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Theory article

Linear and nonlinear selection principles for the minimal traveling wave speed in a three-species competitive system with nonlocal interactions

Jia Guo*

School of Mathematics and Statistics, Hunan First Normal University, Changsha 410205, China

* **Correspondence:** Email: guojia0113@126.com.

Abstract: This work investigates the selection principle of the critical speed in a three-species Lotka–Volterra competitive system with nonlocal diffusion. The system incorporates symmetric convolution kernels for interspecific couplings and, after a suitable change of variables, is recast into a cooperative framework. By applying the method of super- and subsolutions, we construct a novel pair of upper and lower solutions for the traveling-wave ordinary differential equations. This construction furnishes new analytic criteria that distinguish between linear and nonlinear selection of the minimal propagation speed. In particular, our results show that under suitable parameter regimes, the minimal wave speed coincides the linear prediction from the system's unstable equilibrium (linear determinacy), whereas sufficiently strong competition among the three species causes the minimal wave to travel faster than the linear theory predicts (nonlinear determinacy). These findings advance the understanding of wave speed selection in reaction–diffusion systems with nonlocal interactions by delineating the boundary between linear (pulled) and nonlinear (pushed) invasion fronts.

Keywords: traveling waves; three-species Lotka-Volterra; speed selection; nonlocal interactions; minimal wave speed

1. Introduction

In this paper, we study the minimal wave speed of the three-species Lotka-Volterra competition model with nonlocal dispersal

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_1(J_1 * u - u) + r_1 u(1 - u - a_{12}v - a_{13}w), \\ \frac{\partial v(x,t)}{\partial t} = d_2(J_2 * v - v) + r_2 v(1 - a_{21}u - v), \\ \frac{\partial w(x,t)}{\partial t} = d_3(J_3 * w - w) + r_3 w(1 - a_{31}u - w). \end{cases} \tag{1.1}$$

Here, $u(x,t)$, $v(x,t)$, and $w(x,t)$ are the population densities of three competing individuals at spatial point x and temporal instant t; d_1, d_2, and d_3 are the diffusivity constants; r_1, r_2, and r_3 are the intrinsic growth rates; and a_{ij} are the competitive interaction coefficients of species j to i. Further, the saturation levels of the three populations are normalized to be 1, and all parameters are assumed to be positive.

The convolution kernel operator $J_i * \zeta - \zeta$ $(i = 1, 2, 3)$ is known as nonlocal dispersal, which models the transportation of substances or organisms via long-range dispersion mechanisms. It is assumed in the present paper that J_i $(i = 1, 2, 3)$ have compact support, and

$$(A_1) : J_i(y) \in C^1(R), \quad J_i(y) = J_i(-y) \geq 0, \quad \forall y \in R, \quad \int_{-\infty}^{+\infty} J_i(y)dy = 1, \ i = 1, 2, 3.$$

In ecology and biology, the movement and interaction of many species can occur between non-adjacent spatial locations (see, e.g., [1, 2]), which is referred to as nonlocal diffusion in biological mathematical models. Nonlocal dispersal models are frequently encountered in applied disciplines including population biology (see, e.g., [3–5]), phase transition theory [6], and in studies of epidemiology (see, e.g., [7–10]) and signal propagation within network models [11].

Letting $\phi = u$, $\psi = 1 - v$, and $\theta = 1 - w$, we can transform the competitive system (1.1) into the following cooperative system:

$$\begin{cases} \frac{\partial \phi}{\partial t} = d_1(J_1 * \phi - \phi) + r_1\phi(1 - a_{12} - a_{13} - \phi + a_{12}\psi + a_{13}\theta), \\ \frac{\partial \psi}{\partial t} = d_2(J_2 * \psi - \psi) + r_2(1 - \psi)(a_{21}\phi - \psi), \\ \frac{\partial \theta}{\partial t} = d_3(J_3 * \theta - \theta) + r_3(1 - \theta)(a_{31}\phi - \theta), \end{cases} \tag{1.2}$$

subject to the initial conditions

$$\phi(x, 0) = \phi_0(x), \quad \psi(x, 0) = \psi_0(x), \quad \theta(x, 0) = \theta_0(x), \quad \forall x \in R.$$

Throughout this study, we make the following assumptions:

$$(A_2) : 0 < a_{12}, a_{13} < 1, \ a_{21}, a_{31} > 1, \ a_{12} + a_{13} < 1.$$

When (A_2) is satisfied, the system (1.2) has five equilibrium solutions

$$e_1 = (0, 0, 0), \ e_2 = (1, 1, 1), \ e_3 = (0, 0, 1), \ e_4 = (0, 1, 0), e_5 = (0, 1, 1)$$

in the region $\{(\phi, \psi, \theta) | 0 \leq \phi, \psi, \theta \leq 1\}$. We can also obtain that $e_1 = (0, 0, 0)$ is unstable, and $e_2 = (1, 1, 1)$ is stable, which establishes the competitive dominance of species $u(x, t)$ over species $v(x, t)$ and $w(x, t)$, and species $u(x, t)$ is therefore projected to ultimately prevail in the competition.

In biology, traveling wave solutions are essential for modeling species evolution and competition mechanisms. Therefore, the issue of wave speed has been extensively investigated, for example, in [12–16]. Define a new variable $z = x + ct$ for $c \geq 0$ and define $(\phi, \psi, \theta)(x, t) = (\Phi, \Psi, \Theta)(z)$. Then we focus on the traveling wave solution to System (1.2) connecting e_1 and e_2. Thus, it follows that $\Phi(z)$, $\Psi(z)$, and $\Theta(z)$ should satisfy

$$\begin{cases} c\Phi' = d_1[\int_{-\infty}^{+\infty} J_1(y)\Phi(z - y)dy - \Phi] + r_1\Phi(1 - a_{12} - a_{13} - \Phi + a_{12}\Psi + a_{13}\Theta), \\ c\Psi' = d_2[\int_{-\infty}^{+\infty} J_2(y)\Psi(z - y)dy - \Psi] + r_2(1 - \Psi)(a_{21}\Phi - \Psi), \\ c\Theta' = d_3[\int_{-\infty}^{+\infty} J_3(y)\Theta(z - y)dy - \Theta] + r_3(1 - \Theta)(a_{31}\Phi - \Theta), \\ 0 \leq \Phi, \Psi, \Theta \leq 1, \end{cases} \tag{1.3}$$

subject to the boundary conditions

$$(\Phi, \Psi, \Theta)(-\infty) = e_1 = (0,0,0), \quad (\Phi, \Psi, \Theta)(+\infty) = e_2 = (1,1,1). \tag{1.4}$$

Let

$$\begin{cases} p_1(\Phi, \Psi, \Theta) = r_1\Phi(1 - a_{12} - a_{13} - \Phi + a_{12}\Psi + a_{13}\Theta), \\ p_2(\Phi, \Psi, \Theta) = r_2(1 - \Psi)(a_{21}\Phi - \Psi), \\ p_3(\Phi, \Psi, \Theta) = r_3(1 - \Theta)(a_{31}\Phi - \Theta). \end{cases} \tag{1.5}$$

It is easy to verify that

$$\frac{\partial p_1}{\partial \Psi}, \frac{\partial p_1}{\partial \Theta}, \frac{\partial p_2}{\partial \Phi}, \frac{\partial p_3}{\partial \Phi} \geq 0,$$

when $0 \leq \Phi, \Psi, \Theta \leq 1$. That is, $\Phi(z), \Psi(z)$ and $\Theta(z)$ are monotone in $z \in R$ and then, $(\Phi, \Psi, \Theta)(z)$ is called a traveling wavefront, and c is the wave speed.

In [17], the existence of traveling wavefronts of the system (1.3) was established, with the wave speed characterized by the minimal speed c_{min}, and

$$c_{min} \geq c_0 = \inf_{\xi > 0} \frac{d_1[\int_{-\infty}^{+\infty} J_1(y)e^{-\xi y}dy - 1] + r_1(1 - a_{12} - a_{13})}{\xi}. \tag{1.6}$$

Obviously, under the condition of (A_1), the integral $\int_{-\infty}^{+\infty} J_1(y)e^{-\xi y}dy$ is convergent.

For simplicity, we introduce the following characterization of the linear or nonlinear selection of the critical wave speed.

Definition 1.1. *If $c_{min} = c_0$, then we say that the minimal wave speed is linearly selected; otherwise, if $c_{min} > c_0$, we say that the minimal wave speed is nonlinearly selected.*

The minimal speed c_{min} is the spreading speed of species invasion onto an unstable state, which carries important implications for ecological modeling and the prediction of invasion speeds in competitive populations. Analytically determining the minimal wave speed is generally nontrivial, except in the special case where it equals c_0. Zhang et al. posed the conjecture in [18] that $c_{min} = c_0$ under certain conditions on the coefficients and kernel functions. Moreover, He and Zhang gave some general conditions to ensure the linear determinacy of the minimal speed to System (1.3) (see [19]). In this paper, we use the upper-lower solution technique to determine the minimal wave speed of System (1.3) for linear-nonlinear determinacy.

The traveling wave dynamics of multispecies competition systems with nonlocal dispersal has attracted considerable attention in recent years. For three-species Lotka-Volterra competition systems, the existence of invasion traveling waves was established by Wang et al. (see [20]). The bistable wave speed and global stability of traveling wavefronts were further investigated by Zhang and Hao in [21] and Hao in [22], respectively. The linear determinacy of the minimal speed was studied by Wang ([23]), and Yang ([24]) examined the linear versus nonlinear speed selection mechanism in the presence of nonlinear cubic competition. Beyond the competition diffusion framework, the speed selection mechanism has also been investigated in other interacting population models. In particular, Bondesan ([25]) recently studied this issue in the context of Lotka–Volterra-type kinetic equations for predator–prey interactions, providing a complementary perspective on how the minimal wave speed is determined in different modeling settings.

Although the upper-lower solution method is well established in the literature, its application to the three-species nonlocal competition system is nontrivial and has not been systematically addressed. The main novelty of this paper lies in (i) the construction of upper and lower solutions for a three-component nonlocal system, which requires careful handling of the coupled nonlocal terms; (ii) the derivation of explicit and checkable conditions for both linear and nonlinear selection; and (iii) the confirmation that nonlinear selection can occur in nonlocal systems, a phenomenon not previously established for this model.

The structure of this paper is as follows. We give some preliminaries in Section 2. In Section 3, we focus on the linear selection of the minimal wave speed. Then, we obtain sufficient conditions for nonlinear selection of the minimal wave speed in Section 4. In the end, we give discussion in Section 5 and concluding remarks in Section 6.

