



Research article

An iterative method based on GMsFEM for optimal control problem with random inputs

Lingling Ma*

School of Mathematics and Statistics & Hunan Provincial University Key Laboratory for Big Data Analysis and Application, Hunan First Normal University, Changsha 410205, Hunan Province, China

* **Correspondence:** Email: hudalingling@126.com.

Abstract: In this paper, we propose an iterative method for optimal control problems governed by elliptic PDEs based on the generalized multiscale finite element method (GMsFEM). In the iterative method, we first reformulate the original optimal control system into a new system with parameter-independent coefficients and parameter-dependent right-hand sides. Subsequently, we adopt the fixed-point iteration method to solve the reformulated system to get the optimal solutions. To quantify the statistics of the optimal control problems, we need to solve a coupled optimality system for a large number of samples in the stochastic space. Therefore, the computational cost is extremely large, and it may even lead to the “curse-of-dimensionality”. When the diffusion coefficient exhibits multiscale features, the iterative process can be performed in the reduced multiscale space via GMsFEM to enhance computational efficiency. The proposed iterative method can be split into two stages: an offline stage and an online stage. In the offline stage, local multiscale basis functions are constructed from the deterministic optimality system to build the reduced space. In the online stage, the reformulated model is projected onto the reduced space and solved via fixed-point iteration. Finally, we will present two numerical examples to verify the effectiveness of the proposed iterative method.

Keywords: stochastic optimal control problem; iterative method; generalized multiscale finite element method; random inputs

1. Introduction

Optimal control problems constrained by partial differential equations (PDEs) are widely employed in engineering and applied sciences to characterize design problems. For the deterministic optimal control problems, foundational theoretical frameworks and computational approaches have been rigorously developed in works such as [1, 2]. The author [3] discussed the first- and second-order sufficient optimality conditions for the optimal control problems. The work [4] showed how to achieve optimal

accuracy in the generalized finite element method for second-order elliptic PDEs describing fields in heterogeneous media. However, due to incomplete knowledge, intrinsic system variability, or manufacturing imprecisions, the problems are often affected by uncertainties. Consequently, this work is concerned with stochastic optimal control problems where the constraints are given by elliptic PDEs with random inputs. Uncertainties, which are often modeled using random variables or random fields, can arise in practical applications from many sources, such as coefficients in the PDE, boundary conditions, external loads, and the shape of the physical domain. The uncertainty may have a significant impact on the optimal solution. One simple example of stochastic optimal control problems reads:

$$\begin{cases} \mathcal{J}(u, f) = \frac{1}{2} \|u - \hat{u}\|_{\mathcal{L}^2(\mathcal{D})}^2 + \beta \|f\|_{\mathcal{L}^2(\mathcal{D})}^2 \\ \text{s.t. } \mathcal{A}u = f \quad \text{in } \mathcal{D}, \quad u = g(x) \quad \text{on } \partial\mathcal{D}. \end{cases} \quad (1.1)$$

Here, $\mathcal{A}$ denotes a second-order elliptic partial differential operator that depends on the random inputs.

For the numerical simulation of optimization problems constrained by PDEs, there is a choice between discretizing then optimizing and optimizing then discretizing, and different opinions exist as to which route to take (see Collis and Heinkenschloss [5] for more discussion). In this work, we adopt the second framework. In the work [6], the authors have investigated the existence and uniqueness of an optimal solution to stochastic optimal control problems via the finite element (FE) method. The general theory for saddle-point problems that arise from Lagrange multipliers was developed in the work [7]. The paper [8] described a new approach to error control and mesh adaptivity in the discretization of elliptic optimal control problems. From uncoupled optimality conditions, we need to incur a significant computational cost to obtain the optimal solutions of the optimal control problem, particularly for the many-query setting. In the high-dimensional stochastic spaces, the computational cost may lead to the so-called “curse of dimensionality”, i.e., the exponential growth of complexity with respect to the stochastic dimension. This exponential growth renders standard grid-based methods infeasible. Hence, a reduced-order method is needed to solve these optimal control problems.

In the past several years, there have been some efficient methods proposed for solving the optimal control problems constrained by stochastic PDEs. As one of the most straightforward approaches for stochastic problems, the Monte Carlo method [9, 10] generates a large number of independent random samples from the underlying probability distributions, runs the deterministic model for each sample, and then computes statistical estimators directly from the ensemble of outputs. However, this straightforwardness comes with a significant drawback: the convergence is notoriously slow. To achieve acceptable accuracy often demands a huge number of simulations. This computational burden becomes especially critical in “many-query” contexts. The stochastic Galerkin method [11, 12] has been developed as an efficient alternative, overcoming the slow convergence that plagues Monte Carlo-based approaches for stochastic optimal control problems. The works [13, 14] employed the stochastic collocation method [15, 16], a nonintrusive method that features both ease of implementation and fast convergence, thereby avoiding the large tensor-product algebraic systems required by the intrusive stochastic Galerkin method. Although the above methods achieve a higher convergence rate, they require solving the optimality system at each collocation node, which can create a computational burden, particularly in high-dimensional stochastic spaces. To address the computational difficulties and challenges, a number of global model order reduction (MOR) techniques and their variants have been proposed, such as the proper orthogonal decomposition (POD) method [17], the reduced basis (RB)

method [18,19], the ensemble method [20–22], and the multi-modes method [23,24]. Besides these, local model reduction methods such as the multiscale finite element method (MsFEM) [25,26] and mixed GMsFE basis methods [27] have also been proposed. Traditional CFEM often loses key physical information due to oversimplification, whereas the multiscale finite element method embeds mesoscale physical information into the coarse-grid macroscopic model through the sophisticated construction of multiscale basis functions, thereby achieving a better balance between accuracy and efficiency. The generalized multiscale finite element method (GMsFEM) [28, 29] can be viewed as an extension of MsFEM. Its main idea is to identify local dominant modes by solving the following spectral problem in the local region:

$$\mathcal{A}u = \lambda \mathcal{M}u$$

where $\mathcal{M}$ denotes the weighted L^2 -inner operator induced by the multiscale diffusion coefficient. The step is crucial for capturing high-contrast channels and regions. By combining the above two approaches, the works [30, 31] have successfully solved the stochastic optimal control problems constrained by elliptic and parabolic PDEs, respectively.

Except for the approaches discussed above for solving parameterized partial differential equations, the iterative method [32] provides an efficient alternative. The iterative method is built on a core idea: casting the original stochastic convection–diffusion problem into a new fixed-point form. In this new form, the coefficients of the convection and diffusion operators become parameter-independent. All parameter-dependent and solution-dependent terms are then moved to the right-hand side. This allows the costly matrix factorization to be computed only once and then reused for every iteration and every parameter sample, leading to dramatic savings in computational effort.

Inspired by GMsFEM and the iterative method, this work will develop an efficient iterative method based on GMsFEM for solving parameter-dependent optimal control problems. The proposed iterative method begins with a reformulation of the original optimality system into an equivalent form. In the new system, the left-hand side has a parameter-independent diffusion coefficient, while the right-hand side contains a parameter-dependent diffusion coefficient and right-hand sides. This reformulation enables a key computational advantage: the multiscale reduced space, constructed via GMsFEM, needs to be built only once in the offline stage. Then, for any given random sample, the reformulated system is projected onto the low-dimensional multiscale space and solved by fixed-point iteration to obtain the optimal control solution. Moreover, since neither the diffusion coefficients nor the iterations depend on the random parameters, an efficient direct solver and a block Krylov subspace iterative solver can be employed to solve the linear system with multiple right-hand vectors in every iteration. To render the proposed iterative method more intelligible, we show a schematic overview of the Iter-GMs approach in Figure 1.

The structure of the paper is organized as follows. Section 2 serves to set up the preliminaries and notations, describe the stochastic optimal control problems, and assert the global existence and uniqueness of optimal solutions. In Section 3, we describe in detail the iterative algorithm based on FEM for solving the optimal control problem. In Section 4, we elaborate on the step-by-step construction of the Iter-GMs framework for tackling the optimal control problem with random inputs. Several numerical examples are presented in Section 5 to verify the rationality and effectiveness of the proposed iterative method. Finally, the last section outlines some concluding remarks.

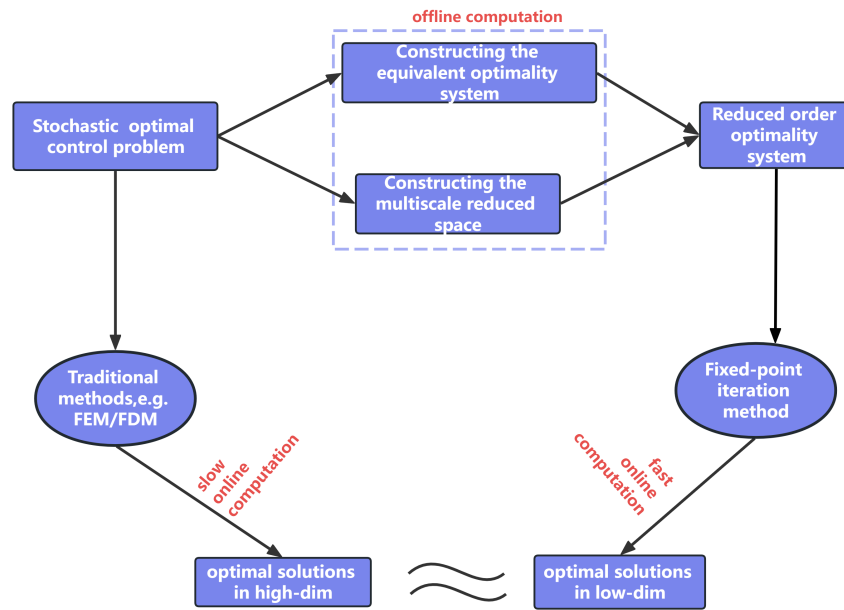


Figure 1. Schema of Iter-GMs method.

2. Stochastic optimal control problems

In this section, we establish the notation employed in the remainder of the paper before introducing the stochastic optimal control problems. Subsequently, we analyze stochastic elliptic optimal control problems using the Lagrangian functional formalism.

2.1. Preliminaries and notations

We begin by establishing the basic probabilistic setup. We assume $(\Omega, \mathcal{F}, \mathcal{P})$ is a complete probability space, where Ω is the event space, $\mathcal{F}$ is the σ -algebra, and $\mathcal{P}$ is the probability measure. To make the problem computationally tractable, we discretize the underlying random field using a finite-dimensional random vector $\eta(\omega) = \{\eta(\omega)_1(\omega), \eta(\omega)_2(\omega), \dots, \eta(\omega)_n(\omega)\}$, i.e., $\eta(\omega)(\cdot) : \Omega \rightarrow \Gamma \subset \mathbb{R}^n$ ($n \geq 1$), which maps the sample space Ω to a subset $\Gamma \subset \mathbb{R}^n$. This vector encapsulates all the stochastic information required for the subsequent sections.

Let $L^2_{\mathcal{P}}(\Omega)$ denote the space of the random variables as

$$L^2_{\mathcal{P}}(\Omega) = \{\eta(\omega) : \omega \in \Omega \rightarrow \eta(\omega) \in \mathbb{R}; \int_{\Omega} \eta(\omega)^2 \mathcal{P}(d\eta(\omega)) < \infty\},$$

equipped with the standard inner product

$$(\eta(\omega), \zeta)_{L^2_{\mathcal{P}}} := \mathbb{E}[\eta(\omega)\zeta] = \int_{\Omega} \eta(\omega)\zeta(\omega) \mathcal{P}(d\omega),$$

and the induced norm $\|\eta(\omega)\|_{L^2_{\mathcal{P}}} := \sqrt{\mathbb{E}[\eta(\omega)^2]}$. With these notations, we define the tensor-product Hilbert space

$$H^s(\mathcal{D}) \otimes L^2_{\mathcal{P}}(\Gamma) := \left\{ v : \mathcal{D} \times \Gamma \rightarrow \mathbb{R} \mid v(\cdot, \eta(\omega)) \in H^s(\mathcal{D}) \text{ for all } \eta(\omega) \in \Gamma, \eta(\omega) \mapsto \|v(\cdot, \eta(\omega))\|_{H^s(\mathcal{D})} \in L^2_{\mathcal{P}}(\Gamma) \right\}.$$

The shorthand notation of the space is denoted by $\mathcal{H}^s(\mathcal{D}) := H^s(\mathcal{D}) \otimes L^2_\rho(\Gamma)$, whose natural norm is given by

$$\|v\|_{\mathcal{H}^s(\mathcal{D})} := \left(\int_{\Gamma} \|v(\cdot, \eta(\omega))\|_{H^s(\mathcal{D})}^2 \rho(\eta(\omega)) d\eta(\omega) \right)^{1/2}.$$

The corresponding Sobolev space with homogeneous Dirichlet boundary condition is denoted by $\mathcal{H}_0^s(\mathcal{D})$. In the special case $s = 0$, we simply write $\mathcal{L}^2(\mathcal{D})$ in place of $\mathcal{H}^0(\mathcal{D})$.

