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Research article

Mean-field backward stochastic Volterra integral equations with jumps and their applications to stochastic optimal control problems

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Abstract: Motivated by the stochastic optimal control problem, in this paper we systematically study a class of mean-field backward stochastic Volterra integral equations with jumps (MF-BSVIEs with jumps, for short). More precisely, by a special kind of BSVIEs with jumps, we present the well-posedness of MF-BSVIEs with jumps in the sense of adapted M-solutions. For completeness, both the well-posedness of mean-field forward stochastic Volterra integral equations with jumps (MF-FSVIEs with jumps, for short) and the comparison theorem for MF-BSVIEs with jumps are directly proved. Based on this, by virtue of the duality principle, we are able to establish Pontryagin's type maximum principle for optimal control problems of MF-FSVIEs with jumps. We illustrate our results with an application to the linear-quadratic optimal control problem of MF-FSVIEs with jumps.

Keywords: mean-field backward stochastic Volterra integral equation with jump; mean-field forward stochastic Volterra integral equation with jump; comparison theorem; stochastic optimal control; maximum principle

1. Introduction

Since stochastic optimal control theory can be used to deal with the problem of optimization under dynamic uncertainty, it has played an essential role in physical, biological, economic, and management systems. For example, Yılmaz et al. [1] applied a stochastic Runge-Kutta technique to study the optimal control of stochastic differential equations, which occurs in many areas of economics and finance and recently in cognitive sciences and neuroscience. A systematic account on the subject can be found in [2, 3]. In fact, to solve an optimal control problem, Bismut [4] initially studied a type of linear backward stochastic differential equations (linear BSDEs, for short). More importantly, Pardoux and Peng [5] successfully established the classical nonlinear BSDE in 1990, which is a generalization of the linear case. It is well known that the general theory of BSDEs has played a central role in mathematical finance, stochastic optimal control, partial differential equations [6–11], and numerous others. In 1994, Tang and Li [12] considered a class of BSDEs with jumps of the following form:

$$Y(t) = \xi + \int_t^T g\left(s, Y(s), Z(s), V(s,\cdot)\right) ds - \int_t^T Z(s)dW(s) - \int_t^T \int_{\mathbb{R}_0} V(s,e)\widetilde{N}(ds,de), \quad t \in [0,T], \tag{1.1}$$

where ξ is the terminal condition of (1.1), $g(\cdot)$ is the generator of (1.1), $W(\cdot)$ is a standard Brownian motion, and $\widetilde{N}(\cdot,\cdot)$ is an independent compensated Poisson random measure. The authors of [12] presented the existence and uniqueness theorem of the adapted solution to (1.1) and explored its application to an optimal control problem with a random jump. Afterward, Quenez and Sulem [13] proved a comparison theorem for BSDEs with jumps under quite weak assumptions, and then presented a representation for dynamic risk measures induced by BSDEs with jumps. Further developments on the subject were made in [14–18], to name a few.

It is worthy of pointing out that the mean-field systems happen to be used to describe physical systems involving a large number of interacting particles (see [19]) and thus have played a major role in the development of statistical mechanics, quantum mechanics, and quantum chemistry. More importantly, Buckdahn et al. [20, 21] first introduced the mean-field BSDEs and deeply investigated the properties of them. Shen and Siu [22] further proposed the existence and uniqueness result of mean-field BSDEs with jumps and solved a bicriteria mean-variance portfolio selection optimal control problem. Along this line, many researchers studied the mean-field BSDEs with jumps from various points of view. For example, Li [23] developed some properties of mean-field forward and backward stochastic differential equations with jumps; Li et al. [24] studied the theory of stochastic linear-quadratic mean-field delayed games with jumps. For this sake, combining the mean-field jump-diffusion BSDEs with the backward stochastic Volterra integral equations (BSVIEs, for short), this paper will focus on a type of mean-field BSVIEs with jumps (MF-BSVIEs with jumps, for short).