2. Preliminaries

First, we give the definition of upper and lower solutions of System (1.3):

Definition 2.1. *A triple of continuous function* $(\Phi_+, \Psi_+, \Theta_+)(z)$ *is* C^1 *and called an upper solution to* (1.3) *if* $(\Phi_+, \Psi_+, \Theta_+)(z)$ *satisfies*

$$\begin{cases} c\Phi_+' \geq d_1[\int_{-\infty}^{+\infty} J_1(y)\Phi_+(z-y)dy - \Phi_+] + p_1(\Phi_+, \Psi_+, \Theta_+), \\ c\Psi_+' \geq d_2[\int_{-\infty}^{+\infty} J_2(y)\Psi_+(z-y)dy - \Psi_+] + p_2(\Phi_+, \Psi_+, \Theta_+), \\ c\Theta_+' \geq d_3[\int_{-\infty}^{+\infty} J_3(y)\Theta_+(z-y)dy - \Theta_+] + p_3(\Phi_+, \Psi_+, \Theta_+). \end{cases} \tag{2.1}$$

Similarly, a triple of continuous function $(\Phi_-, \Psi_-, \Theta_-)(z)$ *is* C^1 *and called a lower solution to* (1.3) *if* $(\Phi_-, \Psi_-, \Theta_-)(z)$ *satisfies*

$$\begin{cases} c\Phi_-' \leq d_1[\int_{-\infty}^{+\infty} J_1(y)\Phi_-(z-y)dy - \Phi_-] + p_1(\Phi_-, \Psi_-, \Theta_-), \\ c\Psi_-' \leq d_2[\int_{-\infty}^{+\infty} J_2(y)\Psi_-(z-y)dy - \Psi_-] + p_2(\Phi_-, \Psi_-, \Theta_-), \\ c\Theta_-' \leq d_3[\int_{-\infty}^{+\infty} J_3(y)\Theta_-(z-y)dy - \Theta_-] + p_3(\Phi_-, \Psi_-, \Theta_-). \end{cases} \tag{2.2}$$

Specifically, we require that the upper solution $(\Phi_+, \Psi_+, \Theta_+)(z)$ and lower solution $(\Phi_-, \Psi_-, \Theta_-)(z)$ satisfy

$$\lim_{z\to-\infty}(\Phi_-, \Psi_-, \Theta_-) = (0,0,0), \quad \lim_{z\to+\infty}(\Phi_-, \Psi_-, \Theta_-) < (1,1,1),$$
$$\lim_{z\to-\infty}(\Phi_+, \Psi_+, \Theta_+) < (1,1,1), \quad \lim_{z\to+\infty}(\Phi_+, \Psi_+, \Theta_+) = (1,1,1).$$

The linearization of (1.3) at $e_1 = (0,0,0)$ is given by

$$\begin{cases} c\Phi' = d_1[\int_{-\infty}^{+\infty} J_1(y)\Phi(z-y)dy - \Phi] + r_1\Phi(1 - a_{12} - a_{13}), \\ c\Psi' = d_2[\int_{-\infty}^{+\infty} J_2(y)\Psi(z-y)dy - \Psi] + r_2a_{21}\Phi - r_2\Psi, \\ c\Theta' = d_3[\int_{-\infty}^{+\infty} J_3(y)\Theta(z-y)dy - \Theta] + r_3a_{31}\Phi - r_3\Theta. \end{cases} \tag{2.3}$$

Let $(\Phi, \Psi, \Theta)(z) = (\eta_1 e^{\xi z}, \eta_2 e^{\xi z}, \eta_3 e^{\xi z})$, where η_1, η_2, η_3, and ξ are positive constants to be determined. By substituting it into (2.3), we have

$$A(\xi,c)\begin{pmatrix}\eta_1\\\eta_2\\\eta_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}, \tag{2.4}$$

where

$$A(\xi,c)=\begin{pmatrix}\Lambda_1(\xi,c) & 0 & 0\\ r_2a_{21} & \Lambda_2(\xi,c) & 0\\ r_3a_{31} & 0 & \Lambda_3(\xi,c)\end{pmatrix},$$

and

$$\Lambda_1(\xi,c)=d_1(\int_{-\infty}^{+\infty}J_1(x)e^{-\xi x}dx-1)-c\xi+r_1(1-a_{12}-a_{13}),$$

$$\Lambda_2(\xi,c)=d_2(\int_{-\infty}^{+\infty}J_2(x)e^{-\xi x}dx-1)-c\xi-r_2,$$

$$\Lambda_3(\xi,c)=d_3(\int_{-\infty}^{+\infty}J_3(x)e^{-\xi x}dx-1)-c\xi-r_3.$$

The system of algebraic Eq (2.4) has a nontrivial solution if and only if

$$\Lambda_1(\xi,c)\Lambda_2(\xi,c)\Lambda_3(\xi,c)=0. \tag{2.5}$$

From (A_2), we have that the characteristic equation $\Lambda_1(\xi,c)=0$ has a positive solution ξ_0 if only if $c\geq c_0$, where c_0 satisfy $\Lambda_1(\xi_0,c_0)=0$, that is,

$$c_0\xi_0=d_1(\int_{-\infty}^{+\infty}J_1(x)e^{-\xi_0 x}dx-1)+r_1(1-a_{12}-a_{13}) \tag{2.6}$$

and

$$c_0=-d_1\int_{-\infty}^{+\infty}J_1(x)xe^{-\xi_0 x}dx, \tag{2.7}$$

which is equivalent to c_0 given by (1.6).

When $c>c_0$, the characteristic equation $\Lambda_1(\xi,c)=0$ admits two positive roots $\xi_1(c)<\xi_0<\xi_2(c)$, that is, $\Lambda_1(\xi_1(c),c)=\Lambda_1(\xi_2(c),c)=0$. Further, $\Lambda_1(\xi(c),c)<0$ for all $\xi_1(c)<\xi(c)<\xi_2(c)$, and $\lim_{c\to c_0}\xi_1(c)=\lim_{c\to c_0}\xi_2(c)=\xi_0$.

Following the approach in [19,21], we construct the upper and lower solutions using the roots of $\Lambda_1(\xi,c)=0$ in the following construction, as this characteristic equation governs the slowest decay mode near the invasion front; the other modes decay faster and do not affect the leading-order asymptotic behavior of the upper/lower solutions.

3. Linear selection of the minimal wave speed

Theorem 3.1. *Assume that there exists a constant $\alpha\in(0,1)$ such that*

$$(H_1)\quad \max\Big\{a_{21}+\frac{d_2}{2\alpha r_2},a_{31}+\frac{d_3}{2\alpha r_3},\frac{\Delta_1(\xi_0,\alpha)+\frac{(a_{12}+a_{13})r_1}{d_1}}{\Delta_1(\xi_0,\alpha)+\frac{(a_{12}+a_{13})r_1}{d_1}+\frac{1}{2}},\frac{\Delta_2(\xi_0,\alpha)-\frac{(1-a_{12}-a_{13})r_1}{d_2}}{\Delta_2(\xi_0,\alpha)+\frac{1}{2}-\frac{(1-a_{12}-a_{13})r_1}{d_2}},$$
$$\frac{\Delta_3(\xi_0,\alpha)-\frac{(1-a_{12}-a_{13})r_1}{d_3}}{\Delta_3(\xi_0,\alpha)+\frac{1}{2}-\frac{(1-a_{12}-a_{13})r_1}{d_3}}\Big\}<\frac{1}{\alpha}-\frac{(d_1+2\alpha r_1)(1-\alpha)}{2r_1(a_{12}+a_{13})\alpha^2}.$$

(H_2) $\Delta_1(\xi_0,\alpha)+\frac{(a_{12}+a_{13})r_1}{d_1}+\frac{1}{2}<0, \Delta_2(\xi_0,\alpha)+\frac{1}{2}-\frac{(1-a_{12}-a_{13})r_1}{d_2}<0,$
$\Delta_3(\xi_0,\alpha)+\frac{1}{2}-\frac{(1-a_{12}-a_{13})r_1}{d_3}<0,$

where

$$\Delta_i(\xi_0,\alpha)=\frac{1}{1-\alpha}\int_{-\infty}^{+\infty}J_i(x)\frac{1-e^{\xi_0 x}}{\frac{\alpha}{1-\alpha}+e^{\xi_0 x}}dx;$$

$i=1,2,3$*; and* ξ_0 *is given by* (2.6) *and* (2.7)*. Then, the minimal wave speed of* (1.3) *is linearly selected.*

Proof. Inspired by [19], we just need to construct a suitable upper solution of (1.3). Define a triple of nondecreasing continuous functions by

$$\Phi_+(z)=\begin{cases}\alpha, & z<z_1,\\ \frac{1}{1+e^{-\xi_1(c)z}}, & z\geq z_1,\end{cases}\qquad \Psi_+(z)=\Theta_+(z)=\begin{cases}\beta_1\Phi_+(z), & z\leq z_2,\\ 1, & z>z_2,\end{cases} \tag{3.1}$$

where $c=c_0+\sigma_1$ for sufficiently small non-negative number σ_1, and $c_0,\xi_1(c)$ are given by Section 2. Combining (H_1) and (H_2), we deduce the existence of β_1 satisfying

$$\begin{aligned}&\max\{a_{21}+\frac{d_2}{2\alpha r_2},a_{31}+\frac{d_3}{2\alpha r_3},\frac{\Delta_1(\xi_0,\alpha)+\frac{(a_{12}+a_{13})r_1}{d_1}}{\Delta_1(\xi_0,\alpha)+\frac{(a_{12}+a_{13})r_1}{d_1}+\frac{1}{2}},\frac{\Delta_2(\xi_0,\alpha)-\frac{(1-a_{12}-a_{13})r_1}{d_2}}{\Delta_2(\xi_0,\alpha)+\frac{1}{2}-\frac{(1-a_{12}-a_{13})r_1}{d_2}},\frac{\Delta_3(\xi_0,\alpha)-\frac{(1-a_{12}-a_{13})r_1}{d_3}}{\Delta_3(\xi_0,\alpha)+\frac{1}{2}-\frac{(1-a_{12}-a_{13})r_1}{d_3}}\}\\&<\beta_1<\frac{2r_1\alpha(a_{12}+a_{13}+\alpha-1)-d_1}{2r_1\alpha^2(a_{12}+a_{13})}.\end{aligned} \tag{3.2}$$

It is easy to have that $\Phi_+(z_1)=\alpha,\Phi_+(z_2)=1/\beta_1$, $z_1<z_2,\beta_1>1$, and $\alpha\beta_1<1$. Next, we prove that $(\Phi_+(z),\Psi_+(z),\Theta_+(z))$ is an upper solution of (1.3). Now, we consider the following three cases:

(i) $z<z_1$**.** In this case, $\Phi_+(z)=\alpha,\Psi_+(z)=\Theta_+(z)=\beta_1\Phi_+(z)=\alpha\beta_1$. Substituting $(\Phi_+(z),\Psi_+(z),\Theta_+(z))$ into (1.3), the first equation of (1.3) becomes

$$\begin{aligned}&c\Phi_+'-d_1[\int_{-\infty}^{+\infty}J_1(x)\Phi_+(z-x)dx-\Phi_+]-r_1\Phi_+(1-a_{12}-a_{13}-\Phi_++a_{12}\Psi_++a_{13}\Theta_+)\\&=-d_1[\int_{-\infty}^{+\infty}J_1(x)\Phi_+(z-x)dx-\alpha]-r_1\alpha(1-a_{12}-a_{13}-\alpha+a_{12}\alpha\beta_1+a_{13}\alpha\beta_1)\\&=-d_1[\int_{-\infty}^{z-z_1}J_1(x)\frac{1}{1+e^{-\xi_1(c)(z-x)}}dx+\int_{z-z_1}^{+\infty}J_1(x)\alpha dx-\alpha]\\&-r_1\alpha(1-a_{12}-a_{13}-\alpha+a_{12}\alpha\beta_1+a_{13}\alpha\beta_1)\\&\geq-d_1[\int_{-\infty}^{z-z_1}J_1(x)dx+\int_{z-z_1}^{+\infty}J_1(x)\alpha dx-\alpha]-r_1\alpha(1-a_{12}-a_{13}-\alpha+a_{12}\alpha\beta_1+a_{13}\alpha\beta_1)\\&\geq-d_1[\frac{1}{2}+\alpha-\alpha]-r_1\alpha(1-a_{12}-a_{13}-\alpha+a_{12}\alpha\beta_1+a_{13}\alpha\beta_1)\\&\geq-\frac{d_1}{2}-r_1\alpha[1-(a_{12}+a_{13})-\alpha+(a_{12}+a_{13})\alpha\beta_1]\\&=r_1\alpha^2(a_{12}+a_{13})[\frac{a_{12}+a_{13}+\alpha-1}{\alpha(a_{12}+a_{13})}-\beta_1-\frac{d_1}{2r_1\alpha^2(a_{12}+a_{13})}]>0\end{aligned} \tag{3.3}$$

by (3.2). For the second equation in (1.3), we obtain that

$$\begin{aligned}&c\Psi_+'-d_2[\int_{-\infty}^{+\infty}J_2(y)\Psi_+(z-y)dy-\Psi_+]-r_2(1-\Psi_+)(a_{21}\Phi_+-\Psi_+)\\&=-d_2[\int_{-\infty}^{+\infty}J_2(y)\Psi_+(z-y)dy-\alpha\beta_1]-r_2(1-\alpha\beta_1)(a_{21}\alpha-\alpha\beta_1)\\&\geq-d_2\frac{1+\alpha\beta_1}{2}\int_{-\infty}^{+\infty}J_2(y)dy+d_2\alpha\beta_1-r_2(1-\alpha\beta_1)(a_{21}\alpha-\alpha\beta_1)\\&=-d_2\frac{1+\alpha\beta_1}{2}+d_2\alpha\beta_1-r_2(1-\alpha\beta_1)(a_{21}\alpha-\alpha\beta_1)\\&=\frac{(\alpha\beta_1-1)d_2}{2}-r_2(1-\alpha\beta_1)(a_{21}\alpha-\alpha\beta_1)\\&=r_2\alpha(1-\alpha\beta_1)(\beta_1-a_{21}-\frac{d_2}{2\alpha r_2})>0\end{aligned} \tag{3.4}$$

by (3.2).

Similarly, we have that the third equation in (1.3) becomes

$$
\begin{aligned}
&c\Theta_+' - d_3[\textstyle\int_{-\infty}^{+\infty} J_3(y)\Theta_+(z-y)dy - \Theta_+] - r_3(1-\Theta_+)(a_{31}\Phi_+ - \Theta_+) \\
&\geq r_3\alpha(1-\alpha\beta_1)(\beta_1 - a_{31} - \tfrac{d_2}{2\alpha r_3}) > 0
\end{aligned}
\tag{3.5}
$$

by (3.2).

(ii) $z_1 \leq z \leq z_2$**.** In this case, $\Phi_+(z) = 1/[1+e^{-\xi_1(c)z}]$, $\Psi_+(z) = \Theta_+(z) = \beta_1\Phi_+(z) = \beta_1/[1+e^{-\xi_1(c)z}]$. Obviously, $\Phi_+'(z) = \xi_1(c)\Phi_+(z)(1-\Phi_+(z))$, and $\Psi_+'(z) = \Theta_+'(z) = \beta_1\Phi_+'(z) = \beta_1\xi_1(c)\Phi_+(z)(1-\Phi_+(z))$. Substituting $(\Phi_+(z), \Psi_+(z), \Theta_+(z))$ into (1.3), the first equation of (1.3) becomes

$$
\begin{aligned}
&c\Phi_+' - d_1[\textstyle\int_{-\infty}^{+\infty} J_1(x)\Phi_+(z-x)dx - \Phi_+] - r_1\Phi_+(1 - a_{12} - a_{13} - \Phi_+ + a_{12}\Psi_+ + a_{13}\Theta_+) \\
&= c\xi_1(c)\Phi_+(1-\Phi_+) - r_1\Phi_+(1-\Phi_+)[1 - \tfrac{a_{12}(1-\Psi_+)}{1-\Phi_+} - \tfrac{a_{13}(1-\Theta_+)}{1-\Phi_+}] \\
&\quad -d_1[\textstyle\int_{-\infty}^{+\infty} J_1(x)\Phi_+(z-x)dx - \Phi_+] \\
&= c\xi_1(c)\Phi_+(1-\Phi_+) - r_1\Phi_+(1-\Phi_+)[1 - \tfrac{a_{12}(1-\Psi_+)}{1-\Phi_+} - \tfrac{a_{13}(1-\Theta_+)}{1-\Phi_+}] \\
&\quad -d_1[\textstyle\int_{-\infty}^{z-z_1} J_1(x)\Phi_+(z-x)dx + \int_{z-z_1}^{+\infty} J_1(x)\Phi_+(z-x)dx - \frac{1}{1+e^{-\xi_1(c)z}}] \\
&= c\xi_1(c)\Phi_+(1-\Phi_+) - r_1\Phi_+(1-\Phi_+)[1 - \tfrac{a_{12}(1-\Psi_+)}{1-\Phi_+} - \tfrac{a_{13}(1-\Theta_+)}{1-\Phi_+}] \\
&\quad -d_1[\textstyle\int_{-\infty}^{z-z_1} J_1(x)\frac{1}{1+e^{-\xi_1(c)(z-x)}}dx + \int_{z-z_1}^{+\infty} J_1(x)\alpha dx - \frac{1}{1+e^{-\xi_1(c)z}}] \\
&= c\xi_1(c)\Phi_+(1-\Phi_+) - r_1\Phi_+(1-\Phi_+)[1 - \tfrac{a_{12}(1-\Psi_+)}{1-\Phi_+} - \tfrac{a_{13}(1-\Theta_+)}{1-\Phi_+}] \\
&\quad -d_1\{\textstyle\int_{-\infty}^{+\infty} J_1(x)[\frac{1}{1+e^{-\xi_1(c)(z-x)}} - \frac{1}{1+e^{-\xi_1(c)z}}]dx + \int_{z-z_1}^{+\infty} J_1(x)[\alpha - \frac{1}{1+e^{-\xi_1(c)(z-x)}}]dx\} \\
&\geq c\xi_1(c)\Phi_+(1-\Phi_+) - r_1\Phi_+(1-\Phi_+) \\
&\quad -d_1\{\textstyle\int_{-\infty}^{+\infty} J_1(x)[\frac{1}{1+e^{-\xi_1(c)(z-x)}} - \frac{1}{1+e^{-\xi_1(c)z}}]dx + \int_{z-z_1}^{+\infty} J_1(x)\alpha dx\} \\
&\geq c\xi_1(c)\Phi_+(1-\Phi_+) - r_1\Phi_+(1-\Phi_+) \\
&\quad -d_1\{\textstyle\int_{-\infty}^{+\infty} J_1(x)[\frac{1}{1+e^{-\xi_1(c)(z-x)}} - \frac{1}{1+e^{-\xi_1(c)z}}]dx + \int_{0}^{+\infty} J_1(x)\alpha dx\} \\
&= c\xi_1(c)\Phi_+(1-\Phi_+) - r_1\Phi_+(1-\Phi_+) - d_1\textstyle\int_{-\infty}^{+\infty} J_1(x)[\frac{1}{1+e^{-\xi_1(c)(z-x)}} - \frac{1}{1+e^{-\xi_1(c)z}}]dx - \frac{\alpha}{2}d_1 \\
&= \Phi_+(1-\Phi_+)\{c\xi_1(c) - r_1 - d_1\textstyle\int_{-\infty}^{+\infty} J_1(x)\frac{(1-e^{\xi_1(c)x})(1+e^{\xi_1(c)z})}{e^{\xi_1(c)z}+e^{\xi_1(c)x}}dx - \frac{\alpha}{2\Phi_+(1-\Phi_+)}d_1\} \\
&\geq \Phi_+(1-\Phi_+)\{c\xi_1(c) - r_1 - d_1\textstyle\int_{-\infty}^{+\infty} J_1(x)\frac{(1-e^{\xi_1(c)x})(1+e^{\xi_1(c)z_1})}{e^{\xi_1(c)z_1}+e^{\xi_1(c)x}}dx - \frac{\alpha}{2\Phi_+(z_1)(1-\Phi_+(z_2))}d_1\} \\
&= \Phi_+(1-\Phi_+)\{c\xi_1(c) - r_1 - d_1\textstyle\int_{-\infty}^{+\infty} J_1(x)\frac{(1-e^{\xi_1(c)x})(1+\frac{\alpha}{1-\alpha})}{\frac{\alpha}{1-\alpha}+e^{\xi_1(c)x}}dx - \frac{\alpha}{2\alpha(1-\frac{1}{\beta_1})}d_1\} \\
&= \Phi_+(1-\Phi_+)\{c\xi_1(c) - r_1 - \tfrac{d_1}{1-\alpha}\textstyle\int_{-\infty}^{+\infty} J_1(x)\frac{1-e^{\xi_1(c)x}}{\frac{\alpha}{1-\alpha}+e^{\xi_1(c)x}}dx - \frac{\beta_1}{2(\beta_1-1)}d_1\} \\
&\to \Phi_+(1-\Phi_+)\{c_0\xi_0 - r_1 - \tfrac{d_1}{1-\alpha}\textstyle\int_{-\infty}^{+\infty} J_1(x)\frac{1-e^{\xi_0 x}}{\frac{\alpha}{1-\alpha}+e^{\xi_0 x}}dx - \frac{\beta_1}{2(\beta_1-1)}d_1\} \quad (c \to c_0) \\
&= \Phi_+(1-\Phi_+)d_1\{\textstyle\int_{-\infty}^{+\infty} J_1(x)e^{-\xi_0 x}dx - 1 - \frac{r_1(a_{12}+a_{13})}{d_1} + \frac{1}{1-\alpha}\int_{-\infty}^{+\infty} J_1(x)\frac{e^{\xi_0 x}-1}{\frac{\alpha}{1-\alpha}+e^{\xi_0 x}}dx - \frac{\beta_1}{2(\beta_1-1)}\} \\
&\geq \Phi_+(1-\Phi_+)d_1\{-\tfrac{r_1(a_{12}+a_{13})}{d_1} + \tfrac{1}{1-\alpha}\textstyle\int_{-\infty}^{+\infty} J_1(x)\frac{e^{\xi_0 x}-1}{\frac{\alpha}{1-\alpha}+e^{\xi_0 x}}dx - \frac{\beta_1}{2(\beta_1-1)}\} \\
&= -\tfrac{\Phi_+(1-\Phi_+)d_1}{\beta_1-1}\{\beta_1[\Delta_1(\xi_0,\alpha) + \tfrac{(a_{12}+a_{13})r_1}{d_1} + \tfrac{1}{2}] - [\Delta_1(\xi_0,\alpha) + \tfrac{(a_{12}+a_{13})r_1}{d_1}]\} > 0
\end{aligned}
\tag{3.6}
$$

by (3.2) and (H_2), where

$$
\Delta_1(\xi_0,\alpha) = \frac{1}{1-\alpha}\int_{-\infty}^{+\infty} J_1(x)\frac{1-e^{\xi_0 x}}{\frac{\alpha}{1-\alpha}+e^{\xi_0 x}}dx.
$$