2.2. Problem definition

Let $\mathcal{D}$ be a bounded domain occupied by a body. The temperature u is controlled by a distributed source f and fixed on the boundary $\partial\mathcal{D}$. In the paper, we consider the specific case of a diffusion problem. The elliptic operator defined in (1.1) is taken as $\mathcal{A}u = -\operatorname{div}(\kappa(x; \eta(\omega))\nabla u)$. Then, the weak form of the constraint can be given as:

$$a(u, v; \eta(\omega)) = (f, v; \eta(\omega)) \quad \forall v \in \mathcal{H}_0^1(\mathcal{D}), \quad (2.1)$$

subject to the Dirichlet boundary condition $u|_{\partial\mathcal{D}} = g(x)$. Here, the bilinear form and linear functional are defined as

$$a(u, v; \eta(\omega)) = \int_{\mathcal{D}} \kappa(x; \eta(\omega)) \nabla u(x) \cdot \nabla v(x) dx,$$

and

$$(f, v; \eta(\omega)) = \int_{\mathcal{D}} f(x; \eta(\omega)) v(x) + \int_{\partial\mathcal{D}} \kappa(x; \eta(\omega)) \nabla g(x) \cdot \nabla v(x) dx.$$

We seek the optimal temperature distribution u_{opt} on $\mathcal{D}$ that matches a given target pattern $\hat{u}$ as closely as possible, with minimal expenditure of the heat source. This leads to the following optimal control problem: minimize the cost functional

$$\mathcal{J}(u, f) = \frac{1}{2} \|u - \hat{u}\|_{\mathcal{L}^2(\mathcal{D})}^2 + \beta \|f\|_{\mathcal{L}^2(\mathcal{D})}^2 \quad (2.2)$$

over $(u, f) \in \mathcal{H}^1(\mathcal{D}) \times \mathcal{L}^2(\mathcal{D})$, subject to the state equation (2.1). Here, β denotes a positive regularization parameter, which serves to penalize large controls and thereby guarantees the existence of a minimizer.

We assume that the bilinear form $a(\cdot, \cdot; \eta(\omega))$ is uniformly coercive and bounded on $H^1(\mathcal{D})$ for all $\eta(\omega) \in \Gamma$, i.e., there exist $\tilde{\alpha}_0, \tilde{\alpha}_1 > 0$ such that

$$\tilde{\alpha}_0 \|v\|_{H^1(\mathcal{D})}^2 \leq a(v, v; \eta(\omega)) \leq \tilde{\alpha}_1 \|v\|_{H^1(\mathcal{D})}^2,$$

for all $v \in H^1(\mathcal{D})$ and $\eta(\omega) \in \Gamma$. This ensures well-posedness of (2.1). Additionally, we assume global Lipschitz continuity of $a(\cdot, \cdot; \eta(\omega))$ in $\mathcal{D} \times \Gamma$ to guarantee spatial regularity of the solution u .

2.3. Lagrangian functional formulation

Following the standard Lagrangian approach [3, 8], we construct the stochastic Lagrangian functional for the constrained optimization problem (2.1) and (2.2):

$$\mathcal{L}(u, f, \lambda; \eta(\omega)) = \frac{1}{2} \|u - \hat{u}\|_{\mathcal{L}^2(\mathcal{D})}^2 + \beta \|f\|_{\mathcal{L}^2(\mathcal{D})}^2 + a(u, \lambda; \eta(\omega)) - (f, \lambda; \eta(\omega)), \quad (2.3)$$

with $\lambda \in \mathcal{H}^1(\mathcal{D})$ serving as the Lagrange multiplier for the PDE constraint. The first-order optimality conditions are then derived by setting the Fréchet derivatives of $\mathcal{L}$ with respect to u , f , and λ to zero at the optimum. This yields the following coupled system:

$$\begin{aligned} \text{State equation:} \quad & a(u, \tilde{u}; \eta(\omega)) = (f, \tilde{u}; \eta(\omega)) \quad \forall \tilde{u} \in \mathcal{H}_0^1(\mathcal{D}), \\ \text{Adjoint equation:} \quad & a(\tilde{\lambda}, \lambda; \eta(\omega)) = -(u - \hat{u}, \tilde{\lambda}; \eta(\omega)) \quad \forall \tilde{\lambda} \in \mathcal{H}_0^1(\mathcal{D}), \\ \text{Gradient equation:} \quad & 2\beta(f, \tilde{f}; \eta(\omega)) = (\tilde{f}, \lambda; \eta(\omega)) \quad \forall \tilde{f} \in \mathcal{L}^2(\mathcal{D}). \end{aligned} \quad (2.4)$$

Here, $a(\tilde{\lambda}, \lambda; \eta(\omega))$ denotes the adjoint bilinear form, and $(\cdot, \cdot; \eta(\omega))$ represents the standard L^2 -inner product.

3. The iterative-FEM framework of optimal control problem

In this section, we develop a finite element based iterative solution strategy for the optimal control problem (2.1) and (2.2). The proposed framework hinges on a reformulation that decouples the diffusion coefficients from the parameter dependence, yielding a parameter-independent left-hand side while shifting the parameter dependence to the right-hand side. This reformulated system is then solved by means of a fixed-point iteration. In addition, to address the multiscale nature of the constraint PDE (2.1), we employ the generalized multiscale finite element method (GMsFEM) to construct a local reduced-order model that efficiently captures the multiscale features. A comprehensive description of the FEM-based iterative procedure is presented in the following subsections.

3.1. The iterative method

We begin by assuming that the diffusion coefficient can be decomposed into two parts:

$$\kappa(x, \eta(\omega)) := \kappa^0(x) + \kappa^1(x, \eta(\omega)),$$

where $\kappa^0(x)$ is the deterministic component exhibiting multiscale characteristics, and $\kappa^1(x, \eta(\omega))$ represents the random fluctuation. The bilinear forms can be rewritten as

$$a(u, \tilde{u}; \eta(\omega)) = a^0(u, \tilde{u}) + a^1(u, \tilde{u}; \eta(\omega)), \quad (3.1)$$

where $a^0(u, \tilde{u})$ is a $\eta(\omega)$ -independent bilinear form, and $a^1(u, \tilde{u}; \eta(\omega))$ is a $\eta(\omega)$ -dependent bilinear form. Consequently, the optimality system (2.4) takes the following equivalent form:

$$\begin{cases} a^0(u, \tilde{u}) = (f, \tilde{u}; \eta(\omega)) - a^1(u, \tilde{u}; \eta(\omega)) & \forall \tilde{u} \in \mathcal{H}_0^1(\mathcal{D}), \\ a^0(\tilde{\lambda}, \lambda) = -(u - \hat{u}, \tilde{\lambda}; \eta(\omega)) - a^1(\tilde{\lambda}, \lambda; \eta(\omega)) & \forall \tilde{\lambda} \in \mathcal{H}_0^1(\mathcal{D}), \\ 2\beta(f, \tilde{f}; \eta(\omega)) = (\tilde{f}, \lambda; \eta(\omega)) & \forall \tilde{f} \in \mathcal{L}^2(\mathcal{D}). \end{cases} \quad (3.2)$$

We now state the iterative algorithm for solving the optimal control problem (2.1) and (2.2).

Algorithm 1 An iterative algorithm in variational setting.

Step 1: For each $\eta(\omega) \in \Omega$, find (u_0, λ_0, f_0) by solving the following problem

$$\begin{cases} a^0(u_0, \tilde{u}) - (f, \tilde{u}) = 0, \quad \forall \tilde{u} \in \mathcal{H}_0^1(\mathcal{D}) \\ a^0(\lambda_0, \tilde{\lambda}) + (u_0, \tilde{\lambda}) = (\hat{u}, \tilde{\lambda}; \eta(\omega)), \quad \forall \tilde{\lambda} \in \mathcal{H}_0^1(\mathcal{D}), \\ 2\beta(f_0, \tilde{f}) - (\tilde{f}, \lambda_0) = 0, \quad \forall \tilde{f} \in \mathcal{L}^2(\mathcal{D}) \end{cases} \quad (3.3)$$

Step 2: For each $\eta(\omega) \in \Omega$, determine $\{(u_n, \lambda_n, f_n)\}_{n \geq 1}$ recursively by solving the problem

$$\begin{cases} a^0(u_n, \tilde{u}) - (f_n, \tilde{u}) = -a^1(u_{n-1}, \tilde{u}; \eta(\omega)), \quad \forall \tilde{u} \in \mathcal{H}_0^1(\mathcal{D}) \\ a^0(\lambda_n, \tilde{\lambda}) + (u_n, \tilde{\lambda}) = -a^1(\lambda_{n-1}, \tilde{\lambda}) + (\hat{u}, \tilde{\lambda}; \eta(\omega)), \quad \forall \tilde{\lambda} \in \mathcal{H}_0^1(\mathcal{D}), \\ 2\beta(f_n, \tilde{f}; \eta(\omega)) - (\tilde{f}, \lambda_n; \eta(\omega)) = 0, \quad \forall \tilde{f} \in \mathcal{L}^2(\mathcal{D}) \end{cases} \quad (3.4)$$

Following the analysis in [32], we enforce a specific condition on the choice of a_0 to ensure the convergence of Algorithm 1.

Lemma 3.1. ([32]) *Criterion for selecting a_0 :*

(a) $a_0(u, v)$ must satisfy bi-linearity, boundedness, and coercivity on the tensor product space $H^1(\mathcal{D}) \times H^1(\mathcal{D})$.

(i) $a_0(u, v)$ is bilinear for on $H^1(\mathcal{D}) \times H^1(\mathcal{D})$.

(ii) There exists a constant $\Lambda_0 > 0$ such that

$$a_0(u, v) \leq \Lambda_0 \|u\|_{H^1} \|v\|_{H^1}, \quad \forall u, v \in H^1(\mathcal{D}).$$

(iii) There exists a constant $\lambda_0 > 0$ such that

$$a_0(v, v) \geq \lambda_0 \|v\|_{H^1}^2, \quad \forall v \in H^1(\mathcal{D}).$$

(b) *Relative dominance:* There exists a $\rho \in (0, 1)$ such that

$$\frac{\|a_1(\eta(\omega))\|_{\mathcal{L}(H^1 \times H^1; \Gamma)}}{\lambda_0} < \rho \quad \forall \eta(\omega) \in \Gamma.$$

3.2. The approximation of optimal control problem based on FEM

We first discretize the spatial domain $\mathcal{D}$ by a uniform fine mesh $\mathcal{T}_h$. On this mesh, we construct the finite element space $V^h(\mathcal{D}) \subset H^1(\mathcal{D})$, with $V_0^h(\mathcal{D})$ denoting the subspace of functions satisfying zero Dirichlet boundary conditions. The mesh parameters are denoted by N_h (vertices), N_e (elements), and $\mathcal{N} := \dim V^h(\mathcal{D})$. The associated stochastic tensor-product space is defined as $\mathcal{V}_0^h(\mathcal{D}) := V_0^h(\mathcal{D}) \otimes L_\rho^2(\Gamma)$. Likewise, let $M_h(\mathcal{D}) \subset L^2(\mathcal{D})$ be finite-dimensional, and define $\mathcal{M}_h(\mathcal{D}) := M_h(\mathcal{D}) \otimes L_\rho^2(\Gamma)$.

