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*Research article*

# Well-posedness of a one-dimensional variational wave system with far field degeneracy

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**Abstract:** This work studied the local well-posedness of the Cauchy problem for a one-dimensional variational wave system arising from the theory of nematic liquid crystals. The system is degenerate and singular at spatial infinity. We applied the method of characteristics and the contraction mapping principle to establish the local existence and uniqueness of classical solutions. A suitable weighted $L^\infty$ norm was introduced to ensure that the solution has no loss of regularity and the existence time of the solution is independent of the spatial variable.

**Keywords:** variational wave system; far field degeneracy; well-posedness; weighted $L^\infty$ norm

## 1. Introduction

Liquid crystals are a phase of matter that combines typical properties of liquids (flow properties) and solids (anisotropy), which makes them widely used in industry. Nematic liquid crystals are one of the simplest and most important members of the liquid crystal family and have been extensively studied theoretically, numerically, and experimentally. Based on the physical characteristics of nematic liquid crystals, the average orientation of molecules can be described by the so-called director field $\mathbf{n}(t, \mathbf{x}) \in \mathbb{S}^2$. In [1], Hunter and Saxton ignored the kinetic energy of the director field and applied the following least action principle to model the propagation of the orientation waves in the director field:

$$\delta \int \Big(\frac{1}{2}\partial_t\mathbf{n} \cdot \partial_t\mathbf{n} - W(\mathbf{n}, \nabla\mathbf{n})\Big)\mathrm{d}\mathbf{x}\mathrm{d}t = 0, \quad \mathbf{n} \cdot \mathbf{n} = 1, \tag{1}$$

where $W(\mathbf{n}, \nabla\mathbf{n})$ is the potential energy density for the director field $\mathbf{n}$, which is given by the well-known Frank-Oseen energy density

$$W(\mathbf{n}, \nabla\mathbf{n}) = \frac{1}{2}k_1(\nabla \cdot \mathbf{n})^2 + \frac{1}{2}k_2(\mathbf{n} \cdot \nabla \times \mathbf{n})^2 + \frac{1}{2}k_3|\mathbf{n} \times (\nabla \times \mathbf{n})|^2. \tag{2}$$

Here $k_1, k_2$, and $k_3$ are the splay, twist, and bend elastic constants of the material, respectively.

For planar deformations of the director field $\mathbf{n}$ depending only on a single spatial variable with $\mathbf{x} = (x, 0, 0)$ and $\mathbf{n} = (\cos u(t,x), \sin u(t,x), 0)$, Hunter and Saxton [1] derived the Euler-Lagrange equation of (1)–(2) as

$$u_{tt} - c_1(u)\left(c_1(u)u_x\right)_x = 0, \tag{3}$$

with $c_1^2(u) = k_3 \cos^2 u + k_1 \sin^2 u$. Equation (3) is called the variational wave equation and has been widely investigated since it was introduced. Interested readers can all see the related works on singularity formation [2], existence of dissipative weak solutions [3–5], existence and uniqueness of conservative weak solutions [6–9], stability and generic regularity of conservative weak solutions [10–12], and a coupled hyperbolic-parabolic system [13–16].

In [17], Ali and Hunter considered three-dimensional deformations and took the director field $\mathbf{n}$ in the following form:

$$\mathbf{n} = (\cos u(t,x), \sin u(t,x) \cos v(t,x), \sin u(t,x) \sin v(t,x)),$$

where $u$, $v$ are spherical polar angles. Then they obtained that the Euler-Lagrange equations of (1)–(2) are

$$\begin{cases} u_{tt} - c_1^2(u)u_{xx} = c_1(u)c_1'(u)u_x^2 - a^2(u)c_2(u)c_2'(u)v_x^2 + a(u)a'(u)\left(v_t^2 - c_2^2(u)v_x^2\right), \\ v_{tt} - c_2^2(u)v_{xx} = 2c_2(u)c_2'(u)u_xv_x - \dfrac{2a'(u)}{a(u)}\left(u_tv_t - c_2^2(u)u_xv_x\right), \end{cases} \tag{4}$$

where

$$c_1^2(u) = k_3 \cos^2 u + k_1 \sin^2 u, \; c_2^2(u) = k_3 \cos^2 u + k_2 \sin^2 u, \; a^2(u) = \sin^2 u. \tag{5}$$

System (4) characterizes the propagation of twist waves of nematic liquid crystal molecules. Under the assumption that the constants $k_i$ $(i = 1, 2, 3)$ are positive (so that $c_1$ and $c_2$ are positive), there exists a series of results on conservative weak solutions to system (4); see, for example, [18–23].

In this paper, we are concerned with system (4) with degenerate wave speeds $c_1$ and $c_2$. Specifically, we consider the case where the bend elastic constant $k_3$ vanishes, which leads to a degradation in the wave speeds $c_1$ and $c_2$. It may seem strange at first glance that the elastic constants disappear or become negative. This is mainly because they are required to be positive for the conventional or uniform nematic liquid crystals, which result in the elastic energy increasing when the director distribution deviates from its uniform equilibrium state. However, for non-uniform nematic liquid crystals, the ground state of the director would no longer be uniformly aligned but would be bent, which leads to the disappearance of or a negative bend elastic constant $k_3$. This characteristic was initially realized by Dozov [24], and was confirmed subsequently in many works; see, among others, [25–28].

Let $k_3 = 0$ and replace $k_1, k_2$ with $k_1^2, k_2^2$ for convenience. Then, (4) with (5) reduces to

$$\begin{cases} u_{tt} - k_1^2 \sin^2 u u_{xx} = k_1^2 \sin u \cos u u_x^2 - 2k_2^2 \sin^3 u \cos u v_x^2 + \sin u \cos u v_t^2, \\ v_{tt} - k_2^2 \sin^2 u v_{xx} = 4k_2^2 \sin u \cos u u_x v_x - \frac{2\cos u}{\sin u} u_t v_t. \end{cases} \tag{6}$$

We particularly point out that, in addition to the degenerate wave speeds $k_1 \sin u$ and $k_2 \sin u$, system (6) also contains a singular nonhomogeneous term that exists in the original system (4) and is not caused by the disappearance of the constant $k_3$. We study the Cauchy problem of (6) with the following initial conditions:

$$u(0,x)=u_0(x),\ v(0,x)=v_0(x),\ u_t(0,x)=u_1(x),\ v_t(0,x)=v_1(x),\ x\in\mathbb{R},$$
$$\sin u_0(x)>0,\quad \lim_{x\to\pm\infty}\sin u_0(x)=0. \tag{7}$$

It is obvious by (7) that problem (6)–(7) is a degenerate Cauchy problem of the nonlinear hyperbolic system with a singular nonhomogeneous term. Although the degradation and singularity only occur at infinity, this problem cannot be solved by the previous theory for the strictly hyperbolic system due to the unboundedness of the coefficients.

The well-posedness of classical solutions to the degenerate hyperbolic equations is one of the important issues in nonlinear partial differential equations. In [29], Manfrin established the local well-posedness in the $C^\infty$ class for the one-dimensional degenerate quasilinear wave equation

$$u_{tt}-a(t,x,u)u^{2k}u_{xx}=f(t,x,u),$$

with initial data $u_0,u_1\in C_0^\infty(\mathbb{R}^n)$, where $k\in\mathbb{N}^+$, $a(t,x,u)\geqslant\lambda_0>0$, and $f(t,x,u)$ are $C^\infty$ functions. We also refer the reader to [30–32] for some revelent works on $C^\infty$. In [33], Dreher proved the local solvability of the $p$-Laplacian wave equation

$$\partial_{tt}u-\partial_x(|\partial_x u|^{p-2}\partial_x u)=0,$$

for $p>5$ and $C_0^k$ initial data with a large natural number $k$. The results in [29–32] are based on the Nash-Moser method, which requires sufficient high regularity on the initial data to cope with the loss of regularity caused by the hyperbolic degeneracy. More specifically, the following energy estimate with the regularity loss was acquired:

$$\|u\|_{H^s}+\|u_t\|_{H^{s-1}}\leqslant C\left(\|u_0\|_{H^{s+r_1}}+\|u_1\|_{H^{s-1+r_2}}\right), \tag{8}$$

where $s\in\mathbb{R}$ and $r_1,r_2\geqslant 0$. To obtain the classical solution without loss of regularity, Sugiyama [34] introduced appropriate weighted $L^\infty$ estimates on the unknown variables and the Riemann variants to study the quasilinear wave equation $\partial_{tt}u=\partial_x(u^{2a}\partial_x u)+F(u)u_x$. By using the method of characteristics, he established the local solvability to its Cauchy problem with far field degeneracy and verified that the solution preserves the initial regularity of the initial data. That means the regularity of the solution obtained by Sugiyama has no loss. Subsequently, Hu and Sugiyama investigated the local solvability of the degenerate initial-boundary value problem to the quasilinear wave equation in [35] and the quasilinear hyperbolic-parabolic coupled system in [36, 37].

We also refer the reader to [38–41] for several recent related works to the present study. Specifically, Gao et al. [39] analyzed one-dimensional wave equations with nonlinear localized damping. They established decay estimates and proved the existence of a global attractor. In [38], Hu and Zeng studied singularity formation in a four-field hyperbolic system from magnetohydrodynamics and demonstrated that smooth solutions can blow up in finite time for certain initial data. Koonce and Metcalfe [40] considered multiple-speed wave systems and obtained an almost global existence result with sharp lifespan estimates in the radial case. In [41], Neal et al. investigated the shock formation in 2D Euler equations and utilized the characteristic method to establish a detailed description of the pre-shock profile.

For the current degenerate Cauchy problem (6)–(7), to overcome the difficulty of regularity loss in the solution, we introduce the corresponding singular weight functions and then derive the weighted $L^\infty$

estimates on the unknown variables $(u, v)$ and Riemann variants. By using the method of characteristics and the contraction mapping principle, we establish the local well-posedness of classical solutions for initial data with a certain degradation rate to infinity. The introduction of a suitable weighted $L^\infty$ norm guarantees the solution without any loss of regularity. Compared with the previous work by Sugiyama [34], the main innovation of the paper lies in the fact that the governing system (6) has four sets of characteristics, $\pm k_1 \sin u$ and $\pm k_2 \sin u$, and contains a singular nonhomogeneous term.

Before stating the conclusion, let us first present the assumptions on the initial data. Let $\beta \geqslant \alpha > 0$ be two arbitrary positive constants and denote

$$\langle x \rangle = (1 + x^2)^{\frac{1}{2}}.$$

Assume that for all $x \in \mathbb{R}$, the initial functions $u_0(x), u_1(x), v_0(x), v_1(x)$ satisfy

$$\begin{aligned}
&(\mathrm{P}_{1,0}): \ 2A_0 \leqslant \langle x \rangle^\alpha \sin u_0(x) \leqslant A_{1,0},\\
&(\mathrm{P}_{2,0}): k_1 \sin u_0(x) \left| u_0'(x) \right|, k_2 \sin u_0(x) \left| v_0'(x) \right| \leqslant A_{2,0} \langle x \rangle^{-\beta},\\
&(\mathrm{P}_{3,0}): |u_1(x)|, |v_1(x)| \leqslant A_{3,0} \langle x \rangle^{-\beta},\\
&(\mathrm{P}_{4,0}): \left| \frac{d}{dx} \left[ u_1(x) + k_1 \sin u_0(x) u_0'(x) \right] \right|, \left| \frac{d}{dx} \left[ v_1(x) + k_2 \sin u_0(x) v_0'(x) \right] \right| \leqslant A_{5,0},
\end{aligned} \tag{9}$$

where $A_0, A_{1,0}, A_{2,0}, A_{3,0}$, and $A_{5,0}$ are positive constants. Denote

$$A_{4,0} = A_{2,0} + A_{3,0},$$

so we have

$$\left| u_1(x) \pm k_1 \sin u_0(x) u_0'(x) \right|, \left| v_1(x) \pm k_2 \sin u_0(x) v_0'(x) \right| \leqslant A_{4,0} \langle x \rangle^{-\beta}, \quad \forall\, x \in \mathbb{R}. \tag{10}$$

Obviously, the assumption $(\mathrm{P}_{1,0})$ indicates that system (6) is degenerate and singular at spatial infinity. By a "degenerate Cauchy problem", we mean that the wave speeds $k_j \sin u$ vanish as $|x| \to \infty$, as indicated by the upper bound in $(P_{1,0})$. The lower bound in $(P_{1,0})$ is necessary to ensure that the hyperbolic degeneracy occurs only at infinity. The uniform weighted estimates are derived with respect to the spatial variable $x$, which guarantees that the local existence time $T$ is uniform in $x$, as stated in Theorem 1. A typical example of admissible initial data is $u_0(x) = \pi/2 + \arctan(x)$, for which $\sin u_0(x) = \langle x \rangle^{-1}$ up to a constant factor, so that the bounds in $(P_{1,0})$ hold with $\alpha = 1$. Such conditions are standard in the study of a single degenerate quasilinear wave equation with a smooth nonhomogeneous term; see Hu and Sugiyama [35].