Inspired by the BSDEs, Lin [25] first considered the existence and uniqueness of the adapted solution to a special kind of BSVIEs. To solve the problem of gluing for the manufacturing polymetric material (see [26]), Yong [27] established the celebrated BSVIEs of the following form:

$$Y(t) = \psi(t) + \int_t^T g(t,s,Y(s),Z(t,s),Z(s,t))ds - \int_t^T Z(t,s)dW(s), \quad t \in [0,T], \tag{1.2}$$

where $g(\cdot)$ and $\psi(\cdot)$ are called the generator and the free term of (1.2), respectively. Since the Itô formula is invalid for the BSVIEs, by introducing the definition of the adapted M-solution, Yong [27] completely solved the well-posedness of BSVIEs. Further, Wang and Yong [28] systematically proved the comparison theorem for the BSVIEs. With an aim of developing stochastic optimal control [29–31], stochastic partial differential equations [32], and risk measurement [33, 34], there have been strong desires to extend the results of BSVIEs. By virtue of the duality principle, Shi et al. [35] proved the well-posedness of mean-field BSVIEs and obtained Pontryagin's type maximum principle for an optimal control of mean-field forward stochastic Volterra integral equations (MF-FSVIEs, for short). Thereafter, Popier [36] was concerned with a type of BSVIEs with jumps in a general filtration. Fu et al. [37] presented the well-posedness, the comparison theorem, and the regularity result of BSVIEs with jumps. More recently, as an application of BSVIEs, Gong and Wang [38] studied a unified treatment of a linear quadratic control problem for stochastic Volterra integral equations. Motivated by [35, 37], our paper will mainly investigate the properties and the applications of MF-BSVIEs with jumps.

Thanks to the profound theoretical significance and extensive applications, such as shown in Example 1 of our paper, we know that the BSVIEs can be used to study the stochastic control problem. Now, in order to solve the optimal control problem of MF-FSVIE with jump (5.1), it is naturally necessary to establish a more general duality principle. For this reason, this paper is devoted to studying the properties and the applications of MF-BSVIEs with jumps. More precisely, we are concerned with the following MF-BSVIEs with jumps:

$$Y(t)=\psi(t)+\int_t^T g\left(\Upsilon(t,s)\right)ds-\int_t^T Z(t,s)dW(s)-\int_t^T\int_{\mathbb{R}_0} V(t,s,e)\widetilde{N}(ds,de),\quad t\in[0,T],\tag{1.3}$$

where

$$\begin{aligned}\Upsilon(t,s):=&\,(t,s,Y(s),Z(t,s),Z(s,t),V(t,s,\cdot),V(s,t,\cdot),\mathbb{E}[Y(s)],\mathbb{E}[Z(t,s)],\mathbb{E}[Z(s,t)],\\&\,\mathbb{E}[V(t,s,\cdot)],\mathbb{E}[V(s,t,\cdot)]),\end{aligned}$$

and $g(\cdot)$ and $\psi(\cdot)$ are called the generator and the free term of (1.3), respectively. The central purpose of this paper is to establish the well-posedness and the comparison theorem of MF-BSVIE with jump (1.3) and propose their application to stochastic optimal controls. Let us outline the main innovation points of the current paper as follows. First, it should be mentioned that there are many special cases in our MF-BSVIE with jump (1.3). For instance, when $g(\cdot)$ is independent of $\mathbb{E}[Y(s)]$, $\mathbb{E}[Z(t,s)]$, $\mathbb{E}[Z(s,t)]$, $\mathbb{E}[V(t,s,\cdot)]$, and $\mathbb{E}[V(s,t,\cdot)]$, MF-BSVIE with jump (1.3) is reduced to the BSVIE with jump studied by [37]. When $\widetilde{N}(\cdot,\cdot)$ is absent and $g(\cdot)$ is independent of $V(t,s,\cdot)$, $V(s,t,\cdot)$, $\mathbb{E}[V(t,s,\cdot)]$, and $\mathbb{E}[V(s,t,\cdot)]$, MF-BSVIE with jump (1.3) is reduced to the mean-field BSVIE given by [35]. Moreover, we establish the well-posedness of the adapted M-solution to (1.3). By applying Lemma 1 in this paper, our method of proving the well-posedness of (1.3) is much simpler than the four-step method in [37]. Besides, both the well-posedness of MF-FSVIEs with jumps and the comparison theorem for MF-BSVIEs with jumps are directly proved, which cover the previous results for BSVIEs. In particular, since [35] did not study the comparison theorem, our result can make up for the lack of the comparison theorem in [35]. Finally, by virtue of the duality principle, we present Pontryagin's type maximum principle for optimal control problems of MF-FSVIEs with jumps. As one of the applications of the principle, we get an optimal control for the linear-quadratic optimal control problem of MF-FSVIEs with jumps. Compared with [35, 37], the area of applications of the MF-BSVIEs with jumps has been extended.