The Galerkin discretization of (2.4) consists of, for any $\eta(\omega) \in \Gamma$, finding $(u_h, f_h, \lambda_h) \in \mathcal{V}_0^h(\mathcal{D}) \times M_h(\mathcal{D}) \times \mathcal{V}_0^h(\mathcal{D})$ such that the following equations hold:

$$\begin{cases} a(u_h, \tilde{u}_h; \eta(\omega)) = (f_h, \tilde{u}_h; \eta(\omega)) & \forall \tilde{u}_h \in \mathcal{V}_0^h(\mathcal{D}), \\ a(\lambda_h, \tilde{\lambda}_h; \eta(\omega)) = -(u_h - \hat{u}, \tilde{\lambda}_h; \eta(\omega)) & \forall \tilde{\lambda}_h \in \mathcal{V}_0^h(\mathcal{D}), \\ 2\beta(f_h, \tilde{f}_h; \eta(\omega)) = (\tilde{f}_h, \lambda_h; \eta(\omega)) & \forall \tilde{f}_h \in \mathcal{M}_h(\mathcal{D}). \end{cases} \quad (3.5)$$

Let $\{\delta_i\}_{i=1}^{N_h}$ and $\{\vartheta_k\}_{k=1}^{N_e}$ be the basis functions for the finite-dimensional spaces $\mathcal{V}_0^h(\mathcal{D})$ and $\mathcal{M}_h(\mathcal{D})$, respectively. Accordingly, the finite element solutions (u_h, f_h, λ_h) admit the following expansions:

$$u_h = \sum_{i=1}^{N_h} u_{h,i} \delta_i, \quad f_h = \sum_{j=1}^{N_e} f_{h,j} \vartheta_j, \quad \lambda_h = \sum_{k=1}^{N_h} \lambda_{h,k} \delta_k. \quad (3.6)$$

The algebraic formulation of (3.5) reads

$$\underbrace{\begin{bmatrix} 2\beta M_{1,h} & 0 & -M_{2,h}^T \\ 0 & M_{3,h} & K_h^T(\eta(\omega)) \\ -M_{2,h} & K_h(\eta(\omega)) & 0 \end{bmatrix}}_{\Lambda_h(\eta(\omega)) \in \mathbb{R}^{(2N_h+N_e) \times (2N_h+N_e)}} \begin{bmatrix} F_h(\eta(\omega)) \\ u_h(\eta(\omega)) \\ \lambda_h(\eta(\omega)) \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{U}(\eta(\omega)) \\ d \end{bmatrix}, \quad (3.7)$$

where

$$(M_{1,h})_{jj'} = (\vartheta_j, \vartheta_{j'}), \quad (M_{2,h})_{kj'} = (\vartheta_{j'}, \delta_k), \quad (M_{3,h})_{ii'} = (\delta_i, \delta_{i'}), \\ (K_h^q)_{kk'} = a^q(\delta_k, \delta_{k'}), \quad (\bar{U}_p)_{k'} = (\bar{u}_p, \delta_{k'}),$$

with the index ranges $1 \leq j, j' \leq N_e$, $1 \leq k, i, i', k' \leq N_h$, $1 \leq p \leq Q_u$, and $1 \leq q \leq Q_a$. The assembled quantities are

$$K_h(\eta(\omega)) = \sum_q Q_a^q(\eta(\omega)) K_h^q, \quad \bar{U}(\eta(\omega)) = \sum_p \bar{u}_p(\eta(\omega)) \bar{U}_p.$$

Here, $u(\eta(\omega))$, $F(\eta(\omega))$, and $\lambda(\eta(\omega))$ denote the coefficient vectors corresponding to the finite element approximations $u_h(\eta(\omega))$, $f_h(\eta(\omega))$, and $\lambda_h(\eta(\omega))$, respectively. The vector d represents the contribution arising from the boundary values of u_h . For notational simplicity, the dimensions of the zero blocks are omitted. In what follows, we will adjust these dimensions to match those of the corresponding nonzero blocks.

Making use of the definitions of a_0 and a_1 , we can rewrite the system as follows:

$$\begin{cases} a^0(u_n, \bar{u}) - (f_n, \bar{u}) = -a^1(u_{n-1}, \bar{u}; \eta(\omega)), \quad \forall \bar{u} \in \mathcal{V}_0^h(\mathcal{D}) \\ a^0(\lambda_n, \bar{\lambda}) + (u_n, \bar{\lambda}) = -a^1(\lambda_{n-1}, \bar{\lambda}) + (\hat{u}, \bar{\lambda}; \eta(\omega)), \quad \forall \bar{\lambda} \in \mathcal{V}_0^h(\mathcal{D}), \\ 2\beta(f_n, \bar{f}; \eta(\omega)) - (\bar{f}, \lambda_n; \eta(\omega)) = 0, \quad \forall \bar{f} \in \mathcal{M}_h(\mathcal{D}) \end{cases} \quad (3.8)$$

We are now ready to present the discrete version of Algorithm 1 for solving Eq (3.8).

Algorithm 2 An iterative algorithm in discrete setting.

Step 1: For each $\eta(\omega) \in \Omega$, find (u_0, λ_0, f_0) by solving the following problem:

$$\begin{cases} a^0(u_0, \tilde{u}) - (f_0, \tilde{u}) = 0, \forall \tilde{u} \in \mathcal{V}_0^h(\mathcal{D}) \\ a^0(\lambda_0, \tilde{\lambda}) + (u_0, \tilde{\lambda}) = (\hat{u}, \tilde{\lambda}; \eta(\omega)), \forall \tilde{\lambda} \in \mathcal{V}_0^h(\mathcal{D}), \\ 2\beta(f_0, \tilde{f}) - (\tilde{f}, \lambda_0) = 0, \forall \tilde{f} \in \mathcal{M}_h(\mathcal{D}) \end{cases} \quad (3.9)$$

Step 2: For each $\eta(\omega) \in \Omega$, solve the following problem recursively to obtain $\{(u_n, \lambda_n, f_n)\}_{n \geq 1}$:

$$\begin{cases} a^0(u_n, \tilde{u}) - (f_n, \tilde{u}) = -a^1(u_{n-1}, \tilde{u}; \eta(\omega)), \forall \tilde{u} \in \mathcal{V}_0^h(\mathcal{D}) \\ a^0(\lambda_n, \tilde{\lambda}) + (u_n, \tilde{\lambda}) = -a^1(\lambda_{n-1}, \tilde{\lambda}) + (\hat{u}, \tilde{\lambda}; \eta(\omega)), \forall \tilde{\lambda} \in \mathcal{V}_0^h(\mathcal{D}), \\ 2\beta(f_n, \tilde{f}; \eta(\omega)) - (\tilde{f}, \lambda_n; \eta(\omega)) = 0, \forall \tilde{f} \in \mathcal{M}_h(\mathcal{D}) \end{cases} \quad (3.10)$$

When dealing with a model that exhibits multiscale features, both the number of degrees of freedom in the FE approximation and the number of iterations must be sufficiently large to achieve accurate results. Moreover, when the model also incorporates uncertainty, Algorithm 2 must be implemented repeatedly. This leads to significant computational cost and memory requirements. To overcome these difficulties, we employ the GMsFEM as a local model reduction technique for the efficient solution of the optimal control problem.

3.3. Matrix formulation based on FEM

Using the definition in (3.1) and the linear representation in (3.6), the matrix representation of (3.9) and (3.10) can be expressed as

$$\begin{bmatrix} 2\beta M_{1,h} & 0 & -M_{2,h}^T \\ 0 & M_{3,h} & (K_h^0)^T \\ -M_{2,h} & K_h^0 & 0 \end{bmatrix} \begin{bmatrix} F_h^0(\eta(\omega)) \\ u_h^0(\eta(\omega)) \\ \lambda_h^0(\eta(\omega)) \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{U}(\eta(\omega)) \\ d \end{bmatrix}, \quad (3.11)$$

and

$$\begin{bmatrix} 2\beta M_{1,h} & 0 & -M_{2,h}^T \\ 0 & M_{3,h} & (K_h^0)^T \\ -M_{2,h} & K_h^0 & 0 \end{bmatrix} \begin{bmatrix} F_h^n(\eta(\omega)) \\ u_h^n(\eta(\omega)) \\ \lambda_h^n(\eta(\omega)) \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{U}(\eta(\omega)) \\ d \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & (K_h^1(\eta(\omega)))^T \\ 0 & K_h^1(\eta(\omega)) & 0 \end{bmatrix} \begin{bmatrix} F_h^{n-1}(\eta(\omega)) \\ u_h^{n-1}(\eta(\omega)) \\ \lambda_h^{n-1}(\eta(\omega)) \end{bmatrix}, \quad (3.12)$$

respectively.

Algorithm 3 presents the matrix-based formulation of the iterative finite element method, which corresponds to the matrix discretization process in Algorithm 2.

Algorithm 3 Detailed procedure of iterative-FEM.

Input: The stochastic optimal control problem (2.2) and (2.1), maximum number of iteration N_{iter} and the error tolerance ϵ .

Output: The iterative-FEM solution $(u_h^n(\eta(\omega)), F_h^n(\eta(\omega)), \lambda_h^n(\eta(\omega)))$.

Offline phase: only perform once

1. Compute full order matrices $M_{1,h}$, $M_{2,h}$, $M_{3,h}$ and K_h^0 independent of $\eta(\omega)$;

2. Assemble and store the full matrix $\begin{bmatrix} 2\beta M_{1,h} & 0 & -M_{2,h}^T \\ 0 & M_{3,h} & (K_h^0)^T \\ -M_{2,h} & K_h^0 & 0 \end{bmatrix}$;

Online phase: for arbitrary sample $\eta(\omega) \in \Gamma$

1. With $n = 0$, solve (3.11) to get the solution coefficient $(u_h^0(\eta(\omega)), F_h^0(\eta(\omega)), \lambda_h^0(\eta(\omega)))$.

2. Renew the load vector $\begin{bmatrix} 0 \\ \overline{U}(\eta(\omega)) \\ d \end{bmatrix}$ and the matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & (K_h^1(\eta(\omega)))^T \\ 0 & K_h^1(\eta(\omega)) & 0 \end{bmatrix}$;

3. Calculate the coefficients $(u_h^n(\eta(\omega)), F_h^n(\eta(\omega)), \lambda_h^n(\eta(\omega)))$ by (3.12);

4. Compute the error $\epsilon_n = \sqrt{\|(u_h^n(\eta(\omega)), F_h^n(\eta(\omega)), \lambda_h^n(\eta(\omega))) - (u_h^{n-1}(\eta(\omega)), F_h^{n-1}(\eta(\omega)), \lambda_h^{n-1}(\eta(\omega)))\|_{L^2}^2}$

5. **while** $n < N_{\text{iter}}$ or $\epsilon_n \leq \epsilon$ **do**

6. $n \rightarrow n + 1$;

7. Return to the steps 3 and 4 in the online phase;

8. **end while**

9. Obtain the iterative-FEM solution $(u_h(\eta(\omega)), F_h(\eta(\omega)), \lambda_h(\eta(\omega)))$.

4. The Iter-GMs method for solving optimal control problem

In this section, we outline the construction of our Iter-GMs (Iterative-GMsFEM) approach for solving optimal control problems in the presence of random inputs. To this end, we first give a short overview of the GMsFEM. For a comprehensive account of the motivation, construction, and technical details of GMsFEM, we refer to [28, 29, 33, 34] and the references therein.

4.1. The Iterative-GMsFEM framework

We start with a uniform coarse partition $\mathcal{T}_H$ of $\mathcal{D}$ with mesh size H , and obtain a fine partition $\mathcal{T}_h$ of size h by refining $\mathcal{T}_H$. The coarse grid has N_c vertices, denoted by $\{N_i\}_{i=1}^{N_c}$. For each vertex N_i , we construct its coarse patch ω_i by collecting all coarse elements that have N_i as a vertex:

$$\omega_i = \bigcup_{K_s \in \mathcal{T}_H} \{K_s : x_i \in \overline{K_s}\}.$$

We illustrate the numerical discretization in Figure 2.