A key technical feature of this paper is the weighted $L^\infty$ framework with the specific weight $\langle x \rangle^\alpha$ chosen to match the decay structure of the far field degeneracy. The upper bound in $(P_{1,0})$ gives $\sin u_0(x) \leq A_{1,0} \langle x \rangle^{-\alpha}$, while the lower bound ensures that the singular coefficient $1/\sin u$ is controlled by the same weight. The choice of $\langle x \rangle^\alpha$ is precisely what allows the nonlinear terms in the governing system of Riemann variables to be estimated uniformly in $x$. The same weight also appears in the decay assumptions on the initial derivatives in $(P_{2,0})$ and $(P_{3,0})$, linking the degeneracy of the wave speeds with the regularity estimates for the solution. Thus the weight chosen in this paper is crucial, as it renders the iteration map contractive for a small time under the weighted $L^\infty$ norm.

The main conclusion of the paper can be stated in the following theorem.

**Theorem 1.** *Assume that the initial data satisfy* $(u_0, u_1), (v_0, v_1) \in C^2(\mathbb{R}) \times C_b^1(\mathbb{R})$*, the decay conditions* (9), (10)*, and the compatibility conditions* (101)*. Then there exists a small positive number* $T > 0$ *such that the degenerate Cauchy problem* (6)–(7) *has a unique classical solution* $(u, v)(t, x) \in C^2([0, T] \times \mathbb{R}) \times C^2([0, T] \times \mathbb{R})$ *on* $[0, T] \times \mathbb{R}$*. In addition, for all* $(t, x) \in [0, T] \times \mathbb{R}$*, the following estimates hold:*

$$\begin{aligned}
&(\mathrm{P}_1): A_0 \leqslant \langle x\rangle^{\alpha} \sin u(t,x) \leqslant A_1,\\
&(\mathrm{P}_2): k_1 \sin u\,|u_x|\,, k_2 \sin u\,|v_x| \leqslant A_2\langle x\rangle^{-\beta}, |u_t|\,, |v_t| \leqslant A_3\langle x\rangle^{-\beta},\\
&(\mathrm{P}_3): |u_t \pm k_1 \sin u\, u_x|\,, |v_t \pm k_2 \sin u\, v_x| \leqslant A_4\langle x\rangle^{-\beta},\\
&(\mathrm{P}_4): |\partial_x (u_t \pm k_1 \sin u u_x)|\,, |\partial_x (v_t \pm k_2 \sin u v_x)| \leqslant A_5,\\
&(\mathrm{P}_5): |\partial_t (u_t \pm k_1 \sin u u_x)|\,, |\partial_t (v_t \pm k_2 \sin u v_x)| \leqslant A_6,
\end{aligned} \tag{11}$$

*for some positive constants* $A_i\,(i = 1, \ldots, 6)$.

**Remark 1.** *Theorem 1 asserts that the solution preserves the regularity of the initial data, that is, its regularity has no loss.*

**Remark 2.** *The existence time of the solution in Theorem 1 is uniform and independent of the spatial variable x, which is mainly attributed to the derived weighted estimates. If the usual estimates are employed, the constants* $A_i$ *will depend on the spatial variable x and then the existence time T may go to zero when x tends to infinity.*

The rest of this paper is organized as follows. In Section 2, we introduce the Riemann variants to transform the second-order equations into a system of first-order equations. Moreover, we provide some estimates of the characteristic curves in this section. Section 3 is devoted to verifying the main theorem. We introduce the singular weight function spaces for the degenerate Cauchy problem in Section 3.1. In Section 3.2, we establish the properties of the solution sequence generated by an iteration mapping. Finally, we show that the solution sequence is convergent and its limit belongs to the selected function space in Section 3.3. These conclusions complete the proof of Theorem 1.

## 2. Preliminaries for the hyperbolic system

### 2.1. The governing system in terms of Riemann variants

We introduce two pairs of unknown variables as follows:

$$\begin{cases} R_1 = u_t + k_1 \sin u u_x, \\ S_1 = u_t - k_1 \sin u u_x, \end{cases} \qquad \begin{cases} R_2 = v_t + k_2 \sin u v_x, \\ S_2 = v_t - k_2 \sin u v_x, \end{cases}$$

so that

$$u_t = \frac{R_1 + S_1}{2}, \quad u_x = \frac{R_1 - S_1}{2k_1 \sin u}, \quad v_t = \frac{R_2 + S_2}{2}, \quad v_x = \frac{R_2 - S_2}{2k_2 \sin u}. \tag{12}$$

Then system (6) can be transformed into a new system of $(u, v, R_1, S_1, R_2, S_2)$:

$$\begin{cases} R_{1t} - k_1 \sin u\, R_{1x} = F_1(u, R_2, S_2) + G(u, R_1, S_1), \\ S_{1t} + k_1 \sin u\, S_{1x} = F_1(u, R_2, S_2) - G(u, R_1, S_1), \\ R_{2t} - k_2 \sin u\, R_{2x} = F_2(u, R_1, S_1, R_2, S_2) + H(u, R_1, S_1, R_2, S_2), \\ S_{2t} + k_2 \sin u\, S_{2x} = F_2(u, R_1, S_1, R_2, S_2) - H(u, R_1, S_1, R_2, S_2), \\ u_t = \frac{R_1 + S_1}{2}, \quad v_t = \frac{R_2 + S_2}{2}, \end{cases} \tag{13}$$

where

$$
\begin{aligned}
F_1 &= -\frac{1}{4}\sin(2u)(R_2 - S_2)^2 + \frac{1}{8}\sin(2u)(R_2 + S_2)^2,\\
G &= \frac{\cos u}{4\sin u}(R_1^2 - S_1^2),\\
F_2 &= \frac{3k_2\cos u}{4k_1\sin u}(R_1 - S_1)(R_2 - S_2) - \frac{\cos u}{2\sin u}(R_1 + S_1)(R_2 + S_2),\\
H &= \frac{\cos u}{4\sin u}(R_1 + S_1)(R_2 - S_2).
\end{aligned} \tag{14}
$$

Let $x_1^{\pm}(t)$ and $x_2^{\pm}(t)$ be the characteristic curves of the equations in (13), that is, $x_j^{\pm}(t)$ are the solutions to the following differential equations:

$$\frac{d}{dt}x_j^{\pm}(t) = \pm k_j \sin u(t, x_j^{\pm}(t)), \quad j = 1, 2. \tag{15}$$

When we emphasize the characteristic curves go through $(s, y)$, we denote $x_j^{\pm}(t)$ by $x_j^{\pm}(t; s, y)$, that is, $x_j^{\pm}(t; s, y)$ satisfy that

$$x_j^{\pm}(t; s, y) = y \pm k_j \int_s^t \sin u(\tau, x_j^{\pm}(\tau; s, y))d\tau, \quad j = 1, 2. \tag{16}$$

In the notation $x_j^{\pm}(\cdot; \cdot, \cdot)$, the first argument denotes the current time, the second denotes the initial time, and the third denotes the initial spatial point.

Along the characteristic curves, the equations of $(R_1, S_1, R_2, S_2)$ in system (13) can be rewritten as

$$
\begin{cases}
\frac{d}{dt}R_1\left(t, x_1^{-}(t)\right) = \left[F_1\left(u, R_2, S_2\right) + G\left(u, R_1, S_1\right)\right]\left(t, x_1^{-}(t)\right),\\
\frac{d}{dt}S_1\left(t, x_1^{+}(t)\right) = \left[F_1\left(u, R_2, S_2\right) - G\left(u, R_1, S_1\right)\right]\left(t, x_1^{+}(t)\right),\\
\frac{d}{dt}R_2\left(t, x_2^{-}(t)\right) = \left[F_2\left(u, R_1, S_1, R_2, S_2\right) + H\left(u, R_1, S_1, R_2, S_2\right)\right]\left(t, x_2^{-}(t)\right),\\
\frac{d}{dt}S_2\left(t, x_2^{+}(t)\right) = \left[F_2\left(u, R_1, S_1, R_2, S_2\right) - H\left(u, R_1, S_1, R_2, S_2\right)\right]\left(t, x_2^{+}(t)\right).
\end{cases} \tag{17}
$$

### 2.2. *Some estimates of characteristic curves*

We prepare some estimates for the characteristic curves. Assume that $u \in C^1([0, T] \times \mathbb{R})$ satisfying

$$A_0 \leqslant \langle x\rangle^{\alpha} \sin u \leqslant A_1, \quad \forall\, (t, x) \in [0, T] \times \mathbb{R}, \tag{18}$$

for $\alpha > 0$, where $A_0$ and $A_1$ are positive constants. Additionally, we suppose that

$$0 \leqslant k_1 \sin u\, |u_x| \leqslant A_2\langle x\rangle^{-\alpha}, \quad \forall\, (t, x) \in [0, T] \times \mathbb{R}, \tag{19}$$

for a constant $A_2 > 0$. It follows by (16) that

$$x - k_j A_1|t - s| \leqslant x_j^{\pm}(t; s, x) \leqslant x + k_j A_1|t - s|, \tag{20}$$

for all $s, t \in [0, T]$ and $x \in \mathbb{R}$.

Next we show a lemma that ensures a uniform Lipschitz continuity of $x_j^{\pm}(t; s, y)$. This lemma is crucial for verifying that a sequence of characteristic curves satisfies the conditions of the Arzelà-Ascoli theorem.

**Lemma 2.1.** *Suppose that* $u \in C^1([0,T] \times \mathbb{R})$. *Let* (18) *and* (19) *be satisfied. Then if* $T > 0$ *is sufficiently small, the following estimate holds:*

$$\left|x_j^{\pm}(t_3; t_1, x_1) - x_j^{\pm}(t_4; t_2, x_2)\right| \leqslant 3(1 + k_j A_1)(|x_1 - x_2| + |t_1 - t_2| + |t_3 - t_4|), \tag{21}$$

*for* $j = 1, 2$, $x_1, x_2 \in \mathbb{R}$, *and* $t_1, t_2, t_3, t_4 \in [0, T]$.

*Proof.* First, we consider the case where $t_3 = t_4 = t \in [0, T]$ and $t \geqslant t_1, t_2$. From (16) and (18), we compute

$$\begin{aligned}
&\left|x_j^{\pm}(t; t_1, x_1) - x_j^{\pm}(t; t_2, x_2)\right| \\
&\leqslant |x_1 - x_2| + k_j \left| \int_{t_1}^{t} \sin u(\tau, x_j^{\pm}(\tau; t_1, x_1)) \mathrm{d}\tau - \int_{t_2}^{t} \sin u(\tau, x_j^{\pm}(\tau; t_2, x_2)) d\tau \right| \\
&\leqslant |x_1 - x_2| + k_j \left| \int_{t_2}^{t_1} \sin u(\tau, x_j^{\pm}(\tau; t_2, x_2)) d\tau \right| \\
&\quad + k_j \left| \int_{t_1}^{t} \left[ \sin u(\tau, x_j^{\pm}(\tau; t_1, x_1)) - \sin u(\tau, x_j^{\pm}(\tau; t_2, x_2)) \right] d\tau \right| \\
&\leqslant |x_1 - x_2| + k_j A_1 \, |t_1 - t_2| \\
&\quad + k_j \int_0^t \left| \sin u(\tau, x_j^{\pm}(\tau; t_1, x_1)) - \sin u(\tau, x_j^{\pm}(\tau; t_2, x_2)) \right| d\tau.
\end{aligned} \tag{22}$$

Thanks to (18) and (19), we know that $u_x$ is bounded. Consequently, the third term on the right-hand side of (22) satisfies the following inequality:

$$\begin{aligned}
&\left| \sin u\left(\tau, x_j^{\pm}(\tau; t_1, x_1)\right) - \sin u\left(\tau, x_j^{\pm}(\tau; t_2, x_2)\right) \right| \\
&= \left| \int_{x_j^{\pm}(\tau; t_2, x_2)}^{x_j^{\pm}(\tau; t_1, x_1)} \cos u(\tau, y) u_x(\tau, y) \, dy \right| \leqslant C \left| x_j^{\pm}(\tau; t_1, x_1) - x_j^{\pm}(\tau; t_2, x_2) \right|,
\end{aligned} \tag{23}$$

for some positive constant $C$. Hence we have

$$\begin{aligned}
&\left|x_j^{\pm}(t; t_1, x_1) - x_j^{\pm}(t; t_2, x_2)\right| \\
&\leqslant (1 + k_j A_1) \left(|x_1 - x_2| + |t_1 - t_2|\right) + C \int_0^t \left| x_j^{\pm}(\tau; t_1, x_1) - x_j^{\pm}(\tau; t_2, x_2) \right| d\tau.
\end{aligned} \tag{24}$$

Applying the Gronwall inequality yields

$$\left|x_j^{\pm}(t; t_1, x_1) - x_j^{\pm}(t; t_2, x_2)\right| \leqslant (1 + k_j A_1) \left(|x_1 - x_2| + |t_1 - t_2|\right) e^{Ct}. \tag{25}$$

For sufficiently small $T$, we obtain $e^{Ct} \leqslant 2$ and then

$$\left|x_j^{\pm}(t; t_1, x_1) - x_j^{\pm}(t; t_2, x_2)\right| \leqslant 2(1 + k_j A_1) \left(|x_1 - x_2| + |t_1 - t_2|\right). \tag{26}$$

The case for $t < t_1$ or $t < t_2$ follows similarly and is omitted for brevity.

