model (2.1) with the boundary condition (2.4). First, we define the following cost functional:

$$J(u) = \frac{1}{2} \int_{\Omega} \beta_1 |p(T) - p_T|^2 dx + \frac{1}{2} \int_0^T \beta_2 |u - u^e|^2 dt, \quad (3.1)$$

where $\beta_i > 0$, $p_T \in L^2(\Omega)$ denotes the desired terminal vegetation density, $u^e \in L^2(0, T)$ denotes the prescribed baseline boundary-regulation level, $i = 1, 2$, $p_T := p_T(x) \in L^2(\Omega)$ is the desired terminal vegetation density, $u^e := u^e(t) \in L^2(0, T)$ is the baseline control, and $\beta_1, \beta_2 > 0$ are weight coefficients.

Remark 3.1. The cost functional in (3.1) consists of two terms with distinct roles. The terminal term

$$\frac{1}{2} \int_{\Omega} \beta_1 |p(T) - p_T|^2 dx$$

measures the deviation of the terminal vegetation distribution from the desired ecological target. The running term

$$\frac{1}{2} \int_0^T \beta_2 |u - u^e|^2 dt$$

penalizes excessive deviation of the irrigation-control coefficient from the prescribed baseline level u^e .

Thus, minimizing $J(u)$ seeks a compromise between vegetation-restoration performance and the cost of sustained boundary intervention. Since $u(t)$ is the coefficient appearing in the Robin boundary condition rather than a direct volumetric water-flow rate, the integral constraint introduced below should be interpreted as a bound on the cumulative irrigation-control effort.

Define the admissible control set as

$$U_{\text{ad}} = \left\{ u \in L^2(0, T) : \int_0^T u(t) dt \leq u_a \text{ and } u_b \leq u \leq u_c \right\} \subset L^\infty(0, T), \quad (3.2)$$

where $u_a > 0$ denotes the maximum admissible cumulative irrigation-control effort over the control horizon, while $u_b, u_c \in L^\infty(0, T)$ denote the lower and upper admissible bounds of the boundary irrigation-control intensity, respectively.

Remark 3.2. Problem (3.3) involves both pointwise control constraints and an integral constraint. The condition

$$u_b(t) \leq u(t) \leq u_c(t)$$

restricts the instantaneous irrigation-control intensity to an admissible operating range. The integral constraint

$$\int_0^T u(t) dt \leq u_a$$

limits the cumulative boundary-regulation effort over the entire control horizon. These constraints prevent unrealistically large or excessively prolonged boundary intervention and provide a mathematical representation of limited irrigation-management capacity.

Define

$$\mathcal{U}_{\text{ad}} = W(0, T) \times W(0, T) \times U_{\text{ad}}.$$

Based on the above definitions, the goal of this chapter is to minimize the control cost while making the terminal vegetation density as close as possible to the desired value. This requires finding an optimal-control strategy that minimizes $J(u)$. The optimal boundary-control problem can then be expressed as

$$\min_{u \in U_{\text{ad}}} J(u) \quad \text{subject to} \quad e(x) = 0. \quad (3.3)$$

Since the integral constraint is considered, a new objective functional needs to be constructed. Before that, we first discuss the existence of an optimal boundary control for problem (3.3).

Lemma 3.3. *There exists an optimal control u^* such that*

$$J(u^*) = \min_{u \in U_{\text{ad}}} J(u)$$

holds.

Proof. From Theorem 2.3, the sequence $\{(p^n, w^n, u^n)\}_{n=1}^\infty$ is bounded in $W(0, T) \times W(0, T) \times L^2(0, T)$. Therefore, the sequence $\{u^n\}_{n=1}^\infty$ is bounded in $L^2(0, T)$, and $\{p^n\}, \{w^n\}$ are bounded in $W(0, T)$. Since $W(0, T)$ is reflexive, there exists a subsequence (still denoted by the same notation) and a limit element $(p^*, w^*, u^*) \in W(0, T) \times W(0, T) \times L^2(0, T)$ such that as $n \rightarrow \infty$,

$$p^n \rightarrow p^* \quad \text{in } W(0, T), \quad w^n \rightarrow w^* \quad \text{in } W(0, T), \quad u^n \rightarrow u^* \quad \text{in } L^2(0, T). \quad (3.4)$$

From (3.4), for any $\zeta \in L^2(0, T; H^1(\Omega))$,

$$\lim_{n \rightarrow \infty} \int_0^T \langle (p^n(t) - p^*(t)), \zeta(t) \rangle_{H^1(\Omega)', H^1(\Omega)} dt = 0,$$

$$\lim_{n \rightarrow \infty} \int_0^T \int_{\Omega} \nabla(p^n(t) - p^*(t)) \cdot \nabla \zeta(t) \, dx dt = 0.$$

Using Hölder's inequality,

$$\begin{aligned} & \int_0^T \int_{\Omega} (gw^n p^n (1 - p^n)(1 + \eta p^n)^2 - gw^* p^* (1 - p^*)(1 + \eta p^*)^2) \zeta \, dx dt \\ &= \int_0^T \int_{\Omega} [gw^n p^n (1 - p^n)(1 + \eta p^n)^2 - gw^* p^n (1 - p^n)(1 + \eta p^n)^2 \\ & \quad + gw^* p^n (1 - p^n)(1 + \eta p^n)^2 - gw^* p^* (1 - p^*)(1 + \eta p^*)^2] \zeta \, dx dt. \end{aligned} \quad (3.5)$$

Since $0 \leq p \leq 1$, the nonlinear terms are smooth and bounded, so they can be decomposed as

$$\int_0^T \int_{\Omega} g(w^n - w^*) p^n (1 - p^n)(1 + \eta p^n)^2 \zeta \, dx dt + \int_0^T \int_{\Omega} gw^* [p^n (1 - p^n)(1 + \eta p^n)^2 - p^* (1 - p^*)(1 + \eta p^*)^2] \zeta \, dx dt.$$

Since $\{p^n\}$ is bounded in $L^2(0, T; H^1(\Omega))$ and $W(0, T)$ is compactly embedded into $L^2(0, T; L^3(\Omega))$ (in three spatial dimensions), we obtain

$$\lim_{n \rightarrow \infty} \int_0^T \int_{\Omega} (w^n - w^*) \zeta \, dx dt = 0, \quad \lim_{n \rightarrow \infty} \int_0^T \int_{\Omega} (p^n - p^*) \zeta \, dx dt = 0.$$

Hence, the nonlinear term converges to zero, and the same argument applies to $-gwp(1 + \eta p)^2$. Moreover, by the trace inequality $\|\phi\|_{L^2(\Gamma_t)} \leq C\|\phi\|_{H^1(\Omega)}$, for every $\xi \in L^2(0, T; H^1(\Omega))$, we have $t \mapsto \int_{\Gamma_t} w(t)\xi(t) \, dS \in L^2(0, T)$, and

$$\lim_{n \rightarrow \infty} \int_0^T \int_{\Gamma_t} (u^n(t)w^n(t) - u^*(t)w^*(t))\xi(t) \, dS \, dt = 0.$$

Thus, $e(p^*, w^*, u^*) = 0$ in $\mathcal{Y}'$. Moreover, since U_{ad} is bounded, closed, and convex, it is weakly closed, so $u^* \in U_{\text{ad}}$. Since J is weakly lower semicontinuous, we have

$$J(u^*) \leq \liminf_{n \rightarrow \infty} J(u^n) = \min_{u \in U_{\text{ad}}} J(u).$$

This completes the proof. $\square$

The admissible control set U_{ad} involves both an integral constraint and bilateral inequality constraints. To handle the integral constraint

$$\int_0^T u(t) \, dt \leq u_a, \quad (3.6)$$

we introduce the following penalized cost functional:

$$J_k(u) = J(u) + \frac{1}{k} K(u), \quad k > 0,$$

where the mapping $K : L^2(0, T) \rightarrow \mathbb{R}$ is defined by

$$K(u) = f\left(\int_0^T u(t) \, dt - u_a\right), \quad f(s) = [s]_+^3,$$

and $[s]_+ = \max\{0, s\}$.

Since the integral constraint is now included in the cost via the penalty term, the admissible control set reduces to

$$U_{\text{ad}}^k = \{u \in L^2(0, T) : u_b \leq u \leq u_c\} \subset L^\infty(0, T).$$

Denote $\mathcal{U}_{\text{ad}}^k = W(0, T) \times W(0, T) \times U_{\text{ad}}^k$. Consider the penalized optimal-control problem

$$\min_{u \in U_{\text{ad}}^k} J_k(u) \quad \text{subject to} \quad e(p, w, u) = 0. \quad (3.7)$$

We now analyze the existence of optimal controls for the penalized problem. First, we establish the smoothness properties of f and K .

Lemma 3.4. *The function $f(s) = [s]_+^3$ is twice differentiable, and its second derivative $f''(s) = 6[s]_+$ is Lipschitz continuous on $\mathbb{R}$. The mapping $K : L^2(0, T) \rightarrow \mathbb{R}$ is weakly continuous, and K is twice continuously Frechet differentiable on $L^2(0, T)$, with a Lipschitz continuous second Frechet derivative. Specifically, the first and second Frechet derivatives of K are given by*

$$\nabla K(u) \delta u = f' \left(\int_0^T u(t) dt - u_a \right) \int_0^T \delta u(t) dt, \quad (3.8)$$

$$\nabla^2 K(u)(\delta u, \tilde{\delta} u) = f'' \left(\int_0^T u(t) dt - u_a \right) \int_0^T \delta u(t) dt \int_0^T \tilde{\delta} u(t) dt, \quad (3.9)$$

for any $\delta u, \tilde{\delta} u \in L^2(0, T)$.

Proof. We first verify the twice differentiability of f . Clearly, $f(s) = 0$ for $s \leq 0$, and $f(s) = s^3$ for $s > 0$. At $s = 0$, we have $f'(0) = \lim_{h \rightarrow 0^+} 3h^2 = 0$ and $f''(0) = \lim_{h \rightarrow 0^+} 6h = 0$, so $f \in C^2(\mathbb{R})$ and $f''(s) = 6[s]_+$, which is Lipschitz continuous since $|6[s]_+ - 6[t]_+| \leq 6|s - t|$.

Next we prove weak continuity of K . Let $\{u_n\}_{n=1}^\infty \subset L^2(0, T)$ converge weakly to u . Then for $\zeta \equiv 1$, we have

$$\lim_{n \rightarrow \infty} \int_0^T u_n(t) dt = \int_0^T u(t) dt.$$

Since f is continuous,

$$\lim_{n \rightarrow \infty} K(u_n) = \lim_{n \rightarrow \infty} f \left(\int_0^T u_n dt - u_a \right) = f \left(\int_0^T u dt - u_a \right) = K(u),$$

so K is weakly continuous.

Now we prove Frechet differentiability. For any $u, \delta u \in L^2(0, T)$, set $L(u) = \int_0^T u dt - u_a$. Then

$$K(u + \delta u) - K(u) = f(L(u + \delta u)) - f(L(u)).$$

By the differentiability of f , there exists $\theta \in (0, 1)$ such that

$$f(L(u + \delta u)) - f(L(u)) = f'(L(u)) L(\delta u) + \frac{1}{2} f''(L(u) + \theta L(\delta u)) (L(\delta u))^2.$$

Note that $|L(\delta u)| \leq \sqrt{T}\|\delta u\|_{L^2(0,T)}$, and f'' is locally bounded, so the remainder term is $O(\|\delta u\|_{L^2}^2)$. Hence,

$$K(u + \delta u) - K(u) = f'(L(u)) \int_0^T \delta u \, dt + o(\|\delta u\|_{L^2(0,T)}),$$

which shows that $\nabla K(u)$ is given by (3.8) and is continuous.

For the second derivative, consider

$$\nabla K(u + \widetilde{\delta u}) \delta u - \nabla K(u) \delta u = [f'(L(u + \widetilde{\delta u})) - f'(L(u))] \int_0^T \delta u \, dt.$$

By the differentiability of f' ,

$$f'(L(u + \widetilde{\delta u})) - f'(L(u)) = f''(L(u)) L(\widetilde{\delta u}) + o(\|\widetilde{\delta u}\|_{L^2}),$$

so

$$\nabla K(u + \widetilde{\delta u}) \delta u - \nabla K(u) \delta u = f''(L(u)) \int_0^T \widetilde{\delta u} \, dt \int_0^T \delta u \, dt + o(\|\widetilde{\delta u}\|_{L^2}),$$

which yields $\nabla^2 K(u)$ as in (3.9). Since f'' is Lipschitz continuous, $\nabla^2 K$ is also Lipschitz continuous: For any $u_1, u_2 \in L^2(0, T)$,

$$\begin{aligned} & |\nabla^2 K(u_1)(\delta u, \widetilde{\delta u}) - \nabla^2 K(u_2)(\delta u, \widetilde{\delta u})| \\ & \leq |f''(L(u_1)) - f''(L(u_2))| \cdot |L(\delta u)| \cdot |L(\widetilde{\delta u})| \\ & \leq 6|L(u_1) - L(u_2)| \cdot \sqrt{T}\|\delta u\|_{L^2} \cdot \sqrt{T}\|\widetilde{\delta u}\|_{L^2} \\ & \leq 6T \sqrt{T}\|u_1 - u_2\|_{L^2} \|\delta u\|_{L^2} \|\widetilde{\delta u}\|_{L^2}. \end{aligned}$$

This completes the proof. $\square$

From Lemma 3.4, K is weakly continuous, so the penalized cost functional J_k is weakly lower semicontinuous on X . Similar to the proof of Lemma 3.3, we obtain the existence of an optimal control for the penalized problem (3.7).

Theorem 3.5. *There exists an optimal control u^* such that*

$$J_k(u^*) = \min_{u \in U_{\text{ad}}^k} J_k(u)$$

holds.