The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries. In Section 3, we are devoted to the study of the well-posedness of MF-BSVIEs with jumps and MF-FSVIEs with jumps. In Section 4, we establish the comparison theorem for MF-BSVIEs with jumps. An application to the optimal control problem driven by MF-FSVIEs with jumps is proposed in Section 5. Finally, Section 6 concludes the paper.

2. Preliminaries

Let $T>0$ be a fixed terminal time and $\mathcal{B}(D)$ be the Borel σ-field generated by any set D. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space with the natural filtration $\mathbb{F}:=\{\mathcal{F}_t;t\in[0,T]\}$ generated by the following two mutually independent stochastic processes and augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$:

(i) $\{W(t);t\in[0,T]\}$ is a one-dimensional standard Brownian motion on $(\Omega,\mathcal{F},\mathbb{P})$;

(*ii*) $\{N(dt,de);(t,e)\in[0,T]\times\mathbb{R}_0\}$ is a Poisson random measure on the product measurable space $([0,T]\times\mathbb{R}_0,\mathcal{B}([0,T])\otimes\mathcal{B}(\mathbb{R}_0))$, where $\mathbb{R}_0:=\mathbb{R}\backslash\{0\}$, with the compensator $\nu'(de,dt):=\nu(de)dt$ such that $\{\widetilde{N}((0,t];A):=(N-\nu')((0,t],A);t\in(0,T]\}$ is a martingale for all $A\in\mathcal{B}(\mathbb{R}_0)$ satisfying $\nu(A)<\infty$. Here, ν is a σ-finite Lévy measure on $(\mathbb{R}_0,\mathcal{B}(\mathbb{R}_0))$ with $\int_{\mathbb{R}_0}(1\wedge e^2)\nu(de)<\infty$.

Suppose that

$$\Delta:=\{(t,s)\in[0,T]^2\mid t\le s\},\quad \Delta^c:=\{(t,s)\in[0,T]^2\mid t>s\}.$$

$\mathcal{P}_{[0,T]}$ denotes the σ-field of $\mathbb{F}$-predictable sets on $\Omega\times[0,T]$, and for any $t\in[0,T]$, we define the following spaces:

$$L^2_{\mathcal{F}_t}(\Omega;\mathbb{R})=\left\{\xi:\Omega\mapsto\mathbb{R}\mid \xi \text{ is } \mathcal{F}_t\text{-measurable, } \mathbb{E}\left[|\xi|^2\right]<\infty\right\},$$

$$L^2_{\mathcal{F}_T}(0,T;\mathbb{R})=\left\{\zeta:\Omega\times[0,T]\mapsto\mathbb{R}\mid \zeta(\cdot)\text{ is } \mathcal{F}_T\otimes\mathcal{B}([0,T])\text{-measurable, } \mathbb{E}\left[\int_0^T|\zeta(t)|^2dt\right]<\infty\right\},$$

$$L^2_{\mathbb{F}}(0,T;\mathbb{R})=\left\{\zeta(\cdot)\in L^2_{\mathcal{F}_T}(0,T;\mathbb{R})\mid \zeta(\cdot)\text{ is càdlàg and } \mathbb{F}\text{-adapted}\right\},$$