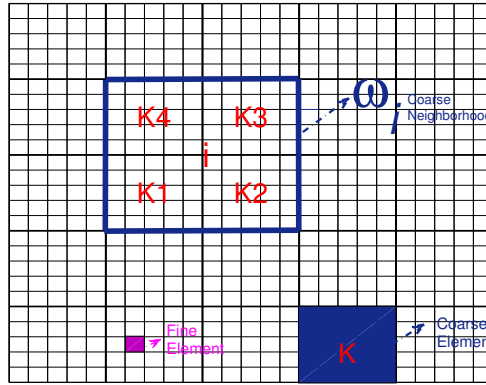


Figure 2. Illustration of discretization configuration.

The GMsFEM involves an offline stage and an online stage. During the offline stage, we first build the snapshot space $V_{\text{snap}}^{(i)}$, from which local reduced basis functions are subsequently derived. The snapshot space may be constructed in various ways [34], including the use of all fine-grid functions, harmonic snapshots, oversampling harmonic snapshots, or force-based snapshots. Here, we adopt the harmonic snapshot approach. Specifically, for each fine-grid function $\delta_j^h(x)$ (with $\delta_j^h(x_k) = \delta_{j,k}$ for $j, k \in J_h(\omega_i)$, where $J_h(\omega_i)$ is the set of fine-grid nodes on $\partial\omega_i$), we define $\zeta_j^{\omega_i}(x)$ as the solution of

$$\mathcal{L}(\zeta_j^{\omega_i}(x)) = 0 \quad \text{in } \omega_i,$$

subject to $\zeta_j^{\omega_i}(x) = \delta_j^h(x)$ on $\partial\omega_i$. The operator $\mathcal{L}$ is the differential operator appearing in the state and adjoint equations of (2.4).

To simplify notation, we drop the explicit dependence on ω_i in the superscript; nevertheless, all local reduced-space calculations are confined to the corresponding coarse subdomain throughout this section. For each ω_i , let l_i be the size of the snapshot space, which is spanned by the functions ζ_j , i.e.,

$$V_{\text{snap}}(\omega_i) = \text{span}\{\zeta_j(x) : 1 \leq j \leq l_i\}.$$

The representation of these snapshot vectors in terms of fine-grid basis functions is encoded in the coefficient matrix

$$R_{\text{snap}} = [\zeta_1, \zeta_2, \dots, \zeta_{l_i}].$$

Having obtained the snapshot space, we then follow an analogous procedure to build the local reduced space $V_{lr}(\mathcal{D})$. This entails solving, for each coarse neighborhood ω_i , a local eigenvalue problem posed on $V_{\text{snap}}(\omega_i)$:

$$A^{lr} \phi_{i,\ell}^{lr} = \lambda_\ell S^{lr} \phi_{i,\ell}^{lr}. \quad (4.1)$$

If the state equation is a diffusion equation with diffusivity $\kappa(x)$, the matrices A^{lr} and S^{lr} take the form

$$A_{mn}^{lr} = \int_{\omega_i} \kappa(x) \nabla \zeta_m \cdot \nabla \zeta_n \, dx, \quad S_{mn}^{lr} = \int_{\omega_i} \kappa(x) \zeta_m \zeta_n \, dx.$$

To accelerate the local eigenvalue solution, we replace $\kappa(x)$ with $\tilde{\kappa}(x) = \kappa(x) \sum_i H^2 |\nabla \chi_i|^2$, where $\{\chi_i\}$ is a partition of unity over $\mathcal{T}_H$ as introduced in [4, 28, 29, 34, 35].

The construction proceeds as follows. Let $\{L_p(x)\}$ be the fine-grid basis spanning $V^h(\mathcal{D})$. Because any snapshot in $V_{\text{snap}}(\omega_i)$ is a linear combination of these L_p , the reduced matrices A^{lr} and S^{lr} satisfy

$$A^{lr} = R_{\text{snap}}^T A R_{\text{snap}}, \quad S^{lr} = R_{\text{snap}}^T S R_{\text{snap}},$$

where A and S are the fine-scale matrices:

$$A_{pq} = \int_{\omega_i} \kappa(x) \nabla L_p \cdot \nabla L_q \, dx, \quad S_{pq} = \int_{\omega_i} \kappa(x) L_p L_q \, dx.$$

Solving (4.1) yields eigenvalues $\lambda_1^i \leq \lambda_2^i \leq \dots$ and eigenvectors $\phi_{i,1}^{lr}, \phi_{i,2}^{lr}, \dots$. These eigenvectors are then projected back onto the local fine grid using the snapshot basis, producing fine-grid eigenfunctions $\rho_{i,\ell}$ on ω_i .

To avoid confusion, we clarify that ζ_m and L_m are spatial functions, while $\phi_{i,\ell}^{lr}$ and $\rho_{i,\ell}$ are vectors of coefficients. The corresponding continuous eigenfunctions of (4.1) are recovered as linear combinations, such as $\sum_m \phi_{i,\ell,m}^{lr} \zeta_m$.

We select the M_i smallest eigenvalues $\lambda_1^{(i)} \leq \dots \leq \lambda_{M_i}^{(i)}$ from (4.1) and the corresponding eigenvectors $\rho_1^{(i)}, \dots, \rho_{M_i}^{(i)}$ for each coarse neighborhood ω_i . These eigenvectors span the local reduced space $V_{lr}^{(i)} := \text{span}\{\rho_\ell^{(i)} : 1 \leq \ell \leq M_i\}$.

The global reduced basis space is then assembled using the partition of unity functions $\{\chi_i\}$:

$$V_{lr}(\mathcal{D}) := \text{span}\{\rho_{i,\ell} : \rho_{i,\ell} = \chi_i \rho_\ell^{(i)}, \, 1 \leq i \leq N_c, \, 1 \leq \ell \leq M_i\}.$$

The total number of multiscale basis functions is $M := \sum_i M_i$, and their fine-grid coefficients are stored in

$$R^l = [\rho_1, \rho_2, \dots, \rho_M]. \quad (4.2)$$

These multiscale basis functions, once constructed in the offline stage, can be reused in the online stage, yielding significant computational savings compared to full fine-grid simulations. It is important to note that the GMsFEM reduction is applied exclusively to the state and adjoint variables; the control variable is discretized on the original fine grid without reduction.

The resulting Iter-GMs algorithm is presented below, which differs from Algorithm 2 in that the finite element discretization space is replaced by the multiscale reduced-order space.

Algorithm 4 The Iter-GMs algorithm.

Step 1 (Initialization): For each $\eta(\omega) \in \Omega$, solve for $(u_0^{\text{Iter-GMs}}, \lambda_0^{\text{Iter-GMs}}, f_0^{\text{Iter-GMs}})$:

$$\begin{cases} a^0(u_0^{\text{Iter-GMs}}, \tilde{u}) = (f_0^{\text{Iter-GMs}}, \tilde{u}), & \forall \tilde{u} \in V_{lr}(\mathcal{D}), \\ a^0(\lambda_0^{\text{Iter-GMs}}, \tilde{\lambda}) = (\hat{u} - u_0^{\text{Iter-GMs}}, \tilde{\lambda}; \eta(\omega)), & \forall \tilde{\lambda} \in V_{lr}(\mathcal{D}), \\ 2\beta(f_0^{\text{Iter-GMs}}, \tilde{f}) = (\tilde{f}, \lambda_0^{\text{Iter-GMs}}), & \forall \tilde{f} \in \mathcal{M}^h(\mathcal{D}). \end{cases} \quad (4.3)$$

Step 2 (Iteration): For $n \geq 1$, recursively compute $\{(u_n^{\text{Iter-GMs}}, \lambda_n^{\text{Iter-GMs}}, f_n^{\text{Iter-GMs}})\}$:

$$\begin{cases} a^0(u_n^{\text{Iter-GMs}}, \tilde{u}) = (f_n^{\text{Iter-GMs}}, \tilde{u}) - a^1(u_{n-1}^{\text{Iter-GMs}}, \tilde{u}; \eta(\omega)), & \forall \tilde{u} \in V_{lr}(\mathcal{D}), \\ a^0(\lambda_n^{\text{Iter-GMs}}, \tilde{\lambda}) = (\hat{u} - u_n^{\text{Iter-GMs}}, \tilde{\lambda}; \eta(\omega)) - a^1(\lambda_{n-1}^{\text{Iter-GMs}}, \tilde{\lambda}; \eta(\omega)), & \forall \tilde{\lambda} \in V_{lr}(\mathcal{D}), \\ 2\beta(f_n^{\text{Iter-GMs}}, \tilde{f}; \eta(\omega)) = (\tilde{f}, \lambda_n^{\text{Iter-GMs}}; \eta(\omega)), & \forall \tilde{f} \in \mathcal{M}^h(\mathcal{D}). \end{cases} \quad (4.4)$$

4.2. Matrix formulation based on GMsFEM.

The previous section introduced the multiscale finite element space $V_{lr}(\mathcal{D})$ (the offline space) together with the corresponding matrix R^l . Using these multiscale basis functions, we now assemble them into an algebraic system for computation,

$$\begin{bmatrix} 2\beta M_{1,h} & 0 & -M_{2,h}^T R^l \\ 0 & (R^l)^T M_{3,h} R^l & (R^l)^T (K_h^0)^T R^l \\ -(R^l)^T M_{2,h} & (R^l)^T K_h^0 R^l & 0 \end{bmatrix} \begin{bmatrix} F_0^{Iter-GMs}(\eta(\omega)) \\ u_0^{Iter-GMs}(\eta(\omega)) \\ \lambda_0^{Iter-GMs}(\eta(\omega)) \end{bmatrix} = \begin{bmatrix} 0 \\ (R^l)^T \bar{U}(\eta(\omega)) \\ (R^l)^T d \end{bmatrix}, \quad (4.5)$$

and

$$\begin{aligned} & \begin{bmatrix} 2\beta M_{1,h} & 0 & -M_{2,h}^T R^l \\ 0 & (R^l)^T M_{3,h} R^l & (R^l)^T (K_h^0)^T R^l \\ -(R^l)^T M_{2,h} & (R^l)^T K_h^0 R^l & 0 \end{bmatrix} \begin{bmatrix} F_n^{Iter-GMs}(\eta(\omega)) \\ u_n^{Iter-GMs}(\eta(\omega)) \\ \lambda_n^{Iter-GMs}(\eta(\omega)) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ (R^l)^T \bar{U}(\eta(\omega)) \\ (R^l)^T d \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & (R^l)^T K_h^1(\eta(\omega)) R^l \\ 0 & (R^l)^T K_h^1(\eta(\omega)) R^l & 0 \end{bmatrix} \begin{bmatrix} F_n^{Iter-GMs}(\eta(\omega)) \\ u_n^{Iter-GMs}(\eta(\omega)) \\ \lambda_n^{Iter-GMs}(\eta(\omega)) \end{bmatrix}. \end{aligned} \quad (4.6)$$

respectively.

We assume that the block diagonal matrix is given by

$$R = [I(R^l)^T (R^l)^T] \in \mathbb{R}^{(N_e+2M) \times (N_e+2N_h)},$$

$$\begin{aligned} \text{define } \Lambda_l &= R \begin{bmatrix} 2\beta M_{1,h} & 0 & -M_{2,h}^T \\ 0 & M_{3,h} & (K_h^0)^T \\ -M_{2,h} & K_h^0 & 0 \end{bmatrix} R^T, \quad \Lambda_r(\eta(\omega)) = R \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & (K_h^1(\eta(\omega)))^T \\ 0 & K_h^1(\eta(\omega)) & 0 \end{bmatrix} R^T, \\ Fr(\eta(\omega)) &= \begin{bmatrix} 0 \\ (R^l)^T \bar{U}(\eta(\omega)) \\ (R^l)^T d \end{bmatrix}. \end{aligned}$$

Then, (4.5)-(4.6) can be rewritten the following matrix form

$$\Lambda_l P_n^{Iter-GMs}(\eta(\omega)) = \begin{cases} Fr(\eta(\omega)), & n = 0, \\ Fr(\eta(\omega)) - \Lambda_r(\eta(\omega)) P_{n-1}^{Iter-GMs}(\eta(\omega)), & n \geq 1, \end{cases} \quad (4.7)$$

where $P_n^{Iter-GMs}(\eta(\omega)) = (F_n^{Iter-GMs}(\eta(\omega)), u_n^{Iter-GMs}(\eta(\omega)), \lambda_n^{Iter-GMs}(\eta(\omega)))^T$. Algorithm 5 describes the Iter-GMsFEM in the matrix formulation.