Lemma 3.6. *Let $\{k_m\}_{m=0}^\infty \subset \mathbb{R}$ be a sequence of positive numbers monotonically decreasing to zero, and let $\{x_m\}_{m=0}^\infty$ be a sequence of optimal solutions to the penalized problem (3.7), i.e., $x_m = (p_m, w_m, u_m)$ satisfies*

$$J_{k_m}(u_m) = \min_{u \in U_{\text{ad}}^{k_m}} J_{k_m}(u), \quad e(p_m, w_m, u_m) = 0.$$

Then the sequence $\{x_m\}$ possesses at least one weak accumulation point $x^ = (p^*, w^*, u^*) \in X$, i.e., there exists a subsequence (still denoted by $\{x_m\}$) such that as $m \rightarrow \infty$,*

$$p_m \rightharpoonup p^*, \quad w_m \rightharpoonup w^* \quad \text{in } W(0, T), \quad u_m \rightharpoonup u^* \quad \text{in } L^2(0, T).$$

Moreover, every such weak accumulation point x^ is an optimal solution to the original problem (3.3).*

Proof. Since $u_m \in U_{\text{ad}}^{k_m} \subset U_{\text{ad}}$, from the definition of the admissible set (3.2), we have $\|u_m\|_{L^2(0,T)}$ uniformly bounded. By Theorem 2.3 and the a priori estimate, the sequence $\{(p_m, w_m)\}$ is bounded in $W(0, T) \times W(0, T)$. Hence, $\{x_m\}$ is bounded in the product space $\mathcal{X} = W(0, T) \times W(0, T) \times L^2(0, T)$. Since $\mathcal{X}$ is reflexive, there exists a subsequence (still denoted by $\{x_m\}$) and an element $x^* = (p^*, w^*, u^*) \in \mathcal{X}$ such that

$$p_m \rightharpoonup p^*, \quad w_m \rightharpoonup w^* \quad \text{in } W(0, T), \quad u_m \rightharpoonup u^* \quad \text{in } L^2(0, T). \quad (3.10)$$

By compactness arguments as used in the proof of Lemma 3.3, from (3.4) we obtain $e(p^*, w^*, u^*) = 0$ and $u^* \in U_{\text{ad}}$. Thus, x^* is a feasible point for the original problem.

Since x_m is optimal for the penalized problem (3.7), for any feasible point $x = (p, w, u) \in \mathcal{U}_{\text{ad}}^{k_m}$ (i.e., $u \in U_{\text{ad}}^{k_m}$ and $e(x) = 0$), we have

$$J_{k_m}(x_m) = J(u_m) + \frac{1}{k_m} K(u_m) \leq J(u) + \frac{1}{k_m} K(u) = J(u). \quad (3.11)$$

In particular, for any feasible point $x \in \mathcal{U}_{\text{ad}}$ of the original problem, since $U_{\text{ad}} \subset U_{\text{ad}}^{k_m}$, inequality holds. Hence,

$$0 \leq K(u_m) \leq k_m(J(u) - J(u_m)), \quad \forall x \in \mathcal{U}_{\text{ad}}. \quad (3.12)$$

From the boundedness of $\{x_m\}$ and the continuity of J (which is bounded on weakly convergent sequences), $J(u_m)$ is bounded. Fix an arbitrary feasible point $x \in \mathcal{U}_{\text{ad}}$, and let $m \rightarrow \infty$; the right-hand side of (3.12) tends to 0, so $\lim_{m \rightarrow \infty} K(u_m) = 0$. By the weak continuity of K (Lemma 3.4) and $u_m \rightharpoonup u^*$, we obtain

$$K(u^*) = \lim_{m \rightarrow \infty} K(u_m) = 0.$$

Since $K(u^*) = 0$ is equivalent to $[\int_0^T u^*(t) dt - u_a]_+^3 = 0$, we have $\int_0^T u^*(t) dt \leq u_a$. Hence, $u^* \in U_{\text{ad}}$, and consequently $x^* \in \mathcal{U}_{\text{ad}}$.

Using the weak lower semicontinuity of J and (3.11), we get

$$J(u^*) \leq \liminf_{m \rightarrow \infty} J(u_m) \leq \liminf_{m \rightarrow \infty} (J(u_m) + \frac{1}{k_m} K(u_m)) = \liminf_{m \rightarrow \infty} J_{k_m}(x_m) \leq J(u), \quad \forall x \in \mathcal{U}_{\text{ad}}.$$

Therefore, $J(u^*) \leq \min_{x \in \mathcal{U}_{\text{ad}}} J(u)$, which means that x^* is an optimal solution to the original problem (3.3). $\square$

This subsection focuses on the first-order necessary optimality conditions for the penalized optimal-control problem (3.7). Before the discussion, we first examine the smoothness properties of the penalized cost functional J_k and the mapping e .

Proposition 3.7. *The penalized cost functional J_k and the mapping e are twice continuously Fréchet differentiable. Moreover, their second Fréchet derivatives are locally Lipschitz continuous on bounded subsets of $\mathcal{X}$.*

Proof. By Lemma 3.4, the penalty term K is twice continuously Fréchet differentiable, and its second Fréchet derivative is Lipschitz continuous, while J is a quadratic functional. We first focus on the first and second derivatives of J_k at $x = (p, w, u) \in \mathcal{X}$. Let $\alpha = (\alpha_p, \alpha_w, \alpha_u)$ and $\tilde{\alpha} = (\tilde{\alpha}_p, \tilde{\alpha}_w, \tilde{\alpha}_u)$. Then

$$\nabla J_k(x)\alpha = \int_{\Omega} \beta_1(p(T) - p_T)\alpha_p(T) dx + \int_0^T \beta_2(u - u^e)\alpha_u dt + \frac{1}{k} \nabla K(u)\alpha_u, \quad (3.13)$$

$$\nabla^2 J_k(x)(\alpha, \tilde{\alpha}) = \int_{\Omega} \beta_1 \alpha_p(T) \tilde{\alpha}_p(T) dx + \int_0^T \beta_2 \alpha_u \tilde{\alpha}_u dt + \frac{1}{k} \nabla^2 K(u)(\alpha_u, \tilde{\alpha}_u), \quad (3.14)$$

where $\nabla K(u)$ and $\nabla^2 K(u)$ are as given in Lemma 3.4, i.e.,

$$\nabla K(u)\alpha_u = f' \left(\int_0^T u dt - u_a \right) \int_0^T \alpha_u dt, \quad \nabla^2 K(u)(\alpha_u, \tilde{\alpha}_u) = f'' \left(\int_0^T u dt - u_a \right) \int_0^T \alpha_u dt \int_0^T \tilde{\alpha}_u dt.$$

Now we consider the operator e . For arbitrary $\alpha, \tilde{\alpha}$, and test functions $\zeta, \xi \in L^2(0, T; H^1(\Omega))$, we have

$$\begin{aligned} & \langle \nabla e_1(x)\alpha, \zeta \rangle \\ &= \int_0^T \left(\langle (\alpha_p)_t(t), \zeta(t) \rangle_{H^1(\Omega)', H^1(\Omega)} + \int_{\Omega} D_p \nabla \alpha_p(t) \cdot \nabla \zeta(t) dx \right. \\ &+ \int_{\Omega} (-gw(t)\alpha_p(t)(1-p(t))(1+\eta p(t))^2 \zeta(t) - gp(t)\alpha_w(t)(1-p(t))(1+\eta p(t))^2 \zeta(t) \\ &\left. - gw(t)p(t)[-(1+\eta p(t))^2 + 2\eta(1-p(t))(1+\eta p(t))]\alpha_p(t)\zeta(t)) dx + \int_{\Omega} \alpha_p(t)\zeta(t) dx \right) dt, \end{aligned} \quad (3.15)$$

$$\begin{aligned} \langle \nabla e_2(x)\alpha, \xi \rangle &= \int_0^T \left[\langle (\alpha_w)_t(t), \xi(t) \rangle_{H^1(\Omega)', H^1(\Omega)} + \int_{\Omega} \delta_w \nabla \alpha_w(t) \cdot \nabla \xi(t) dx - \gamma \int_{\Omega} \nabla \alpha_p(t) \cdot \nabla \xi(t) dx \right. \\ &+ \int_{\Omega} [\nu \alpha_w(t)(1-\rho p(t)) - \nu \rho w(t)\alpha_p(t) + g\alpha_w(t)p(t)(1+\eta p(t))^2 \\ &+ gw(t)\alpha_p(t)(1+\eta p(t))^2 + 2g\eta w(t)p(t)(1+\eta p(t))\alpha_p(t)] \xi(t) dx \\ &\left. + \delta_w(a-u(t)) \int_{\Gamma_t} \alpha_w(t)\xi(t) dS - \delta_w \alpha_u(t) \int_{\Gamma_t} w(t)\xi(t) dS \right] dt, \end{aligned} \quad (3.16)$$

and

$$\nabla e_3(x)\alpha = \alpha_p(0), \quad \nabla e_4(x)\alpha = \alpha_w(0). \quad (3.17)$$

The second derivatives of e are given by

$$\begin{aligned} \langle \nabla^2 e_1(x)(\alpha, \tilde{\alpha}), \zeta \rangle &= -g \int_0^T \int_{\Omega} \left\{ [(1-2p)(1+\eta p)^2 + 2\eta p(1-p)(1+\eta p)](\alpha_w \tilde{\alpha}_p + \tilde{\alpha}_w \alpha_p) \right. \\ &\left. + w[-2(1+\eta p)^2 + 4\eta(1-2p)(1+\eta p) + 2\eta^2 p(1-p)]\alpha_p \tilde{\alpha}_p \right\} \zeta dx dt, \end{aligned} \quad (3.18)$$

$$\begin{aligned} \langle \nabla^2 e_2(x)(\alpha, \tilde{\alpha}), \xi \rangle &= \int_0^T \int_{\Omega} \left\{ -\nu \rho (\alpha_p \tilde{\alpha}_w + \tilde{\alpha}_p \alpha_w) + g[(1+\eta p)^2 + 2\eta p(1+\eta p)](\alpha_w \tilde{\alpha}_p + \tilde{\alpha}_w \alpha_p) \right. \\ &\left. + gw[4\eta(1+\eta p) + 2\eta^2 p]\alpha_p \tilde{\alpha}_p \right\} \xi dx dt - \delta_w \int_0^T \int_{\Gamma_t} (\tilde{\alpha}_u \alpha_w + \alpha_u \tilde{\alpha}_w) \xi dS dt, \end{aligned} \quad (3.19)$$

$$\nabla^2 e_3(x)(\alpha, \tilde{\alpha}) = 0, \quad \nabla^2 e_4(x)(\alpha, \tilde{\alpha}) = 0. \quad (3.20)$$

To prove the first-order Fréchet differentiability of e_1 , we estimate the remainder. Using Hölder's inequality and the Gagliardo–Nirenberg inequality, we obtain

$$\begin{aligned} & \|e_1(x + \alpha) - e_1(x) - \nabla e_1(x)\alpha\|_{L^2(0,T;H^1(\Omega)')} \\ &= \left\| \int_0^T \int_{\Omega} \left(-g\alpha_w\alpha_p(1-p)(1+\eta p)^2\zeta - g w \alpha_p^2 [-(1+\eta p)^2 + 2\eta(1-p)(1+\eta p)]\zeta \right. \right. \\ &\quad \left. \left. - g\alpha_p\alpha_w(1-p)(1+\eta p)^2\zeta + O(\|\alpha\|_X^3) \right) dx dt \right\| \\ &\leq C(\|\alpha_p\|_{W(0,T)}^2 + \|\alpha_w\|_{W(0,T)}^2) \|\zeta\|_{L^2(0,T;H^1(\Omega))}, \end{aligned}$$

where the constant C depends on the model parameters and Sobolev embedding constants. Consequently,

$$\lim_{\|\alpha\|_X \rightarrow 0} \frac{1}{\|\alpha\|_X} \|e_1(x + \alpha) - e_1(x) - \nabla e_1(x)\alpha\|_{L^2(0,T;H^1(\Omega)')} = 0.$$

Similarly, for e_2 , we have

$$\begin{aligned} & \|e_2(x + \alpha) - e_2(x) - \nabla e_2(x)\alpha\|_{L^2(0,T;H^1(\Omega)')} \\ &= \sup_{\|\xi\| \leq 1} \left| \int_0^T \int_{\Omega} g \left(\frac{1}{2} \frac{\partial^2 F}{\partial p^2} (\alpha_p)^2 + \frac{\partial^2 F}{\partial p \partial w} \alpha_p \alpha_w \right) \xi dx dt + \delta_w \int_0^T \int_{\Gamma_i} \alpha_u \alpha_w \xi dS dt \right| \\ &\leq C \int_0^T \int_{\Omega} (|\alpha_p \alpha_w| + |\alpha_p|^2) |\xi| dx dt + \delta_w \int_0^T \int_{\Gamma_i} |\alpha_u \alpha_w \xi| dS dt \\ &\leq C \left(\int_0^T \|\alpha_p\|_{L^4}^2 \|\alpha_w\|_{L^4}^2 dt \right)^{1/2} \left(\int_0^T \|\xi\|_{L^2}^2 dt \right)^{1/2} + C \left(\int_0^T \|\alpha_p\|_{L^4}^4 dt \right)^{1/2} \left(\int_0^T \|\xi\|_{L^2}^2 dt \right)^{1/2} \\ &\quad + \delta_w \int_0^T \|\alpha_u\|_{L^2(\Gamma_i)} \|\alpha_w\|_{L^2(\Gamma_i)} \|\xi\|_{L^2(\Gamma_i)} dt \\ &\leq C(\|\alpha_p\|_{L^4(0,T;L^4)}^2 \|\alpha_w\|_{L^4(0,T;L^4)}^2)^{1/2} \|\xi\|_{L^2(0,T;L^2)} + C\|\alpha_p\|_{L^4(0,T;L^4)}^2 \|\xi\|_{L^2(0,T;L^2)} \\ &\quad + \delta_w \|\alpha_u\|_{L^2(0,T)} \left(\int_0^T \|\alpha_w\|_{L^2(\Gamma_i)}^4 dt \right)^{1/4} \left(\int_0^T \|\xi\|_{L^2(\Gamma_i)}^4 dt \right)^{1/4} \\ &\leq C(\|\alpha_p\|_{W(0,T)} \|\alpha_w\|_{W(0,T)} + \|\alpha_p\|_{W(0,T)}^2) \|\xi\|_{L^2(0,T;H^1)} + C\|\alpha_u\|_{L^2(0,T)} \|\alpha_w\|_{W(0,T)} \|\xi\|_{L^2(0,T;H^1)} \\ &\leq C(\|\alpha_p\|_{W(0,T)}^2 + \|\alpha_w\|_{W(0,T)}^2 + \|\alpha_u\|_{L^2(0,T)}^2) \|\xi\|_{L^2(0,T;H^1)}. \end{aligned}$$

Hence, the limit vanishes, so (3.15) and (3.16) are the first Fréchet derivatives of $e_i(x)$ in the direction α , $i = 1, 2$. Since e_3, e_4 are linear bounded operators, (3.17) are their Fréchet derivatives.