$$\begin{aligned}L^2_{\mathbb{F}}(\Delta;\mathbb{R})=\{&\zeta:\Omega\times\Delta\mapsto\mathbb{R}\mid\zeta(\cdot,\cdot)\text{ is }\mathcal{F}_T\otimes\mathcal{B}(\Delta)\text{-measurable, and for any } t\in[0,T],\\ &s\mapsto\zeta(t,s)\text{ is } \mathbb{F}\text{-predictable on } [t,T],\ \mathbb{E}\left[\int_0^T\int_t^T|\zeta(t,s)|^2dsdt\right]<\infty\right\},\end{aligned}$$

$$\begin{aligned}L^2_{\mathbb{F}}([0,T]^2;\mathbb{R})=\Big\{&\zeta:\Omega\times[0,T]^2\mapsto\mathbb{R}\mid\zeta(\cdot,\cdot)\text{ is }\mathcal{F}_T\otimes\mathcal{B}([0,T]^2)\text{-measurable, and for any } t\in[0,T],\\ &s\mapsto\zeta(t,s)\text{ is } \mathbb{F}\text{-predictable on } [0,T],\ \mathbb{E}\left[\int_0^T\int_0^T|\zeta(t,s)|^2dsdt\right]<\infty\Big\},\end{aligned}$$

$$L^2_{\nu}(\mathbb{R})=\left\{\vartheta:\mathbb{R}_0\mapsto\mathbb{R}\mid\vartheta(\cdot)\text{ is }\mathcal{B}(\mathbb{R}_0)\text{-measurable, } \|\vartheta\|^2_\nu:=\int_{\mathbb{R}_0}|\vartheta(e)|^2\nu(de)<\infty\right\},$$

$$\begin{aligned}L^2_N(0,T;\mathbb{R})=\{&\vartheta:\Omega\times[0,T]\times\mathbb{R}_0\mapsto\mathbb{R}\mid\vartheta(\cdot,\cdot)\text{ is }\mathcal{P}_{[0,T]}\otimes\mathcal{B}(\mathbb{R}_0)\text{-measurable,}\\ &\mathbb{E}\left[\int_0^T\|\vartheta(t,\cdot)\|^2_\nu dt\right]<\infty\right\},\end{aligned}$$

$$\begin{aligned}L^2_N(\Delta;\mathbb{R})=\{&\vartheta:\Omega\times\Delta\times\mathbb{R}_0\mapsto\mathbb{R}\mid\vartheta(\cdot,\cdot,\cdot)\text{ is }\mathcal{F}_T\otimes\mathcal{B}(\Delta)\otimes\mathcal{B}(\mathbb{R}_0)\text{-measurable, and for any } t\in[0,T],\\ &s\mapsto\vartheta(t,s,\cdot)\text{ is } \mathbb{F}\text{-predictable on } [t,T],\ \mathbb{E}\left[\int_0^T\int_t^T\|\vartheta(t,s,\cdot)\|^2_\nu dsdt\right]<\infty\right\},\end{aligned}$$

$$\begin{aligned}L^2_N([0,T]^2;\mathbb{R})=\Big\{&\vartheta:\Omega\times[0,T]^2\times\mathbb{R}_0\mapsto\mathbb{R}\mid\vartheta(\cdot,\cdot,\cdot)\text{ is }\mathcal{F}_T\otimes\mathcal{B}([0,T]^2)\otimes\mathcal{B}(\mathbb{R}_0)\text{-measurable,}\\ &\text{and for any } t\in[0,T],\ s\mapsto\vartheta(t,s,\cdot)\text{ is } \mathbb{F}\text{-predictable on } [0,T],\\ &\mathbb{E}\left[\int_0^T\int_0^T\|\vartheta(t,s,\cdot)\|^2_\nu dsdt\right]<\infty\Big\}.\end{aligned}$$

Clearly, for any $0\le t\le T$, $L^2_{\mathbb{F}}(0,t;\mathbb{R})$, $L^2_{\mathbb{F}}(t,T;\mathbb{R})$, $L^2_{\mathbb{F}}(\Delta^c;\mathbb{R})$, $L^2_N(\Delta^c;\mathbb{R})$, $L^\infty_{\mathbb{F}}([0,T]^2;\mathbb{R})$, and $L^\infty_N([0,T]^2;\mathbb{R})$ have similar definitions.