Here, we have used

$$
\begin{aligned}
\textstyle\int_{-\infty}^{+\infty} J_1(x)e^{-\xi_0 x}dx &= \textstyle\int_{-\infty}^{0} J_1(x)e^{-\xi_0 x}dx + \int_{0}^{+\infty} J_1(x)e^{-\xi_0 x}dx \\
&= \textstyle\int_{0}^{+\infty} J_1(x)(e^{\xi_0 x} + e^{-\xi_0 x})dx \geq 2\int_{0}^{+\infty} J_1(x)dx = 1.
\end{aligned}
\tag{3.7}
$$

For the second equation in (1.3), we obtain that

$$
\begin{aligned}
& c\Psi_+' - d_2[\int_{-\infty}^{+\infty} J_2(x)\Psi_+(z-x)dx - \Psi_+] - r_2(1-\Psi_+)(a_{21}\Phi_+ - \Psi_+) \\
&= c\beta_1\xi(c)\Phi_+(1-\Phi_+) - \tfrac{r_2(a_{21}-\beta_1)(1-\Psi_+)}{\beta_1(1-\Phi_+)} - d_2[\int_{-\infty}^{z-z_2} + \int_{z-z_2}^{z-z_1} + \int_{z-z_1}^{+\infty} J_2(x)\Psi_+(z-x)dx - \Psi_+] \\
&= c\beta_1\xi(c)\Phi_+(1-\Phi_+) - \tfrac{r_2(a_{21}-\beta_1)(1-\Psi_+)}{\beta_1(1-\Phi_+)} \\
&\quad -d_2[\int_{-\infty}^{z-z_2} J_2(x)dx + \int_{z-z_2}^{z-z_1} J_2(x)\tfrac{\beta_1}{1+e^{-\xi(c)(z-x)}}dx + \int_{z-z_1}^{+\infty} J_2(x)\alpha\beta_1 dx - \tfrac{\beta_1}{1+e^{-\xi(c)z}}] \\
&= c\beta_1\xi(c)\Phi_+(1-\Phi_+) - \tfrac{r_2(a_{21}-\beta_1)(1-\Psi_+)}{\beta_1(1-\Phi_+)} - d_2\{\int_{-\infty}^{z-z_2} J_2(x)(1 - \tfrac{\beta_1}{1+e^{-\xi(c)(z-x)}})dx \\
&\quad + \int_{-\infty}^{+\infty} J_2(x)[\tfrac{\beta_1}{1+e^{-\xi(c)(z-x)}} - \tfrac{\beta_1}{1+e^{-\xi(c)z}}]dx + \int_{z-z_1}^{+\infty} J_2(x)[\alpha\beta_1 - \tfrac{\beta_1}{1+e^{-\xi(c)(z-x)}}]dx\} \\
&\geq c\beta_1\xi(c)\Phi_+(1-\Phi_+) - d_2\{\int_{-\infty}^{+\infty} J_2(x)[\tfrac{\beta_1}{1+e^{-\xi(c)(z-x)}} - \tfrac{\beta_1}{1+e^{-\xi(c)z}}]dx + \int_{z-z_1}^{+\infty} J_2(x)\alpha\beta_1 dx\} \\
&\geq c\beta_1\xi(c)\Phi_+(1-\Phi_+) - d_2\{\int_{-\infty}^{+\infty} J_2(x)[\tfrac{\beta_1}{1+e^{-\xi(c)(z-x)}} - \tfrac{\beta_1}{1+e^{-\xi(c)z}}]dx + \int_{0}^{+\infty} J_2(x)\alpha\beta_1 dx\} \\
&= c\beta_1\xi(c)\Phi_+(1-\Phi_+) - d_2\{\int_{-\infty}^{+\infty} J_2(x)[\tfrac{\beta_1}{1+e^{-\xi(c)(z-x)}} - \tfrac{\beta_1}{1+e^{-\xi(c)z}}]dx + \tfrac{\alpha\beta_1}{2}\} \\
&= \beta_1\Phi_+(1-\Phi_+)\{c\xi(c) - d_2\int_{-\infty}^{+\infty} J_2(x)[\tfrac{1}{1+e^{-\xi(c)(z-x)}} - \tfrac{1}{1+e^{-\xi(c)z}}]dx - \tfrac{\alpha d_2}{2\Phi_+(1-\Phi_+)}\} \\
&\geq \beta_1\Phi_+(1-\Phi_+)\{c\xi(c) - d_2\int_{-\infty}^{+\infty} J_2(x)\tfrac{(1-e^{\xi(c)x})(1+e^{\xi(c)z_1})}{e^{\xi(c)z_1}+e^{\xi(c)x}}dx - \tfrac{\alpha d_2}{2\Phi_+(z_1)(1-\Phi_+(z_2))}\} \\
&= \beta_1\Phi_+(1-\Phi_+)\{c\xi(c) - d_2\int_{-\infty}^{+\infty} J_2(x)\tfrac{(1-e^{\xi(c)x})(1+\frac{\alpha}{1-\alpha})}{\frac{\alpha}{1-\alpha}+e^{\xi(c)x}}dx - \tfrac{\alpha d_2}{2\alpha(1-\frac{1}{\beta_1})}\} \\
&= \beta_1\Phi_+(1-\Phi_+)\{c\xi(c) - \tfrac{d_2}{1-\alpha}\int_{-\infty}^{+\infty} J_2(x)\tfrac{1-e^{\xi(c)x}}{\frac{\alpha}{1-\alpha}+e^{\xi(c)x}}dx - \tfrac{\beta_1 d_2}{2(\beta_1-1)}\} \\
&\to \beta_1\Phi_+(1-\Phi_+)\{c_0\xi_0 - \tfrac{d_2}{1-\alpha}\int_{-\infty}^{+\infty} J_2(x)\tfrac{1-e^{\xi_0 x}}{\frac{\alpha}{1-\alpha}+e^{\xi_0 x}}dx - \tfrac{\beta_1 d_2}{2(\beta_1-1)}\} \quad (c \to c_0) \\
&= \beta_1\Phi_+(1-\Phi_+)\{d_1(\int_{-\infty}^{+\infty} J_1(x)e^{-\xi_0 x}dx - 1) + r_1(1-a_{12}-a_{13}) \\
&\quad -\tfrac{d_2}{1-\alpha}\int_{-\infty}^{+\infty} J_2(x)\tfrac{1-e^{\xi_0 x}}{\frac{\alpha}{1-\alpha}+e^{\xi_0 x}}dx - \tfrac{\beta_1 d_2}{2(\beta_1-1)}\} \\
&\geq \beta_1\Phi_+(1-\Phi_+)\{r_1(1-a_{12}-a_{13}) - \tfrac{d_2}{1-\alpha}\int_{-\infty}^{+\infty} J_2(x)\tfrac{1-e^{\xi_0 x}}{\frac{\alpha}{1-\alpha}+e^{\xi_0 x}}dx - \tfrac{\beta_1 d_2}{2(\beta_1-1)}\} \\
&= -\tfrac{\Phi_+(1-\Phi_+)d_2\beta_1}{\beta_1-1}\{\beta_1[\Delta_2(\xi_0,\alpha) + \tfrac{1}{2} - \tfrac{(1-a_{12}-a_{13})r_1}{d_2}] - [\Delta_2(\xi_0,\alpha) - \tfrac{(1-a_{12}-a_{13})r_1}{d_2}]\} > 0
\end{aligned}
\tag{3.8}
$$

by (3.2), (3.7) and (H_2), where

$$\Delta_2(\xi_0,\alpha) = \frac{1}{1-\alpha}\int_{-\infty}^{+\infty} J_2(x)\frac{1-e^{\xi_0 x}}{\frac{\alpha}{1-\alpha}+e^{\xi_0 x}}dx.$$

Here, we have used $\beta_1\Phi_+ \geq 1$ for $z > z_2$, that is, $\beta_1/[1+e^{-\xi(c)z}] \geq 1$.

Similarly, the third equation in (1.3) becomes

$$
\begin{aligned}
& c\Theta_+' - d_3[\int_{-\infty}^{+\infty} J_3(x)\Theta_+(z-x)dx - \Theta_+] - r_3(1-\Theta_+)(a_{31}\Phi_+ - \Theta_+) \\
&\geq -\tfrac{\Phi_+(1-\Phi_+)d_3\beta_1}{\beta_1-1}\{\beta_1[\Delta_3(\xi_0,\alpha) + \tfrac{1}{2} - \tfrac{(1-a_{12}-a_{13})r_1}{d_3}] - [\Delta_3(\xi_0,\alpha) - \tfrac{(1-a_{12}-a_{13})r_1}{d_3}]\} > 0
\end{aligned}
\tag{3.9}
$$

by (3.2), (3.7), and (H_2), where

$$\Delta_3(\xi_0,\alpha) = \frac{1}{1-\alpha}\int_{-\infty}^{+\infty} J_3(x)\frac{1-e^{\xi_0 x}}{\frac{\alpha}{1-\alpha}+e^{\xi_0 x}}dx.$$

(iii) $z > z_2$**.** In this case, $\Phi_+(z) = 1/[1+e^{-\xi_1(c)z}]$, $\Psi_+(z) = \Theta_+(z) = 1$. Obviously, $\Phi_+'(z) = \xi_1(c)\Phi_+(z)(1-$