Next, we consider the case where $t_3 \neq t_4$ to continue the proof of (21). The left-hand side of (21) can be written as

$$\left|x_j^{\pm}(t_3; t_1, x_1) - x_j^{\pm}(t_4; t_2, x_2)\right|$$

$$
\begin{aligned}
&\leqslant \left|x_j^{\pm}(t_3;t_1,x_1)-x_j^{\pm}(t_3;t_2,x_2)\right|+\left|x_j^{\pm}(t_3;t_2,x_2)-x_j^{\pm}(t_4;t_2,x_2)\right|\\
&\leqslant 2(1+k_jA_1)\left(|x_1-x_2|+|t_1-t_2|\right)+k_j\int_{t_4}^{t_3}\left|\sin u(\tau,x_j^{\pm}(\tau;t_2,x_2))\right|d\tau\\
&\leqslant 3(1+k_jA_1)\left(|x_1-x_2|+|t_1-t_2|+|t_3-t_4|\right),
\end{aligned}\tag{27}
$$

which leads to the desired inequality (21). □

The following lemma is used to show the boundedness of the derivatives of $R_1$, $S_1$, $R_2$, and $S_2$.

**Lemma 2.2.** *Let $u \in C^1([0,T]\times\mathbb{R})$. Suppose that inequalities* (18) *and* (19) *hold. Then the characteristic curves $x_j^{\pm}(t;s,x)$ are differentiable with respect to $x$ and $\partial_x x_j^{\pm}(t;s,x)$ satisfies, with $(t,x)\in[0,T]\times\mathbb{R}$ and $s\in[0,T]$ for small $T>0$,*

$$
\begin{cases}
\frac{d}{ds}\partial_x x_j^{\pm}(s;t,x)=\pm k_j\cos u u_x(s,x_j^{\pm}(s;t,x))\partial_x x_j^{\pm}(s;t,x),\\
\partial_x x_j^{\pm}(t;t,x)=1,
\end{cases}\tag{28}
$$

*and*

$$
\left|\partial_x x_j^{\pm}(s;t,x)\right|\leqslant 2,\tag{29}
$$

*where $j=1,2$.*

*Proof.* From (15) and (16), we know that the characteristic curves $x_j^{\pm}(s;t,x)$ satisfy

$$
\begin{cases}
\frac{d}{ds}x_j^{\pm}(s;t,x)=\pm k_j\sin u(s,x_j^{\pm}(s;t,x)),\\
x_j^{\pm}(t;t,x)=x.
\end{cases}\tag{30}
$$

Since $u\in C^1([0,T]\times\mathbb{R})$, by the classical theory of hyperbolic partial differential equations (see, e.g., Li and Yu [42]), we know that $x_j^{\pm}(s;t,x)$ are differentiable with respect to $x$. Taking the partial derivative of (30) with respect to $x$, we can obtain (28). We now estimate $\partial_x x_j^{\pm}(s;t,x)$. Without loss of generality, we consider the case where $t\geqslant s$ to show (29). Due to the boundedness of $u_x$, by (18) and (19), we calculate

$$
\begin{aligned}
&\left|\partial_x x_j^{\pm}(s;t,x)\right|\\
\leqslant\ & 1+k_j\int_s^t\left|\cos u(\tau,x_j^{\pm}(\tau;t,x))\right|\left|u_x(\tau,x_j^{\pm}(\tau;t,x))\right|\left|\partial_x x_j^{\pm}(\tau;t,x)\right|d\tau\\
\leqslant\ & 1+C\int_s^t\left|\partial_x x_j^{\pm}(\tau;t,x)\right|d\tau,
\end{aligned}\tag{31}
$$

from which, along with the Gronwall inequality, we gain (29) for sufficiently small $T>0$. □

## 3. The proof of the main theorem

In this section, we demonstrate Theorem 1 by employing the contraction mapping principle within the chosen function space.

### 3.1. *The function spaces*

For $\beta \geqslant \alpha > 0$, we denote

$$\begin{aligned} A_0 \leqslant \langle x\rangle^{\alpha} \sin f(t,x) &\leqslant A_1, && (32)\\ k_1 \sin f(t,x)\, |f_x(t,x)| &\leqslant A_2 \langle x\rangle^{-\beta}, && (33)\\ k_2 \sin u(t,x)\, |f_x(t,x)| &\leqslant A_2 \langle x\rangle^{-\beta}, && (34)\\ |f_t(t,x)| &\leqslant A_3 \langle x\rangle^{-\beta}, && (35) \end{aligned}$$

or

$$\begin{aligned} |f(t,x)| &\leqslant A_3 \langle x\rangle^{-\beta}, && (36)\\ |f_x(t,x)| &\leqslant A_4, && (37)\\ |f_t(t,x)| &\leqslant A_5, && (38) \end{aligned}$$

where $A_j$ $(j = 1, \dots, 5)$ are positive constants. We define sets of $C^1$ functions $U$, $V$, $T_{r,1}$, $T_{s,1}$, $T_{r,2}$, $T_{s,2}$ as follows:

$$\begin{aligned} U &= \left\{ f \in C_b^1 \mid f(0,x) = u_0(x) \text{ and (32), (33), and (35) hold} \right\},\\ V &= \left\{ f \in C_b^1 \mid f(0,x) = v_0(x) \text{ and (34), (35), and } u \in U \text{ hold} \right\},\\ T_{r,1} &= \left\{ f \in C_b^1 \mid f(0,x) = R_{1,0}(x) \text{ and (36), (37), and (38) hold} \right\},\\ T_{s,1} &= \left\{ f \in C_b^1 \mid f(0,x) = S_{1,0}(x) \text{ and (36), (37), and (38) hold} \right\},\\ T_{r,2} &= \left\{ f \in C_b^1 \mid f(0,x) = R_{2,0}(x) \text{ and (36), (37), and (38) hold} \right\},\\ T_{s,2} &= \left\{ f \in C_b^1 \mid f(0,x) = S_{2,0}(x) \text{ and (36), (37), and (38) hold} \right\}, \end{aligned}$$

where $(u_0, v_0, R_{1,0}, S_{1,0}, R_{2,0}, S_{2,0}) \in C_b^1 \times C_b^1 \times C_b^1 \times C_b^1 \times C_b^1 \times C_b^1$ are given functions. In this paper, $C_b^1$ is a set of continuous and bounded functions whose derivatives are also bounded. Then, we define $\Pi$ in the following function space:

$$\Pi = U \times V \times T_{r,1} \times T_{s,1} \times T_{r,2} \times T_{s,2}.$$

### 3.2. *The properties of the solution sequence*

For given functions $(\bar{u}, \bar{v}, \bar{R}_1, \bar{S}_1, \bar{R}_2, \bar{S}_2) \in \Pi$, we consider the first-order linear hyperbolic equations:

$$\begin{cases} R_{1t} - k_1 \sin \bar{u}\, R_{1x} = F_1(\bar{u}, \bar{R}_2, \bar{S}_2) + G(\bar{u}, \bar{R}_1, \bar{S}_1),\\ S_{1t} + k_1 \sin \bar{u}\, S_{1x} = F_1(\bar{u}, \bar{R}_2, \bar{S}_2) - G(\bar{u}, \bar{R}_1, \bar{S}_1),\\ R_{2t} - k_2 \sin \bar{u}\, R_{2x} = F_2(\bar{u}, \bar{R}_1, \bar{S}_1, \bar{R}_2, \bar{S}_2) + H(\bar{u}, \bar{R}_1, \bar{S}_1, \bar{R}_2, \bar{S}_2),\\ S_{2t} + k_2 \sin \bar{u}\, S_{2x} = F_2(\bar{u}, \bar{R}_1, \bar{S}_1, \bar{R}_2, \bar{S}_2) - H(\bar{u}, \bar{R}_1, \bar{S}_1, \bar{R}_2, \bar{S}_2), \end{cases} \tag{39}$$

with the initial conditions

$$\begin{aligned} &(R_1(0,x), S_1(0,x), R_2(0,x), S_2(0,x))\\ &= (R_{1,0}, S_{1,0}, R_{2,0}, S_{2,0}) \in C_b^1 \times C_b^1 \times C_b^1 \times C_b^1. \end{aligned} \tag{40}$$

We set

$$u(t,x) = u_0(x) + \int_0^t \frac{R_1 + S_1}{2}(s,x)\,ds,$$
$$v(t,x) = v_0(x) + \int_0^t \frac{R_2 + S_2}{2}(s,x)\,ds. \tag{41}$$

Using the method of characteristics, we find that (39) with $C^1$ initial data has a unique and time-global solution such that $R_1, S_1, R_2, S_2 \in C^1([0,T]\times\mathbb{R})$ for any fixed $T > 0$. Furthermore, it follows by (41) that $u, v \in C^1$.

Namely, we can define the map

$$\Phi : \Pi \to C^1 \times C^1 \times C^1 \times C^1 \times C^1 \times C^1,$$

such that $\Phi(\bar{u}, \bar{v}, \bar{R}_1, \bar{S}_1, \bar{R}_2, \bar{S}_2) = (u, v, R_1, S_1, R_2, S_2)$. We take four positive numbers $A_0, A_{1,0}, A_{3,0}, A_{4,0}$ satisfying

$$2A_0 \leqslant \langle x\rangle^{\alpha} \sin u_0(x) \leqslant \frac{A_{1,0}}{2}, \quad \forall\, x \in \mathbb{R}, \tag{42}$$

$$\left\|\langle x\rangle^{\beta} R_{j,0}\right\|_{L^\infty} + \left\|\langle x\rangle^{\beta} S_{j,0}\right\|_{L^\infty} \leqslant \frac{A_{3,0}}{4}, \tag{43}$$

$$\left\|R'_{j,0}\right\|_{L^\infty} + \left\|S'_{j,0}\right\|_{L^\infty} \leqslant \frac{A_{4,0}}{8}, \tag{44}$$

where $j = 1, 2$ and the constants $A_{2,0}$ and $A_{5,0}$ will be specified later. Moreover, we assume that

$$\left\|\langle x\rangle^{\beta} k_1 \sin u_0\, u'_0\right\|_{L^\infty},\ \left\|\langle x\rangle^{\beta} k_2 \sin u_0\, v'_0\right\|_{L^\infty} \leqslant B_1, \tag{45}$$

for a positive constant $B_1$.

In the following, we prove that $(u, v, R_1, S_1, R_2, S_2) \in \Pi$ and $\Phi$ is a contraction mapping in the topology of $L^\infty$ for sufficiently small $T$. Although the spaces $U, V, T_{r,j}, T_{s,j}$ are not closed in the $L^\infty$ topology, we shall show that $\Phi$ maps $\Pi$ into itself in Lemma 3.1 below and $\Phi$ is a contraction in the $L^\infty$-norm in Lemma 3.2. Then we obtain a unique fixed point for the iterative sequence in $L^\infty$. Moreover, we show in Lemma 3.3 that the limit function still belongs to $\Pi$. Furthermore, we will improve the regularity of the limit function to $u \in C^2$, $v \in C^2$, which means that it is the smooth solution to the original degenerate Cauchy problem (6)–(7).

Next, we present the following lemmas.

**Lemma 3.1.** *Let* $(u_0, v_0, R_{1,0}, S_{1,0}, R_{2,0}, S_{2,0}) \in C^1 \times C^1 \times C^1 \times C^1 \times C^1 \times C^1$ *satisfying* (42)–(45). *Suppose that* $(\bar{u}, \bar{v}, \bar{R}_1, \bar{S}_1, \bar{R}_2, \bar{S}_2) \in \Pi$. *Then* $\Phi(\bar{u}, \bar{v}, \bar{R}_1, \bar{S}_1, \bar{R}_2, \bar{S}_2) = (u, v, R_1, S_1, R_2, S_2) \in \Pi$ *for sufficiently small* $T > 0$.