3. Well-posedness of the solution

In this section, let us establish the well-posedness of the solution to MF-BSVIEs with jumps and MF-FSVIEs with jumps, respectively.

3.1. Well-posedness of MF-BSVIEs with jumps

In this subsection, we are devoted to the study of the well-posedness of adapted M-solutions to MF-BSVIEs with jumps, and their forms are as follows:

$$Y(t) = \psi(t) + \int_t^T g(\Upsilon(t,s))\, ds - \int_t^T Z(t,s)dW(s) - \int_t^T \int_{\mathbb{R}_0} V(t,s,e)\widetilde{N}(ds,de), \quad t \in [0,T], \tag{3.1}$$

where

$$\begin{aligned}\Upsilon(t,s) := (&t, s, Y(s), Z(t,s), Z(s,t), V(t,s,\cdot), V(s,t,\cdot), \mathbb{E}[Y(s)], \mathbb{E}[Z(t,s)], \mathbb{E}[Z(s,t)],\\ &\mathbb{E}[V(t,s,\cdot)], \mathbb{E}[V(s,t,\cdot)]).\end{aligned}$$

Here, $W(\cdot)$ is a standard Brownian motion, $\widetilde{N}(\cdot,\cdot)$ is an independent compensated Poisson random measure, $T > 0$ is a fixed terminal time, and $\psi(\cdot) \in L^2_{\mathcal{F}_T}(0,T;\mathbb{R})$. $g : \Omega\times\Delta\times(\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times L^2_\nu(\mathbb{R})\times L^2_\nu(\mathbb{R}))^2 \mapsto \mathbb{R}$ is $\mathcal{F}_T\otimes\mathcal{B}(\Delta\times(\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times L^2_\nu(\mathbb{R})\times L^2_\nu(\mathbb{R}))^2)$-measurable such that $s \mapsto g(t,s,y,z,z',\upsilon,\upsilon',\overline{y},\overline{z},\overline{z'},\overline{\upsilon},\overline{\upsilon'})$ is $\mathbb{F}$-progressively measurable for all $(t,y,z,z',\upsilon,\upsilon',\overline{y},\overline{z},\overline{z'},\overline{\upsilon},\overline{\upsilon'}) \in [0,T]\times(\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times L^2_\nu(\mathbb{R})\times L^2_\nu(\mathbb{R}))^2$.

In what follows, let us make the following assumptions on the generator g of MF-BSVIE with jump (3.1):

(A1) There exists a constant $C > 0$ such that for almost all $(\omega,t,s) \in \Omega\times\Delta$, and for any $y_1, y_2, z_1, z_2, z'_1, z'_2, \overline{y}_1, \overline{y}_2, \overline{z}_1, \overline{z}_2, \overline{z'}_1, \overline{z'}_2 \in \mathbb{R}$, $\upsilon_1, \upsilon_2, \upsilon'_1, \upsilon'_2, \overline{\upsilon}_1, \overline{\upsilon}_2, \overline{\upsilon'}_1, \overline{\upsilon'}_2 \in L^2_\nu(\mathbb{R})$,

$$\begin{aligned}&\left|g(\omega,t,s,y_1,z_1,z'_1,\upsilon_1,\upsilon'_1,\overline{y}_1,\overline{z}_1,\overline{z'}_1,\overline{\upsilon}_1,\overline{\upsilon'}_1) - g(\omega,t,s,y_2,z_2,z'_2,\upsilon_2,\upsilon'_2,\overline{y}_2,\overline{z}_2,\overline{z'}_2,\overline{\upsilon}_2,\overline{\upsilon'}_2)\right|\\ &\le C\Big(|y_1-y_2| + |z_1-z_2| + |z'_1-z'_2| + \|\upsilon_1-\upsilon_2\|_\nu + \|\upsilon'_1-\upsilon'_2\|_\nu + |\overline{y}_1-\overline{y}_2| + |\overline{z}_1-\overline{z}_2| + |\overline{z'}_1-\overline{z'}_2|\\ &\quad + \|\overline{\upsilon}_1-\overline{\upsilon}_2\|_\nu + \|\overline{\upsilon'}_1-\overline{\upsilon'}_2\|_\nu\Big).\end{aligned}$$