$\Phi_+(z))$. Substituting $(\Phi_+(z), \Psi_+(z), \Theta_+(z))$ into (1.3), the first equation of (1.3) becomes

$$\begin{aligned}
& c\Phi_+' - d_1[\int_{-\infty}^{+\infty} J_1(x)\Phi_+(z-x)dx - \Phi_+] - r_1\Phi_+(1 - a_{12} - a_{13} - \Phi_+ + a_{12}\Psi_+ + a_{13}\Theta_+) \\
&= c\xi_1(c)\Phi_+(1-\Phi_+) - r_1\Phi_+(1 - a_{12} - a_{13} - \Phi_+ + a_{12} + a_{13}) \\
&\quad -d_1[\int_{-\infty}^{+\infty} J_1(x)\Phi_+(z-x)dx - \Phi_+] \\
&\geq c\xi_1(c)\Phi_+(1-\Phi_+) - r_1\Phi_+(1-\Phi_+) - d_1\int_{-\infty}^{+\infty} J_1(x)[\tfrac{1}{1+e^{-\xi_1(c)(z-x)}} - \tfrac{1}{1+e^{-\xi_1(c)z}}]dx - \tfrac{\alpha d_1}{2} \\
&\geq \Phi_+(1-\Phi_+)\{c\xi_1(c) - r_1 - d_1\int_{-\infty}^{+\infty} J_1(x)\tfrac{(1-e^{\xi_1(c)x})(e^{\xi_1(c)z_1}+1)}{e^{\xi_1(c)z_1}+e^{\xi_1(c)x}}dx \\
&- \tfrac{\alpha d_1}{2\Phi_+(z_1)(1-\Phi_+(z_2))}\} \\
&= \Phi_+(1-\Phi_+)\{c\xi_1(c) - r_1 - d_1\int_{-\infty}^{+\infty} J_1(x)\tfrac{(1-e^{\xi_1(c)x})(\frac{\alpha}{1-\alpha}+1)}{\frac{\alpha}{1-\alpha}+e^{\xi_1(c)x}}dx - \tfrac{\alpha d_1}{2\alpha(1-\frac{1}{\beta_1})}\} \\
&= \Phi_+(1-\Phi_+)\{c\xi_1(c) - r_1 - \tfrac{d_1}{1-\alpha}\int_{-\infty}^{+\infty} J_1(x)\tfrac{1-e^{\xi_1(c)x}}{\frac{\alpha}{1-\alpha}+e^{\xi_1(c)x}}dx - \tfrac{\beta_1 d_1}{2(\beta_1-1)}\} \\
&\to \Phi_+(1-\Phi_+)\{c_0\xi_0 - r_1 - \tfrac{d_1}{1-\alpha}\int_{-\infty}^{+\infty} J_1(x)\tfrac{1-e^{\xi_0 x}}{\frac{\alpha}{1-\alpha}+e^{\xi_0 x}}dx - \tfrac{\beta_1 d_1}{2(\beta_1-1)}\} \quad (c \to c_0) \\
&= \Phi_+(1-\Phi_+)\{d_1[\int_{-\infty}^{+\infty} J_1(x)e^{-\xi_0 x}dx - 1] + r_1(1 - a_{12} - a_{13}) - r_1 \\
&\quad - \tfrac{d_1}{1-\alpha}\int_{-\infty}^{+\infty} J_1(x)\tfrac{1-e^{\xi_0 x}}{\frac{\alpha}{1-\alpha}+e^{\xi_0 x}}dx - \tfrac{\beta_1 d_1}{2(\beta_1-1)}\} \\
&\geq -d_1\Phi_+(1-\Phi_+)\{\tfrac{r_1(a_{12}+a_{13})}{d_1} + \tfrac{1}{1-\alpha}\int_{-\infty}^{+\infty} J_1(x)\tfrac{1-e^{\xi_0 x}}{\frac{\alpha}{1-\alpha}+e^{\xi_0 x}}dx + \tfrac{\beta_1}{2(\beta_1-1)}\} \\
&= -d_1\Phi_+(1-\Phi_+)\{\tfrac{r_1(a_{12}+a_{13})}{d_1} + \Delta_1(\xi_0,\alpha) + \tfrac{\beta_1}{2(\beta_1-1)}\} \\
&= -\tfrac{\Phi_+(1-\Phi_+)d_1}{\beta_1-1}\{\beta_1[\Delta_1(\xi_0,\alpha) + \tfrac{(a_{12}+a_{13})r_1}{d_1} + \tfrac{1}{2}] - [\Delta_1(\xi_0,\alpha) + \tfrac{(a_{12}+a_{13})r_1}{d_1}]\} > 0
\end{aligned} \tag{3.10}$$

by (3.8), (3.2), (3.7), and (H_2).

It is easy to see that the second equation of (1.3) becomes

$$c\Psi_+' - d_2[\int_{-\infty}^{+\infty} J_2(x)\Psi_+(z-x)dx - \Psi_+] - r_2(1-\Psi_+)(a_{21}\Phi_+ - \Psi_+) = 0, \tag{3.11}$$

and the third equation of (1.3) becomes

$$c\Theta_+' - d_3[\int_{-\infty}^{+\infty} J_3(x)\Theta_+(z-x)dx - \Theta_+] - r_3(1-\Theta_+)(a_{31}\Phi_+ - \Theta_+) = 0. \tag{3.12}$$

In summary, we have the conclusion that $(\Phi_+(z), \Psi_+(z), \Theta_+(z))$ is an upper solution of (1.3), as σ_1 is sufficiently close to 0, and c is sufficiently close to c_0. Therefore, we complete the proof.

4. Nonlinear selection of the minimal wave speed

To investigate the nonlinear selection mechanism via the upper-lower solution technique, we proceed to construct a lower solution $(\Phi_-(z), \Psi_-(z), \Theta_-(z))$ that behaves like $e^{\xi_2(c)z}$ as $z \to +\infty$, (exhibiting a faster decay). The lemma below provides a justification.

Lemma 4.1. *For $c_1 > c_0$, assume $(\Phi_-, \Psi_-, \Theta_-)(z^*) \geq 0$ is a lower solution to the partial differetial system*

$$\begin{cases}
\frac{\partial\phi}{\partial t} = d_1(J_1 * \phi - \phi) + r_1\phi(1 - a_{12} - a_{13} - \phi + a_{12}\psi + a_{13}\theta), \\
\frac{\partial\psi}{\partial t} = d_2(J_2 * \psi - \psi) + r_2(1-\psi)(a_{21}\phi - \psi), \\
\frac{\partial\theta}{\partial t} = d_3(J_3 * \theta - \theta) + r_3(1-\theta)(a_{31}\phi - \theta),
\end{cases}$$

where $z^* = x + c_1 t$. Furthermore, suppose that $\Phi_-(z^*)$ is monotonic and satisfies $\limsup_{z^*\to+\infty}\Phi_-(z^*) < 1$, $\Phi_-(z^*) \sim e^{\xi_2 z^*}$ as $z^* \to +\infty$, where ξ_2 is defined in Section 2. Then, no traveling wave solution exists for (1.3) *with speed c in $[c_0, c_1)$.*

For the proof of Lemma 4.1, we refer interesting readers to Lemma 2.8 in [26], and related results can be found in [19, 22], so the proof process is omitted here.

Theorem 4.2. *Assume that there exists a constant* $\alpha \in (0,1)$ *such that*

$$(H_3) \quad \max\{a_{12}+a_{13}+\tfrac{d_1}{2\alpha r_1}, \tfrac{\Omega_1(\xi_0,\alpha)}{\Omega_1(\xi_0,\alpha)+\frac{1}{2}}, \tfrac{\Omega_2(\xi_0,\alpha)+r_2}{\Omega_2(\xi_0,\alpha)+r_2+\frac{d_2}{2}}, \tfrac{\Omega_3(\xi_0,\alpha)+r_3}{\Omega_3(\xi_0,\alpha)+r_3+\frac{d_3}{2}}\}$$
$$< \tfrac{1}{\alpha} - \max\{\tfrac{(1-\alpha)(d_2+2\alpha r_2)}{2\alpha^2 r_2 a_{21}}, \tfrac{(1-\alpha)(d_3+2\alpha r_3)}{2\alpha^2 r_3 a_{31}}\}.$$

$$(H_4) \quad \Omega_1(\xi_0,\alpha)+\tfrac{1}{2}<0,\ \ \Omega_2(\xi_0,\alpha)+r_2+\tfrac{d_2}{2}<0,\ \ \Omega_3(\xi_0,\alpha)+r_3+\tfrac{d_3}{2}<0,$$

where

$$\Omega_i(\xi_0,\alpha) = d_1\Big(\int_{-\infty}^{+\infty} J_1(x)e^{-\xi_0 x}dx - 1\Big) + r_1(1-a_{12}-a_{13}) - \frac{d_i}{\alpha}\int_{-\infty}^{+\infty} J_i(x)\frac{1-e^{\xi_0 x}}{\frac{1-\alpha}{\alpha}+e^{\xi_0 x}}dx;$$

$i = 1,2,3$*; and* ξ_0 *is given by* (2.6) *and* (2.7)*. Then, the minimal wave speed of* (1.3) *is nonlinearly selected.*

Proof. Inspired by [26], we just need to construct a suitable lower solution of (1.3). Define a triple of nondecreasing continuous functions by

$$\Psi_-(z) = \Theta_-(z) = \begin{cases} \frac{1}{1+e^{-\xi_2(c)z}}, & z \le z_3, \\ 1-\alpha, & z > z_3, \end{cases} \quad \Phi_-(z) = \begin{cases} 0, & z < z_4, \\ 1-\beta_2(1-\Psi_-(z)), & z \ge z_4, \end{cases} \tag{4.1}$$

where $c = c_0 + \sigma_1$ for sufficiently small non-negative number σ_1, and c_0, $\xi_2(c)$ are given by Section 2. Hypotheses (H_3) and (H_4) enable us to choose β_2 subject to the condition that

$$\begin{aligned} &\max\{a_{12}+a_{13}+\tfrac{d_1}{2\alpha r_1}, \tfrac{\Omega_1(\xi_0,\alpha)}{\Omega_1(\xi_0,\alpha)+\frac{1}{2}}, \tfrac{\Omega_2(\xi_0,\alpha)+r_2}{\Omega_2(\xi_0,\alpha)+r_2+\frac{d_2}{2}}, \tfrac{\Omega_3(\xi_0,\alpha)+r_3}{\Omega_3(\xi_0,\alpha)+r_3+\frac{d_3}{2}}\} \\ &<\beta_2 < \tfrac{1}{\alpha} - \max\{\tfrac{(1-\alpha)(d_2+2\alpha r_2)}{2\alpha^2 r_2 a_{21}}, \tfrac{(1-\alpha)(d_3+2\alpha r_3)}{2\alpha^2 r_3 a_{31}}\}. \end{aligned} \tag{4.2}$$

It is easy to have that $\Psi_-(z_3) = \Theta_-(z_3) = 1-\alpha, \Psi_-(z_4) = \Theta_-(z_4) = 1-1/\beta_2$, $z_4 < z_3, \beta_2 > 1$. Next, we prove that $(\Phi_-(z), \Psi_-(z), \Theta_-(z))$ is a lower solution of (1.3). Now, we consider the following three cases:

(i) $z < z_4$**.** In this case, $\Phi_-(z) = 0$, $\Psi_-(z) = \Theta_-(z) = 1/[1+e^{-\xi_2(c)z}]$. Obviously, $\Psi'_-(z) = \Theta'_-(z) = \xi_2(c)\Psi_-(z)(1-\Psi_-(z))$. Substituting $(\Phi_-(z), \Psi_-(z), \Theta_-(z))$ into (1.3), the first equation of (1.3) becomes

$$\begin{aligned} &c\Phi'_- - d_1\Big[\int_{-\infty}^{+\infty} J_1(x)\Phi_-(z-x)dx - \Phi_-\Big] - r_1\Phi_-(1-a_{12}-a_{13}-\Phi_-+a_{12}\Psi_-+a_{13}\Theta_-) \\ &= -d_1\int_{-\infty}^{+\infty} J_1(x)\Phi_+(z-x)dx \le 0. \end{aligned} \tag{4.3}$$

It is easy to see that the second equation of (1.3) becomes

$$\begin{aligned}
&c\Psi_-' - d_2[\int_{-\infty}^{+\infty} J_2(x)\Psi_-(z-x)dx - \Psi_-] - r_2(1-\Psi_-)(a_{21}\Phi_- - \Psi_-)\\
&= c\xi_2(c)\Psi_-(1-\Psi_-) - d_2[\int_{-\infty}^{z-z_3} J_2(x)(1-\alpha)dx + \int_{z-z_3}^{+\infty} J_2(x)\tfrac{1}{1+e^{-\xi_2(c)(z-x)}}dx - \tfrac{1}{1+e^{-\xi_2(c)z}}]\\
&\quad + r_2\Psi_-(1-\Psi_-)\\
&= c\xi_2(c)\Psi_-(1-\Psi_-) + r_2\Psi_-(1-\Psi_-)\\
&\quad - d_2[\int_{-\infty}^{+\infty} J_2(x)(\tfrac{1}{1+e^{-\xi_2(c)(z-x)}} - \tfrac{1}{1+e^{-\xi_2(c)z}})dx + \int_{-\infty}^{z-z_3} J_2(x)(1-\alpha-\tfrac{1}{1+e^{-\xi_2(c)(z-x)}})dx]\\
&\le c\xi_2(c)\Psi_-(1-\Psi_-) + r_2\Psi_-(1-\Psi_-)\\
&\quad - d_2[\int_{-\infty}^{+\infty} J_2(x)(\tfrac{1}{1+e^{-\xi_2(c)(z-x)}} - \tfrac{1}{1+e^{-\xi_2(c)z}})dx - \int_{-\infty}^{z-z_3} J_2(x)\alpha dx]\\
&\le c\xi_2(c)\Psi_-(1-\Psi_-) + r_2\Psi_-(1-\Psi_-) - d_2[\int_{-\infty}^{+\infty} J_2(x)(\tfrac{1}{1+e^{-\xi_2(c)(z-x)}} - \tfrac{1}{1+e^{-\xi_2(c)z}})dx - \tfrac{\alpha}{2}]\\
&= \Psi_-(1-\Psi_-)\{c\xi_2(c) + r_2 - d_2\int_{-\infty}^{+\infty} J_2(x)\tfrac{(1-e^{\xi_2(c)x})(1+e^{\xi_2(c)z})}{e^{\xi_2(c)z}+e^{\xi_2(c)x}}dx + \tfrac{\alpha}{2\Psi_-(1-\Psi_-)}d_2\}\\
&\le \Psi_-(1-\Psi_-)\{c\xi_2(c) + r_2 - d_2\int_{-\infty}^{+\infty} J_2(x)\tfrac{(1-e^{\xi_2(c)x})(1+e^{\xi_2(c)z_3})}{e^{\xi_2(c)z_3}+e^{\xi_2(c)x}}dx + \tfrac{\alpha}{2\Psi_-(z_4)(1-\Psi_-(z_3))}d_2\}\\
&= \Psi_-(1-\Psi_-)\{c\xi_2(c) + r_2 - d_2\int_{-\infty}^{+\infty} J_2(x)\tfrac{(1-e^{\xi_2(c)x})(1+\frac{1-\alpha}{\alpha})}{\frac{1-\alpha}{\alpha}+e^{\xi_2(c)x}}dx + \tfrac{\alpha}{2(1-\frac{1}{\beta_2})\alpha}d_2\}\\
&= \Psi_-(1-\Psi_-)\{c\xi_2(c) + r_2 - \tfrac{d_2}{\alpha}\int_{-\infty}^{+\infty} J_2(x)\tfrac{1-e^{\xi_2(c)x}}{\frac{1-\alpha}{\alpha}+e^{\xi_2(c)x}}dx + \tfrac{\beta_2}{2(\beta_2-1)}d_2\}\\
&\to \Psi_-(1-\Psi_-)\{c_0\xi_0 + r_2 - \tfrac{d_2}{\alpha}\int_{-\infty}^{+\infty} J_2(x)\tfrac{1-e^{\xi_0 x}}{\frac{1-\alpha}{\alpha}+e^{\xi_0 x}}dx + \tfrac{\beta_2}{2(\beta_2-1)}d_2\} \quad (c\to c_0)\\
&= \Psi_-(1-\Psi_-)\{d_1(\int_{-\infty}^{+\infty} J_1(x)e^{-\xi_0 x}dx - 1) + r_1(1-a_{12}-a_{13}) + r_2\\
&\quad - \tfrac{d_2}{\alpha}\int_{-\infty}^{+\infty} J_2(x)\tfrac{1-e^{\xi_0 x}}{\frac{1-\alpha}{\alpha}+e^{\xi_0 x}}dx + \tfrac{\beta_2}{2(\beta_2-1)}d_2\}\\
&= \tfrac{\Psi_-(1-\Psi_-)}{\beta_2-1}\{\beta_2[\Omega_2(\xi_0,\alpha) + r_2 + \tfrac{d_2}{2}] - [\Omega_2(\xi_0,\alpha) + r_2]\}\\
&\le 0
\end{aligned} \tag{4.4}$$

by (4.2), (H_3), and (H_4).

Similarly, the third equation in (1.3) becomes

$$\begin{aligned}
&c\Theta_-' - d_3[\int_{-\infty}^{+\infty} J_3(x)\Theta_-(z-x)dx - \Theta_-] - r_3(1-\Theta_-)(a_{31}\Phi_- - \Theta_-)\\
&\le \Theta_-(1-\Theta_-)\{c\xi_2(c) + r_3 - \tfrac{d_3}{\alpha}\int_{-\infty}^{+\infty} J_3(x)\tfrac{1-e^{\xi_2(c)x}}{\frac{1-\alpha}{\alpha}+e^{\xi_2(c)x}}dx + \tfrac{\beta_2}{2(\beta_2-1)}d_3\}\\
&\to \Theta_-(1-\Theta_-)\{c_0\xi_0 + r_3 - \tfrac{d_3}{\alpha}\int_{-\infty}^{+\infty} J_3(x)\tfrac{1-e^{\xi_0 x}}{\frac{1-\alpha}{\alpha}+e^{\xi_0 x}}dx + \tfrac{\beta_2}{2(\beta_2-1)}d_3\} \quad (c\to c_0)\\
&= \Theta_-(1-\Theta_-)\{d_1(\int_{-\infty}^{+\infty} J_1(x)e^{-\xi_0 x}dx - 1) + r_1(1-a_{12}-a_{13}) + r_3\\
&\quad - \tfrac{d_3}{\alpha}\int_{-\infty}^{+\infty} J_3(x)\tfrac{1-e^{\xi_0 x}}{\frac{1-\alpha}{\alpha}+e^{\xi_0 x}}dx + \tfrac{\beta_2}{2(\beta_2-1)}d_3\}\\
&= \tfrac{\Theta_-(1-\Theta_-)}{\beta_2-1}\{\beta_2[\Omega_3(\xi_0,\alpha) + r_3 + \tfrac{d_3}{2}] - [\Omega_2(\xi_0,\alpha) + r_3]\}\\
&\le 0
\end{aligned} \tag{4.5}$$

by (4.2), (H_3) and (H_4).

(ii) $z_4 \le z \le z_3$**.** In this case,

$$\Phi_-(z) = 1 - \beta_2(1 - \frac{1}{1+e^{-\xi_2(c)z}}), \quad \Psi_-(z) = \Theta_-(z) = \frac{1}{1+e^{-\xi_2(c)z}}.$$

Obviously, $\Psi_-'(z) = \Theta_-'(z) = \xi_2(c)\Psi_-(z)(1-\Psi_-(z))$, and $\Phi_-'(z) = \beta_2\Psi_-'(z) = \beta_2\xi_2(c)\Psi_-(z)(1-\Psi_-(z))$.

Substituting $(\Phi_-(z), \Psi_-(z), \Theta_-(z))$ into (1.3), the first equation of (1.3) becomes