*Proof.* From the method of characteristics, we can see that the solution of (39) can be written as

$$\begin{cases} R_1(t,x) = R_1(0,x_1^-(0)) + \displaystyle\int_0^t \Big[F_1(\bar{u},\bar{R}_2,\bar{S}_2) + G(\bar{u},\bar{R}_1,\bar{S}_1)\Big](s,x_1^-(s))ds, \\ S_1(t,x) = S_1(0,x_1^+(0)) + \displaystyle\int_0^t \Big[F_1(\bar{u},\bar{R}_2,\bar{S}_2) - G(\bar{u},\bar{R}_1,\bar{S}_1)\Big](s,x_1^+(s))ds, \\ R_2(t,x) = R_2(0,x_2^-(0)) + \displaystyle\int_0^t \Big[F_2(\bar{u},\bar{R}_1,\bar{S}_1,\bar{R}_2,\bar{S}_2) + H(\bar{u},\bar{R}_1,\bar{S}_1,\bar{R}_2,\bar{S}_2)\Big](s,x_2^-(s))ds, \\ S_2(t,x) = S_2(0,x_2^+(0)) + \displaystyle\int_0^t \Big[F_2(\bar{u},\bar{R}_1,\bar{S}_1,\bar{R}_2,\bar{S}_2) - H(\bar{u},\bar{R}_1,\bar{S}_1,\bar{R}_2,\bar{S}_2)\Big](s,x_2^+(s))ds, \end{cases} \tag{46}$$

where the characteristic curves for the linear equation (39) are defined as follows:

$$\frac{d}{dt}x_j^\pm(t) = \pm k_j \sin\bar{u}(t,x_j^\pm(t)), \quad j=1,2,$$

with initial data $x_j^\pm(t) = x$. From this expression, we have that $(u,v,R_1,S_1,R_2,S_2) \in (C^1)^6$. Next, we divide the proof of the lemma into seven steps. For convenience, we use $C(\cdot,\ldots,\cdot)$ to denote the positive constants, which may vary across different expressions but exclusively depend on the parameters specified within the brackets.

**Step 1.** The estimates of $R_j$ and $S_j$. From (20), if $T$ is small, we have

$$\frac{\langle x\rangle^\beta}{2} \leqslant \Big\langle x_j^-(s;t,x)\Big\rangle^\beta \leqslant 2\langle x\rangle^\beta. \tag{47}$$

From (46), (47), (32), (36), and (43), we have that if $T$ is small,

$$\begin{aligned} &\Big|\langle x\rangle^\beta R_1(t,x)\Big| \\ &\leqslant 2\left\|\langle x_1^-(0)\rangle^\beta R_1(0,\cdot)\right\|_{L^\infty} + \left|2\langle x_1^-(s)\rangle^\beta \int_0^t \left[-\frac{1}{4}\sin 2\bar{u}(\bar{R}_2-\bar{S}_2)^2 + \frac{1}{8}\sin 2\bar{u}(\bar{R}_2+\bar{S}_2)^2 \right.\right. \\ &\qquad \left.\left. + \frac{1}{4}\frac{\cos\bar{u}}{\sin\bar{u}}(\bar{R}_1^2 - \bar{S}_1^2)\right]ds\right| \\ &\leqslant \frac{A_{3,0}}{2} + \int_0^t \left[\langle x_1^-(s)\rangle^{2\beta}\left|\bar{R}_2^2 + 2\bar{R}_2\bar{S}_2 + \bar{S}_2^2\right| + \frac{\langle x_1^-(s)\rangle^{\beta+\alpha}}{2A_0\langle x_1^-(s)\rangle^\alpha}\left|\bar{R}_1^2 + \bar{S}_1^2\right|\right]ds \\ &\leqslant \frac{A_{3,0}}{2} + \int_0^t \langle x_1^-(s)\rangle^{2\beta}\left(|\bar{R}_2|^2 + 2|\bar{R}_2||\bar{S}_2| + |\bar{S}_2|^2 + \frac{1}{2A_0}(|\bar{R}_1|^2 + |\bar{S}_1|^2)\right)ds \\ &\leqslant \frac{A_{3,0}}{2} + T\Big(\|\langle x\rangle^\beta\bar{R}_2\|_{L^\infty}^2 + 2\|\langle x\rangle^\beta\bar{R}_2\|_{L^\infty}\|\langle x\rangle^\beta\bar{S}_2\|_{L^\infty} + \|\langle x\rangle^\beta\bar{S}_2\|_{L^\infty}^2\Big) \\ &\qquad + \frac{T}{2A_0}\Big(\|\langle x\rangle^\beta\bar{R}_1\|_{L^\infty}^2 + \|\langle x\rangle^\beta\bar{S}_1\|_{L^\infty}^2\Big) \\ &\leqslant \frac{A_{3,0}}{2} + C(A_0,A_3)T \leqslant A_{3,0} =: A_3, \end{aligned}$$

where $C(A_0,A_3) := 4A_3^2 + \frac{A_3^2}{A_0}$, and similarly,

$$|\langle x\rangle^\beta R_2(t,x)| \leqslant \frac{A_{3,0}}{2} + C(A_0,A_3)T \leqslant A_{3,0} = A_3,$$

provided that $C(A_0,A_3)T \leqslant \frac{A_{3,0}}{2}$. Hence we obtain that

$$\left\|\langle x\rangle^{\beta}R_j\right\|_{L^\infty} \leqslant A_3. \tag{48}$$

Similarly, we have that

$$\left\|\langle x\rangle^{\beta}S_j\right\|_{L^\infty} \leqslant A_3. \tag{49}$$

**Step 2.** The estimates of $R_{jx}$ and $S_{jx}$. Denote $V_j = R_{jx}, W_j = S_{jx}$ and $\overline{V}_j = \overline{R}_{jx}, \overline{W}_j = \overline{S}_{jx}$. Differentiating both sides of Eq (46) with respect to $x$, we can obtain the governing system of $(V_1,W_1,V_2,W_2)$. For example, the equations for $V_1$ and $V_2$ are

$$\begin{aligned}
V_1(t,x) &= V_{1,0}(x_1^-(0;t,x))\partial_x x_1^-(0;t,x)\\
&+\int_0^t \partial_x x_1^-(s;t,x)\left(F_{1u}\bar{u}_x + F_{1R_2}\bar{V}_2 + F_{1S_2}\bar{W}_2\right)(s,x_1^-(s;t,x))ds\\
&+\int_0^t \partial_x x_1^-(s;t,x)\left(G_u\bar{u}_x + G_{R_1}\bar{V}_1 + G_{S_1}\bar{W}_1\right)(s,x_1^-(s;t,x))ds
\end{aligned} \tag{50}$$

and

$$\begin{aligned}
V_2(t,x) &= V_{2,0}(x_2^-(0;t,x))\partial_x x_2^-(0;t,x)\\
&+\int_0^t \partial_x x_2^-(s;t,x)\left(F_{2u}\bar{u}_x + F_{2R_1}\bar{V}_1 + F_{2S_1}\bar{W}_1 + F_{2R_2}\bar{V}_2 + F_{2S_2}\bar{W}_2\right)(s,x_2^-(s;t,x))ds\\
&+\int_0^t \partial_x x_2^-(s;t,x)\left(H_u\bar{u}_x + H_{R_1}\bar{V}_1 + H_{S_1}\bar{W}_1 + H_{R_2}\bar{V}_2 + H_{S_2}\bar{W}_2\right)(s,x_2^-(s;t,x))ds,
\end{aligned} \tag{51}$$

where we denote $(V_{j,0},W_{j,0}) = (R'_{j,0}(\cdot),S'_{j,0}(\cdot))$ for $j=1,2$ and $I_u, I_{R_i}, I_{S_i}$ $(I = F_j,G,H; i=1,2)$ are partial derivatives of $I_j = I_j(u,R_1,S_1,R_2,S_2)$ with respect to $u$, $R_j$, and $S_j$, respectively. From Lemma 2.2, we obtain that $|\partial_x x_j^{\pm}(s;t,x)|$ is bounded by 2 if $T$ is small. Therefore, by (44), we can derive that

$$\begin{aligned}
|V_1(t,x)| \leqslant \frac{A_{4,0}}{4} &+ 2\int_0^t \left|F_{1u}\bar{u}_x + F_{1R_2}\bar{V}_2 + F_{1S_2}\bar{W}_2\right|(s,x_1^-(s;t,x))ds\\
&+ 2\int_0^t \left|G_u\bar{u}_x + G_{R_1}\bar{V}_1 + G_{S_1}\bar{W}_1\right|(s,x_1^-(s;t,x))ds.
\end{aligned} \tag{52}$$

From (32), (36), (37), and $\alpha \leqslant \beta$, we have the following estimations:

$$\begin{aligned}
\left|F_{1R_2}\bar{V}_2\right| + \left|F_{1S_2}\bar{W}_2\right| &\leqslant \left|F_{1R_2}\right|\left|\bar{V}_2\right| + \left|F_{1S_2}\right|\left|\bar{W}_2\right|\\
&\leqslant \left|\frac{1}{4}\sin 2\bar{u}(3\bar{S}_2 - \bar{R}_2)\right|\left|\bar{V}_2\right| + \left|\frac{1}{4}\sin 2\bar{u}(3\bar{R}_2 - \bar{S}_2)\right|\left|\bar{W}_2\right|\\
&\leqslant \frac{3}{2}\sin\bar{u}(\left|\bar{R}_2\right| + \left|\bar{S}_2\right|)(\left|\bar{V}_2\right| + \left|\bar{W}_2\right|)\\
&\leqslant 6A_1A_3A_4\langle x\rangle^{-\alpha-\beta} \leqslant C(A_1,A_3,A_4),
\end{aligned} \tag{53}$$

where $C(A_1,A_3,A_4) := 6A_1A_3A_4$.

In addition,

$$\begin{aligned}\left|G_{R_1}\bar{V}_1\right| + \left|G_{S_1}\bar{W}_1\right| &\leqslant \left|G_{R_1}\right|\left|\bar{V}_1\right| + \left|G_{S_1}\right|\left|\bar{W}_1\right| \\ &\leqslant \left|\frac{\cos\bar{u}}{2\sin\bar{u}}\right|\left|\bar{R}_1\right|\left|\bar{V}_1\right| + \left|\frac{\cos\bar{u}}{2\sin\bar{u}}\right|\left|\bar{S}_1\right|\left|\bar{W}_1\right| \\ &\leqslant \frac{\langle x\rangle^{\alpha}}{\langle x\rangle^{\alpha}\sin\bar{u}}A_3\langle x\rangle^{-\beta}A_4 \\ &\leqslant \frac{A_3A_4}{A_0}\langle x\rangle^{\alpha-\beta} \leqslant C(A_0,A_3,A_4),\end{aligned} \tag{54}$$

where $C(A_0,A_3,A_4) := \frac{A_3A_4}{A_0}$.

From (32),(33), (36), and $\alpha \leqslant \beta$, $|F_{1u}\bar{u}_x|$ and $|G_u\bar{u}_x|$ can be estimated as

$$\begin{aligned}|F_{1u}\bar{u}_x| \leq |F_{1u}|\,|\bar{u}_x| &\leqslant \left|\frac{1}{4}\cos 2\bar{u}(-\bar{R}_2^2 + 6\bar{R}_2\bar{S}_2 - \bar{S}_2^2)\right||\bar{u}_x| \\ &\leqslant \left(\left|\bar{R}_2\right|^2 + 6\left|\bar{R}_2\right|\left|\bar{S}_2\right| + \left|\bar{S}_2\right|^2\right)\frac{A_2\langle x\rangle^{\alpha-\beta}}{k_1\langle x\rangle^{\alpha}\sin\bar{u}} \\ &\leqslant \frac{8A_2A_3^2}{k_1A_0}\langle x\rangle^{\alpha-3\beta} \leqslant C(A_0,A_2,A_3)\end{aligned} \tag{55}$$

and

$$\begin{aligned}|G_u\bar{u}_x| \leq |G_u|\,|\bar{u}_x| &\leqslant \frac{1}{4\sin^2\bar{u}}\left(\left|\bar{R}_1\right|^2 + \left|\bar{S}_1\right|^2\right)\frac{A_2\langle x\rangle^{-\beta}}{k_1\sin\bar{u}} \\ &\leqslant \frac{\langle x\rangle^{2\alpha}}{\langle x\rangle^{2\alpha}\sin^2\bar{u}}\cdot 2A_3^2\langle x\rangle^{-2\beta}\frac{A_2\langle x\rangle^{\alpha-\beta}}{k_1\langle x\rangle^{\alpha}\sin\bar{u}} \\ &\leqslant \frac{A_2A_3^2}{2k_1A_0^3}\langle x\rangle^{3\alpha-3\beta} \leqslant C(A_0,A_2,A_3).\end{aligned} \tag{56}$$

Putting (53)–(56) to (52), we obtain

$$\begin{aligned}|V_1(t,x)| &\leqslant \frac{A_{4,0}}{4} + 2\left(C(A_1,A_3,A_4) + C(A_0,A_2,A_3)\right)T \\ &\quad + 2\left(C(A_0,A_3,A_4) + C(A_0,A_2,A_3)\right)T \\ &\leqslant \frac{A_{4,0}}{4} + C(A_0,A_1,A_2,A_3,A_4)T \leqslant A_{4,0} =: A_4,\end{aligned} \tag{57}$$

provided that $C(A_0,A_1,A_2,A_3,A_4)T \leqslant \frac{3A_{4,0}}{4}$.