Furthermore,
$$\mathbb{E}\left[\int_0^T \int_t^T |g_0(t,s)|^2 dsdt\right] < \infty,$$
where $g_0(t,s) := g(t,s,0,0,0,0,0,0,0,0,0,0)$.

Definition 1. *A triple of processes* $(Y(\cdot), Z(\cdot,\cdot), V(\cdot,\cdot,\cdot)) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}) \times L^2_{\mathbb{F}}([0,T]^2;\mathbb{R}) \times L^2_N([0,T]^2;\mathbb{R})$ *is called an adapted M-solution of MF-BSVIE with jump (3.1) if (3.1) holds in the usual Itô's sense for almost everywhere all* $t \in [0,T]$*, and for any* $S \in [0,T]$*, the following relation holds:*

$$Y(t) = \mathbb{E}[Y(t)|\mathcal{F}_S] + \int_S^t Z(t,s)dW(s) + \int_S^t \int_{\mathbb{R}_0} V(t,s,e)\widetilde{N}(ds,de), \quad a.e.\ t \in [S,T]. \tag{3.2}$$

By virtue of Definition 1, we introduce some spaces to be used throughout this paper. For any $\beta > 0$, let $\mathcal{H}^2_\beta[0,T]$ denote the Banach space of all triple of processes $(Y(\cdot), Z(\cdot,\cdot), V(\cdot,\cdot,\cdot)) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}) \times L^2_{\mathbb{F}}([0,T]^2;\mathbb{R}) \times L^2_N([0,T]^2;\mathbb{R})$ such that

$$\begin{aligned}&\|(Y(\cdot), Z(\cdot,\cdot), V(\cdot,\cdot,\cdot))\|^2_{\mathcal{H}^2_\beta[0,T]}\\ &:= \mathbb{E}\left[\int_0^T e^{\beta t}|Y(t)|^2 dt + \int_0^T\int_0^T e^{\beta s}|Z(t,s)|^2 dsdt + \int_0^T\int_0^T e^{\beta s}\|V(t,s,\cdot)\|^2_\nu dsdt\right] < \infty.\end{aligned}$$