For the second derivatives, let

$$x_i = (p_i, w_i, u_i) \in X, \quad i = 1, 2,$$

belong to a bounded subset of X . From (3.18) and (3.19), the coefficients appearing in $\nabla^2 e_1(x)$ and $\nabla^2 e_2(x)$ are polynomial functions of the state variables p and w . By the boundedness of the state variables and the smoothness of these polynomial coefficients, they are Lipschitz continuous on bounded subsets of the state space.

Hence, using Hölder's inequality, the Gagliardo–Nirenberg inequality, and the trace inequality, for arbitrary $\alpha, \tilde{\alpha} \in X$, we obtain

$$\begin{aligned} & \|(\nabla^2 e_i(x_1) - \nabla^2 e_i(x_2))(\alpha, \tilde{\alpha})\|_{Y'} \\ & \leq C\|x_1 - x_2\|_X \|\alpha\|_X \|\tilde{\alpha}\|_X, \quad i = 1, 2, \end{aligned}$$

where $C > 0$ depends on the bounded subset of X but is independent of α and $\tilde{\alpha}$. Since

$$\nabla^2 e_3(x) = \nabla^2 e_4(x) = 0,$$

it follows that

$$\|\nabla^2 e(x_1) - \nabla^2 e(x_2)\|_{\mathcal{L}^2(X, Y')} \leq C\|x_1 - x_2\|_X.$$

Therefore, $\nabla^2 e$ is locally Lipschitz continuous on bounded subsets of X , and hence e is twice continuously Fréchet differentiable. This completes the proof. $\square$

Remark 3.8. In the above derivation, the treatment of nonlinear terms relies on the boundedness of the vegetation density p and soil moisture w , which is guaranteed by Theorem 2.3. The twice continuous Fréchet differentiability of the penalty term $K(u)$ follows from Lemma 3.4, which provides a theoretical foundation for using Newton-type methods to solve the optimal-control problem. In particular, the term $\frac{1}{k} \nabla K(u)$ in the first-order condition reflects the influence of the integral constraint on the control, while the term $\frac{1}{k} \nabla^2 K(u)$ in the second-order condition ensures the convexity of the objective functional under appropriate conditions.

Proposition 3.9. For every $x = (p, w, u) \in \mathcal{X}$, the linear operator $\nabla_{(p,w)} e(x) : W(0, T) \times W(0, T) \rightarrow \mathcal{Y}'$ is bijective. Moreover, there exists a constant $C_4 > 0$ such that for all $\alpha = (\alpha_p, \alpha_w, \alpha_u) \in \mathcal{N}(\nabla e(x))$ (the nullspace of $\nabla e(x)$),

$$\|\alpha_p\|_{W(0,T)}^2 + \|\alpha_w\|_{W(0,T)}^2 \leq C_4 \|\alpha_u\|_{L^2(0,T)}^2. \quad (3.21)$$

Proof. The bijectivity of $\nabla_{(p,w)} e(x)$ follows from the unique solvability of the linearized system (3.22) and (3.23) for arbitrary right-hand sides; this is standard and can be proved by the same variational arguments as in Theorem 2.3 (see also [17]). We therefore focus on the estimate (3.21).

Let $\alpha = (\alpha_p, \alpha_w, \alpha_u)$ be an element of $\mathcal{N}(\nabla e(x))$; then (α_p, α_w) satisfy the homogeneous linearized equations with $h_1 = h_2 = 0$ and $\psi_1 = \psi_2 = 0$.

$$\alpha_p(0) = 0, \quad \alpha_w(0) = 0,$$

and for all $\zeta, \xi \in H^1(\Omega)$ and almost every $t \in (0, T)$,

$$\begin{aligned} & \langle (\alpha_p)_t, \zeta \rangle_{H^1(\Omega)', H^1(\Omega)} + \int_{\Omega} \nabla \alpha_p \cdot \nabla \zeta \, dx \\ & = \int_{\Omega} \left(gw\alpha_p(1-p)(1+\eta p)^2 \zeta + gp\alpha_w(1-p)(1+\eta p)^2 \zeta \right. \\ & \quad \left. + gwp[-(1+\eta p)^2 + 2\eta(1-p)(1+\eta p)]\alpha_p \zeta - \alpha_p \zeta \right) dx, \end{aligned} \quad (3.22)$$

$$\langle (\alpha_w)_t, \xi \rangle_{H^1(\Omega)', H^1(\Omega)} + \int_{\Omega} \delta_w \nabla \alpha_w \cdot \nabla \xi \, dx$$

$$\begin{aligned}
&= \int_{\Omega} \left(-v\alpha_w(1-\rho p)\xi + vwp\alpha_p\xi - g\alpha_wp(1+\eta p)^2\xi - gw\alpha_p(1+\eta p)^2\xi \right. \\
&\quad \left. - 2g\eta wp(1+\eta p)\alpha_p\xi \right) dx + \gamma \int_{\Omega} \nabla\alpha_p \cdot \nabla\xi \, dx \\
&\quad + \delta_w(u(t) - a) \int_{\Gamma_t} \alpha_w\xi \, dS + \delta_w\alpha_u(t) \int_{\Gamma_t} w\xi \, dS.
\end{aligned} \tag{3.23}$$

Choose $\zeta = \alpha_p$ in (3.22) and $\xi = \alpha_w$ in (3.23), and add the two equations. We obtain

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left(\|\alpha_p\|_{L^2}^2 + \|\alpha_w\|_{L^2}^2 \right) + \|\nabla\alpha_p\|_{L^2}^2 + \delta_w \|\nabla\alpha_w\|_{L^2}^2 - \gamma \int_{\Omega} \nabla\alpha_p \cdot \nabla\alpha_w \, dx \\
&= \delta_w(u - a) \int_{\Gamma_t} (\alpha_w)^2 \, dS + \delta_w\alpha_u \int_{\Gamma_t} w\alpha_w \, dS \\
&\quad + \int_{\Omega} \left[(gw(1-p)(1+\eta p)^2 + gwp[-(1+\eta p)^2 + 2\eta(1-p)(1+\eta p)] - 1)(\alpha_p)^2 \right. \\
&\quad \left. + (gp(1-p)(1+\eta p)^2 + v\rho w - gw(1+\eta p)^2 - 2g\eta wp(1+\eta p))\alpha_p\alpha_w \right. \\
&\quad \left. - (v(1-\rho p) + gp(1+\eta p)^2)(\alpha_w)^2 \right] dx.
\end{aligned} \tag{3.24}$$

For any $\varepsilon_0 > 0$, Young's inequality gives

$$\gamma \left| \int_{\Omega} \nabla\alpha_p \cdot \nabla\alpha_w \, dx \right| \leq \varepsilon_0 \|\nabla\alpha_p\|_{L^2}^2 + \frac{\gamma^2}{4\varepsilon_0} \|\nabla\alpha_w\|_{L^2}^2. \tag{3.25}$$

Since p and w are bounded, all reaction coefficients in (3.24) are bounded. Moreover, by the trace inequality and Young's inequality, for any $\varepsilon > 0$,

$$\begin{aligned}
&\left| \delta_w(u - a) \int_{\Gamma_t} (\alpha_w)^2 \, dS \right| + \left| \delta_w\alpha_u \int_{\Gamma_t} w\alpha_w \, dS \right| \\
&\leq \varepsilon \|\alpha_w\|_{H^1(\Omega)}^2 + C_\varepsilon(1 + |u|^2) \|\alpha_w\|_{L^2(\Omega)}^2 + C_\varepsilon |\alpha_u|^2.
\end{aligned} \tag{3.26}$$

Assume $\gamma^2 < 4\delta_w$, and choose

$$\frac{\gamma^2}{4\delta_w} < \varepsilon_0 < 1.$$

Then

$$D_* = \min \left\{ 1 - \varepsilon_0, \delta_w - \frac{\gamma^2}{4\varepsilon_0} \right\} > 0.$$

Combining (3.24) and (3.26) and choosing $\varepsilon > 0$ sufficiently small, we obtain

$$\frac{d}{dt} \left(\|\alpha_p\|_{L^2}^2 + \|\alpha_w\|_{L^2}^2 \right) + D_* \left(\|\alpha_p\|_{H^1}^2 + \|\alpha_w\|_{H^1}^2 \right) \leq C(1 + |u|^2) \left(\|\alpha_p\|_{L^2}^2 + \|\alpha_w\|_{L^2}^2 \right) + C|\alpha_u|^2. \tag{3.27}$$

Since $\alpha_p(0) = \alpha_w(0) = 0$, Gronwall's inequality yields

$$\|\alpha_p\|_{L^\infty(0,T;L^2)}^2 + \|\alpha_w\|_{L^\infty(0,T;L^2)}^2 + \|\alpha_p\|_{L^2(0,T;H^1)}^2 + \|\alpha_w\|_{L^2(0,T;H^1)}^2 \leq C\|\alpha_u\|_{L^2(0,T)}^2.$$

Using (3.22) and (3.23), the time derivatives satisfy the same type of estimate in $L^2(0, T; H^1(\Omega)')$. Hence,

$$\|\alpha_p\|_{W(0,T)}^2 + \|\alpha_w\|_{W(0,T)}^2 \leq C_4 \|\alpha_u\|_{L^2(0,T)}^2,$$

which proves (3.21). This completes the proof. $\square$

For each $k > 0$, consider the Lagrange functional $L_k : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$ associated with the penalized optimal-control problem (3.7):

$$\begin{aligned} L_k(x, \rho) &= J_k(x) + \langle e(x), \rho \rangle_{\mathcal{Y}', \mathcal{Y}} \\ &= J_k(x) + \int_0^T (\langle p_t(t), h_1(t) \rangle_{H^1(\Omega)', H^1(\Omega)} + \langle w_t(t), h_2(t) \rangle_{H^1(\Omega)', H^1(\Omega)}) dt \\ &\quad + \int_0^T \int_{\Omega} (D_p \nabla p \cdot \nabla h_1 + \delta_w \nabla w \cdot \nabla h_2) dx dt - \gamma \int_0^T \int_{\Omega} \nabla p \cdot \nabla h_2 dx dt \\ &\quad + \int_0^T \int_{\Gamma_t} \delta_w (a - u) w h_2 dS dt \\ &\quad + \int_0^T \int_{\Omega} (-gwp(1-p)(1+\eta p)^2 h_1 + ph_1) dx dt \\ &\quad + \int_0^T \int_{\Omega} (-lh_2 + vw(1-\rho p)h_2 + gwp(1+\eta p)^2 h_2) dx dt \\ &\quad + \int_{\Omega} ((p(0) - \phi_1)\xi_1 + (w(0) - \phi_2)\xi_2) dx, \end{aligned}$$

where $\rho = (h_1, h_2, \xi_1, \xi_2) \in \mathcal{Y}$. From [17], it is known that L_k is twice continuously Fréchet differentiable and its second Fréchet derivative is Lipschitz continuous on $\mathcal{X}$.