Performing the same procedure, we estimate $|V_2(t,x)|$ from (51):

$$\begin{aligned}&|V_2(t,x)| \\ &\leqslant \frac{A_{4,0}}{4} + 2\int_0^t \left|F_{2u}\bar{u}_x + F_{2R_1}\bar{V}_1 + F_{2S_1}\bar{W}_1 + F_{2R_2}\bar{V}_2 + F_{2S_2}\bar{W}_2\right|(s, x_2^-(s;t,x))ds \\ &\quad + 2\int_0^t \left|H_u\bar{u}_x + H_{R_1}\bar{V}_1 + H_{S_1}\bar{W}_1 + H_{R_2}\bar{V}_2 + H_{S_2}\bar{W}_2\right|(s, x_2^-(s;t,x))ds.\end{aligned} \tag{58}$$

From (32), (36), (37), and $\alpha \leqslant \beta$, we have the following estimates:

$$\begin{aligned}
&\left|F_{2R_1}\bar{V}_1\right| + \left|F_{2S_1}\bar{W}_1\right| + \left|F_{2R_2}\bar{V}_2\right| + \left|F_{2S_2}\bar{W}_2\right| \\
\leqslant\, &\left|F_{2R_1}\right|\left|\bar{V}_1\right| + \left|F_{2S_1}\right|\left|\bar{W}_1\right| + \left|F_{2R_2}\right|\left|\bar{V}_2\right| + \left|F_{2S_2}\right|\left|\bar{W}_2\right| \\
\leqslant\, &\left[\left|\frac{3k_2\cos\bar{u}}{4k_1\sin\bar{u}}\right|\left(\left|\bar{R}_2\right| + \left|\bar{S}_2\right|\right) + \left|\frac{\cos\bar{u}}{2\sin\bar{u}}\right|\left(\left|\bar{R}_2\right| + \left|\bar{S}_2\right|\right)\right]\left(\left|\bar{V}_1\right| + \left|\bar{W}_1\right|\right) \\
&+ \left[\left|\frac{3k_2\cos\bar{u}}{4k_1\sin\bar{u}}\right|\left(\left|\bar{R}_1\right| + \left|\bar{S}_1\right|\right) + \left|\frac{\cos\bar{u}}{2\sin\bar{u}}\right|\left(\left|\bar{R}_1\right| + \left|\bar{S}_1\right|\right)\right]\left(\left|\bar{V}_2\right| + \left|\bar{W}_2\right|\right) \\
\leqslant\, &C\frac{1}{\sin\bar{u}}\left(\left|\bar{R}_2\right| + \left|\bar{S}_2\right|\right)\left(\left|\bar{V}_1\right| + \left|\bar{W}_1\right|\right) + C\frac{1}{\sin\bar{u}}\left(\left|\bar{R}_1\right| + \left|\bar{S}_1\right|\right)\left(\left|\bar{V}_2\right| + \left|\bar{W}_2\right|\right) \\
\leqslant\, &C\frac{\langle x\rangle^{\alpha}}{\langle x\rangle^{\alpha}\sin\bar{u}}A_3\langle x\rangle^{-\beta}A_4 \leqslant C(A_0, A_3, A_4)\langle x\rangle^{\alpha-\beta} \leqslant C(A_0, A_3, A_4)
\end{aligned} \tag{59}$$

and

$$\begin{aligned}
&\left|H_{R_1}\bar{V}_1\right| + \left|H_{S_1}\bar{W}_1\right| + \left|H_{R_2}\bar{V}_2\right| + \left|H_{S_2}\bar{W}_2\right| \\
\leqslant\, &\left|H_{R_1}\right|\left|\bar{V}_1\right| + \left|H_{S_1}\right|\left|\bar{W}_1\right| + \left|H_{R_2}\right|\left|\bar{V}_2\right| + \left|H_{S_2}\right|\left|\bar{W}_2\right| \\
\leqslant\, &\frac{1}{\sin\bar{u}}(\left|\bar{R}_2\right| + \left|\bar{S}_2\right|)(\left|\bar{V}_1\right| + \left|\bar{W}_1\right|) + \frac{1}{\sin\bar{u}}(\left|\bar{R}_1\right| + \left|\bar{S}_1\right|)(\left|\bar{V}_2\right| + \left|\bar{W}_2\right|) \\
\leqslant\, &\frac{8\langle x\rangle^{\alpha}}{\langle x\rangle^{\alpha}\sin\bar{u}}A_3\langle x\rangle^{-\beta}A_4 \leqslant C(A_0, A_3, A_4)\langle x\rangle^{\alpha-\beta} \leqslant C(A_0, A_3, A_4).
\end{aligned} \tag{60}$$

Furthermore, $|F_{2u}\bar{u}_x|$ and $|H_u\bar{u}_x|$ can be estimated from (32), (33), (36), and $\alpha \leqslant \beta$:

$$\begin{aligned}
&|F_{2u}\bar{u}_x| \leqslant |F_{2u}|\,|\bar{u}_x| \\
\leqslant\, &\left|-\frac{3k_2}{4k_1\sin^2\bar{u}}(\bar{R}_1 - \bar{S}_1)(\bar{R}_2 - \bar{S}_2) + \frac{1}{2\sin^2\bar{u}}(\bar{R}_1 + \bar{S}_1)(\bar{R}_2 + \bar{S}_2)\right|\frac{A_2\langle x\rangle^{-\beta}}{k_1\sin\bar{u}} \\
\leqslant\, &C\frac{1}{\sin^2\bar{u}}(\left|\bar{R}_1\right| + \left|\bar{S}_1\right|)(\left|\bar{R}_2\right| + \left|\bar{S}_2\right|)\frac{A_2\langle x\rangle^{-\beta}}{k_1\sin\bar{u}} \\
\leqslant\, &C\frac{\langle x\rangle^{2\alpha}}{\langle x\rangle^{2\alpha}\sin^2\bar{u}}A_3^2\langle x\rangle^{-2\beta}\frac{A_2\langle x\rangle^{\alpha-\beta}}{k_1\langle x\rangle^{\alpha}\sin\bar{u}} \\
\leqslant\, &C(A_0, A_2, A_3)\langle x\rangle^{3\alpha-3\beta} \leqslant C(A_0, A_2, A_3)
\end{aligned} \tag{61}$$

and

$$\begin{aligned}
&|H_u\bar{u}_x| \leqslant |H_u|\,|\bar{u}_x| \leqslant \left|\frac{1}{4\sin^2\bar{u}}(\bar{R}_1 + \bar{S}_1)(\bar{R}_2 - \bar{S}_2)\right||\bar{u}_x|\,\frac{A_2\langle x\rangle^{-\beta}}{k_1\sin\bar{u}} \\
\leqslant\, &\frac{\langle x\rangle^{2\alpha}}{4\langle x\rangle^{2\alpha}\sin^2\bar{u}}(\left|\bar{R}_1\right| + \left|\bar{S}_1\right|)(\left|\bar{R}_2\right| + \left|\bar{S}_2\right|)\frac{A_2\langle x\rangle^{\alpha-\beta}}{k_1\langle x\rangle^{\alpha}\sin\bar{u}} \\
\leqslant\, &C(A_0, A_2, A_3)\langle x\rangle^{3\alpha-3\beta} \leqslant C(A_0, A_2, A_3).
\end{aligned} \tag{62}$$

Inserting (59)–(62) in (58), we find that

$$|V_2(t,x)| \leqslant \frac{A_{4,0}}{4} + 2\left(C(A_0, A_3, A_4) + C(A_0, A_2, A_3)\right)T$$

$$
\begin{aligned}
&+2\left(C(A_0,A_3,A_4)+C(A_0,A_2,A_3)\right)T\\
&\leqslant \frac{A_{4,0}}{4}+C(A_0,A_2,A_3,A_4)T\leqslant A_{4,0}=A_4,
\end{aligned}\tag{63}
$$

provided that $C(A_0,A_2,A_3,A_4)T \leqslant \frac{3A_{4,0}}{4}$. Similarly, we have the estimates for $W_1, W_2$:

$$
|W_1(t,x)|,|W_2(t,x)|\leqslant A_4. \tag{64}
$$

**Step 3.** The estimates of $R_{jt}$ and $S_{jt}$. In fact, based on (39), (32), (36), (57), (63), and (64), we obtain that

$$
\begin{aligned}
&|R_{1t}(t,x)|+|S_{1t}(t,x)|\\
&\leqslant |k_1\sin\bar{u}R_{1x}|+|k_1\sin\bar{u}S_{1x}|+2\left|F_1(\bar{u},\bar{R}_2,\bar{S}_2)\right|+2\left|G(\bar{u},\bar{R}_1,\bar{S}_1)\right|\\
&\leqslant k_1\sin\bar{u}\left(|R_{1x}|+|S_{1x}|\right)+2\left|-\frac{1}{4}\sin 2\bar{u}(\bar{R}_2-\bar{S}_2)^2+\frac{1}{8}\sin 2\bar{u}(\bar{R}_2+\bar{S}_2)^2\right|\\
&\qquad+2\left|\frac{\cos\bar{u}}{4\sin\bar{u}}(\bar{R}_1^2-\bar{S}_1^2)\right|\\
&\leqslant 2k_1A_1A_4+C(A_1)\left(\left|\bar{R}_2\right|^2+\left|\bar{S}_2\right|^2\right)+C(A_0,A_3)\langle x\rangle^{\alpha-2\beta}\\
&\leqslant 2k_1A_1A_4+C(A_1,A_3)\langle x\rangle^{-2\beta}+C(A_0,A_3)\langle x\rangle^{\alpha-2\beta}\\
&\leqslant C(A_0,A_1,A_3,A_4)
\end{aligned}\tag{65}
$$

and

$$
\begin{aligned}
&|R_{2t}(t,x)|+|S_{2t}(t,x)|\\
&\leqslant |k_2\sin\bar{u}R_{2x}|+|k_2\sin\bar{u}S_{2x}|+2\left|F_2(\bar{u},\bar{R}_1,\bar{S}_1,\bar{R}_2,\bar{S}_2)\right|+2\left|H(\bar{u},\bar{R}_1,\bar{S}_1,\bar{R}_2,\bar{S}_2)\right|\\
&\leqslant k_2\sin\bar{u}(|R_{2x}|+|S_{2x}|)+2\left|\frac{3k_2\cos\bar{u}}{4k_1\sin\bar{u}}(\bar{R}_1-\bar{S}_1)(\bar{R}_2-\bar{S}_2)-\frac{\cos\bar{u}}{2\sin\bar{u}}(\bar{R}_1+\bar{S}_1)(\bar{R}_2+\bar{S}_2)\right|\\
&\qquad+2\left|\frac{\cos\bar{u}}{4\sin\bar{u}}(\bar{R}_1+\bar{S}_1)(\bar{R}_2-\bar{S}_2)\right|\\
&\leqslant 2k_2A_1A_4+C\frac{\langle x\rangle^{\alpha}}{\langle x\rangle^{\alpha}\sin\bar{u}}(\left|\bar{R}_1\right|+\left|\bar{S}_1\right|)(\left|\bar{R}_2\right|+\left|\bar{S}_2\right|)\\
&\leqslant 2k_2A_1A_4+C(A_0,A_3)\langle x\rangle^{\alpha-2\beta}\leqslant C(A_0,A_1,A_3,A_4).
\end{aligned}\tag{66}
$$

Then one can choose the constant $A_5$ such that

$$
|R_{1t}(t,x)|,|S_{1t}(t,x)|,|R_{2t}(t,x)|,|S_{2t}(t,x)|\leqslant A_5.
$$

According to the above estimates, we have that $(R_1,S_1,R_2,S_2)\in T_{r,1}\times T_{s,1}\times T_{r,2}\times T_{s,2}$.