5. Numerical experiments

We present several numerical examples to illustrate the efficiency of the Iter-GMs method for stochastic optimal control problems constrained by elliptic PDEs. Relative errors and computational performance are reported for the state, control, and adjoint variables. In Section 5.1, we consider a problem with random diffusion coefficient and random target function, and discuss the influence of the coarse mesh size H and the regularization parameter β . In Section 5.2, the Iter-GMs method is applied to a problem where both the diffusion coefficient and the target function are high-dimensional random fields. The convergence behavior of the iterative scheme is verified by plotting the relative error against iteration steps.

Algorithm 5 Detailed procedure of Iter-GMs method.

Input: The stochastic optimal control problem (2.1) and (2.2), the maximum number of iterations N_{iter} , and the error tolerance ε .

Output: The Iter-GMs solution $P^{\text{Iter-GMs}}(\eta(\omega))$.

Offline phase: only perform once

1. Compute full order matrices $M_{1,h}$, $M_{2,h}$, $M_{2,h}$ and K_h^0 independent of $\eta(\omega)$;
2. Calculate projective matrix R by GMSFEM;
3. Assemble the reduced stiffness matrix Λ_l ;

Online phase: for arbitrary sample $\eta(\omega) \in \Omega$

1. With $n = 0$, solve the Eq (4.7) to get the solution coefficient $P_0^{\text{Iter-GMs}}(\eta(\omega))$;
 2. Renew the load vector $F_r(\eta(\omega))$ and the matrix $\Lambda_r(\eta(\omega))$;
 3. Calculate the coefficient $P_n^{\text{Iter-GMs}}(\eta(\omega))$ by Eq (4.7);
 4. Compute the error $\varepsilon_n = \|P_n^{\text{Iter-GMs}}(\eta(\omega)) - P_{n-1}^{\text{Iter-GMs}}(\eta(\omega))\|_\infty$;
 5. **While** $n < N_{\text{iter}}$ or $\varepsilon_n \geq \varepsilon$
 6. $n \rightarrow n + 1$;
 7. Return to the steps 3 and 4 in the online phase;
 8. **End While**
 9. Obtain the Iter-GMs solution $P^{\text{Iter-GMs}}(\eta(\omega))$ by matrix R .
-

5.1. Stochastic optimal control problem with one-dimensional random inputs

We take the physical domain to be the two-dimensional unit square $\mathcal{D} = [0, 1]^2$. The model problem is the following optimal control problem subject to a stochastic diffusion equation:

$$\begin{cases} \min_{u,f} J(u, f) = \frac{1}{2} \|u - \hat{u}\|_{\mathcal{L}^2(\mathcal{D})}^2 + \beta \|f\|_{\mathcal{L}^2(\mathcal{D})}^2, \\ -\nabla \cdot (\kappa(x, \eta(\omega)) \nabla u) = f(x, \eta(\omega)) \quad \text{in } \mathcal{D}, \\ u = g(x) \quad \text{on } \partial\mathcal{D}. \end{cases} \quad (5.1)$$

The random coefficient κ and the target function $\hat{u}$ are defined as

$$\kappa(x, \eta(\omega)) = \kappa_1(x) + (1 + \eta(\omega)^2) \kappa_2(x), \quad (5.2)$$

$$\hat{u}(x, \eta(\omega)) = x_1 x_2 (x_1 + 1)(x_2 - 1)(1 + \eta(\omega)^2) + x_1^2 x_2 (x_1 - 1)(x_2 + 1)(2 + \cos \eta(\omega)). \quad (5.3)$$

In this problem, $x = (x_1, x_2)$ denotes the position in the unit square $\mathcal{D}$, and $\eta(\omega)$ is a uniform random variable on $[0, 1]$. The two deterministic functions $\kappa_1(x)$ and $\kappa_2(x)$ are independent of $\eta(\omega)$. The coefficient $\kappa_1(x)$ is of high contrast, and its distribution on the domain is depicted in Figure 3.

In this numerical experiment, the reference solutions $(u_{\text{ref}}, f_{\text{ref}}, \lambda_{\text{ref}})$ are computed on a 120×120 uniform fine grid using the standard finite element method, while the reduced-order solutions $(u_{\text{mr}}, f_{\text{mr}}, \lambda_{\text{mr}})$ are obtained on a 10×10 coarse mesh, where the subscript *mr* indicates either GMSFEM

or Iter-GMs. The relative errors for the state, control, and adjoint variables are defined as follows:

$$\begin{cases} e_u^{mr} = \frac{1}{N} \sum_{i=1}^N \frac{\|u_{ref}(x, \eta_i(\omega)) - u_{mr}(x, \eta_i(\omega))\|_{L^2(\mathcal{D})}}{\|u_{ref}(x, \eta_i(\omega))\|_{L^2(\mathcal{D})}}, \\ e_f^{mr} = \frac{1}{N} \sum_{i=1}^N \frac{\|f_{ref}(x, \eta_i(\omega)) - f_{mr}(x, \eta_i(\omega))\|_{L^2(\mathcal{D})}}{\|f_{ref}(x, \eta_i(\omega))\|_{L^2(\mathcal{D})}}, \\ e_\lambda^{mr} = \frac{1}{N} \sum_{i=1}^N \frac{\|\lambda_{ref}(x, \eta_i(\omega)) - \lambda_{mr}(x, \eta_i(\omega))\|_{L^2(\mathcal{D})}}{\|\lambda_{ref}(x, \eta_i(\omega))\|_{L^2(\mathcal{D})}}. \end{cases} \quad (5.4)$$

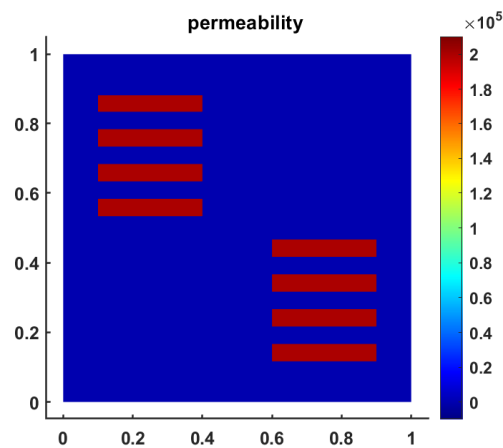


Figure 3. High-contrast function $\kappa_1(x)$.

In Figure 4, we depict the mean of the target function $\hat{u}(x, \eta(\omega))$.

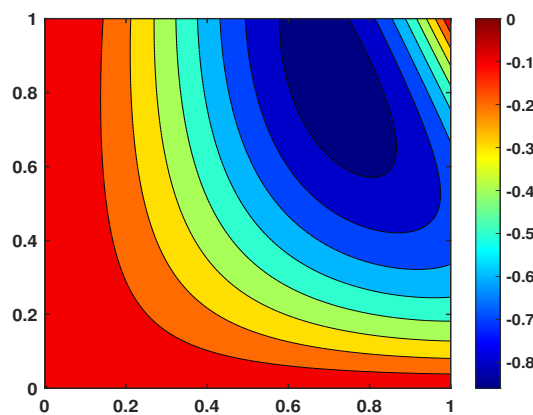


Figure 4. The mean of target function $\hat{u}(x, \eta(\omega))$.

To test the approximation capability of the proposed iterative method, we randomly pick 500 parameter points and compute their average L^2 errors. The numerical results for the state variable u and the control variable f are presented in Figure 5, which were obtained with the parameters $\beta = 2 \times 10^{-7}$, $\varepsilon = 10^{-8}$, $N_{\text{iter}} = 30$, and $L = 5$.

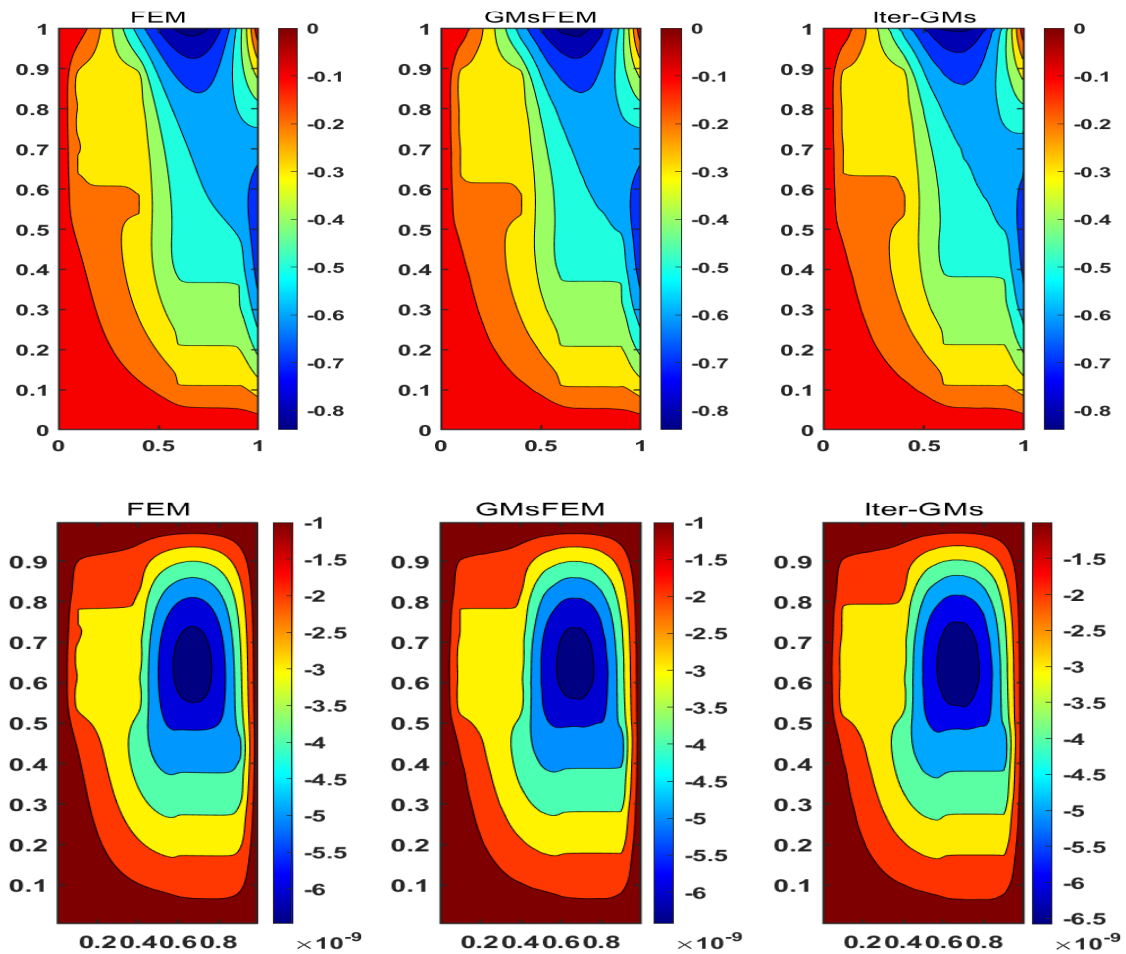


Figure 5. The mean of $u(x, \eta(\omega))$ (1st row) and $f(x, \eta(\omega))$ (2nd row) by different methods.

The means of the solutions $u(x, \eta(\omega))$ and $f(x, \eta(\omega))$ obtained by different methods are plotted in the first and second rows, respectively. The figure shows that the moments of $u(x, \eta(\omega))$ and $f(x, \eta(\omega))$ computed using GMSFEM and the proposed iterative method are in good agreement with the reference results. Furthermore, we will find that the trends of the state variables are consistent with the trends of the target function. The consistency between the trend of state variables and that of the target function implies that the currently applied control action is effectively driving the objective function in the desired direction; in other words, the control strategy is aligned with the optimization goal. Thus, our proposed iterative method is effective.

Table 1. Relative L^2 errors of the state variable u for varying coarse mesh size H .