Based on the above results, we now derive the first-order necessary optimality conditions for (3.7). Since (3.7) is a non-convex optimization problem, it may possess different local minima; we therefore do not directly study its global solution. Let $\bar{x}^k = (\bar{p}^k, \bar{w}^k, \bar{u}^k) \in \mathcal{U}_{\text{ad}}^k$ be a fixed local solution and define $\tilde{A}^k = \tilde{A}_-^k \cup \tilde{A}_+^k$ with

$$\tilde{A}_-^k = \{t \in [0, T] : \tilde{u}^k(t) = u_b(t)\}, \quad \tilde{A}_+^k = \{t \in [0, T] : \tilde{u}^k(t) = u_c(t)\}.$$

Theorem 3.10. *Let*

$$\bar{x}^k = (\bar{p}^k, \bar{w}^k, \bar{u}^k)$$

be a local solution of the penalized problem. Then there exists a Lagrange multiplier

$$\bar{\rho}^k = (\bar{\lambda}_p^k, \bar{\lambda}_w^k, \bar{\xi}_1^k, \bar{\xi}_2^k)$$

such that the adjoint variables satisfy

$$\begin{aligned} -(\bar{\lambda}_p^k)_t - \Delta \bar{\lambda}_p^k &= -\gamma \Delta \bar{\lambda}_w^k + \left[g\bar{w}^k((1-2\bar{p}^k)(1+\eta\bar{p}^k)^2 + 2\eta\bar{p}^k(1-\bar{p}^k)(1+\eta\bar{p}^k)) - 1 \right] \bar{\lambda}_p^k \\ &\quad + \left[v\rho\bar{w}^k - g\bar{w}^k(1+\eta\bar{p}^k)^2 - 2g\eta\bar{w}^k\bar{p}^k(1+\eta\bar{p}^k) \right] \bar{\lambda}_w^k, \end{aligned} \quad (3.28)$$

and

$$\begin{aligned} -(\bar{\lambda}_w^k)_t - \delta_w \Delta \bar{\lambda}_w^k &= g\bar{p}^k(1-\bar{p}^k)(1+\eta\bar{p}^k)^2 \bar{\lambda}_p^k \\ &\quad - \left[v(1-\rho\bar{p}^k) + g\bar{p}^k(1+\eta\bar{p}^k)^2 \right] \bar{\lambda}_w^k. \end{aligned} \quad (3.29)$$

The boundary conditions are

$$\begin{cases} \frac{\partial \tilde{\lambda}_w^k}{\partial n} = 0, & (x, t) \in \Gamma_r \times (0, T), \\ \frac{\partial \tilde{\lambda}_w^k}{\partial n} = (\tilde{u}^k - a)\tilde{\lambda}_w^k, & (x, t) \in \Gamma_t \times (0, T), \\ \frac{\partial \tilde{\lambda}_p^k}{\partial n} - \gamma \frac{\partial \tilde{\lambda}_w^k}{\partial n} = 0, & (x, t) \in \Gamma \times (0, T). \end{cases} \quad (3.30)$$

The terminal conditions are

$$\tilde{\lambda}_p^k(T) = -\beta_1(\tilde{p}^k(T) - p_T), \quad \tilde{\lambda}_w^k(T) = 0,$$

and

$$\tilde{\xi}_1^k = \tilde{\lambda}_p^k(0), \quad \tilde{\xi}_2^k = \tilde{\lambda}_w^k(0).$$

Moreover, there exists $\tilde{\phi}^k \in L^2(0, T)$ such that

$$\beta_2(\tilde{u}^k - u^e) + \frac{1}{k} f' \left(\int_0^T \tilde{u}^k(t) dt - u_a \right) - \int_{\Gamma_t} \delta_w \tilde{w}^k \tilde{\lambda}_w^k dS + \tilde{\phi}^k(t) = 0 \quad (3.31)$$

for almost every $t \in (0, T)$.

Proof. By Proposition 3.9, the linearized operator $\nabla_{(p,w)} e(\tilde{x}^k)$ is bijective. Consequently, there exists a unique Lagrange multiplier $\tilde{\rho}^k \in \mathcal{Y}$ such that

$$\nabla_{(p,w)} L_k(\tilde{x}^k, \tilde{\rho}^k) = 0 \quad \text{in } W(0, T) \times W(0, T). \quad (3.32)$$

Expanding (3.32), for all $(\alpha_p, \alpha_w) \in W(0, T) \times W(0, T)$, we obtain

$$\begin{aligned} 0 &= \nabla_{(p,w)} L_k(\tilde{x}^k, \tilde{\rho}^k)(\alpha_p, \alpha_w) \\ &= \int_{\Omega} \beta_1(\tilde{p}^k(T) - p_T) \alpha_p(T) dx + \int_{\Omega} (\alpha_p(0) \tilde{\xi}_1^k + \alpha_w(0) \tilde{\xi}_2^k) dx \\ &\quad + \int_0^T (\langle (\alpha_p)_t, \tilde{\lambda}_p^k \rangle + \langle (\alpha_w)_t, \tilde{\lambda}_w^k \rangle) dt \\ &\quad + \int_0^T \int_{\Omega} (\nabla \alpha_p \cdot \nabla \tilde{\lambda}_p^k + \delta_w \nabla \alpha_w \cdot \nabla \tilde{\lambda}_w^k - \gamma \nabla \alpha_p \cdot \nabla \tilde{\lambda}_w^k) dx dt \\ &\quad + \int_0^T \int_{\Omega} \left[-g \tilde{w}^k \alpha_p (1 - \tilde{p}^k)(1 + \eta \tilde{p}^k)^2 - g \tilde{p}^k \alpha_w (1 - \tilde{p}^k)(1 + \eta \tilde{p}^k)^2 \right. \\ &\quad \left. - g \tilde{w}^k \tilde{p}^k (-(1 + \eta \tilde{p}^k)^2 + 2\eta(1 - \tilde{p}^k)(1 + \eta \tilde{p}^k)) \alpha_p + \alpha_p \right] \tilde{\lambda}_p^k dx dt \\ &\quad + \int_0^T \int_{\Omega} \left[v \alpha_w (1 - \rho \tilde{p}^k) - v \rho \tilde{w}^k \alpha_p + g \alpha_w \tilde{p}^k (1 + \eta \tilde{p}^k)^2 \right. \\ &\quad \left. + g \tilde{w}^k \alpha_p (1 + \eta \tilde{p}^k)^2 + 2g\eta \tilde{w}^k \tilde{p}^k (1 + \eta \tilde{p}^k) \alpha_p \right] \tilde{\lambda}_w^k dx dt \\ &\quad + \delta_w \int_0^T \int_{\Gamma_t} (a - \tilde{u}^k) \alpha_w \tilde{\lambda}_w^k dS dt. \end{aligned} \quad (3.33)$$

First, choose $\alpha_p(t) = \chi(t)\zeta$ and $\alpha_w(t) = \chi(t)\xi$, where $\chi \in C_c^\infty(0, T)$ and $\zeta, \xi \in H^1(\Omega)$. Applying integration by parts in time and Green's formula in space to (3.33), we obtain for almost every $t \in (0, T)$

$$\begin{aligned} 0 = & \int_{\Omega} \left[-(\tilde{\lambda}_p^k)_t - \Delta \tilde{\lambda}_p^k + \gamma \Delta \tilde{\lambda}_w^k \right. \\ & + \left(-g\tilde{w}^k(1 - \tilde{p}^k)(1 + \eta\tilde{p}^k)^2 - g\tilde{w}^k\tilde{p}^k[-(1 + \eta\tilde{p}^k)^2 + 2\eta(1 - \tilde{p}^k)(1 + \eta\tilde{p}^k)] + 1 \right) \tilde{\lambda}_p^k \\ & + \left(-\nu\rho\tilde{w}^k + g\tilde{w}^k(1 + \eta\tilde{p}^k)^2 + 2g\eta\tilde{w}^k\tilde{p}^k(1 + \eta\tilde{p}^k) \right) \tilde{\lambda}_w^k \Big] \zeta \, dx \\ & + \int_{\Omega} \left[-(\tilde{\lambda}_w^k)_t - \delta_w \Delta \tilde{\lambda}_w^k - g\tilde{p}^k(1 - \tilde{p}^k)(1 + \eta\tilde{p}^k)^2 \tilde{\lambda}_p^k \right. \\ & + \left(\nu(1 - \rho\tilde{p}^k) + g\tilde{p}^k(1 + \eta\tilde{p}^k)^2 \right) \tilde{\lambda}_w^k \Big] \xi \, dx + \int_{\partial\Omega} \left(\frac{\partial \tilde{\lambda}_p^k}{\partial n} - \gamma \frac{\partial \tilde{\lambda}_w^k}{\partial n} \right) \zeta \, dS \\ & + \delta_w \int_{\partial\Omega} \frac{\partial \tilde{\lambda}_w^k}{\partial n} \xi \, dS + \delta_w \int_{\Gamma_t} (a - \tilde{u}^k) \tilde{\lambda}_w^k \xi \, dS. \end{aligned}$$

Since ζ and ξ can be varied independently and $\partial\Omega = \Gamma_r \cup \Gamma_t$, we obtain the adjoint equations (3.28) and (3.29) and the boundary conditions (3.30). In particular,

$$\frac{\partial \tilde{\lambda}_p^k}{\partial n} - \gamma \frac{\partial \tilde{\lambda}_w^k}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

and

$$\frac{\partial \tilde{\lambda}_w^k}{\partial n} = 0 \quad \text{on } \Gamma_r, \quad \frac{\partial \tilde{\lambda}_w^k}{\partial n} = (\tilde{u}^k - a) \tilde{\lambda}_w^k \quad \text{on } \Gamma_t.$$

To obtain the terminal conditions, take $\alpha_p(t) = \chi(t)\zeta$ and $\alpha_w(t) = \chi(t)\xi$, where $\chi \in C^1([0, T])$, $\chi(0) = 0$, and $\chi(T)$ is arbitrary. After integration by parts and using the adjoint equations and boundary conditions, the remaining terminal terms are

$$\int_{\Omega} [\tilde{\lambda}_p^k(T) + \beta_1(\tilde{p}^k(T) - p_T)] \alpha_p(T) \, dx + \int_{\Omega} \tilde{\lambda}_w^k(T) \alpha_w(T) \, dx = 0.$$

Since $\alpha_p(T)$ and $\alpha_w(T)$ are arbitrary,

$$\tilde{\lambda}_p^k(T) = -\beta_1(\tilde{p}^k(T) - p_T), \quad \tilde{\lambda}_w^k(T) = 0.$$

Similarly, choosing variations that are arbitrary at $t = 0$ gives

$$\tilde{\xi}_1^k = \tilde{\lambda}_p^k(0), \quad \tilde{\xi}_2^k = \tilde{\lambda}_w^k(0).$$

Finally, consider the variation with respect to the control. Since $\tilde{x}^k$ is a local solution, for all $u \in U_{\text{ad}}^k$,

$$\nabla_u L_k(\tilde{x}^k, \tilde{\rho}^k)(u - \tilde{u}^k) \geq 0.$$

Computing the variation gives

$$\nabla_u L_k(\tilde{x}^k, \tilde{\rho}^k)(u - \tilde{u}^k) = \int_0^T \beta_2(\tilde{u}^k - u^e)(u - \tilde{u}^k) \, dt + \frac{1}{k} f' \left(\int_0^T \tilde{u}^k \, dt - u_a \right) \int_0^T (u - \tilde{u}^k) \, dt - \int_0^T \int_{\Gamma_t} \delta_w \tilde{w}^k \tilde{\lambda}_w^k (u - \tilde{u}^k) \, dS \, dt.$$

Define $\tilde{\phi}^k \in L^2(0, T)$ by

$$-\langle \tilde{\phi}^k, u - \tilde{u}^k \rangle_{L^2(0, T)} = \nabla_u L_k(\tilde{x}^k, \tilde{\rho}^k)(u - \tilde{u}^k).$$

Then

$$\beta_2(\tilde{u}^k - u^e) + \frac{1}{k} f' \left(\int_0^T \tilde{u}^k dt - u_a \right) - \int_{\Gamma_t} \delta_w \tilde{w}^k \tilde{\lambda}_w^k dS + \tilde{\phi}^k(t) = 0,$$

which is equivalent to (3.31). Since $\tilde{\phi}^k \in N_{U_{\text{ad}}^k}(\tilde{u}^k)$, the corresponding sign conditions follow. This completes the proof. $\square$

Corollary 3.11. *There exists a positive constant $C_{\tilde{\lambda}} > 0$ such that*

$$\|\tilde{\lambda}_p^k\|_{L^2(0, T; L^4(\Omega))} + \|\tilde{\lambda}_w^k\|_{L^2(0, T; L^4(\Omega))} \leq C_{\tilde{\lambda}} \|\tilde{p}^k(T) - p_T\|_{L^2(\Omega)}. \quad (3.34)$$

4. Second-order optimality conditions

The previous section established the first-order necessary optimality conditions, which are the focus of most existing studies on optimal control of ecosystem models [30, 31]. However, second-order optimality conditions are equally important, as second-order sufficient conditions can help avoid local optimization traps when designing full-cycle sustainable strategies. Before proving the second-order sufficient optimality condition for system (2.2), we recall the definition of the second-order sufficient optimality condition.

Theorem 4.1. *Assume that*

$$\kappa = \min \left\{ \frac{\beta_2}{2}, \frac{\beta_2}{4C_4} - C_1 C_{\tilde{\lambda}} \|\tilde{p}^k(T) - p_T\|_{L^2(\Omega)} - C_2 C_{\tilde{\lambda}}^2 \|\tilde{p}^k(T) - p_T\|_{L^2(\Omega)}^2 \right\} > 0.$$

Then the second-order sufficient optimality condition

$$\nabla_x^2 L_k(\tilde{x}^k, \tilde{\rho}^k)(\delta x, \delta x) \geq \kappa \|\delta x\|_X^2, \quad \delta x \in C(\tilde{x}^k),$$

holds.