$$\begin{aligned}
&c\Phi_-' - d_1[\int_{-\infty}^{+\infty} J_1(x)\Phi_-(z-x)dx - \Phi_-] - r_1\Phi_-(1 - a_{12} - a_{13} - \Phi_- + a_{12}\Psi_- + a_{13}\Theta_-)\\
&= c\beta_2\xi_2(c)\Psi_-(1-\Psi_-) - d_1[\int_{-\infty}^{+\infty} J_1(x)\Phi_-(z-x)dx - \Phi_-] - r_1\Phi_-(1-\Psi_-)(\beta_2 - a_{12} - a_{13})\\
&= c\beta_2\xi_2(c)\Psi_-(1-\Psi_-) - r_1\Phi_-(1-\Psi_-)(\beta_2 - a_{12} - a_{13}) - d_1\int_{-\infty}^{+\infty} J_1(x)\Phi_-(z-x)dx + d_1\Phi_-\\
&= c\beta_2\xi_2(c)\Psi_-(1-\Psi_-) - r_1\Phi_-(1-\Psi_-)(\beta_2 - a_{12} - a_{13})\\
&\quad -d_1[\int_{-\infty}^{z-z_3} J_1(x)\Phi_-(z-x)dx + \int_{z-z_3}^{z-z_4} J_1(x)\Phi_-(z-x)dx + \int_{z-z_4}^{+\infty} J_1(x)\Phi_-(z-x)dx] + d_1\Phi_-\\
&= c\beta_2\xi_2(c)\Psi_-(1-\Psi_-) - r_1\Phi_-(1-\Psi_-)(\beta_2 - a_{12} - a_{13}) - d_1\int_{-\infty}^{z-z_3} J_1(x)(1-\alpha\beta_2)dx\\
&\quad -d_1\int_{z-z_3}^{z-z_4} J_1(x)[1-\beta_2(1-\tfrac{1}{1+e^{-\xi_2(c)(z-x)}})]dx + d_1\int_{-\infty}^{+\infty} J_1(x)[1-\beta_2(1-\tfrac{1}{1+e^{-\xi_2(c)z}})]dx\\
&= c\beta_2\xi_2(c)\Psi_-(1-\Psi_-) - r_1\Phi_-(1-\Psi_-)(\beta_2 - a_{12} - a_{13}) - d_1\int_{-\infty}^{z-z_3} J_1(x)\beta_2[1-\alpha-\tfrac{1}{1+e^{-\xi_2(c)z}}]dx\\
&\quad -d_1\int_{z-z_3}^{z-z_4} J_1(x)\beta_2[\tfrac{1}{1+e^{-\xi_2(c)(z-x)}} - \tfrac{1}{1+e^{-\xi_2(c)z}}]dx + d_1\int_{z-z_4}^{+\infty} J_1(x)[1-\beta_2(1-\tfrac{1}{1+e^{-\xi_2(c)z}})]dx\\
&= c\beta_2\xi_2(c)\Psi_-(1-\Psi_-) - r_1\Phi_-(1-\Psi_-)(\beta_2 - a_{12} - a_{13})\\
&\quad -d_1\int_{-\infty}^{+\infty} J_1(x)\beta_2[\tfrac{1}{1+e^{-\xi_2(c)(z-x)}} - \tfrac{1}{1+e^{-\xi_2(c)z}}]dx - d_1\int_{-\infty}^{z-z_3} J_1(x)\beta_2[1-\alpha-\tfrac{1}{1+e^{-\xi_2(c)(z-x)}}]dx\\
&\quad +d_1\int_{z-z_4}^{+\infty} J_1(x)[1-\beta_2(1-\tfrac{1}{1+e^{-\xi_2(c)(z-x)}})]dx\\
&\le c\beta_2\xi_2(c)\Psi_-(1-\Psi_-) - r_1\Phi_-(1-\Psi_-)(\beta_2 - a_{12} - a_{13})\\
&\quad -d_1\int_{-\infty}^{+\infty} J_1(x)\beta_2[\tfrac{1}{1+e^{-\xi_2(c)(z-x)}} - \tfrac{1}{1+e^{-\xi_2(c)z}}]dx + d_1\alpha\beta_2\int_{-\infty}^{z-z_3} J_1(x)dx\\
&\le c\beta_2\xi_2(c)\Psi_-(1-\Psi_-) - r_1\Phi_-(1-\Psi_-)(\beta_2 - a_{12} - a_{13})\\
&\quad -d_1\beta_2\int_{-\infty}^{+\infty} J_1(x)[\tfrac{1}{1+e^{-\xi_2(c)(z-x)}} - \tfrac{1}{1+e^{-\xi_2(c)z}}]dx + \tfrac{d_1\alpha\beta_2}{2}\\
&= \beta_2\Psi_-(1-\Psi_-)\{c\xi_2(c) - \tfrac{r_1[\beta_2-(a_{12}+a_{13})\Phi_-]}{\beta_2\Psi_-} - d_1\int_{-\infty}^{+\infty} J_1(x)\tfrac{(1-e^{\xi_2(c)x})(1+e^{\xi_2(c)z})}{e^{\xi_2(c)z}+e^{\xi_2(c)x}}dx + \tfrac{\alpha d_1}{2\Psi_-(1-\Psi_-)}\}\\
&\le \beta_2\Psi_-(1-\Psi_-)\{c\xi_2(c) + \tfrac{\alpha d_1}{2\Psi_-(z_4)(1-\Psi_-(z_3))} - d_1\int_{-\infty}^{+\infty} J_1(x)\tfrac{(1-e^{\xi_2(c)x})(1+e^{\xi_2(c)z_3})}{e^{\xi_2(c)z_3}+e^{\xi_2(c)x}}dx\}\\
&= \beta_2\Psi_-(1-\Psi_-)\{c\xi_2(c) + \tfrac{\beta_2}{2(\beta_2-1)} - \tfrac{d_1}{\alpha}\int_{-\infty}^{+\infty} J_1(x)\tfrac{1-e^{\xi_2(c)x}}{\frac{1-\alpha}{\alpha}+e^{\xi_2(c)x}}dx\}\\
&\to \beta_2\Psi_-(1-\Psi_-)\{c_0\xi_0 + \tfrac{\beta_2}{2(\beta_2-1)} - \tfrac{d_1}{\alpha}\int_{-\infty}^{+\infty} J_1(x)\tfrac{1-e^{\xi_0 x}}{\frac{1-\alpha}{\alpha}+e^{\xi_0 x}}dx\} \quad (c\to c_0)\\
&= \beta_2\Psi_-(1-\Psi_-)\{d_1(\int_{-\infty}^{+\infty} J_1(x)e^{-\xi_0 x}dx - 1) + r_1(1-a_{12}-a_{13}) + \tfrac{\beta_2}{2(\beta_2-1)}\\
&\quad -\tfrac{d_1}{\alpha}\int_{-\infty}^{+\infty} J_1(x)\tfrac{1-e^{\xi_0 x}}{\frac{1-\alpha}{\alpha}+e^{\xi_0 x}}dx\}\\
&= \tfrac{\beta_2\Psi_-(1-\Psi_-)}{\beta_2-1}\{\beta_2[\Omega_1(\xi_0,\alpha)+\tfrac{1}{2}] - \Omega_1(\xi_0,\alpha)\}\\
&\le 0
\end{aligned} \tag{4.6}$$

by (4.2), (H_3), and (H_4).

It is easy to see that the second equation of (1.3) becomes

$$\begin{aligned}
&c\Psi_-' - d_2[\int_{-\infty}^{+\infty} J_2(x)\Psi_-(z-x)dx - \Psi_-] - r_2(1-\Psi_-)(a_{21}\Phi_- - \Psi_-)\\
&\le c\Psi_-' - d_2\int_{-\infty}^{+\infty} J_2(x)[\tfrac{1}{1+e^{-\xi_2(c)(z-x)}} - \tfrac{1}{1+e^{-\xi_2(c)z}}]dx + \tfrac{\alpha d_2}{2} - r_2(1-\Psi_-)(a_{21}\Phi_- - \Psi_-)\\
&= c\xi_2(c)\Psi_-(1-\Psi_-) - r_2(1-\Psi_-)(a_{21}\Phi_- - \Psi_-) + \tfrac{\alpha d_2}{2}\\
&\quad -d_2\int_{-\infty}^{+\infty} J_2(x)[\tfrac{1}{1+e^{-\xi_2(c)(z-x)}} - \tfrac{1}{1+e^{-\xi_2(c)z}}]dx\\
&= \Psi_-(1-\Psi_-)\{c\xi_2(c) + r_2 - r_2a_{21}\tfrac{\Phi_-}{\Psi_-} + \tfrac{\alpha d_2}{2\Psi_-(1-\Psi_-)} - d_2\int_{-\infty}^{+\infty} J_2(x)\tfrac{(1-e^{\xi_2(c)x})(1+e^{\xi_2(c)z})}{e^{\xi_2(c)z}+e^{\xi_2(c)x}}dx\}\\
&\le \Psi_-(1-\Psi_-)\{c\xi_2(c) + r_2 + \tfrac{\alpha d_2}{2\Psi_-(z_4)(1-\Psi_-(z_3))} - d_2\int_{-\infty}^{+\infty} J_2(x)\tfrac{(1-e^{\xi_2(c)x})(1+e^{\xi_2(c)z_3})}{e^{\xi_2(c)z_3}+e^{\xi_2(c)x}}dx\}\\
&= \Psi_-(1-\Psi_-)\{c\xi_2(c) + r_2 + \tfrac{d_2\beta_2}{2(\beta_2-1)} - d_2\int_{-\infty}^{+\infty} J_2(x)\tfrac{(1-e^{\xi_2(c)x})(1+\frac{1-\alpha}{\alpha})}{\frac{1-\alpha}{\alpha}+e^{\xi_2(c)x}}dx\}\\
&\to \Psi_-(1-\Psi_-)\{c_0\xi_0 + r_2 + \tfrac{d_2\beta_2}{2(\beta_2-1)} - \tfrac{d_2}{\alpha}\int_{-\infty}^{+\infty} J_2(x)\tfrac{1-e^{\xi_0 x}}{\frac{1-\alpha}{\alpha}+e^{\xi_0 x}}dx\} \quad (c\to c_0)
\end{aligned}$$

$$
\begin{aligned}
&= \Psi_-(1-\Psi_-)\{d_1(\int_{-\infty}^{+\infty} J_1(x)e^{-\xi_0 x}dx - 1) + r_1(1-a_{12}-a_{13}) + r_2 + \tfrac{d_2\beta_2}{2(\beta_2-1)} \\
&\quad -\tfrac{d_2}{\alpha}\int_{-\infty}^{+\infty} J_2(x)\tfrac{1-e^{\xi_0 x}}{\frac{1-\alpha}{\alpha}+e^{\xi_0 x}}dx\} \\
&= \tfrac{\Psi_-(1-\Psi_-)}{\beta_2-1}\{\beta_2[\Omega_2(\xi_0,\alpha) + r_2 + \tfrac{d_2}{2}] - [\Omega_2(\xi_0,\alpha) + r_2]\} \\
&\leq 0
\end{aligned}
\tag{4.7}
$$

by (4.2), (H_3), and (H_4).

Similarly, the third equation in (1.3) becomes

$$
\begin{aligned}
&c\Theta_-' - d_3[\int_{-\infty}^{+\infty} J_3(x)\Theta_-(z-x)dx - \Theta_-] - r_3(1-\Theta_-)(a_{31}\Phi_- - \Theta_-) \\
&\leq \Theta_-(1-\Theta_-)\{c\xi_2(c) + r_3 + \tfrac{d_3\beta_2}{2(\beta_2-1)} - d_3\int_{-\infty}^{+\infty} J_3(x)\tfrac{(1-e^{\xi_2(c)x})(1+\frac{1-\alpha}{\alpha})}{\frac{1-\alpha}{\alpha}+e^{\xi_2(c)x}}dx\} \\
&\to \Theta_-(1-\Theta_-)\{c_0\xi_0 + r_3 + \tfrac{d_3\beta_2}{2(\beta_2-1)} - \tfrac{d_3}{\alpha}\int_{-\infty}^{+\infty} J_3(x)\tfrac{1-e^{\xi_0 x}}{\frac{1-\alpha}{\alpha}+e^{\xi_0 x}}dx\} \quad (c \to c_0) \\
&= \Theta_-(1-\Theta_-)\{d_1(\int_{-\infty}^{+\infty} J_1(x)e^{-\xi_0 x}dx - 1) + r_1(1-a_{12}-a_{13}) + r_3 + \tfrac{d_3\beta_2}{2(\beta_2-1)} \\
&\quad -\tfrac{d_3}{\alpha}\int_{-\infty}^{+\infty} J_3(x)\tfrac{1-e^{\xi_0 x}}{\frac{1-\alpha}{\alpha}+e^{\xi_0 x}}dx\} \\
&= \tfrac{\Theta_-(1-\Theta_-)}{\beta_2-1}\{\beta_2[\Omega_3(\xi_0,\alpha) + r_3 + \tfrac{d_3}{2}] - [\Omega_3(\xi_0,\alpha) + r_3]\} \\
&\leq 0
\end{aligned}
\tag{4.8}
$$

by (4.2), (H_3), and (H_4).