Coarse mesh size	$H = 1/2$	$H = 1/4$	$H = 1/6$	$H = 1/8$	$H = 1/10$	$H = 1/12$
e_u^{GMS}	3.7787E-01	6.2556E-02	6.0131E-02	8.1250E-02	4.8136E-05	6.0139E-05
$e_u^{Iter-GMs}$	3.8323E-01	6.5142E-02	6.1034E-02	7.8785E-02	8.3887E-04	8.1617E-04

To investigate how the coarse mesh size affects the error, we test a range of values in the example:

$H = \{1/2, 1/4, 1/6, 1/10, 1/12\}$. Tables 1 and 2 give the relative L^2 errors for $u(x, \eta(\omega))$ and $f(x, \eta(\omega))$, respectively.

From the table, we observe that the errors in the approximations of the state variable $u(x, \eta(\omega))$ and control variable $f(x, \eta(\omega))$ decrease monotonically as the coarse mesh size is refined.

Table 2. Relative L^2 errors of the control variable f vs. different coarse mesh size H .

Coarse mesh size	$H = 1/2$	$H = 1/4$	$H = 1/6$	$H = 1/8$	$H = 1/10$	$H = 1/12$
e_f^{GMs}	5.2869E-01	3.9231E-01	2.9810E-01	2.9317E-01	1.9340E-04	2.9098E-04
$e_f^{Iter-GMs}$	5.2963E-01	3.8435E-01	2.8952E-01	2.8156E-01	6.3212E-04	5.7037E-04

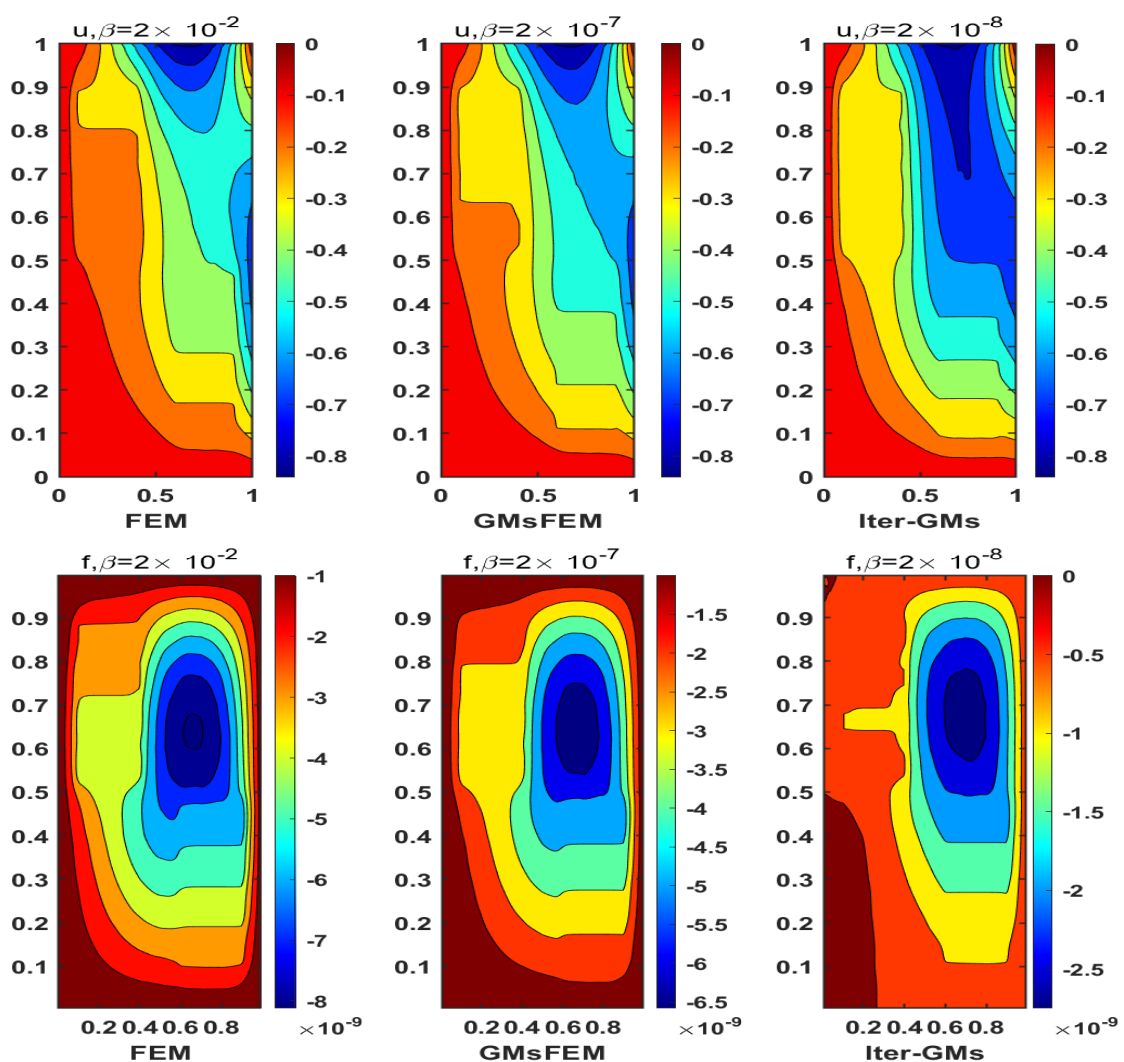


Figure 6. Means of u (top) and f (bottom) computed by different methods.

The choice of the regularization parameter β significantly influences the solution of the stochastic optimal control problem with an elliptic PDE constraint. In particular, a smaller β leads to a weaker penalty on the control, and consequently the state stays closer to the desired state. Conversely, increas-

ing β raises the contribution of the control term to the objective functional, thereby hindering the state variable from attaining the target in the relative L^2 -norm. Figure 6 displays the contour lines of the state and control variables for three distinct β values.

In the Table 3, we list the minimal value of the cost functional $J_{\min}(u, f)$ for the three values of β .

Table 3. The minimal value of cost functional $J(u, f)$ for different values of β .

β	$J_{\min}^{ref}$	$J_{\min}^{GMs}$	$J_{\min}^{Iter-GMs}$
2×10^{-2}	1.5987E-02	1.1063E-02	3.7152E-03
2×10^{-7}	1.6021E-02	1.1206E-02	3.8005E-03
2×10^{-8}	1.7409E-02	1.2331E-02	4.1767E-03

It is evident from the figures and the table that selecting an appropriate regularization parameter is essential for the optimal solutions.

We then set the regularization parameter β and the number of local basis functions for each coarse element, and select five global basis functions. Using FEM, the minimal value of the cost functional $J_{\min}(u, f)$ is 1.1063E-02. This values obtained by model reduction method are reported in Table 4 with different mesh sizes.

As shown in the table, the minimum values of the cost functional approximate the reference values monotonically as the coarse mesh becomes finer. This trend indicates that a finer coarse mesh leads to a lower optimal cost, which is consistent with the expectation that mesh refinement improves the accuracy of the numerical solution.

Table 4. Relative L^2 errors of the optimal cost $J(u, f)$ for varying coarse mesh size H .

Coarse mesh size	$H = 1/2$	$H = 1/4$	$H = 1/6$	$H = 1/8$	$H = 1/10$	$H = 1/12$
$J_{\min}^{GMs}$	5.7341E-02	1.9692E-02	1.7158E-02	1.6484E-02	1.1206E-02	1.1147E-02
$J_{\min}^{Iter-GMs}$	5.8219E-02	2.0846E-02	1.8317E-02	1.7524E-02	1.2331E-02	1.2271E-02

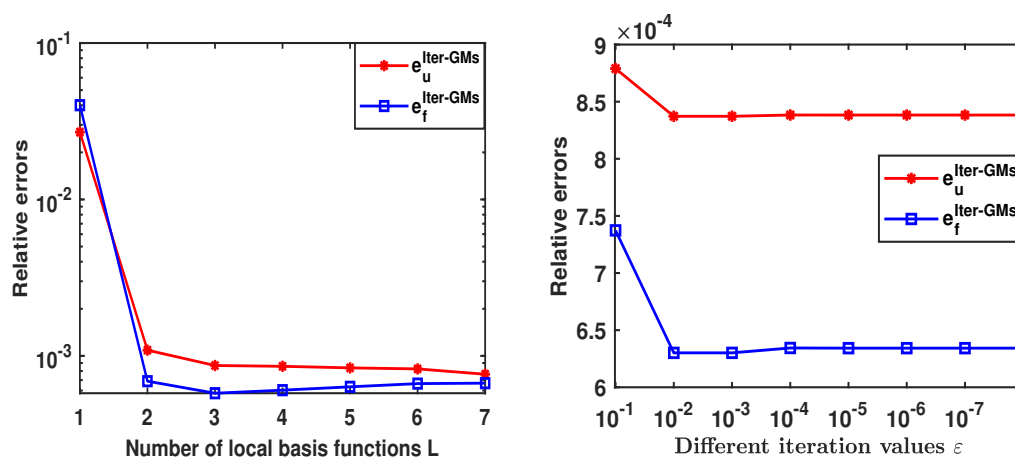


Figure 7. Relative errors vs. number of local basis functions or iteration values.

Figure 7 shows, in its left subfigure, the evolution of the relative error against the number of local basis functions L , and in its right panel, the relative error versus the iteration threshold ε . Both panels show that the error decreases with decreasing L and ε , and converges to a steady value after reaching a certain accuracy, demonstrating the convergence stability of the proposed iterative method.

5.2. High-dimensional stochastic optimal control problem

In the second case, we impose the stochastic diffusion equation as the constraint, with the computational domain identical to that of Case I. The model is as follows:

$$\begin{cases} \min_{u,f} J = \frac{1}{2} \|u(x, \eta(\omega)) - \hat{u}(x, \eta(\omega))\|_{\mathcal{L}^2(\mathcal{D})}^2 + \beta \|f(x, \eta(\omega))\|_{\mathcal{L}^2(\mathcal{D})}^2, \\ \text{s.t.} \quad -\operatorname{div}(\kappa(x, \eta(\omega)) \nabla u) = f(x, \eta(\omega)) \quad \text{in } \mathcal{D} \\ u|_{\partial\mathcal{D}} = g(x), \end{cases} \quad (5.5)$$

with the diffusion coefficient $\kappa(x, \eta(\omega))$ and the target function specified as

$$\begin{cases} \kappa(x, \eta(\omega)) = \kappa_1(x) + (1 + \eta_1(\omega) + 2\sin(\eta_3(\omega) + \eta_4(\omega)\eta_5(\omega)))\kappa_2(x), \\ \hat{u}(x, \eta(\omega)) = x_1 x_2^2 (x_1 - 1)^2 (x_2 - 1) (1 + \eta_1(\omega)^2) + \sin(\pi x_1) (1 - \cos(\pi x_2)) (2 + \cos(\eta_1(\omega) + \eta_2(\omega))), \end{cases}$$

respectively. The random variable $\eta(\omega) := (\eta_1(\omega), \eta_2(\omega), \eta_3(\omega), \eta_4(\omega), \eta_5(\omega))$. We set each $\eta_i(\omega)$ ($i = 1, \dots, 5$) to be beta-distributed with $\text{Beta}(2, 2)$. The coefficient $\kappa_1(x)$, being independent of $\eta(\omega)$, has a high-contrast nature. Its distribution is visualized in Figure 8.

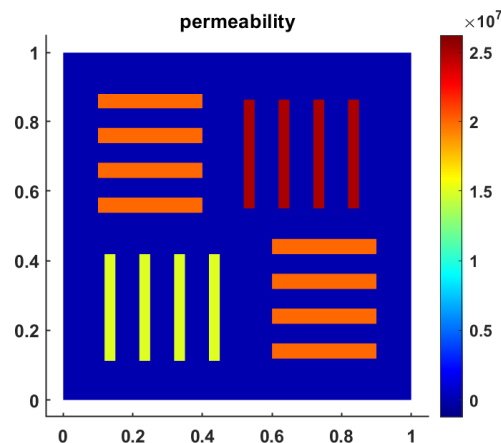


Figure 8. High-contrast function $\kappa_1(x)$.

Compared to the contrast function shown in Section 5.1, this highly oscillatory contrast function not only possesses horizontal channels, but also vertical channels. In other words, it exhibits oscillatory variations along both spatial directions, which allows it to capture anisotropic features in the medium.