**Step 4.** The estimates of $\sin u$. From (41), (42), (48), (49), and $\alpha\leqslant\beta$, it follows that for a sufficiently small $T$,

$$
\begin{aligned}
\langle x\rangle^{\alpha}\sin u(t,x)&=\langle x\rangle^{\alpha}\sin\left(u_0+\int_0^t\frac{R_1+S_1}{2}ds\right)\\
&\geqslant 2A_0\cos(A_3T)-\int_0^t\frac{\langle x\rangle^{\alpha}\left(|R_1|+|S_1|\right)}{2}ds
\end{aligned}
$$

$$\geqslant 2A_0\cos(A_3T) - A_3T\langle x\rangle^{\alpha-\beta} \geqslant A_0 \tag{67}$$

and

$$\begin{aligned}\langle x\rangle^{\alpha}\sin u(t,x) &= \langle x\rangle^{\alpha}\sin\left(u_0 + \int_0^t \frac{R_1+S_1}{2}ds\right)\\ &\leqslant \frac{A_{1,0}}{2} + \int_0^t \frac{\langle x\rangle^{\alpha}(|R_1|+|S_1|)}{2}ds \leqslant \frac{A_{1,0}}{2} + A_3T\langle x\rangle^{\alpha-\beta} \leqslant A_{1,0} =: A_1,\end{aligned} \tag{68}$$

provided that $2\cos(A_3T) - 1 \geq \frac{A_3}{A_0}T$ and $A_3T \leqslant \frac{A_{1,0}}{2}$.

**Step 5.** The estimates of $k_1\sin uu_x$ and $k_2\sin uv_x$. First, it follows directly by (41) that

$$\begin{aligned}k_1\sin u\,|u_x| &= k_1\sin u\left|u_0'(x) + \int_0^t \frac{R_{1x}+S_{1x}}{2}ds\right|\\ &\leqslant k_1\sin u\left|u_0'(x)\right| + k_1\sin u\left|\int_0^t \frac{R_{1x}+S_{1x}}{2}ds\right|.\end{aligned} \tag{69}$$

From (42) and $\alpha \leqslant \beta$, we infer that

$$\langle x\rangle^{-\beta} \leqslant \langle x\rangle^{-\alpha} \leqslant \frac{\sin u_0(x)}{2A_0}.$$

Thus, from (41), (48), and (49), we obtain

$$\begin{aligned}\sin u &= \sin u_0\cos\left(\int_0^t \frac{R_1+S_1}{2}ds\right) + \cos u_0\sin\left(\int_0^t \frac{R_1+S_1}{2}ds\right)\\ &\leqslant \sin u_0 + A_3T\langle x\rangle^{-\beta} \leqslant \sin u_0 + A_3T\frac{\sin u_0}{2A_0} \leqslant C(A_0,A_3)\sin u_0.\end{aligned} \tag{70}$$

By using (45) and (70), we can estimate the first term in (69) as follows:

$$\begin{aligned}k_1\sin u\left|u_0'(x)\right| &\leqslant k_1C(A_0,A_3)\sin u_0\left|u_0'(x)\right|\\ &\leqslant C(A_0,A_3)B_1\langle x\rangle^{-\beta} \leqslant C(B_1,A_0,A_3).\end{aligned} \tag{71}$$

Next, we estimate the second term in (69):

$$\begin{aligned}k_1\sin u\left|\int_0^t \frac{R_{1x}+S_{1x}}{2}ds\right| &= k_1\sin u\left|\int_0^t \frac{R_{1t}-S_{1t}-2G}{2k_1\sin\bar{u}}ds\right|\\ \leqslant k_1\sin u\left|\int_0^t \frac{R_{1t}-S_{1t}}{2k_1\sin\bar{u}}ds\right| &+ k_1\sin u\left|\int_0^t \frac{G}{k_1\sin\bar{u}}ds\right|.\end{aligned} \tag{72}$$

Since $\bar{u} \in U$, it follows from (35) that

$$|\bar{u}_t(t,x)| \leq A_3\langle x\rangle^{-\beta}.$$

Integrating this estimate in time gives

$$|\bar{u}(t,x) - u_0(x)| \leq \int_0^t |\bar{u}_t(s,x)|\,ds \leq A_3t\langle x\rangle^{-\beta} \leq A_3T\langle x\rangle^{-\beta}.$$

Using the Lipschitz continuity of the sine function, we obtain

$$|\sin\bar{u}(t,x)-\sin u_0(x)|\le|\bar{u}(t,x)-u_0(x)|\le A_3T\langle x\rangle^{-\beta},$$

which implies the following estimates:

$$\sin u_0-A_3T\langle x\rangle^{-\beta}\leqslant\sin\bar{u}\leqslant\sin u_0+A_3T\langle x\rangle^{-\beta}. \tag{73}$$

Hence, with the help of (32), it holds that

$$\frac{1}{2}\leqslant 1-\frac{A_3T}{2A_0}\leqslant\frac{\sin\bar{u}}{\sin u_0}\leqslant 1+\frac{A_3T}{2A_0}\leqslant 2, \tag{74}$$

for small $T$. By applying (43), (48), (49), (70), and (74), we can obtain the following estimate:

$$\begin{aligned}
&k_1\sin u\left|\int_0^t\frac{R_{1t}-S_{1t}}{2k_1\sin\bar{u}}ds\right|\\
&\leqslant k_1\sin u\left|\int_0^t\frac{1}{2k_1\sin\bar{u}}d(R_1-S_1)\right|\\
&\leqslant C(A_0,A_3)\sin u_0\left|\frac{R_1-S_1}{2\sin\bar{u}}-\frac{R_{1,0}-S_{1,0}}{2\sin\bar{u}_0}+\int_0^t\frac{(R_1-S_1)\cos\bar{u}\,u_t}{2\sin^2\bar{u}}ds\right|\\
&\leqslant C(A_0,A_3)\sin u_0\left(\frac{|R_1|+|S_1|}{2\sin\bar{u}}+\frac{|R_{1,0}|+|S_{1,0}|}{2\sin\bar{u}_0}+\int_0^t\frac{|R_1|^2+|S_1|^2}{2\sin^2\bar{u}}ds\right)\\
&\leqslant C(A_0,A_3)\langle x\rangle^{-\beta}\leqslant C(A_0,A_3)
\end{aligned} \tag{75}$$

and

$$\begin{aligned}
k_1\sin u\left|\int_0^t\frac{G}{k_1\sin\bar{u}}ds\right|&\leqslant k_1\sin u\int_0^t\frac{\left|\bar{R}_1\right|^2+\left|\bar{S}_1\right|^2}{4k_1\sin^2\bar{u}}ds\\
&\leqslant C(A_0,A_3)\sin u_0\int_0^t\frac{\left(|\bar{R}_1|^2+\left|\bar{S}_1\right|^2\right)\langle x\rangle^{2\alpha}}{4\sin^2\bar{u}\langle x\rangle^{2\alpha}}ds\\
&\leqslant C(A_0,A_3)\langle x\rangle^{2\alpha-2\beta}\leqslant C(A_0,A_3).
\end{aligned} \tag{76}$$

Applying (71), (75), and (76) to (69), we obtain

$$\left\|k_1\langle x\rangle^{-\beta}\sin u\,u_x\right\|_{L^\infty}\leqslant C(B_1,A_0,A_3)+C(A_0,A_3)=:A_2. \tag{77}$$

The estimation for $k_2\sin u|v_x|$ follows a similar approach to the one described above.

**Step 6.** The estimates of $u_t$ and $v_t$. We can directly derive from (41), (48), and (49) that

$$\left\|\langle x\rangle^{\beta}u_t\right\|_{L^\infty}\leqslant\left\|\frac{\langle x\rangle^{-\beta}(R_1+S_1)}{2}\right\|_{L^\infty}\leqslant A_3 \tag{78}$$

and

$$\left\|\langle x\rangle^{\beta}v_t\right\|_{L^\infty}\leqslant\left\|\frac{\langle x\rangle^{-\beta}(R_2+S_2)}{2}\right\|_{L^\infty}\leqslant A_3. \tag{79}$$

**Step 7.** The Lipschitz continuity of $(u, v, R_1, S_1, R_2, S_2)$. To complete the proof that $(u, v, R_1, S_1, R_2, S_2) \in \Pi$, we need to show that $(u, v, R_1, S_1, R_2, S_2)$ is Lipschitz continuous. From (57) and (63)–(66), we can obviously check that $R_j$ and $S_j$ $(j = 1, 2)$ satisfy the following uniform Lipschitz estimate:

$$\begin{aligned}
&\left|R_j(t,x) - R_j(s,y)\right| + \left|S_j(t,x) - S_j(s,y)\right| \\
&\leqslant \left|R_j(t,x) - R_j(s,x)\right| + \left|R_j(s,x) - R_j(s,y)\right| + \left|S_j(t,x) - S_j(s,x)\right| + \left|S_j(s,x) - S_j(s,y)\right| \\
&\leqslant \int_s^t \left|R_{jt}(\tau,x)\right| d\tau + \int_y^x \left|R_{jx}(s,z)\right| \mathrm{d}z + \int_s^t \left|S_{jt}(\tau,x)\right| d\tau + \int_y^x \left|S_{jx}(s,z)\right| dz \\
&\leqslant 2A_5|t-s| + 2A_4|x-y| \leqslant 2(A_4 + A_5)(|t-s| + |x-y|).
\end{aligned} \tag{80}$$

Next we check that $u$ is Lipschitz continuous. From (78), (67), and (77), we gain

$$\begin{aligned}
|u(t_1,x_1) - u(t_2,x_2)| &\leqslant |u(t_1,x_1) - u(t_2,x_1)| + |u(t_2,x_1) - u(t_2,x_2)| \\
&\leqslant \int_{t_2}^{t_1} |u_t(\tau,x_1)|\, d\tau + \int_{x_2}^{x_1} |u_x(t_2,z)|\, dz \\
&\leqslant C(A_3)\langle x\rangle^{-\beta}|t_1 - t_2| + C(A_0,A_2)|x_1 - x_2| \\
&\leqslant C(A_0,A_2,A_3)(|t_1 - t_2| + |x_1 - x_2|).
\end{aligned} \tag{81}$$

Similarly, we can also show that $v$ is Lipschitz continuous.

The proof imposes several smallness conditions on $T$ in Steps 1–4:

$$C(A_0,A_3)T \leq \frac{A_{3,0}}{2}, \quad C(A_0,A_1,A_2,A_3,A_4)T \leq \frac{3A_{4,0}}{4}, \quad C(A_0,A_2,A_3,A_4)T \leq \frac{3A_{4,0}}{4},$$

$$2\cos(A_3T) - 1 \geq \frac{A_3}{A_0}T, \qquad A_3T \leq \frac{A_{1,0}}{2}.$$

Choosing

$$T := \min\left\{\frac{A_{3,0}}{2C(A_0,A_3)}, \frac{3A_{4,0}}{4C(A_0,A_1,A_2,A_3,A_4)}, \frac{3A_{4,0}}{4C(A_0,A_2,A_3,A_4)}, T_*, \frac{A_{1,0}}{2A_3}\right\},$$

where $T_* > 0$ satisfies $2\cos(A_3T_*) - 1 \geq \frac{A_3}{A_0}T_*$, ensures that all estimates hold simultaneously.

The proof of the lemma is complete. □

**Lemma 3.2.** *Under the same assumptions on Lemma 3.1, $\Phi$ is a contraction mapping in the topology of the $L^\infty$-norm with small $T > 0$ and satisfies that*

$$\begin{aligned}
&\|u^{(1)} - u^{(2)}\|_{L^\infty} + \|v^{(1)} - v^{(2)}\|_{L^\infty} + \|R_1^{(1)} - R_1^{(2)}\|_{L^\infty} + \|S_1^{(1)} - S_1^{(2)}\|_{L^\infty} \\
&\quad + \|R_2^{(1)} - R_2^{(2)}\|_{L^\infty} + \|S_2^{(1)} - S_2^{(2)}\|_{L^\infty} \\
&\leq \frac{1}{2}\Big(\|\bar{u}^{(1)} - \bar{u}^{(2)}\|_{L^\infty} + \|\bar{v}^{(1)} - \bar{v}^{(2)}\|_{L^\infty} + \|\bar{R}_1^{(1)} - \bar{R}_1^{(2)}\|_{L^\infty} + \|\bar{S}_1^{(1)} - \bar{S}_1^{(2)}\|_{L^\infty} \\
&\quad + \|\bar{R}_2^{(1)} - \bar{R}_2^{(2)}\|_{L^\infty} + \|\bar{S}_2^{(1)} - \bar{S}_2^{(2)}\|_{L^\infty}\Big),
\end{aligned} \tag{82}$$

*where* $\left(u^{(i)}, v^{(i)}, R_1^{(i)}, S_1^{(i)}, R_2^{(i)}, S_2^{(i)}\right) = \Phi\left(\bar{u}^{(i)}, \bar{v}^{(i)}, \bar{R}_1^{(i)}, \bar{S}_1^{(i)}, \bar{R}_2^{(i)}, \bar{S}_2^{(i)}\right)$ *for* $i = 1, 2$.

*Proof.* Set $\tilde{I} = I^{(1)} - I^{(2)}$ $(I = u, v, R_1, S_1, R_2, S_2)$.