Proof. Consider

$$\begin{aligned} \nabla_x^2 L_k(\tilde{x}^k, \tilde{\rho}^k)(\delta x, \delta x) &= \int_{\Omega} \beta_1 |\delta p(T)|^2 dx + \frac{1}{k} \nabla^2 K(\tilde{u}^k)(\delta u, \delta u) + \int_0^T \beta_2 |\delta u|^2 dt \\ &\quad - g \int_0^T \int_{\Omega} \left[\tilde{w}^k \left(-2(1 + \eta \tilde{p}^k)^2 + 4\eta(1 - 2\tilde{p}^k)(1 + \eta \tilde{p}^k) + 2\eta^2 \tilde{p}^k(1 - \tilde{p}^k) \right) (\delta p)^2 \right. \\ &\quad \left. + 2 \left((1 - 2\tilde{p}^k)(1 + \eta \tilde{p}^k)^2 + 2\eta \tilde{p}^k(1 - \tilde{p}^k)(1 + \eta \tilde{p}^k) \right) \delta p \delta w \right] \tilde{\lambda}_p^k dx dt \\ &\quad + \int_0^T \int_{\Omega} \left[g \tilde{w}^k \left(4\eta(1 + \eta \tilde{p}^k) + 2\eta^2 \tilde{p}^k \right) (\delta p)^2 \right. \\ &\quad \left. + 2 \left(-v\rho + g(1 + \eta \tilde{p}^k)^2 + 2g\eta \tilde{p}^k(1 + \eta \tilde{p}^k) \right) \delta p \delta w \right] \tilde{\lambda}_w^k dx dt \\ &\quad - 2\delta_w \int_0^T \int_{\Gamma_t} \delta u \delta w \tilde{\lambda}_w^k dS dt. \end{aligned} \quad (4.1)$$

Obviously,

$$\nabla^2 K(\tilde{u}^k)(\delta u, \delta u) = 6 \left[\int_0^T \tilde{u}^k dt - u_a \right]_+ \left(\int_0^T \delta u dt \right)^2 \geq 0. \quad (4.2)$$

Since $\tilde{p}^k$ and $\tilde{w}^k$ are bounded, using Hölder's inequality and the Gagliardo–Nirenberg inequality, we obtain

$$\begin{aligned} & \left| g \int_0^T \int_{\Omega} \left[\tilde{w}^k \left(-2(1 + \eta \tilde{p}^k)^2 + 4\eta(1 - 2\tilde{p}^k)(1 + \eta \tilde{p}^k) + 2\eta^2 \tilde{p}^k(1 - \tilde{p}^k) \right) (\delta p)^2 \right. \right. \\ & \quad \left. \left. + 2 \left((1 - 2\tilde{p}^k)(1 + \eta \tilde{p}^k)^2 + 2\eta \tilde{p}^k(1 - \tilde{p}^k)(1 + \eta \tilde{p}^k) \right) \delta p \delta w \right] \tilde{\lambda}_p^k dx dt \right| \\ & \leq C_1 \|\tilde{\lambda}_p^k\|_{L^2(0,T;L^4(\Omega))} \left(\|\delta p\|_{W(0,T)}^2 + \|\delta w\|_{W(0,T)}^2 \right). \end{aligned}$$

Similarly,

$$\begin{aligned} & \left| \int_0^T \int_{\Omega} \left[g \tilde{w}^k \left(4\eta(1 + \eta \tilde{p}^k) + 2\eta^2 \tilde{p}^k \right) (\delta p)^2 \right. \right. \\ & \quad \left. \left. + 2 \left(-\nu \rho + g(1 + \eta \tilde{p}^k)^2 + 2g\eta \tilde{p}^k(1 + \eta \tilde{p}^k) \right) \delta p \delta w \right] \tilde{\lambda}_w^k dx dt \right| \\ & \leq C_2 \|\tilde{\lambda}_w^k\|_{L^2(0,T;L^4(\Omega))} \left(\|\delta p\|_{W(0,T)}^2 + \|\delta w\|_{W(0,T)}^2 \right). \end{aligned}$$

By Hölder's inequality, the trace theorem, and Young's inequality,

$$2\delta_w \left| \int_0^T \int_{\Gamma_t} \delta u \delta w \tilde{\lambda}_w^k dS dt \right| \leq \frac{\beta_2}{4} \|\delta u\|_{L^2(0,T)}^2 + C_3 \|\tilde{\lambda}_w^k\|_{W(0,T)}^2 \|\delta w\|_{W(0,T)}^2, \quad (4.3)$$

where $C_3 > 0$ depends on δ_w and the trace constant.

Combining (4.1)–(4.3) and using (4.2), we obtain

$$\begin{aligned} \nabla_x^2 L_k(\tilde{x}^k, \tilde{p}^k)(\delta x, \delta x) & \geq \frac{3\beta_2}{4} \|\delta u\|_{L^2(0,T)}^2 \\ & \quad - C_1 \|\tilde{\lambda}_p^k\|_{L^2(0,T;L^4)} \left(\|\delta p\|_{W(0,T)}^2 + \|\delta w\|_{W(0,T)}^2 \right) \\ & \quad - C_2 \|\tilde{\lambda}_w^k\|_{L^2(0,T;L^4)} \left(\|\delta p\|_{W(0,T)}^2 + \|\delta w\|_{W(0,T)}^2 \right) - C_3 \|\tilde{\lambda}_w^k\|_{W(0,T)}^2 \|\delta w\|_{W(0,T)}^2. \end{aligned} \quad (4.4)$$

By Corollary 3.11,

$$\|\tilde{\lambda}_p^k\|_{L^2(0,T;L^4)} + \|\tilde{\lambda}_w^k\|_{L^2(0,T;L^4)} \leq C_{\lambda} \|\tilde{p}^k(T) - p_T\|_{L^2(\Omega)}, \quad (4.5)$$

and

$$\|\tilde{\lambda}_w^k\|_{W(0,T)}^2 \leq C_{\lambda}^2 \|\tilde{p}^k(T) - p_T\|_{L^2(\Omega)}^2. \quad (4.6)$$

Moreover, Proposition 3.9 gives

$$\|\delta p\|_{W(0,T)}^2 + \|\delta w\|_{W(0,T)}^2 \leq C_4 \|\delta u\|_{L^2(0,T)}^2. \quad (4.7)$$

Equivalently,

$$\|\delta u\|_{L^2(0,T)}^2 \geq \frac{1}{C_4} \left(\|\delta p\|_{W(0,T)}^2 + \|\delta w\|_{W(0,T)}^2 \right).$$

Splitting

$$\frac{3\beta_2}{4}\|\delta u\|_{L^2}^2 = \frac{\beta_2}{2}\|\delta u\|_{L^2}^2 + \frac{\beta_2}{4}\|\delta u\|_{L^2}^2$$

and using (4.5)–(4.7), we obtain

$$\begin{aligned} \nabla_x^2 L_k(\tilde{x}^k, \tilde{\rho}^k)(\delta x, \delta x) &\geq \frac{\beta_2}{2}\|\delta u\|_{L^2(0,T)}^2 + \left[\frac{\beta_2}{4C_4} - C_1 C_{\tilde{\lambda}} \|\tilde{\rho}^k(T) - p_T\|_{L^2(\Omega)} - C_2 C_{\tilde{\lambda}}^2 \|\tilde{\rho}^k(T) - p_T\|_{L^2(\Omega)}^2 \right] \\ &\quad \times (\|\delta p\|_{W(0,T)}^2 + \|\delta w\|_{W(0,T)}^2), \end{aligned}$$

where the constants C_1 and C_2 have been enlarged, if necessary, to absorb the constants appearing in (4.4).

Therefore,

$$\nabla_x^2 L_k(\tilde{x}^k, \tilde{\rho}^k)(\delta x, \delta x) \geq \kappa (\|\delta u\|_{L^2(0,T)}^2 + \|\delta p\|_{W(0,T)}^2 + \|\delta w\|_{W(0,T)}^2) = \kappa \|\delta x\|_X^2,$$

where

$$\kappa = \min \left\{ \frac{\beta_2}{2}, \frac{\beta_2}{4C_4} - C_1 C_{\tilde{\lambda}} \|\tilde{\rho}^k(T) - p_T\|_{L^2(\Omega)} - C_2 C_{\tilde{\lambda}}^2 \|\tilde{\rho}^k(T) - p_T\|_{L^2(\Omega)}^2 \right\} > 0.$$

Thus, the second-order sufficient optimality condition is satisfied. $\square$ $\square$

Remark 4.2. The second-order sufficient condition is closely related to the deviation between the vegetation density $p(T)$ at the terminal time T and the desired state p_T . When setting a target for the vegetation density to be achieved within a fixed time, practical factors must be fully considered. If the target is set too high, far beyond the current ecological carrying capacity and water resource availability, it will impose a heavy burden on the ecosystem. Therefore, setting a reasonable target is a crucial prerequisite for effective ecological restoration.

5. Numerical simulation

5.1. Numerical setting and implementation

To ensure the complete reproducibility of the numerical experiments, the computational domain, model parameters, initial and boundary conditions, spatial and temporal discretizations, control parameterization, and optimization settings are specified explicitly.

The numerical experiments are performed on the one-dimensional domain

$$\Omega = (0, L), \quad L = 200,$$

over the time interval

$$0 \leq t \leq T, \quad T = 200.$$

Although the theoretical analysis is formulated for a general bounded domain, the one-dimensional setting provides a transparent framework for examining the effects of boundary regulation and cross-diffusion.

Unless otherwise stated, the model parameters are chosen as

$$g = 0.4571, \quad \eta = 3.1, \quad l = 1.5179, \quad v = 1.4286,$$

$$\rho = 0.7, \quad \delta_w = 125, \quad \gamma = 20, \quad a = 1.$$

The corresponding parameter values and their roles in the model are summarized in Table 1.

The initial vegetation and soil-water distributions are prescribed by

$$p(x, 0) = 0.45 + 0.05 \cos\left(\frac{2\pi x}{L}\right),$$

and

$$w(x, 0) = 0.70 + 0.05 \cos\left(\frac{2\pi x}{L}\right).$$

In the one-dimensional setting, the boundary conditions become

$$p_x(0, t) = p_x(L, t) = 0,$$

and

$$w_x(0, t) = 0, \quad w_x(L, t) = (u(t) - a)w(L, t).$$

Thus, $u = a$ corresponds to zero net diffusive water exchange through the controlled boundary, whereas $u > a$ represents inward water supply.

The spatial domain is divided into N uniform subintervals with grid points

$$x_i = i\Delta x, \quad \Delta x = \frac{L}{N}, \quad i = 0, 1, \dots, N.$$

Hence, the numerical state contains $N + 1$ spatial degrees of freedom for each of p and w . The second spatial derivatives at interior grid points are approximated by the standard second-order central finite-difference formula

$$\frac{\partial^2 y}{\partial x^2}(x_i, t) \approx \frac{y_{i+1}(t) - 2y_i(t) + y_{i-1}(t)}{\Delta x^2}, \quad y = p, w.$$

The homogeneous Neumann conditions are imposed using ghost-point elimination. In particular, at $x = 0$ and $x = L$,

$$p_{-1} = p_1, \quad p_{N+1} = p_{N-1},$$

which gives

$$p_{xx}|_{x=0} \approx \frac{2(p_1 - p_0)}{\Delta x^2}, \quad p_{xx}|_{x=L} \approx \frac{2(p_{N-1} - p_N)}{\Delta x^2}.$$

The same treatment is used for w at the left boundary. At the controlled boundary $x = L$, let

$$\kappa_j = u_j - a.$$

The Robin condition is discretized by the centered ghost-point formula

$$\frac{w_{N+1} - w_{N-1}}{2\Delta x} = \kappa_j w_N,$$

and therefore

$$w_{N+1} = w_{N-1} + 2\Delta x \kappa_j w_N.$$

Consequently, the boundary approximation for the water diffusion term is

$$w_{xx}|_{x=L} \approx \frac{2w_{N-1}}{\Delta x^2} + \left(-\frac{2}{\Delta x^2} + \frac{2\kappa_j}{\Delta x}\right)w_N.$$

This treatment is used consistently for every control interval.

For the temporal discretization, the state equations are advanced using a second-order semi-implicit Crank–Nicolson/Adams–Bashforth of order two (CNAB2) scheme. Define the reaction terms

$$F_p(p, w) = gwp(1 - p)(1 + \eta p)^2 - p,$$

and

$$F_w(p, w) = l - vw(1 - \rho p) - gwp(1 + \eta p)^2.$$

For $n \geq 1$, the vegetation equation is discretized as

$$\left(I - \frac{\Delta t}{2}L_p\right)p^{n+1} = \left(I + \frac{\Delta t}{2}L_p\right)p^n + \Delta t\left(\frac{3}{2}F_p^n - \frac{1}{2}F_p^{n-1}\right),$$

where L_p denotes the finite-difference Laplacian incorporating the homogeneous Neumann boundary conditions.

For the soil-water equation,

$$\left(I - \frac{\Delta t}{2}\delta_w L_w^{(j)}\right)w^{n+1} = \left(I + \frac{\Delta t}{2}\delta_w L_w^{(j)}\right)w^n - \frac{\Delta t}{2}\gamma L_p(p^{n+1} + p^n) + \Delta t\left(\frac{3}{2}F_w^n - \frac{1}{2}F_w^{n-1}\right),$$

where $L_w^{(j)}$ is the Robin finite-difference Laplacian associated with the control value u_j in the j th control interval. Thus, the ordinary diffusion terms are treated by Crank–Nicolson, the nonlinear reaction terms by Adams–Bashforth 2, and the cross-diffusion term by a time-centered discretization. The first time step is obtained using the same implicit diffusion operators together with the current reaction terms to initialize the subsequent CNAB2 stepping.

At each time step, the resulting sparse linear systems are solved using linear unit (LU) decomposition. For a time-dependent control, one Robin diffusion matrix is constructed for each control interval, so that the dependence of the boundary operator on the control is explicitly included in the state solver.

For the subsequent optimal-control simulations, the boundary control is parameterized as a piecewise-constant function,

$$u(t) = u_j, \quad t \in [(j-1)\Delta t_c, j\Delta t_c), \quad j = 1, \dots, n_c,$$

with

$$n_c = 20, \quad \Delta t_c = \frac{T}{n_c} = 10.$$

The control is assigned to each time step according to the midpoint of the time step. Specifically, for

$$t_{n+\frac{1}{2}} = \left(n - \frac{1}{2}\right)\Delta t,$$

the corresponding control interval is determined by

$$j = \left\lceil \frac{t_{n+\frac{1}{2}}}{\Delta t_c} \right\rceil + 1.$$

Thus, the continuous control is represented numerically by the 20-dimensional vector

$$\mathbf{u} = (u_1, \dots, u_{20})^T.$$

The baseline optimal-control parameters are

$$\beta_1 = 1, \quad \beta_2 = 1, \quad u^e = 1,$$

with the pointwise bounds

$$1.00 \leq u_j \leq 1.10, \quad j = 1, \dots, 20,$$

and the integral constraint

$$\int_0^T u(t) dt \leq 210.$$

Because the control is piecewise constant, the integral constraint is implemented exactly as

$$\Delta t_c \sum_{j=1}^{20} u_j \leq 210.$$

The initial guess for the optimization is

$$u_j^{(0)} = 1.05, \quad j = 1, \dots, 20,$$

which satisfies the integral constraint with equality.