The reference solutions $(u_{ref}, f_{ref}, \lambda_{ref})$ are computed on a 160×160 uniform fine grid, which results in $N_f = 25,921$ finite element degrees of freedom. The proposed iterative method is then applied to solve problem (5.5). The error assessment follows the same L^2 -norm formulas given in (5.4), and we use 500 random samples in the numerical evaluation.

We choose $L = 5$ local basis functions and set $\beta = 3 \times 10^{-9}$ as the regularization parameter. Under these settings, the mean and standard deviation of the state and control variables are shown in Figures 9 and 10 for both the reference and the reduced-order models. The coarse mesh size used in this case is $H = 1/10$.

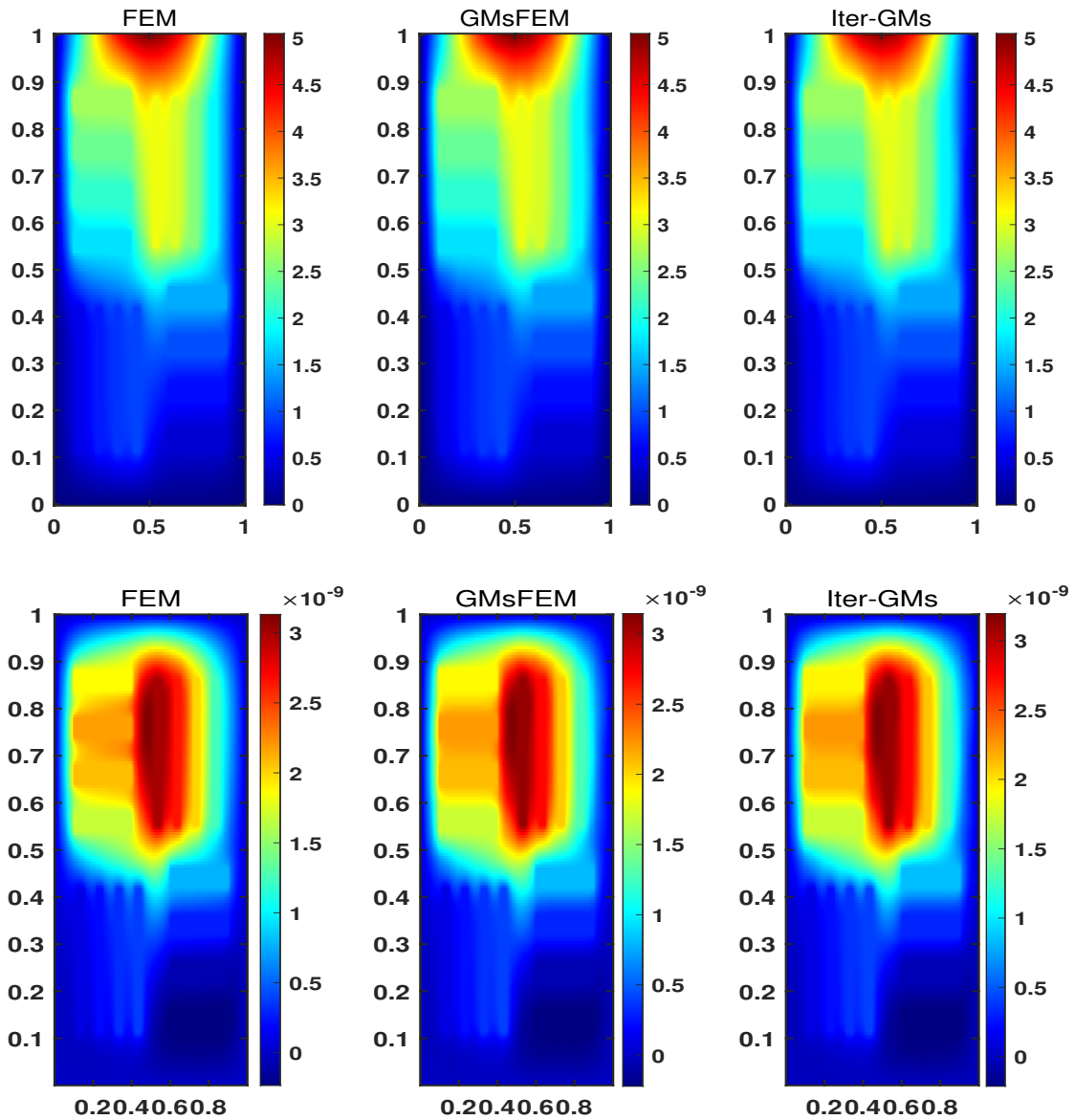


Figure 9. The mean of $u(x, \eta(\omega))$ (1st row) and $f(x, \eta(\omega))$ (2nd row) by different methods.

It can be observed from the figure that, on one hand, the mean and standard deviation of the state variable u obtained via different methods are nearly identical; on the other hand, the moments of the control variable f produced by the proposed iterative method are in good agreement with those of the reference solution.

Based on the reference, GMsFEM and the iterative method, Figure 12 depicts the probability density estimates of $u(x, \eta(\omega))$ and $f(x, \eta(\omega))$ at one random location.

For verifying the convergence condition, we choose the infimum of $\kappa_0(x)$ as 200, corresponding to the background conductivity shown in Figure 8. In this experiment, we will change the supremum of $\kappa_1^1(x, \eta(\omega))$ as $\{5.1723, 7.6283, 195.3998, 310.3620\}$. The relative L^2 -norm errors corresponding to different ratios of $\overline{\kappa^2}$ to $\underline{\kappa^1}$ are reported in Table 5.

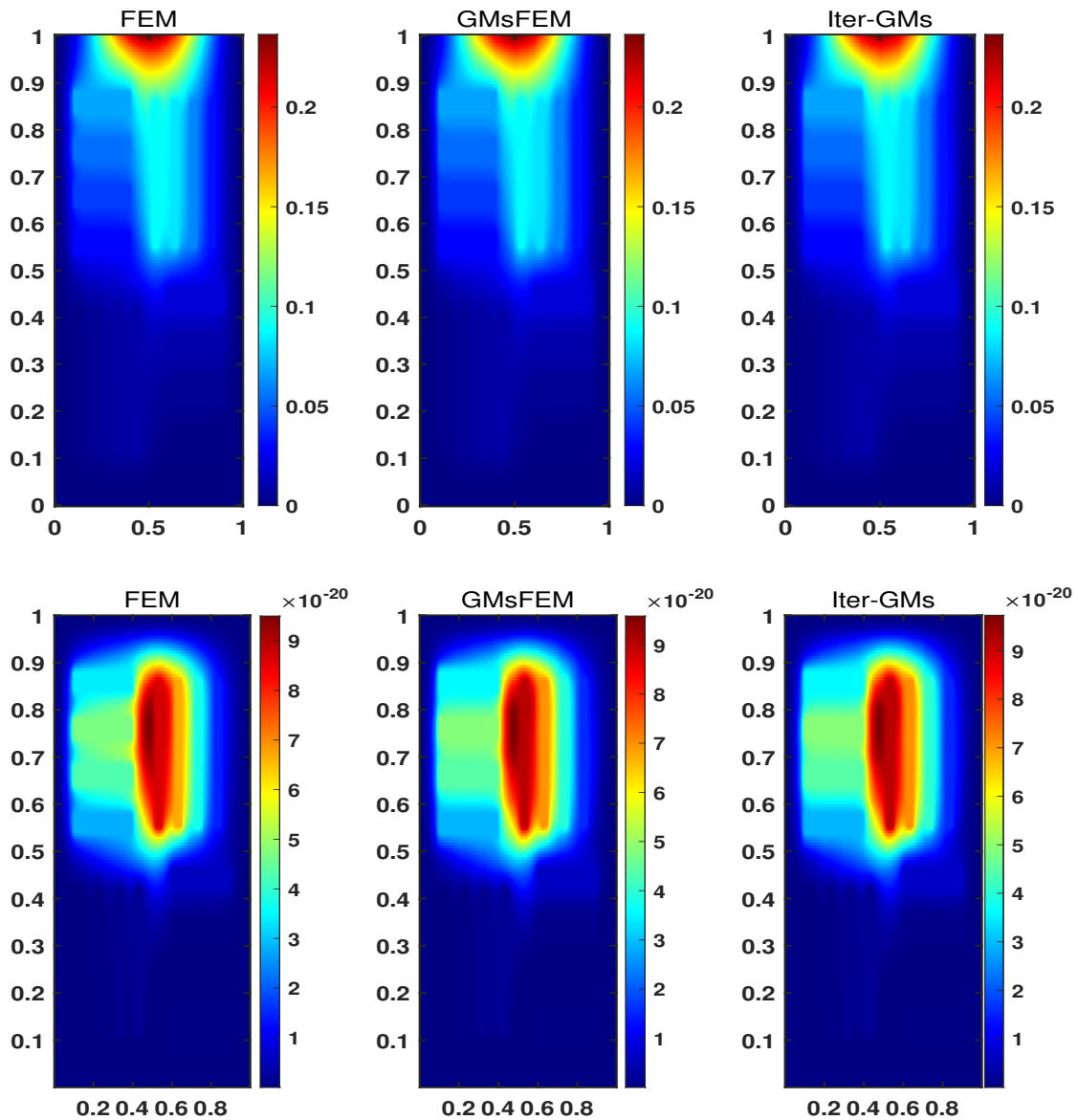


Figure 10. Standard deviations of state variable $u(x, \eta(\omega))$ and control variable $f(x, \eta(\omega))$.

It is observed that for ratios below unity, a smaller value of $\overline{\kappa^2}/\underline{\kappa^1}$ leads to a corresponding reduction in the L^2 -errors for both the state and the control variables. This is consistent with the relative-dominance condition in Lemma 3.1: a smaller ratio yields a smaller $\rho < 1$, which tightens the spectral-radius bound of the iteration matrix and thus enhances the decay rate of the energy-norm error. Consequently, the L^2 -errors decrease accordingly as the ratio becomes smaller.

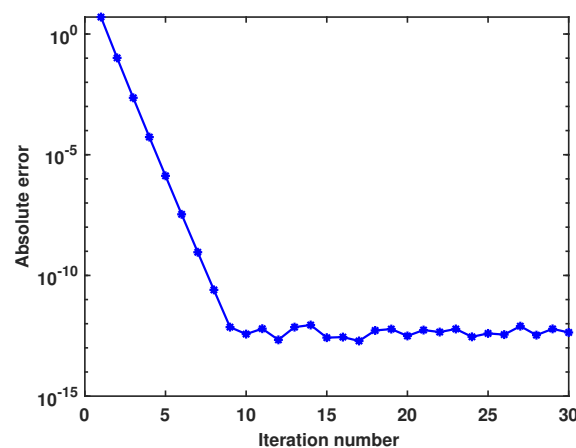
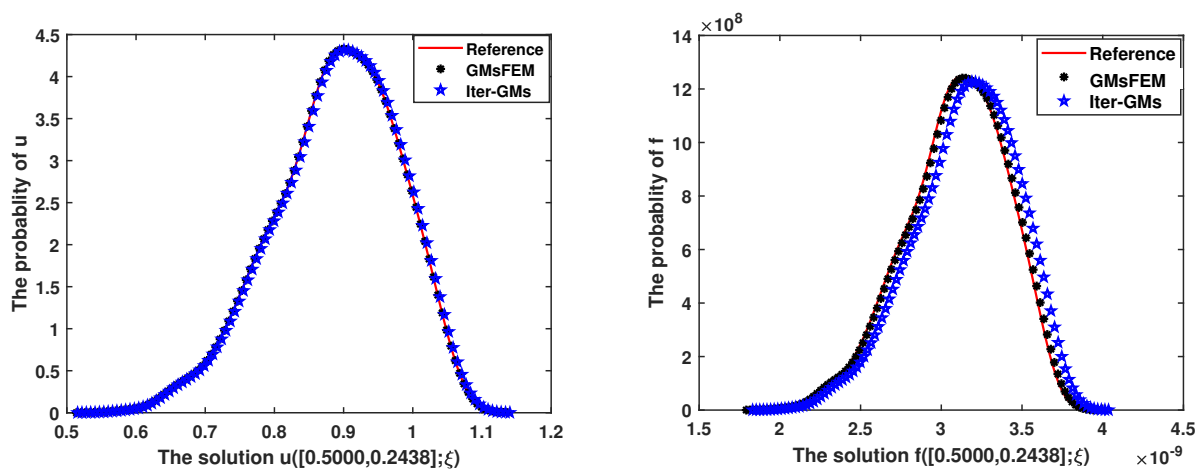
Table 5. History of convergence with 5 basis functions.