**Step 1.** The estimates of $\tilde{R}_1$ and $\tilde{S}_1$. From (39), we obtain

$$\begin{aligned}\tilde{R}_{1t} - k_1 \sin \bar{u}^{(1)} \tilde{R}_{1x} = {} & F_1\left(\bar{u}^{(1)}, \bar{R}_2^{(1)}, \bar{S}_2^{(1)}\right) - F_1\left(\bar{u}^{(2)}, \bar{R}_2^{(2)}, \bar{S}_2^{(2)}\right) \\ & + G\left(\bar{u}^{(1)}, \bar{R}_1^{(1)}, \bar{S}_1^{(1)}\right) - G\left(\bar{u}^{(2)}, \bar{R}_1^{(2)}, \bar{S}_1^{(2)}\right) \\ & + k_1\left(\sin \bar{u}^{(1)} - \sin \bar{u}^{(2)}\right) R_{1x}^{(2)}. \end{aligned} \tag{83}$$

For $(t, x) \in [0, T] \times \mathbb{R}$, let $x_1^-(s) := x_1^-(s; t, x)$ be defined by

$$\begin{cases} \dfrac{d}{ds} x_1^-(s) = -k_1 \sin \bar{u}^{(1)}(s, x_1^-(s)), \\ x_1^-(t) = x. \end{cases}$$

Integrating (83) along $x_1^-(s)$ from 0 to $t$ gives

$$\begin{aligned}\tilde{R}_1(t, x) = {} & \int_0^t \left[F_1\left(\bar{u}^{(1)}, \bar{R}_2^{(1)}, \bar{S}_2^{(1)}\right) - F_1\left(\bar{u}^{(2)}, \bar{R}_2^{(2)}, \bar{S}_2^{(2)}\right)\right](s, x_1^-(s))\, ds \\ & + \int_0^t \left[G\left(\bar{u}^{(1)}, \bar{R}_1^{(1)}, \bar{S}_1^{(1)}\right) - G\left(\bar{u}^{(2)}, \bar{R}_1^{(2)}, \bar{S}_1^{(2)}\right)\right](s, x_1^-(s))\, ds \\ & + \int_0^t k_1\left(\sin \bar{u}^{(1)} - \sin \bar{u}^{(2)}\right) R_{1x}^{(2)}(s, x_1^-(s))\, ds. \end{aligned} \tag{84}$$

We now estimate the first term of (84):

$$\begin{aligned} & \int_0^t \left[F_1\left(\bar{u}^{(1)}, \bar{R}_2^{(1)}, \bar{S}_2^{(1)}\right) - F_1\left(\bar{u}^{(2)}, \bar{R}_2^{(2)}, \bar{S}_2^{(2)}\right)\right] ds \\ \leqslant {} & \int_0^t \left|\left(\sin 2\bar{u}^{(1)} - \sin 2\bar{u}^{(2)}\right)\left(\bar{R}_2^{(1)} - \bar{S}_2^{(1)}\right)^2\right| ds \\ & + \int_0^t \left|\sin 2\bar{u}^{(2)}\left[\left(\bar{R}_2^{(1)} - \bar{S}_2^{(1)}\right)^2 - \left(\bar{R}_2^{(2)} - \bar{S}_2^{(2)}\right)^2\right]\right| ds \\ & + \int_0^t \left|\left(\sin 2\bar{u}^{(1)} - \sin 2\bar{u}^{(2)}\right)\left(\bar{R}_2^{(1)} + \bar{S}_2^{(1)}\right)^2\right| ds \\ & + \int_0^t \left|\sin 2\bar{u}^{(2)}\left[\left(\bar{R}_2^{(1)} + \bar{S}_2^{(1)}\right)^2 - \left(\bar{R}_2^{(2)} + \bar{S}_2^{(2)}\right)^2\right]\right| ds. \end{aligned} \tag{85}$$

Equation (36) suggests that

$$\begin{aligned} & \int_0^t \left|\left(\sin 2\bar{u}^{(1)} - \sin 2\bar{u}^{(2)}\right)\left(\bar{R}_2^{(1)} \pm \bar{S}_2^{(1)}\right)^2\right| ds \\ \leqslant {} & \int_0^t \left|\bar{u}^{(1)} - \bar{u}^{(2)}\right| \left(\left|\bar{R}_2^{(1)}\right| + \left|\bar{S}_2^{(1)}\right|\right)^2 ds \leqslant CT \left\|\bar{u}^{(1)} - \bar{u}^{(2)}\right\|_{L^\infty}. \end{aligned} \tag{86}$$

Additionally, we compute

$$\int_0^t \left|\sin 2\bar{u}^{(2)}\left[\left(\bar{R}_2^{(1)} \pm \bar{S}_2^{(1)}\right)^2 - \left(\bar{R}_2^{(2)} \pm \bar{S}_2^{(2)}\right)^2\right]\right| ds$$

$$
\begin{aligned}
&\leq C\int_0^t \Big[\left(\left|\bar{R}_2^{(1)}\right|+\left|\bar{R}_2^{(2)}\right|\right)\left|\bar{R}_2^{(1)}-\bar{R}_2^{(2)}\right|+\left(\left|\bar{S}_2^{(1)}\right|+\left|\bar{S}_2^{(2)}\right|\right)\left|\bar{S}_2^{(1)}-\bar{S}_2^{(2)}\right| \\
&\quad +2\left|\bar{S}_2^{(1)}\right|\left|\bar{R}_2^{(1)}-\bar{R}_2^{(2)}\right|+2\left|\bar{R}_2^{(2)}\right|\left|\bar{S}_2^{(1)}-\bar{S}_2^{(2)}\right|\Big]\,ds \\
&\leqslant CT\left(\left\|\bar{R}_2^{(1)}-\bar{R}_2^{(2)}\right\|_{L^\infty}+\left\|\bar{S}_2^{(1)}-\bar{S}_2^{(2)}\right\|_{L^\infty}\right).
\end{aligned}
\tag{87}
$$

Inserting (86) and (87) into (85) arrives at

$$
\begin{aligned}
&\left|\int_0^t \left[F_1(\bar{u}^{(1)},\bar{R}_2^{(1)},\bar{S}_2^{(1)})-F_1(\bar{u}^{(2)},\bar{R}_2^{(2)},\bar{S}_2^{(2)})\right]ds\right| \\
&\leqslant CT\left(\left\|\bar{u}^{(1)}-\bar{u}^{(2)}\right\|_{L^\infty}+\left\|\bar{R}_2^{(1)}-\bar{R}_2^{(2)}\right\|_{L^\infty}+\left\|\bar{S}_2^{(1)}-\bar{S}_2^{(2)}\right\|_{L^\infty}\right).
\end{aligned}
\tag{88}
$$

For the second term of (84), we have

$$
\begin{aligned}
&\int_0^t \left[G\left(\bar{u}^{(1)},\bar{R}_1^{(1)},\bar{S}_1^{(1)}\right)-G\left(\bar{u}^{(2)},\bar{R}_1^{(2)},\bar{S}_1^{(2)}\right)\right]ds \\
&\leqslant \left|\int_0^t \left(\frac{\cos\bar{u}^{(1)}}{4\sin\bar{u}^{(1)}}-\frac{\cos\bar{u}^{(2)}}{4\sin\bar{u}^{(2)}}\right)\left[\left(\bar{R}_1^{(1)}\right)^2-\left(\bar{S}_1^{(1)}\right)^2\right]ds\right| \\
&\quad +\left|\int_0^t \frac{\cos\bar{u}^{(2)}}{4\sin\bar{u}^{(2)}}\left(\left[\left(\bar{R}_1^{(1)}\right)^2-\left(\bar{S}_1^{(1)}\right)^2\right]-\left[\left(\bar{R}_1^{(2)}\right)^2-\left(\bar{S}_1^{(2)}\right)^2\right]\right)ds\right|.
\end{aligned}
\tag{89}
$$

One finds by (36) that

$$
\begin{aligned}
&\left|\int_0^t \left(\frac{\cos\bar{u}^{(1)}}{4\sin\bar{u}^{(1)}}-\frac{\cos\bar{u}^{(2)}}{4\sin\bar{u}^{(2)}}\right)\left[\left(\bar{R}_1^{(1)}\right)^2-\left(\bar{S}_1^{(1)}\right)^2\right]ds\right| \\
&\leqslant \int_0^t \frac{\left|\sin\left(\bar{u}^{(2)}-\bar{u}^{(1)}\right)\right|}{\sin\bar{u}^{(1)}\sin\bar{u}^{(2)}}\left(\left|\bar{R}_1^{(1)}\right|^2+\left|\bar{S}_1^{(1)}\right|^2\right)ds \\
&\leqslant \int_0^t C\langle x\rangle^{2\alpha-2\beta}\left|\bar{u}^{(1)}-\bar{u}^{(2)}\right|ds \leqslant CT\left\|\bar{u}^{(1)}-\bar{u}^{(2)}\right\|_{L^\infty}
\end{aligned}
\tag{90}
$$

and

$$
\begin{aligned}
&\left|\int_0^t \frac{\cos\bar{u}^{(2)}}{4\sin\bar{u}^{(2)}}\left\{\left[\left(\bar{R}_1^{(1)}\right)^2-\left(\bar{S}_1^{(1)}\right)^2\right]-\left[\left(\bar{R}_1^{(2)}\right)^2-\left(\bar{S}_1^{(2)}\right)^2\right]\right\}ds\right| \\
&\leqslant \int_0^t C\langle x\rangle^{\alpha}\left[\left(\left|\bar{R}_1^{(1)}\right|+\left|\bar{R}_1^{(2)}\right|\right)\left|\bar{R}_1^{(1)}-\bar{R}_1^{(2)}\right|+\left(\left|\bar{S}_1^{(1)}\right|+\left|\bar{S}_1^{(2)}\right|\right)\left|\bar{S}_1^{(1)}-\bar{S}_1^{(2)}\right|\right]ds \\
&\leqslant CT\left(\left\|\bar{R}_1^{(1)}-\bar{R}_1^{(2)}\right\|_{L^\infty}+\left\|\bar{S}_1^{(1)}-\bar{S}_1^{(2)}\right\|_{L^\infty}\right).
\end{aligned}
\tag{91}
$$

Substituting (90) and (91) into (89) leads to

$$
\begin{aligned}
&\left|\int_0^t \left[G(\bar{u}^{(1)},\bar{R}_1^{(1)},\bar{S}_1^{(1)})-G(\bar{u}^{(2)},\bar{R}_1^{(2)},\bar{S}_1^{(2)})\right]ds\right| \\
&\leqslant CT\left(\left\|\bar{u}^{(1)}-\bar{u}^{(2)}\right\|_{L^\infty}+\left\|\bar{R}_1^{(1)}-\bar{R}_1^{(2)}\right\|_{L^\infty}+\left\|\bar{S}_1^{(1)}-\bar{S}_1^{(2)}\right\|_{L^\infty}\right).
\end{aligned}
\tag{92}
$$

Finally, by (57), one can estimate the third term of (84):

$$\left|\int_0^t k_1\left(\sin\bar{u}^{(1)}-\sin\bar{u}^{(2)}\right)R_{1x}^{(2)}ds\right|\leqslant\int_0^t k_1\left|\sin\bar{u}^{(1)}-\sin\bar{u}^{(2)}\right|\left|R_{1x}^{(2)}\right|ds$$
$$\leqslant\int_0^t C\left|\bar{u}^{(1)}-\bar{u}^{(2)}\right|ds\leqslant CT\left\|\bar{u}^{(1)}-\bar{u}^{(2)}\right\|_{L^\infty}. \tag{93}$$

We put (88), (92), and (93) into (84) to deduce

$$\begin{aligned}\left\|\tilde{R}_1(t,x)\right\|_{L^\infty}\leqslant\frac{1}{12}\Big(&\left\|\bar{u}^{(1)}-\bar{u}^{(2)}\right\|_{L^\infty}+\left\|\bar{v}^{(1)}-\bar{v}^{(2)}\right\|_{L^\infty}+\left\|\bar{R}_1^{(1)}-\bar{R}_1^{(2)}\right\|_{L^\infty}\\&+\left\|\bar{S}_1^{(1)}-\bar{S}_1^{(2)}\right\|_{L^\infty}+\left\|\bar{R}_2^{(1)}-\bar{R}_2^{(2)}\right\|_{L^\infty}+\left\|\bar{S}_2^{(1)}-\bar{S}_2^{(2)}\right\|_{L^\infty}\Big),\end{aligned} \tag{94}$$

for sufficiently small $T$.

The estimate in (94) is also valid for the terms $\widetilde{I}(t,x)$ ($I=u,v,S_1,R_2,S_2$). Summing these estimates yields (82). Therefore, we find that $\Phi$ is a contraction mapping for sufficiently small $T>0$. The proof of the lemma is finished. □

*3.3. The existence and uniqueness of solutions*

In this subsection, we complete the proof of Theorem 1. We show the existence and uniqueness of classical solutions to the degenerate Cauchy problem, and verify that the solution has no loss of regularity.