The desired terminal vegetation state $p_T(x)$ is generated numerically using the same state solver and the constant control

$$u(t) \equiv 1.10.$$

Since

$$\int_0^T 1.10 dt = 220 > 210,$$

this control is used only to construct the prescribed target and is not required to satisfy the admissibility constraint of the subsequent optimal-control problem.

For each trial control vector $\mathbf{u}$, the state system is integrated over $[0, T]$, and the objective functional is evaluated numerically. The spatial integral in the terminal tracking term is approximated using the trapezoidal rule, while the control term is evaluated directly from its piecewise-constant representation:

$$J_h(\mathbf{u}) = \frac{\beta_1}{2} \text{trapz} \left[(p^M - p_T)^2 \right] + \frac{\beta_2}{2} \Delta t_c \sum_{j=1}^{20} (u_j - u^e)^2,$$

where p^M denotes the numerical vegetation state at $t = T$.

The finite-dimensional constrained optimization problem is solved using the MATLAB function `fmincon` with the sequential quadratic programming (SQP) algorithm. No analytical gradient is supplied to the optimizer; the objective derivatives with respect to the control variables are approximated using forward finite differences with step size 10^{-4} . The maximum number of iterations and function evaluations are set to 50 and 1500, respectively. The stopping tolerances are

$$\text{OptimalityTolerance} = 10^{-5}, \quad \text{StepTolerance} = 10^{-7}, \quad \text{ConstraintTolerance} = 10^{-8}.$$

Unless otherwise stated, the baseline numerical resolution is

$$N = 400, \quad \Delta x = 0.5, \quad \Delta t = 0.01.$$

Thus, the baseline simulations use 401 spatial grid points and 20,000 time steps over the interval $[0, 200]$. For time-history visualization, the numerical state is stored every 20 time steps, corresponding to a storage interval of 0.2; this storage frequency does not affect the time integration itself.

For post-processing, spatial averages are evaluated using the trapezoidal rule. The physical inward water input at the controlled boundary is calculated from

$$q_{\text{in}}(t) = \delta_w \max\{u(t) - a, 0\}w(L, t),$$

and its cumulative value is obtained by trapezoidal integration in time. A numerical safety check is also applied during the time integration to terminate a computation if non-finite values or unbounded numerical states are detected.

Table 1. Baseline parameter values used in the numerical simulations.

Parameter	Value	Description	Role in the model
g	0.4571	Vegetation growth coefficient	Vegetation growth and water uptake
η	3.1	Root-to-shoot feedback parameter	Enhancement of water uptake by vegetation
l	1.5179	Water-input parameter	Background water supply
ν	1.4286	Water-loss coefficient	Evaporation and water loss
ρ	0.7	Vegetation shading coefficient	Reduction of evaporation by vegetation
δ_w	125	Soil-water diffusion coefficient	Fickian redistribution of soil water
γ	20	Cross-diffusion coefficient	Vegetation-induced water redistribution
a	1	Baseline boundary-exchange coefficient	Natural boundary water loss
L	200	Domain length	Size of the one-dimensional ecosystem
T	200	Final simulation time	Duration of the numerical experiment

5.2. Error analysis and convergence verification

To validate the accuracy and reliability of the numerical method, we perform systematic spatial- and temporal-refinement tests following the numerical validation procedures adopted for reaction–diffusion systems [32, 33]. Since an exact analytical solution of the present nonlinear vegetation–water system is unavailable, sufficiently refined numerical solutions are employed as reference solutions. For a sufficiently smooth function y , the second-order central-difference approximation used in Section 5.1 satisfies

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2} = \frac{\partial^2 y}{\partial x^2}(x_i, t) + O(\Delta x^2),$$

and the CNAB2 scheme combines the Crank–Nicolson approximation for the diffusion terms with the second-order Adams–Bashforth approximation for the nonlinear reaction terms. Therefore, both the spatial and temporal discretizations are expected to exhibit second-order accuracy.

For a numerical solution obtained with spatial mesh size h , the relative L^2 errors of the vegetation and soil-water variables at the final time are defined by

$$E_p(h) = \frac{\|p_h(\cdot, T) - p_{\text{ref}}(\cdot, T)\|_{L^2(\Omega)}}{\|p_{\text{ref}}(\cdot, T)\|_{L^2(\Omega)}}, \quad E_w(h) = \frac{\|w_h(\cdot, T) - w_{\text{ref}}(\cdot, T)\|_{L^2(\Omega)}}{\|w_{\text{ref}}(\cdot, T)\|_{L^2(\Omega)}},$$

and the combined relative error is defined as

$$E_{p,w}(h) = \left[\frac{\|p_h - p_{\text{ref}}\|_{L^2(\Omega)}^2 + \|w_h - w_{\text{ref}}\|_{L^2(\Omega)}^2}{\|p_{\text{ref}}\|_{L^2(\Omega)}^2 + \|w_{\text{ref}}\|_{L^2(\Omega)}^2} \right]^{1/2}.$$

Following the convergence-rate criterion used in [33], the observed order between two successive resolutions h_1 and h_2 is evaluated by

$$r = \frac{\log(E(h_1)/E(h_2))}{\log(h_1/h_2)},$$

with the same definition applied to the temporal convergence test by replacing h with Δt .

For the spatial convergence test, the solutions obtained with

$$N = 100, \quad 200, \quad 400, \quad 800$$

are compared with a reference solution computed using $N_{\text{ref}} = 1600$. For the temporal convergence test, the time steps

$$\Delta t = 0.04, \quad 0.02, \quad 0.01, \quad 0.005$$

are successively refined while the spatial resolution is kept sufficiently fine to minimize the influence of the spatial discretization error. A numerical solution computed with a further refined time step is used as the temporal reference solution. The resulting errors and observed convergence orders are reported in Tables 2 and 3.

Table 2. Spatial errors and observed convergence orders.

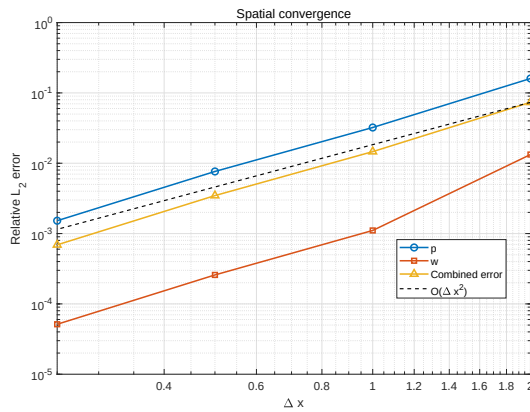
N	Δx	E_p	r_p	E_w	r_w	$E_{p,w}$	$r_{p,w}$
100	2.000	1.6020×10^{-1}	—	1.3292×10^{-2}	—	7.3500×10^{-2}	—
200	1.000	3.2212×10^{-2}	2.3142	1.1068×10^{-3}	3.5861	1.4619×10^{-2}	2.3299
400	0.500	7.6394×10^{-3}	2.0761	2.5808×10^{-4}	2.1005	3.4668×10^{-3}	2.0762
800	0.250	1.5233×10^{-3}	2.3263	5.1335×10^{-5}	2.3298	6.9127×10^{-4}	2.3263

Table 3. Temporal errors and observed convergence orders.

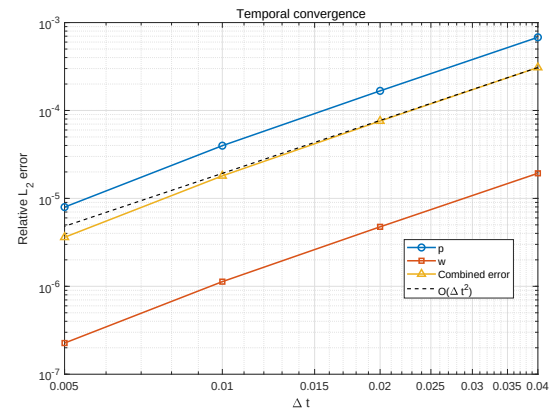
Δt	E_p	r_p	E_w	r_w	$E_{p,w}$	$r_{p,w}$
0.04000	6.7984×10^{-4}	—	1.9299×10^{-5}	—	3.0830×10^{-4}	—
0.02000	1.6727×10^{-4}	2.0230	4.7490×10^{-6}	2.0228	7.5856×10^{-5}	2.0230
0.01000	3.9783×10^{-5}	2.0720	1.1299×10^{-6}	2.0715	1.8041×10^{-5}	2.0720
0.00500	7.9624×10^{-6}	2.3209	2.2626×10^{-7}	2.3201	3.6109×10^{-6}	2.3209

Tables 2 and 3 show that the numerical errors decrease systematically under spatial and temporal refinement. For the spatial discretization, the observed orders of p and the combined state remain close to two on the refined meshes. Although w exhibits a higher apparent order between the two coarsest meshes, its observed orders on the finer meshes remain close to the expected second-order behavior. For the temporal discretization, the observed convergence orders of p , w , and the combined state are consistently close to two. In particular, the combined spatial error decreases from 7.3500×10^{-2} at

$N = 100$ to 6.9127×10^{-4} at $N = 800$, while the combined temporal error decreases from 3.0830×10^{-4} at $\Delta t = 0.04$ to 3.6109×10^{-6} at $\Delta t = 0.005$. These results are consistent with the expected second-order accuracy of the numerical discretization.



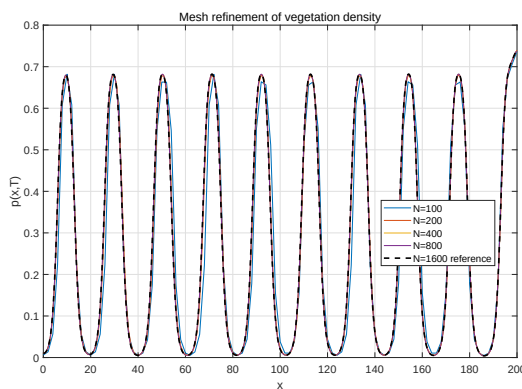
(a) Spatial convergence.



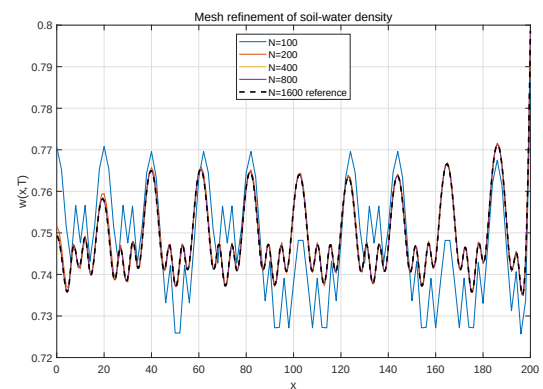
(b) Temporal convergence.

Figure 1. Convergence of the numerical discretization. The relative L^2 errors of the vegetation density p , soil-water density w , and the combined state decrease under spatial and temporal refinement. The dashed lines indicate second-order reference slopes.

The log–log plots in Figure 1 provide a visual confirmation of the convergence behavior reported in the tables. To further examine mesh independence and verify that the observed spatial structures are not numerical artifacts, the final vegetation and soil-water profiles obtained using different spatial resolutions are compared in Figure 2.



(a) Vegetation density $p(x, T)$.



(b) Soil-water density $w(x, T)$.

Figure 2. Final-state profiles obtained with different spatial resolutions. The solution corresponding to $N = 1600$ is used as the reference solution.

Figure 2 shows visible differences for the coarse mesh $N = 100$, whereas the solutions obtained with $N = 400$, 800 , and 1600 are nearly indistinguishable. In particular, both the vegetation distribution and the enhanced soil-water density near the controlled boundary persist under mesh refinement, indicating

that the observed spatial structures are not numerical artifacts. Taken together, the spatial and temporal convergence tests provide a quantitative assessment of the numerical error, while the mesh-independence comparison confirms the robustness of the computed spatial structures. Accordingly, unless otherwise stated, all subsequent simulations are performed using

$$N = 400, \quad \Delta x = 0.5, \quad \Delta t = 0.01.$$

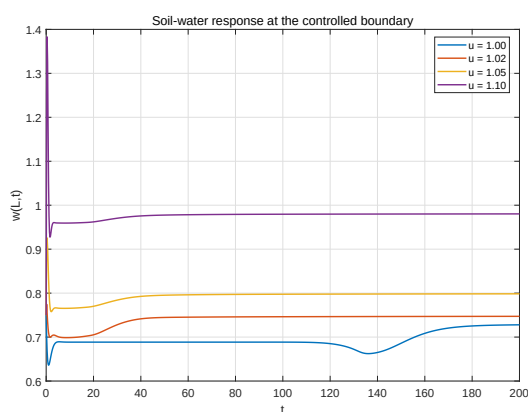
At this resolution, the combined spatial relative error is approximately 3.47×10^{-3} , while the temporal discretization error is substantially smaller, providing a suitable balance between numerical accuracy and computational cost.

5.3. Effects of constant boundary regulation

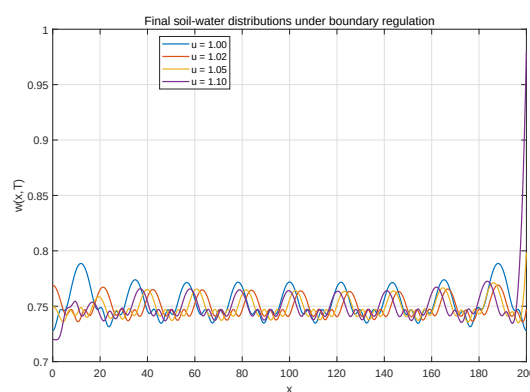
We next examine the response of the vegetation–water system to constant boundary regulation. With $a = 1$, four control levels are considered:

$$u(t) \equiv 1.00, \quad 1.02, \quad 1.05, \quad 1.10.$$

The case $u = a = 1$ corresponds to zero net boundary exchange, whereas $u > a$ represents increasing inward irrigation.