(iii) $z > z_3$**.** In this case, $\Phi_-(z) = 1-\alpha\beta_2$, $\Psi_-(z) = \Theta_-(z) = 1-\alpha$. Substituting $(\Phi_-(z), \Psi_-(z), \Theta_-(z))$ into (1.3), the first equation of (1.3) becomes

$$
\begin{aligned}
&c\Phi_-' - d_1[\int_{-\infty}^{+\infty} J_1(x)\Phi_-(z-x)dx - \Phi_-] - r_1\Phi_-(1-a_{12}-a_{13}-\Phi_- + a_{12}\Psi_- + a_{13}\Theta_-) \\
&\leq d_1\tfrac{1-\alpha\beta_2}{2} - r_1(1-\alpha\beta_2)[1-a_{12}-a_{13}-(1-\alpha\beta_2) + a_{12}(1-\alpha) + a_{13}(1-\alpha)] \\
&= d_1\tfrac{1-\alpha\beta_2}{2} - \alpha r_1(1-\alpha\beta_2)(\beta_2 - a_{12} - a_{13}) \\
&= \alpha r_1(1-\alpha\beta_2)(a_{12} + a_{13} + \tfrac{d_1}{2\alpha r_1} - \beta_2) \leq 0
\end{aligned}
\tag{4.9}
$$

by (4.2).

Similarly, the second equation of (1.3) becomes

$$
\begin{aligned}
&c\Psi_-' - d_2[\int_{-\infty}^{+\infty} J_2(x)\Psi_-(z-x)dx - \Psi_-] - r_2(1-\Psi_-)(a_{21}\Phi_- - \Psi_-) \\
&\leq d_2\tfrac{1-\alpha}{2} - \alpha r_2[a_{21}(1-\alpha\beta_2) - (1-\alpha)] \\
&= \alpha^2 r_2 a_{21}[\beta_2 + \tfrac{(1-\alpha)(d_2+2\alpha r_2)}{2\alpha^2 r_2 a_{21}} - \tfrac{1}{\alpha}] \\
&\leq 0
\end{aligned}
\tag{4.10}
$$

by (4.2).

By a same way for the third equation of (1.3), we have that

$$
\begin{aligned}
&c\Theta_-' - d_3[\int_{-\infty}^{+\infty} J_3(x)\Theta_-(z-x)dx - \Theta_-] - r_3(1-\Theta_-)(a_{31}\Phi_- - \Theta_-) \\
&\leq \alpha^2 r_3 a_{31}[\beta_2 + \tfrac{(1-\alpha)(d_3+2\alpha r_3)}{2\alpha^2 r_3 a_{31}} - \tfrac{1}{\alpha}] \\
&\leq 0
\end{aligned}
\tag{4.11}
$$

by (4.2).

Based on all the above analysis, we can conclude that $(\Phi_-(z), \Psi_-(z), \Theta_-(z))$ is a lower solution of (1.3), as σ_1 is sufficiently close to 0, and c is sufficiently close to c_0. Therefore, we complete the proof.

5. Discussion

The minimal propagation speed in reaction-diffusion systems can be determined either by the linear dynamics at the leading edge of the wave or by the full nonlinear system dynamics. In this section, we discuss the linear selection of the minimal wave speed; that is, the minimal wave speed equals c_0. We note that the asymptotic spreading speed, defined by solutions of the initial value problem, may not coincide with the minimal wave speed in general; a rigorous proof of their equivalence is left for future investigation. In this regime, infinitesimal perturbations at the invasion front grow and effectively "pull" the wave forward, making the linear theory determinative. By contrast, nonlinear selection (associated with pushed waves) arises when the slowest stable traveling wave travels faster than the linear prediction, due to cooperative or competitive nonlinear interactions in the wave's interior. In such cases, the wave is "pushed" to a higher speed by the nonlinear growth of the invading population behind the leading edge, and the linear theory alone underestimates the actual minimal speed. This dichotomy between pulled and pushed fronts has been well documented in simpler reaction-diffusion equations, and the present study extends this understanding to a three-species nonlocal Lotka-Volterra competition system.

In our analysis, we derived explicit conditions under which each selection mechanism governs the minimal wave speed. Using an upper-lower solution approach, we obtained inequalities $(H_1) - (H_2)$ as sufficient conditions for linear selection, and complementary inequalities $(H_3) - (H_4)$ as conditions for nonlinear selection. Intuitively, these criteria encapsulate how the nonlocal competition and diffusion parameters influence the wave dynamics. Linear selection is ensured when the influence of the resident competitor on the invading species is sufficiently weak at low densities, and the nonlocal dispersal effects (modeled by the kernel integrals) do not substantially accelerate the front. Under these conditions, the minimal wave speed equals the value c_0, predicted by the linearized growth rate at the leading edge analogous to the classical Fisher–Kolmogorov–Petrovsky–Piskunov (Fisher-KPP) speed. Conversely, when the nonlinear interactions are strong enough (for instance, if the invader's advantage in the high-density region or the long-range competitive effects are significant), the wave attains a higher speed than c_0. In this nonlinear selection regime, the standard linear criterion is violated, and the slowest invading front is a pushed wave traveling at a speed strictly greater than the linear prediction. The derived conditions $(H_3) - (H_4)$ precisely characterize this regime in terms of model parameters and kernel integrals, delineating when the invasion speed is elevated by nonlinear competition effects. These results provide a theoretical basis for distinguishing pulled versus pushed propagation in the nonlocal competition model, and they clarify the parameter ranges in which each mechanism operates.

This work builds upon and significantly extends a rich body of literature on traveling wave speeds in reaction-diffusion competition models. Early studies on local diffusion models, such as those by Hosono [27] first identified the possibility of both linear (pulled) and nonlinear (pushed) wave speed selection in twospecies Lotka-Volterra competition systems. Hosono's numerical and analytical explorations demonstrated that by tuning parameters, a transition could be made between a front whose speed is given by the linear theory and one requiring nonlinear effects to achieve propagation. These pioneering results highlighted the phenomenon of speed selection but left open the challenge of predicting which mechanism dominates for a given set of parameters. Subsequent analytical works focused primarily on establishing linear determinacy conditions. For instance, Weinberger et al. ([28]) proved that for certain two-species competition models, the spreading speed is linearly determined by the unstable state's dynamics. In related cooperative (monotone) systems, Li et al. ([29]) likewise showed that the minimal wave speed equals the slowest wave predicted by linear analysis. These studies, alongside results by Huang and colleagues, solidified the theoretical foundation for pulled wave behavior (linear selection) in reaction-diffusion systems. Weinberger's general framework further provided sufficient conditions under which a spreading speed is linearly determinate, offering broad criteria applicable to monotone systems and reinforcing the importance of linear mechanisms in many ecological invasions.

Against this backdrop, the key innovation of our study is the rigorous analysis of linear and nonlinear speed selection in a nonlocal competition context, an area that had remained less explored due to its mathematical complexity. He and Zhang ([19]) made important strides by examining linear determinacy in the nonlocal Lotka-Volterra competition model, proving that under certain conditions, the minimal wave speed is linear determined. In contrast, our study is, to the best of our knowledge, one of the first to address linear and nonlinear minimal wave speed selection in a nonlocal competition model. By considering symmetric dispersal kernels, we have generalized the speed selection theory to incorporate long-distance interactions, which are increasingly recognized in ecological systems. The use of upper and lower solution constructions in our proofs is another novel aspect of this work. Specifically, we apply the comparison principle for monotone systems to construct explicit upper and lower solutions that yield checkable conditions for linear versus nonlinear speed selection. This approach not only verifies the existence of pushed fronts in the nonlocal model (thereby confirming that nonlinear selection can occur beyond the local case), but also pinpoints the parameter regimes where this happens. Compared with earlier comparison-based theories, which often require additional restrictions or rely on different analytical frameworks, our method provides more explicit and easily verifiable criteria.

Furthermore, our results bridge and extend the earlier findings of Weinberger, Li, and others by showing that many of the classical insights on linear versus nonlinear spreading speeds continue to hold under nonlocal dynamics, albeit with modified criteria. In particular, when the nonlocal interaction kernels become very narrow (approaching local interactions), our linear selection condition recovers the known linear determinacy results ([30]), consistent with the criteria of Weinberger ([16]) and Weinberger et al. ([28]). Meanwhile, in regimes where those classical criteria fail, we have identified the compensating nonlinear effects (stemming from strong competition or broad kernels) that push the wave faster. This comprehensive treatment of both selection mechanisms in a unified nonlocal framework is a clear improvement over previous studies that examined only one mechanism or were restricted to local models. In summary, the present work not only corroborates earlier research such as Hosono's observations of dual regimes and Weinberger and Li's linear determinacy theory but also advances it by providing a broader and more applicable theory of minimal wave speed selection for complex ecological models. Our findings pave the way for analyzing a wider class of integro-differential equations in ecology and beyond, demonstrating how nonlinear interactions and dispersal patterns together determine the pace of biological invasions.

In summary, our results establish the linear speed selection mechanism under assumptions (H_1) – (H_2) and the nonlinear speed selection mechanism under assumptions (H_3) – (H_4). However, these assumptions are standard and mild in the study of reaction-diffusion competition systems, and are expected to be satisfied by a broad class of commonly used growth functions. A systematic construction of explicit examples is left for future investigation. Despite this limitation, we believe the theoretical framework presented here provides a useful foundation for further investigation.

6. Concluding remarks

In this work, we have studied a three-species Lotka-Volterra competition model with nonlocal dispersal and established a rigorous characterization of its minimal traveling wave speed. The modeling framework incorporated symmetric nonlocal kernels to describe dispersal, and a change of variables was used to convert the competitive system into a cooperative form. This conversion allowed us to employ upper and lower solution constructions in conjunction with comparison principles to tackle the challenges posed by the nonlocal terms. By carefully designing two pairs of upper and lower solutions for the traveling wave equations, we derived sufficient conditions under which the minimal wave speed is determined by the linearized dynamics versus when it is enhanced by nonlinear effects. These analytical techniques enabled us to overcome the difficulties of nonlocal coupling and to rigorously verify the existence of traveling wave solutions at critical speeds.

The criteria we obtained for linear versus nonlinear speed selection generalize earlier results from local diffusion models to the nonlocal dispersal setting, reaffirming that the classic pulled/pushed dichotomy persists even when competition and dispersal occur over spatial scales. These findings contribute to the broader theory of reaction-diffusion equations with nonlocal interactions by specifying how dispersal kernels and competition parameters govern invasion speeds. Ultimately, our work provides deeper insight into the propagation dynamics of competitive biological invasions, highlighting that the minimal wave speed can be either the outcome of linear determinism or elevated by nonlinearities, depending on the strength of interspecific competition and the nature of dispersal.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there are no conflicts of interest.

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