For any $\beta > 0$, let $\mathcal{M}^2_\beta[0,T]$ denote the Banach space of all triple of processes $(Y(\cdot), Z(\cdot,\cdot), V(\cdot,\cdot,\cdot)) \in \mathcal{H}^2_\beta[0,T]$ such that (3.2) holds. Obviously, for any $(Y(\cdot), Z(\cdot,\cdot), V(\cdot,\cdot,\cdot)) \in \mathcal{M}^2_\beta[0,T]$, it follows from (3.2) that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{\beta t}|Y(t)|^2 dt + \int_0^T\int_0^T e^{\beta s}|Z(t,s)|^2 dsdt + \int_0^T\int_0^T e^{\beta s}\|V(t,s,\cdot)\|^2_\nu dsdt\right]\\
&\le \mathbb{E}\left[\int_0^T e^{\beta t}|Y(t)|^2 dt + \int_0^T\int_t^T e^{\beta s}|Z(t,s)|^2 dsdt + \int_0^T\int_t^T e^{\beta s}\|V(t,s,\cdot)\|^2_\nu dsdt\right.\\
&\quad\left. + \int_0^T e^{\beta t}\int_0^t |Z(t,s)|^2 dsdt + \int_0^T e^{\beta t}\int_0^t \|V(t,s,\cdot)\|^2_\nu dsdt\right]\\
&= \mathbb{E}\left[\int_0^T e^{\beta t}|Y(t)|^2 dt + \int_0^T\int_t^T e^{\beta s}|Z(t,s)|^2 dsdt + \int_0^T\int_t^T e^{\beta s}\|V(t,s,\cdot)\|^2_\nu dsdt\right]\\
&\quad + \mathbb{E}\left[\int_0^T e^{\beta t}\left(\int_0^t Z(t,s)dW(s) + \int_0^t\int_{\mathbb{R}_0} V(t,s,e)\widetilde{N}(ds,de)\right)^2 dt\right]\\
&= \mathbb{E}\left[\int_0^T e^{\beta t}|Y(t)|^2 dt + \int_0^T\int_t^T e^{\beta s}|Z(t,s)|^2 dsdt + \int_0^T\int_t^T e^{\beta s}\|V(t,s,\cdot)\|^2_\nu dsdt\right]\\
&\quad + \mathbb{E}\left[\int_0^T e^{\beta t}\left(Y(t) - \mathbb{E}[Y(t)]\right)^2 dt\right]\\
&\le 2\mathbb{E}\left[\int_0^T e^{\beta t}|Y(t)|^2 dt + \int_0^T\int_t^T e^{\beta s}|Z(t,s)|^2 dsdt + \int_0^T\int_t^T e^{\beta s}\|V(t,s,\cdot)\|^2_\nu dsdt\right],
\end{aligned} \tag{3.3}$$

where we use the fact that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{\beta t}\left(Y(t) - \mathbb{E}[Y(t)]\right)^2 dt\right] = \mathbb{E}\left[\int_0^T e^{\beta t}\left(|Y(t)|^2 - 2Y(t)\mathbb{E}[Y(t)] + |\mathbb{E}[Y(t)]|^2\right) dt\right]\\
&= \int_0^T e^{\beta t}\left(\mathbb{E}[|Y(t)|^2] - 2\mathbb{E}[Y(t)\mathbb{E}[Y(t)]] + \mathbb{E}[|\mathbb{E}[Y(t)]|^2]\right) dt\\
&= \int_0^T e^{\beta t}\left(\mathbb{E}[|Y(t)|^2] - |\mathbb{E}[Y(t)]|^2\right) dt \le \mathbb{E}\left[\int_0^T e^{\beta t}|Y(t)|^2 dt\right].
\end{aligned} \tag{3.4}$$

For this, we define an equivalent norm for $\mathcal{M}^2_\beta[0,T]$ as follows:

$$\begin{aligned}
&\|(Y(\cdot), Z(\cdot,\cdot), V(\cdot,\cdot,\cdot))\|^2_{\mathcal{M}^2_\beta[0,T]}\\
&:= \mathbb{E}\left[\int_0^T e^{\beta t}|Y(t)|^2 dt + \int_0^T\int_t^T e^{\beta s}|Z(t,s)|^2 dsdt + \int_0^T\int_t^T e^{\beta s}\|V(t,s,\cdot)\|^2_\nu dsdt\right].
\end{aligned} \tag{3.5}$$

In order to prove the well-posedness of adapted M-solutions to MF-BSVIE with jump (3.1), we first consider a special case when g is a map from $\Omega \times \Delta$ to $\mathbb{R}$.

Lemma 1. *For any $\psi(\cdot) \in L^2_{\mathcal{F}_T}(0,T;\mathbb{R})$ and $g(\cdot,\cdot) \in L^2_{\mathbb{F}}(\Delta;\mathbb{R})$, there exists a unique adapted M-solution $(Y(\cdot), Z(\cdot,\cdot), V(\cdot,\cdot,\cdot)) \in \mathcal{M}^2_\beta[0,T]$ solving the following BSVIE with jump:*

$$Y(t) = \psi(t) + \int_t^T g(t,s)ds - \int_t^T Z(t,s)dW(s) - \int_t^T\int_{\mathbb{R}_0} V(t,s,e)\widetilde{N}(ds,de), \quad t \in [0,T], \tag{3.6}$$