(a) Temporal response of the soil-water density at the controlled boundary.



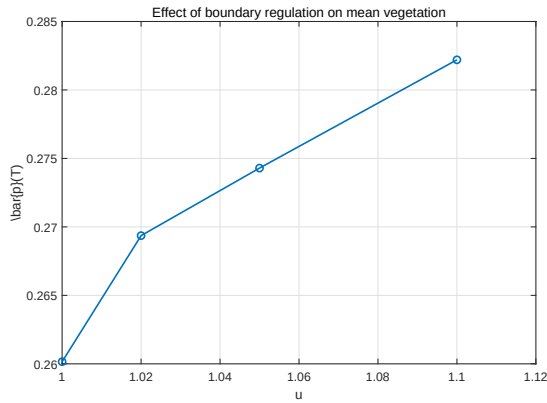
(b) Final spatial distributions of the soil-water density.

Figure 3. Effects of the constant boundary control on the soil-water density for different values of u .

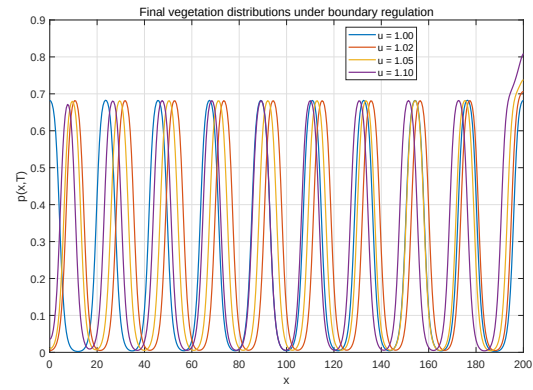
Figure 3 illustrates the direct response of soil water to the boundary regulation. Increasing u produces a higher soil-water density at the controlled boundary $x = L$. After the initial transient, $w(L, t)$ approaches a nearly stationary level that increases with the irrigation intensity. The final spatial profiles further show that the influence of irrigation is most pronounced near the controlled boundary. The interior water distribution is also modified through its coupling with vegetation and cross-diffusion. Hence, the Robin boundary regulation produces a spatially nonuniform water response rather than a uniform increase throughout the domain.

The corresponding vegetation response is shown in Figure 4. To quantify the overall vegetation level, we define

$$\bar{p}(T) = \frac{1}{L} \int_0^L p(x, T) dx.$$



(a) Spatially averaged vegetation density at the final time.



(b) Final spatial distributions of the vegetation density.

Figure 4. Effects of the constant boundary control on the vegetation density.

Figure 4(a) shows that $\bar{p}(T)$ increases as the boundary-control intensity is increased from $u = 1$ to $u = 1.10$. Thus, although the control acts directly only through the soil-water boundary condition, the additional water supply improves the overall vegetation state through the reaction coupling between w and p .

The effect is not simply a uniform increase in vegetation density. As shown in Figure 4(b), the positions of the vegetation peaks and troughs also change with u . Therefore, boundary irrigation modifies both the spatially averaged vegetation level and the spatial organization of the vegetation pattern.

To quantify the physical water supplied through the controlled boundary, we define the cumulative inward water input by

$$I(u) = \int_0^T \delta_w(u - a)w(L, t) dt, \quad u \geq a.$$

The corresponding irrigation-efficiency index is defined as

$$E_I(u) = \frac{\bar{p}_u(T) - \bar{p}_a(T)}{I(u)},$$

where $\bar{p}_a(T)$ denotes the final mean vegetation density for the zero-net-exchange case $u = a$.

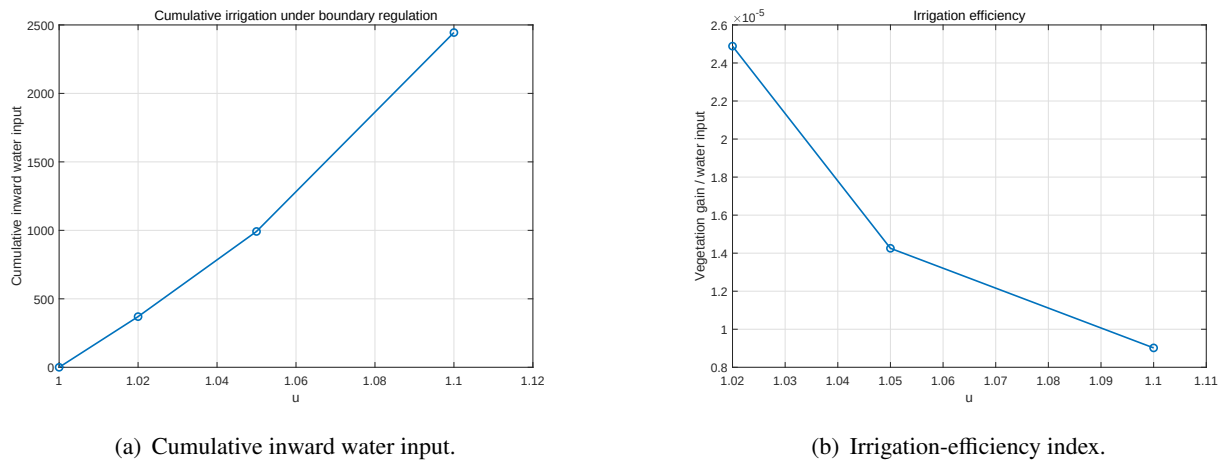


Figure 5. Cumulative water input and irrigation efficiency under different constant boundary-control intensities.

Figure 5 reveals a clear trade-off between vegetation improvement and cumulative water input. The cumulative inward water input increases rapidly as u increases, whereas the improvement in the mean vegetation density is much more moderate. Consequently, the irrigation-efficiency index decreases over the investigated control range.

These results indicate that stronger boundary irrigation does not lead to a proportional improvement in the vegetation state. Therefore, simply selecting the largest admissible value of u is not necessarily an efficient strategy, which motivates the time-dependent optimal-control problem considered below.

5.4. Optimal boundary-control strategy

We next solve the time-dependent optimal boundary-control problem. The parameters in the objective functional and the admissible control set are chosen as

$$a = u^e = 1, \quad \beta_1 = \beta_2 = 1,$$

with

$$1 \leq u(t) \leq 1.10, \quad \int_0^T u(t) dt \leq 210.$$

The desired terminal vegetation state $p_T(x)$ is generated by the state system under the constant control

$$u(t) \equiv 1.10.$$

Since

$$\int_0^{200} 1.10 dt = 220 > 210,$$

this trajectory is used only to construct the desired terminal state and is not admissible in the optimization problem.

For the numerical optimization, the interval $[0, T]$ is divided into 20 equal temporal subintervals, and the control is approximated by a piecewise-constant function. The resulting finite-dimensional constrained optimization problem is solved using a sequential quadratic programming method.

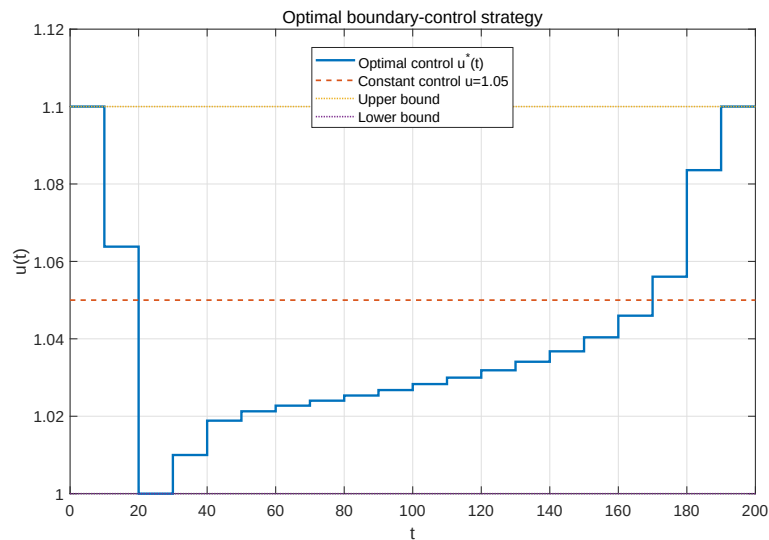


Figure 6. Temporal variation of the optimal boundary control $u^*(t)$. The constant control $u = 1.05$ and the admissible upper and lower bounds are shown for comparison.

Figure 6 shows that the computed optimal control is clearly time dependent. Relatively strong regulation is applied during the initial stage, followed by a temporary reduction, while the control gradually increases again during the intermediate and terminal stages. The computed strategy remains within the prescribed pointwise bounds and satisfies the integral constraint.

Compared with the constant strategy $u = 1.05$, the result indicates that the effectiveness of boundary regulation depends not only on the overall control effort but also on its temporal allocation.

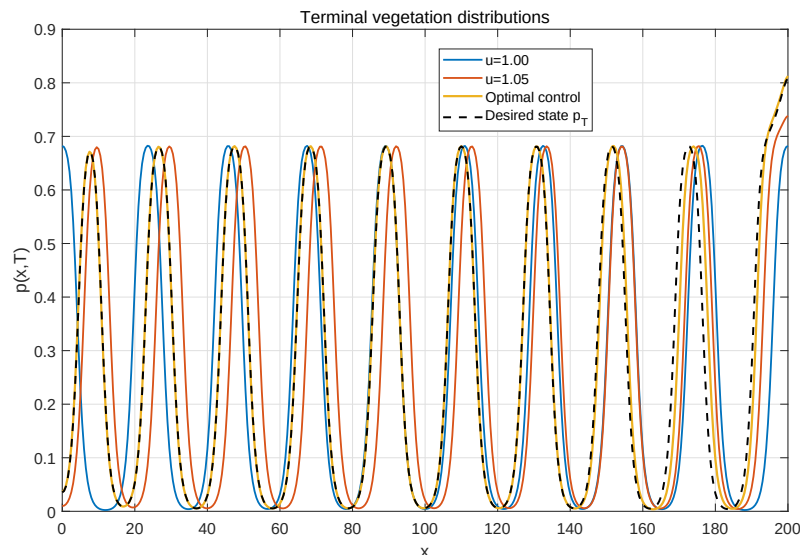


Figure 7. Terminal vegetation distributions under different control strategies. The results corresponding to $u = 1$, $u = 1.05$, and the optimal control $u^*(t)$ are compared with the desired terminal state $p_T(x)$.

Figure 7 compares the terminal vegetation distributions under the different control strategies. The profiles corresponding to $u = 1$ and the constant control $u = 1.05$ exhibit noticeable spatial mismatches with the desired state $p_T(x)$. In contrast, the terminal vegetation distribution generated by the optimal control follows the target much more closely in both the locations and amplitudes of the vegetation peaks and troughs.

This result is consistent with the terminal-tracking term

$$\|p(\cdot, T) - p_T\|_{L^2(\Omega)}^2$$

in the objective functional. Hence, increasing the spatially averaged vegetation density alone does not necessarily imply improved control performance; the complete spatial distribution of the terminal vegetation state must also be taken into account.

The objective value obtained with $u^*(t)$ is substantially smaller than those corresponding to the constant controls $u = 1$ and $u = 1.05$. Therefore, temporally redistributing the available boundary regulation provides a more effective strategy for approaching the desired terminal vegetation state than prescribing a constant irrigation intensity.

5.5. Effect of cross-diffusion on the optimal control

Finally, we investigate whether cross-diffusion modifies the optimal temporal allocation of the boundary regulation. The optimal-control problem is solved independently for

$$\gamma = 0, \quad \gamma = 10, \quad \gamma = 20,$$

while all other model parameters, control constraints, and the desired terminal state are kept unchanged. For each value of γ , the optimal control is recomputed rather than applying the same prescribed control trajectory to different state systems.

The two representative cases $\gamma = 0$ and $\gamma = 20$ are compared in Figure 8.

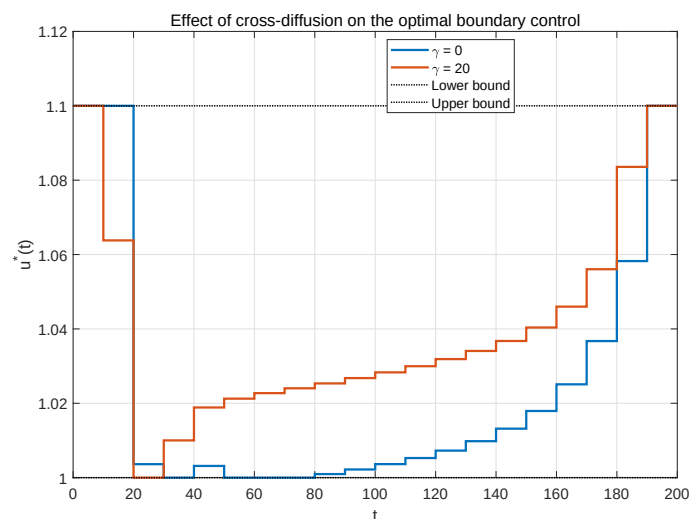


Figure 8. Optimal boundary-control strategies obtained with and without cross-diffusion. The control is approximated by a piecewise-constant function over 20 equal temporal subintervals.

Figure 8 shows a clear difference between the optimal strategies with and without cross-diffusion. For $\gamma = 0$, the optimal control remains close to the lower admissible bound over a relatively long intermediate interval and increases mainly during the later stage. In contrast, for $\gamma = 20$, the control begins to increase earlier and grows more gradually throughout the intermediate and late stages.