$\overline{\kappa^2}/\kappa^1$	$e_u^{Iter-GMs}$	$e_f^{Iter-GMs}$
0.0259	2.4446E-04	1.5489E-03
0.0381	3.4539E-04	2.2802E-03
0.9770	8.9363E-02	3.3444E-01
1.5518	1.2363E+07	1.1858E+08

To assess the performance of the proposed iterative method, we define the absolute error in the infinity norm as

$$E_{\text{abs}} = \max_{v \in \{u, f, \lambda\}} \|v^{k+1} - v^k\|_{\infty}.$$

We list in Figure 11 the absolute errors of the variables against the iteration number to illustrate their convergence behavior.

**Figure 11.** Absolute errors of different iteration number.**Figure 12.** Probability density of state variable $u(x, \eta(\omega))$ and control variable $f(x, \eta(\omega))$.

As can be seen from the figure, the absolute error is steadily reduced as the number of iterations increases, and it eventually stabilizes.

In Figure 12, we display the probability densities of the state variable u and the control variable f (both functions of x and $\eta(\omega)$).

As can be seen from the figures, the probability densities obtained by the Iter-GMs method agree well with those of the reference solution and the GMsFEM solution.

In Table 6, we describe the CPU time for FEM and the proposed Iter-GMs method.

Table 6. The CPU time for different methods.

Computational setting	FE method	Iter – GMs method
Total number of basis functions	25291	500
Offline time for reduced space	/	2.0252E+01 s
Online average time for optimal solutions	5.1602E+00 s	2.7203E+00 s
Total time for all optimal solutions	2.5801E+03 s	1.3733E+03 s

Compared with standard FE method, we can find: (1) the number of basis functions is reduced by approximately 98%, indicating that the Iter-GMs method reduces the degrees of freedom of the optimal control problem; (2) Despite the extra offline time required for building the reduced-order space, the iterative method reduces the average online solution time from 5.16 to 2.72 s and the total time from 2.5801E+03 to 1.3733E+03 s, achieving an improvement in computational efficiency of nearly twofold. This shows that the Iter-GMs method offers a reduction in computational cost while preserving the desired level of accuracy.

6. Conclusions

In this paper, we have presented an Iter-GMsFEM for solving stochastic optimal control problems constrained by elliptic PDE with random inputs. Our approach leverages the benefits of iterative methods combined with GMsFEM, offering an effective solution strategy for such stochastic problems. We first transform the original optimality system into an equivalent form with a parameter-independent multiscale coefficient and exchange term, at the cost of introducing a parameter-dependent right-hand side. The resulting system is then solved via a fixed-point iteration. Due to the highly heterogeneous nature and high-contrast regions of the problem, applying iterative methods directly to the full model would incur substantial computational burden and memory requirements. To address this, we construct a lower-dimensional space using GMsFEM to accelerate the iterative process. The entire solution procedure is divided into two stages: offline and online. Owing to the reformulation, the multiscale basis functions are computed only once in the offline phase. The online phase is highly efficient, as it leverages the reuse of precomputed matrix inverses and model reduction techniques. The numerical experiments presented in this section demonstrate the effectiveness of the proposed iterative method for stochastic optimal control problems with random inputs. The obtained results confirm that the method achieves a satisfactory trade-off between solution accuracy and computational effort.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was funded by the National Natural Science Foundation of China (Grant No. 12101217) and the Natural Science Foundation of Hunan Province (Grant No. 2022JJ40113).

Conflict of interest

The authors declare there is no conflict of interest.

References

1. J. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Grundlehren der mathematischen Wissenschaften, Springer-Verlag, Berlin–Heidelberg, 1971.
2. R. Glowinski, J. Lions, Exact and approximate controllability for distributed parameter systems, *Acta Numer.*, **3** (1994), 269–378. <https://doi.org/10.1017/S0962492900002452>
3. H. Maurer, First and second order sufficient optimality conditions in mathematical programming and optimal control, in *Mathematical Programming at Oberwolfach*, **14** (1981), 163–177. <https://doi.org/10.1007/BFb0120927>
4. I. Babuška, R. Lipton, Optimal local approximation spaces for generalized finite element methods with application to multiscale problems, *Multiscale Model. Simul.*, **9** (2011), 373–406. <https://doi.org/10.1137/100791051>
5. S. Collis, M. Heinkenschloss, Analysis of the streamline upwind/Petrov Galerkin method applied to the solution of optimal control problems, *CAAM Technical Report TR02-01*, Rice University, Houston, TX, March 2002.
6. L. Hou, J. Lee, H. Manouzi, Finite element approximations of stochastic optimal control problems constrained by stochastic elliptic PDEs, *J. Math. Anal. Appl.*, **384** (2011), 87–103. <https://doi.org/10.1016/J.JMAA.2010.07.036>
7. F. Brezzi, On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers, *R.A.I.R.O. Anal. Numér.*, **8** (1974), 129–151. <https://doi.org/10.1051/m2an/197408R201291>
8. R. Becker, H. Kapp, R. Rannacher, Adaptive finite element methods for optimal control of partial differential equations: Basic concept, *SIAM J. Control Optim.*, **39** (2000), 113–132. <https://doi.org/10.1137/S0363012999351097>
9. M. Kalos, P. Whitlock, *Monte Carlo Methods*, 2nd edition, Wiley-VCH, Weinheim, 2008. <https://doi.org/10.1002/9783527626212>
10. C. Graham, D. Talay, *Stochastic Simulation and Monte Carlo Methods: Mathematical Foundations of Stochastic Simulation*, Springer, 2013.
11. M. Gunzburger, H. Lee, J. Lee, Error estimates of stochastic optimal Neumann boundary control problems, *SIAM J. Numer. Anal.*, **49** (2011), 1532–1552. <https://doi.org/10.1137/100801731>

12. E. Rosseel, G. Wells, Optimal control with stochastic PDE constraints and uncertain controls, *Comput. Methods Appl. Mech. Engrg.*, **213–216** (2012), 152–167. <https://doi.org/10.1016/j.cma.2011.11.026>
13. P. Chen, A. Quarteroni, G. Rozza, Stochastic optimal Robin boundary control problems of advection-dominated elliptic equations, *SIAM J. Numer. Anal.*, **51** (2013), 2700–2722. <https://doi.org/10.1137/120884158>
14. D. Kouri, M. Heinkenschloss, D. Ridzal, B. Waanders, A trust-region algorithm with adaptive stochastic collocation for PDE optimization under uncertainty, *SIAM J. Sci. Comput.*, **35** (2013), A1847–A1879. <https://doi.org/10.1137/120892362>
15. I. Babuška, F. Nobile, R. Tempone, A stochastic collocation method for elliptic partial differential equations with random input data, *SIAM Rev.*, **52** (2010), 317–355. <https://doi.org/10.1137/100786356>
16. F. Nobile, R. Tempone, C. Webster, A sparse grid stochastic collocation method for partial differential equations with random input data, *SIAM J. Numer. Anal.*, **46** (2008), 2309–2345. <https://doi.org/10.1137/060663660>
17. A. Manzoni, *Reduced Models for Optimal Control, Shape Optimization and Inverse Problems in Haemodynamics*, Ph.D thesis, EPFL, 2012.
18. M. Grepl, Y. Maday, N. Nguyen, A. Patera, Efficient reduced-basis treatment of nonaffine and nonlinear partial differential equations, *ESAIM Math. Model. Numer. Anal.*, **41** (2007), 575–605. <https://doi.org/10.1051/m2an:2007031>
19. P. Chen, A. Quarteroni, Weighted reduced basis method for stochastic optimal control problems with elliptic PDE constraint, *SIAM/ASA J. Uncertain. Quantif.*, **2** (2014), 364–396. <https://doi.org/10.1137/130940517>
20. M. Gunzburger, N. Jiang, Z. Wang, An efficient algorithm for simulating ensembles of parameterized flow problems, *IMA J. Numer. Anal.*, **39** (2019), 1180–1205. <https://doi.org/10.1093/imanum/dry029>
21. N. Jiang, W. Layton, An algorithm for fast calculation of flow ensembles, *Int. J. Uncertain. Quantif.*, **4** (2014), 273–301. <https://doi.org/10.1615/Int.J.UncertaintyQuantification.2014007691>
22. Y. Luo, Z. Wang, A multilevel Monte Carlo ensemble scheme for random parabolic PDEs, *SIAM J. Sci. Comput.*, **41** (2019), A622–A642. <https://doi.org/10.1137/18M1174635>
23. X. Feng, J. Lin, C. Lorton, An efficient numerical method for acoustic wave scattering in random media, *SIAM/ASA J. Uncertain. Quantif.*, **3** (2015), 790–822. <https://doi.org/10.1137/140958232>
24. X. Feng, C. Lorton, An efficient Monte Carlo interior penalty discontinuous Galerkin method for elastic wave scattering in random media, *Comput. Methods Appl. Mech. Engrg.*, **315** (2017), 141–168. <https://doi.org/10.1016/j.cma.2016.10.036>
25. T. Hou, X. Wu, A multiscale finite element method for elliptic problems in composite materials and porous media, *J. Comput. Phys.*, **134** (1997), 169–189. <https://doi.org/10.1006/jcph.1997.5682>
26. L. Jiang, Y. Efendiev, V. Ginting, Multiscale methods for parabolic equations with continuum spatial scales, *Discrete Contin. Dyn. Syst. Ser. B*, **8** (2007), 833–859. <https://doi.org/10.3934/dcdsb.2007.8.833>

27. L. Jiang, Q. Li, Model's sparse representation based on reduced mixed GMsFE basis methods, *J. Comput. Phys.*, **338** (2017), 285–312. <https://doi.org/10.1016/j.jcp.2017.02.055>
28. Y. Efendiev, J. Galvis, T. Hou, Generalized multiscale finite element methods (GMsFEM), *J. Comput. Phys.*, **251** (2013), 116–135. <https://doi.org/10.1016/j.jcp.2013.04.045>
29. Y. Efendiev, J. Galvis, X. Wu, Multiscale finite element methods for high-contrast problems using local spectral basis functions, *J. Comput. Phys.*, **230** (2011), 937–955. <https://doi.org/10.1016/j.jcp.2010.09.026>
30. L. Ma, Q. Li, L. Jiang, Local–global model reduction method for stochastic optimal control problems constrained by partial differential equations, *Comput. Methods Appl. Mech. Engrg.*, **339** (2018), 514–541. <https://doi.org/10.1016/j.cma.2018.05.012>
31. L. Jiang, L. Ma, A hybrid model reduction method for stochastic parabolic optimal control problems, *Comput. Methods Appl. Mech. Engrg.*, **370** (2020), 113244. <https://doi.org/10.1016/j.cma.2020.113244>
32. X. Feng, Y. Luo, L. Vo, Z. Wang, An efficient iterative method for solving parameter-dependent and random convection–diffusion problems, *J. Sci. Comput.*, **90** (2022), 72. <https://doi.org/10.1007/s10915-021-01737-z>
33. I. Babuška, J. Osborn, Generalized finite element methods: Their performance and their relation to mixed methods, *SIAM J. Numer. Anal.*, **20** (1983), 510–536. <https://doi.org/10.1137/0720034>
34. E. Chung, Y. Efendiev, T. Hou, Adaptive multiscale model reduction with generalized multiscale finite element methods, *J. Comput. Phys.*, **320** (2016), 69–95. <https://doi.org/10.1016/j.jcp.2016.04.054>
35. I. Babuška, U. Banerjee, J. Osborn, Generalized finite element methods—main ideas, results and perspective, *Int. J. Comput. Methods*, **1** (2004), 67–103. <https://doi.org/10.1142/S0219876204000083>



AIMS Press

©2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>)