**Lemma 3.3.** *Let the assumptions in Lemma 3.1 hold. For sufficiently small $T>0$, there uniquely exist $(u,v,R_1,S_1,R_2,S_2)\in U\times V\times T_{r,1}\times T_{s,1}\times T_{r,2}\times T_{s,2}$ and $x_j^\pm(s)=x_j^\pm(s;t,x)$ satisfying the following equations:*

$$\begin{cases}R_1(t,x)=R_1\left(0,x_1^-(0)\right)+\displaystyle\int_0^t\left[F_1\left(u,R_2,S_2\right)+G\left(u,R_1,S_1\right)\right]\left(s,x_1^-(s)\right)ds,\\ S_1(t,x)=S_1\left(0,x_1^+(0)\right)+\displaystyle\int_0^t\left[F_1\left(u,R_2,S_2\right)-G\left(u,R_1,S_1\right)\right]\left(s,x_1^+(s)\right)ds,\\ R_2(t,x)=R_2\left(0,x_2^-(0)\right)+\displaystyle\int_0^t\left[F_2\left(u,R_1,S_1,R_2,S_2\right)+H\left(u,R_1,S_1,R_2,S_2\right)\right]\left(s,x_2^-(s)\right)ds,\\ S_2(t,x)=S_2\left(0,x_2^+(0)\right)+\displaystyle\int_0^t\left[F_2\left(u,R_1,S_1,R_2,S_2\right)-H\left(u,R_1,S_1,R_2,S_2\right)\right]\left(s,x_2^+(s)\right)ds\end{cases} \tag{95}$$

*and*

$$u=u_0(x)+\int_0^t\frac{R_1+S_1}{2}(s,x)ds,\quad v=v_0(x)+\int_0^t\frac{R_2+S_2}{2}(s,x)ds. \tag{96}$$

*Moreover, the characteristics $x_j^\pm(s;t,x)$ satisfy*

$$x_j^\pm(s;t,x)=x\pm k_j\int_t^s\sin u\left(\tau,x_j^\pm(\tau;t,x)\right)d\tau,\quad j=1,2. \tag{97}$$

*Proof.* We fix $K \geqslant 1$ arbitrarily. From Lemma 3.2, we can define a sequence

$$\{u_n, v_n, R_{1,n}, S_{1,n}, R_{2,n}, S_{2,n}\}_{n\in\mathbb{N}} \in U \times V \times T_{r,1} \times T_{s,1} \times T_{r,2} \times T_{s,2},$$

such that

$$(u_{n+1}, v_{n+1}, R_{1,n+1}, S_{1,n+1}, R_{2,n+1}, S_{2,n+1}) = \Phi\left(u_n, v_n, R_{1,n}, S_{1,n}, R_{2,n}, S_{2,n}\right),$$

with the initial term $(u_0, v_0, R_{1,0}, S_{1,0}, R_{2,0}, S_{2,0})$. According to Lemma 3.1, we know that $(u_n, v_n, R_{1,n}, S_{1,n}, R_{2,n}, S_{2,n})$ converges to a vector function $(u, v, R_1, S_1, R_2, S_2)$ in the topology of $L^\infty$.

For each $n \in \mathbb{N}$, define the characteristic curves $x^\pm_{j,n}(s; t, x)$ associated with $u_n$ by

$$x^\pm_{j,n}(s; t, x) = x \pm k_j \int_t^s \sin u_n(\tau, x^\pm_{j,n}(\tau; t, x))\, d\tau, \quad j = 1, 2. \tag{98}$$

These curves are uniquely defined on $[0, T]$ for every $(t, x) \in [0, T] \times \mathbb{R}$. We now prove the convergence of the characteristic curves. Fix $K \geq 1$. Consider the sequence of functions

$$(s, t, x) \longmapsto x^\pm_{j,n}(s; t, x), \qquad (s, t, x) \in [0, T] \times [0, T] \times [-K, K].$$

By (20), this sequence is uniformly bounded. Moreover, it is equicontinuous in each of the three variables: equicontinuity in $s$ follows from the uniform bound $|\partial_s x^\pm_{j,n}| \leq k_j A_1$, equicontinuity in $x$ follows from Lemma 2.2, and equicontinuity in $t$ follows from Lemma 2.1. Therefore, by the Arzelà–Ascoli theorem, there exists a subsequence, depending on $K$, that converges uniformly on $[0, T] \times [0, T] \times [-K, K]$ to some limit $x^\pm_j(s; t, x)$. To eliminate the dependence on $K$, we apply Cantor's diagonal argument. For each $m \in \mathbb{N}$, choose a subsequence that converges uniformly on $[0, T] \times [0, T] \times [-m, m]$; taking the diagonal subsequence yields a single sequence, still denoted by $x^\pm_{j,n}$, that converges locally uniformly on $[0, T] \times [0, T] \times \mathbb{R}$ to $x^\pm_j(s; t, x)$. Passing to the limit $n \to \infty$ in (98), using the local uniform convergence of $x^\pm_{j,n}$ and the convergence $u_n \to u$ in $L^\infty$, we obtain that $x^\pm_j$ satisfies

$$x^\pm_j(s; t, x) = x \pm k_j \int_t^s \sin u(\tau, x^\pm_j(\tau; t, x))\, d\tau, \quad j = 1, 2.$$

Furthermore, by (80)–(81), as $n \to \infty$,

$$(u_n, v_n, R_{1,n}, S_{1,n}, R_{2,n}, S_{2,n})(t, x^\pm_{j,n}(t)) \longrightarrow (u, v, R_1, S_1, R_2, S_2)(t, x^\pm_j(t)).$$

Consequently, passing to the limit in the integral representation for the $n$-th approximation yields (95) and the characteristic curves satisfy (97).

Now, we confirm that $(u, v, R_1, S_1, R_2, S_2) \in U \times V \times T_{r,1} \times T_{s,1} \times T_{r,2} \times T_{s,2}$. Since the estimates in Lemmas 3.1 and 3.2 are uniform in $n$, they pass to the limit. From the Lipschitz continuity, $u, v, R_1, S_1, R_2, S_2$ are differentiable almost everywhere. Performing the same way as in the proof of Lemma 3.1, we can achieve

$$k_1 \sin u(t, x)\, |u_x(t, x)|\,,\ k_2 \sin u(t, x)\, |v_x(t, x)| \leqslant A_2 \langle x \rangle^{-\beta}, \tag{99}$$

$$|u_t(t, x)|\,,\ |v_t(t, x)| \leqslant A_3 \langle x \rangle^{-\beta}. \tag{100}$$

Thus we have $u \in U$ and $v \in V$.

To demonstrate the boundedness of $R_{jx}$ and $S_{jx}$, one differentiates (95) with respect to $x$ to acquire

$$\begin{aligned} V_1(t,x) &= V_{1,0}(x_1^-(0;t,x))\partial_x x_1^-(0;t,x) \\ &\quad + \int_0^t \partial_x x_1^-(s;t,x)\left(F_{1u}u_x + F_{1R_2}V_2 + F_{1S_2}W_2\right)(s, x_1^-(s;t,x))ds \\ &\quad + \int_0^t \partial_x x_1^-(s;t,x)\left(G_u u_x + G_{R_1}V_1 + G_{S_1}W_1\right)(s, x_1^-(s;t,x))ds, \end{aligned}$$

where $(V_1, W_1, V_2, W_2) = (R_{1x}, S_{1x}, R_{2x}, S_{2x})$. Using the same argument as in the proof of (52) and (58), we can obtain the boundedness of $V_j$ and $W_j$. The estimates of $R_{jt}$ and $S_{jt}$ can be shown by similarity as in (65) and (66). Thus it concludes that $(u, v, R_1, S_1, R_2, S_2) \in U \times V \times T_{r,1} \times T_{s,1} \times T_{r,2} \times T_{s,2}$. The uniqueness can be shown in the same way as in the proof of Lemma 3.2. The proof of Lemma 3.3 is complete. □

In the discussions so far, we do not assume any relations between $u_0$, $v_0$ , $R_{j,0}$, and $S_{j,0}$. To show that $(u, v)$ in (96) is a solution of (6), we suppose that

$$u_0' = \frac{R_{1,0} - S_{1,0}}{2k_1 \sin u_0}, \quad u_1 = \frac{R_{1,0} + S_{1,0}}{2}, \quad v_0' = \frac{R_{2,0} - S_{2,0}}{2k_2 \sin u_0}, \quad v_1 = \frac{R_{2,0} + S_{2,0}}{2}. \tag{101}$$

Moreover, we shall improve the regularity of the solution if $(u_0, u_1), (v_0, v_1) \in C^2 \times C^1$. The following lemma completes the proof of Theorem 1.

**Lemma 3.4.** *Let the assumptions in Lemma 3.3 hold. We further assume that* $(u_0, u_1)$ *and* $(v_0, v_1)$ *are in* $C^2 \times C_b^1$ *and satisfy* (101). *Then the functions u and v defined in* (41) *are* $C^2$ *on* $[0, T] \times \mathbb{R}$ *and form a classical solution to the degenerate Cauchy problem* (6)–(7).

*Proof.* From the Lipschitz continuity of $R_1, S_1, R_2, S_2$, these are differentiable almost everywhere and satisfy that

$$\begin{cases} R_{1t} - k_1 \sin u R_{1x} = F_1(u, R_2, S_2) + G(u, R_1, S_1), \\ S_{1t} + k_1 \sin u S_{1x} = F_1(u, R_2, S_2) - G(u, R_1, S_1), \\ R_{2t} - k_2 \sin u R_{2x} = F_2(u, R_1, S_1, R_2, S_2) + H(u, R_1, S_1, R_2, S_2), \\ S_{2t} + k_2 \sin u S_{2x} = F_2(u, R_1, S_1, R_2, S_2) - H(u, R_1, S_1, R_2, S_2). \end{cases} \tag{102}$$

Since $u$ is also differentiable almost everywhere, we differentiate (41) to obtain

$$u_x = u_0'(x) + \int_0^t \frac{R_{1x} + S_{1x}}{2} ds, \quad u_t = \frac{R_1 + S_1}{2}. \tag{103}$$

From the first and second equations of (102), one has

$$R_{1x} + S_{1x} = \frac{1}{k_1 \sin u}\left(R_{1t} - S_{1t} - 2G(u, R_1, S_1)\right). \tag{104}$$

Thus, by (101), (103), and (14), we can compute that

$$u_x = u_0'(x) + \int_0^t \frac{R_{1x} + S_{1x}}{2} ds$$

$$
\begin{aligned}
&= u_0'(x) + \frac{1}{2}\int_0^t \frac{R_{1t} - S_{1t} - 2G(u, R_1, S_1)}{k_1 \sin u} ds \\
&= u_0'(x) + \frac{1}{2}\int_0^t \frac{1}{k_1 \sin u} d(R_1 - S_1) - \frac{1}{2}\int_0^t \frac{\cos u (R_1^2 - S_1^2)}{2k_1 \sin^2 u} ds \\
&= u_0'(x) + \frac{1}{2}\left[\frac{R_1 - S_1}{k_1 \sin u} - \frac{R_{1,0} - S_{1,0}}{k_1 \sin u_0} + \int_0^t \frac{(R_1 - S_1)\cos u \cdot u_t}{k_1 \sin^2 u} ds\right] \\
&\quad - \frac{1}{2}\int_0^t \frac{\cos u (R_1^2 - S_1^2)}{2k_1 \sin^2 u} ds \\
&= u_0'(x) + \frac{1}{2}\left[\frac{R_1 - S_1}{k_1 \sin u} - 2u_0'(x) + \int_0^t \frac{\cos u (R_1^2 - S_1^2)}{2k_1 \sin^2 u} ds\right] \\
&\quad - \frac{1}{2}\int_0^t \frac{\cos u (R_1^2 - S_1^2)}{2k_1 \sin^2 u} ds \\
&= \frac{R_1 - S_1}{2k_1 \sin u}. \qquad (105)
\end{aligned}
$$

Similarly, it follows that

$$
v_x = \frac{R_2 - S_2}{2k_2 \sin u}. \qquad (106)
$$

Combining (102), (96), (105), and (106), we acquire that the functions $(u, v)$ satisfy (6). Finally, applying Theorem 4 in Douglis [43], one can obtain the continuity of $W_j = R_{jx}$ and $V_j = S_{jx}$ for $j = 1, 2$. Thanks to the equations of $R_j$ and $S_j$ $(j = 1, 2)$, we conclude that $R_{jt}$ and $S_{jt}$ are also continuous, which lead to the continuity of $u_{xx}, u_{tx}, u_{tt}$ and $v_{xx}, v_{tx}, v_{tt}$. Therefore, we have that $u, v \in C^2([0, T] \times \mathbb{R})$. □

Now, we have completed the proof of Theorem 1.

## Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare that they have no conflict of interest.

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