The staircase shape results from the piecewise-constant numerical parameterization of the control and should not be interpreted as an instantaneous physical jump in irrigation intensity. Each horizontal segment represents the optimized boundary-control intensity over one temporal subinterval.

The quantitative results for the three values of γ are summarized in Table 4.

Table 4. Influence of the cross-diffusion coefficient on the optimal-control performance.

γ	$J(u^*)$	$\ p(T) - p_T\ _{L^2}$	$\bar{p}(T)$	$\int_0^T u^*(t) dt$
0	2.3866	2.1006	0.29587	204.87
10	1.3307	1.4939	0.28132	206.69
20	0.39918	0.57709	0.28100	208.00

Table 4 shows that both the optimal objective value and the terminal tracking error depend strongly on the cross-diffusion coefficient. Interestingly, the case $\gamma = 0$ yields a larger spatially averaged vegetation density than the cases with positive cross-diffusion, while its terminal tracking error is substantially larger. This further confirms that the present optimization problem does not simply maximize the mean vegetation density, but instead seeks to reproduce the complete prescribed spatial vegetation distribution.

The comparison demonstrates that cross-diffusion affects not only the intrinsic spatial dynamics of the vegetation–water system but also the temporal allocation of the optimal boundary regulation. Changing γ modifies the internal redistribution of soil water associated with spatial vegetation heterogeneity and therefore changes the control-to-state relationship

$$u(t) \longrightarrow w(x, t) \longrightarrow p(x, t).$$

Consequently, a boundary-control strategy that is optimal for the system without cross-diffusion is generally not optimal for the cross-diffusive system.

It should be emphasized that the common desired terminal state $p_T(x)$ used in the above comparison is generated from the baseline system with $\gamma = 20$. Therefore, the smaller tracking error obtained for $\gamma = 20$ should not be interpreted as a general conclusion that increasing the cross-diffusion coefficient necessarily improves the controllability of the vegetation–water system. Rather, the present comparison demonstrates that, for a fixed prescribed target, changing the cross-diffusion coefficient modifies both the attainable tracking performance and the corresponding optimal temporal regulation strategy.

6. Concluding remarks

In this paper, we have developed a theoretical and numerical framework for the optimal boundary control of a vegetation–water reaction–diffusion system with cross-diffusion in arid and semi-arid ecosystems. Starting from a classical vegetation–water model, an effective cross-diffusion term $-\gamma \nabla^2 p$ was introduced into the soil-water equation to represent vegetation-induced soil-water redistribution at

the ecosystem scale. A Robin-type boundary condition was further imposed on a part of the boundary to describe external water exchange associated with management measures such as irrigation and drainage.

For the state system, we established the existence and uniqueness of weak solutions together with the corresponding a priori estimates under suitable assumptions on the cross-diffusion coefficient and the admissible control. Based on the resulting control-to-state mapping, an optimal boundary-control problem was formulated by combining a terminal vegetation-tracking objective with a control-cost term, subject to pointwise control bounds and an integral constraint on the cumulative boundary-regulation effort. The existence of an optimal control was established, and the control-to-state mapping was shown to be twice continuously Fréchet differentiable. Furthermore, by means of the Lagrange multiplier approach, we derived the adjoint system and the first-order necessary optimality conditions, and obtained second-order sufficient optimality conditions on the corresponding critical cone.

The numerical results further illustrate the effectiveness and relevance of the proposed boundary-control framework. Grid and time-step refinement tests confirm the convergence and numerical reliability of the adopted discretization. For constant boundary regulation, increasing the irrigation intensity enhances the soil-water availability near the controlled boundary and increases the spatially averaged terminal vegetation density, while also modifying the spatial organization of the vegetation distribution. At the same time, the required cumulative water input increases rapidly, and the vegetation gain per unit water input decreases, indicating diminishing irrigation efficiency as the control intensity increases. This trade-off provides a direct motivation for optimizing the temporal distribution of the boundary regulation.

The computed time-dependent optimal control significantly improves the terminal tracking of the prescribed vegetation state compared with constant-control strategies. The results show that the control resource need not be distributed uniformly over the entire time horizon; instead, an appropriate temporal redistribution of the boundary regulation can produce a substantially better terminal vegetation distribution under the prescribed control constraints. In addition, comparison of the cases with and without cross-diffusion demonstrates that the cross-diffusion mechanism changes the optimal temporal allocation of the boundary control. This confirms that vegetation-induced internal water redistribution and externally imposed boundary regulation are dynamically coupled through the vegetation–water system.

The present work provides a basis for further studies of boundary management in spatial ecological systems. Future research may incorporate stochastic environmental forcing and parameter uncertainty, consider more realistic spatially heterogeneous boundary interventions, and develop efficient adjoint-based numerical optimization algorithms for large-scale problems. Further calibration of the model and control parameters using field or experimental data would also be valuable for assessing the practical applicability of the proposed framework.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The research was supported by the National Natural Science Foundation of China (Nos. 12561092, 12261069, and 12161068) and the Natural Science Foundation of Ningxia (No. 2026AAC030095).

Conflict of interest

The authors declare there are no conflicts of interest.

References

1. S. Kefi, M. Rietkerk, C. L. Alados, Y. Pueyo, V. P. Papanastasis, A. ElAich, et al., Spatial vegetation patterns and imminent desertification in Mediterranean arid ecosystems, *Nature*, **449** (2007), 213–217. <https://doi.org/10.1038/nature06111>
2. M. Rietkerk, R. Bastiaansen, S. Banerjee, J. van de Koppel, M. Baudena, A. Doelman, Evasion of tipping in complex systems through spatial pattern formation, *Science*, **374** (2021), eabj0359. <https://doi.org/10.1126/science.abj0359>
3. M. Scheffer, J. Bascompte, W. A. Brock, V. Brovkin, S. R. Carpenter, V. Dakos, et al., Early-warning signals for critical transitions, *Nature*, **461** (2009), 53–59. <https://doi.org/10.1038/nature08227>
4. C. A. Klausmeier, Regular and irregular patterns in semi-arid vegetation, *Science*, **284** (1999), 1826–1828. <https://doi.org/10.1126/science.284.5421.1826>
5. T. M. Scanlon, K. K. Caylor, S. A. Levin, I. Rodriguez-Iturbe, Positive feedbacks promote power-law clustering of Kalahari vegetation, *Nature*, **449** (2007), 209–212. <https://doi.org/10.1038/nature06060>
6. E. Gilad, J. von Hardenberg, A. Provenzale, M. Shachak, E. Meron, A mathematical model of plants as ecosystem engineers, *J. Theor. Biol.*, **244** (2007), 680–691. <https://doi.org/10.1016/j.jtbi.2006.08.006>
7. G. Grifò, Vegetation patterns in the hyperbolic Klausmeier model with secondary seed dispersal, *Mathematics*, **11** (2023), 1084. <https://doi.org/10.3390/math11051084>
8. G. Consolo, G. Grifò, Turing vegetation patterns in flat arid environments with finite soil carrying capacity, *Ric. Mat.*, **74** (2025), 235–247. <https://doi.org/10.1007/s11587-023-00783-z>
9. G. Consolo, G. Grifò, Eckhaus instability of stationary patterns in hyperbolic reaction–diffusion models on large finite domains, *Partial Differ. Equations Appl.*, **3** (2022), 57. <https://doi.org/10.1007/s42985-022-00193-0>
10. G. Grifò, C. Currò, G. Valenti, Travelling waves in dryland ecology: Continuous and discontinuous connections in a hyperbolic vegetation model, *Nonlinear Dyn.*, **113** (2025), 15295–15319. <https://doi.org/10.1007/s11071-025-10925-7>
11. G. Grifò, A. Iuorio, F. Veerman, Far-from-equilibrium traveling pulses in sloped semiarid environments driven by autotoxicity effects, *SIAM J. Appl. Math.*, **85** (2025), 188–209. <https://doi.org/10.1137/24M1669499>
12. G. Consolo, C. Currò, G. Grifò, G. Valenti, Vegetation pattern formation and transition in dryland ecosystems with finite soil resources and inertia, *Physica D*, **476** (2025), 134601. <https://doi.org/10.1016/j.physd.2025.134601>
13. F. Li, R. Yang, L. Liu, Synergistic effects of feedback regulation and vegetation internal competition on vegetation patterns in semi-arid environments, *Chaos Solitons Fractals*, **199** (2025), 116912. <https://doi.org/10.1016/j.chaos.2025.116912>

14. Y. Zhang, M. Zhang, E. Buhe, J. H. He, K. A. Gepreel, Effects of local grazing and cross-diffusion on vegetation-water systems and modeling for desertification prevention, *Appl. Math. Model.*, **151** (2026), 116502. <https://doi.org/10.1016/j.apm.2025.116502>
15. G. Guo, Q. Qin, H. Cao, Y. Jia, D. Pang, Pattern formation of a spatial vegetation system with cross-diffusion and nonlocal delay, *Chaos Solitons Fractals*, **181** (2024), 114622. <https://doi.org/10.1016/j.chaos.2024.114622>
16. G. Guo, S. Zhao, D. Pang, Y. Su, Stability and cross-diffusion-driven instability for a water-vegetation model with the infiltration feedback effect, *Z. Angew. Math. Phys.*, **75** (2024), 33. <https://doi.org/10.1007/s00033-023-02167-7>
17. R. Griesse, S. Volkwein, A primal-dual active set strategy for optimal boundary control of a nonlinear reaction-diffusion system, *SIAM J. Control Optim.*, **44** (2005), 467–494. <https://doi.org/10.1137/S0363012903438696>
18. J. Jiang, Z. Tan, J. Zhou, Optimal boundary control of viscoelastic fluids equations, *J. Math. Phys.*, **67** (2026), 031501. <https://doi.org/10.1063/5.0311050>
19. C. Duarte-Leiva, Y. R. Linares, S. Lorca, E. Mallea-Zepeda, Optimal control problem related to the non-stationary Boussinesq system with Navier-slip boundary conditions, *Math. Control Relat. Fields*, (2026), In press. <https://doi.org/10.3934/mcrf.2026014>
20. J. K. Kazaku, M. M. Kahilu, J. K. K. Kasongo, D. Dochain, Dynamical analysis and observer-based boundary optimal control of a counterflow heat exchanger, *IMA J. Math. Control Inf.*, **43** (2026), dnag006. <https://doi.org/10.1093/imamci/dnag006>
21. P. Schäfer Aguilar, S. Ulbrich, Convergence of numerical adjoint schemes arising from optimal boundary control problems of hyperbolic conservation laws, *SIAM J. Control Optim.*, **64** (2026), 335–362. <https://doi.org/10.1137/23M1560975>
22. P. Colli, J. Sprekels, Second-order optimality conditions for the sparse optimal control of nonviscous Cahn–Hilliard systems, *ESAIM Control Optim. Calc. Var.*, **32** (2026), 8. <https://doi.org/10.1051/cocv/2025096>
23. G. Q. Sun, L. F. Hou, L. Li, Z. Jin, H. Wang, Spatial dynamics of a vegetation model with uptake-diffusion feedback in an arid environment, *J. Math. Biol.*, **85** (2022), 50. <https://doi.org/10.1007/s00285-022-01825-0>
24. M. M. Caldwell, T. E. Dawson, J. H. Richards, Hydraulic lift: Consequences of water efflux from the roots of plants, *Oecologia*, **113** (1998), 151–161. <https://doi.org/10.1007/s004420050363>
25. I. Prieto, C. Armas, F. I. Pugnaire, Water release through plant roots: New insights into its consequences at the plant and ecosystem level, *New Phytol.*, **193** (2012), 830–841. <https://doi.org/10.1111/j.1469-8137.2011.04039.x>
26. R. B. Neumann, Z. G. Cardon, The magnitude of hydraulic redistribution by plant roots: A review and synthesis of empirical and modeling studies, *New Phytol.*, **194** (2012), 337–352. <https://doi.org/10.1111/j.1469-8137.2012.04088.x>
27. A. R. Kacimov, Yu. V. Obnosov, Modelling of 2-D seepage from aquifer towards stream via clogged bed: The Toth–Trefftz legacy conjugated, *Adv. Water Resour.*, **131** (2019), 103372. <https://doi.org/10.1016/j.advwatres.2019.07.002>

28. R. Sarmah, I. Sonkar, S. R. Chavan, Analytical solutions for predicting seepage in a layered ditch drainage system under Dirichlet and lagging Robin boundary conditions, *Hydrol. Sci. J.*, **67** (2022), 1917–1940. <https://doi.org/10.1080/02626667.2022.2101891>
29. M. Chen, Z. Hu, Q. Zheng, H. M. Srivastava, Dynamics analysis of a spatiotemporal SI model, *Alex. Eng. J.*, **74** (2023), 705–714. <https://doi.org/10.1016/j.aej.2023.05.044>
30. Q. Liu, H. Xiang, M. Zhou, Optimal control strategies for an ecological model including infection and competition, *J. Franklin Inst.*, **359** (2022), 3444–3465. <https://doi.org/10.1016/j.jfranklin.2022.03.022>
31. H. Xiang, B. Liu, Z. Fang, Optimal control strategies for a new ecosystem governed by reaction-diffusion equations, *J. Math. Anal. Appl.*, **467** (2018), 270–291. <https://doi.org/10.1016/j.jmaa.2018.07.001>
32. H. L. Zhang, Z. Y. Li, Investigating the impact of fractional parameters on stability and bifurcation in the time-fractional Swift–Hohenberg model, *Int. J. Bifurcation Chaos*, **36** (2026), 2650185. <https://doi.org/10.1142/S0218127426501853>
33. H. L. Zhang, X. Y. Li, Z. Y. Li, Analysis and simulation of the effect of fractional parameters on dynamic behavior for a fractional-in-time three-species reaction-diffusion model, *Networks Heterogen. Media*, **21** (2026), 426–445. <https://doi.org/10.3934/nhm.2026020>



AIMS Press

©2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>)