and the following estimate holds:

$$\begin{aligned}&\mathbb{E}\left[\int_0^T e^{\beta t}|Y(t)|^2dt+\int_0^T\int_t^T e^{\beta s}|Z(t,s)|^2dsdt+\int_0^T\int_t^T e^{\beta s}\|V(t,s,\cdot)\|_\nu^2dsdt\right]\\&\leq\mathbb{E}\left[\int_0^T e^{\beta T}|\psi(t)|^2dt\right]+\frac{1}{\beta}\mathbb{E}\left[\int_0^T\int_t^T e^{\beta s}|g(t,s)|^2dsdt\right],\end{aligned}\tag{3.7}$$

where $\beta > 0$ is an arbitrary constant.

Proof. According to the equivalent norm defined by (3.5) in $\mathcal{M}_\beta^2[0,T]$, we know that the norm $\|\cdot\|_{\mathcal{M}_\beta^2[0,T]}$ with $\beta = 0$ given in [37] is equivalent to the norm $\|\cdot\|_{\mathcal{M}_\beta^2[0,T]}$ with $\beta > 0$ given in our paper. Thus, from Lemma 3.1 in [37], it is clear that (3.6) admits a unique adapted M-solution $(Y(\cdot), Z(\cdot,\cdot), V(\cdot,\cdot,\cdot)) \in \mathcal{M}_\beta^2[0,T]$.

In the sequel, we shall prove that the estimate (3.7) holds. In fact, for any $\psi(\cdot) \in L^2_{\mathcal{F}_T}(0,T;\mathbb{R})$ and $g(\cdot,\cdot) \in L^2_{\mathbb{F}}(\Delta;\mathbb{R})$, by the Fubini theorem, it holds that

$$\mathbb{E}\left[|Y(t)|^2+\int_t^T|g(t,s)|^2ds+\int_t^T|Z(t,s)|^2ds+\int_t^T\|V(t,s,\cdot)\|_\nu^2ds\right]<\infty,\quad\forall\, t\in[0,T].\tag{3.8}$$

For any fixed $t \in [0,T]$, we define

$$M_t(s):=Y(t)-\int_t^s g(t,r)dr+\int_t^s Z(t,r)dW(r)+\int_t^s\int_{\mathbb{R}_0}V(t,r,e)\widetilde{N}(dr,de),\quad s\in[t,T].$$

For each $t \in [0,T]$, it immediately follows from (3.8) that $\{M_t(s), s \in [t,T]\}$ is a semimartingale in $L^2_{\mathbb{F}}(t,T;\mathbb{R})$. Note that $M_t(T) = \psi(t)$ and $M_t(t) = Y(t)$ for any $t \in [0,T]$. Applying the Itô formula for $s \mapsto e^{\beta(s-t)}|M_t(s)|^2$ on $[t,T]$, we have the following:

$$\begin{aligned}e^{\beta(s-t)}|M_t(s)|^2&=|Y(t)|^2+\int_t^s\beta e^{\beta(r-t)}|M_t(r)|^2dr+\int_t^s 2e^{\beta(r-t)}M_t(r)\left(-g(t,r)dr+Z(t,r)dW(r)\right)\\&\quad+\int_t^s e^{\beta(r-t)}|Z(t,r)|^2dr+\int_t^s\int_{\mathbb{R}_0}e^{\beta(r-t)}|V(t,r,e)|^2\nu(de)dr\\&\quad+\int_t^s\int_{\mathbb{R}_0}e^{\beta(r-t)}\left(|M_t(r-)+V(t,r,e)|^2-|M_t(r-)|^2\right)\widetilde{N}(dr,de),\quad\forall s\in[t,T].\end{aligned}$$

Letting $s = T$ and taking the expectation, we obtain that

$$\begin{aligned}&\mathbb{E}\left[e^{\beta t}|Y(t)|^2+\int_t^T e^{\beta s}|Z(t,s)|^2ds+\int_t^T e^{\beta s}\|V(t,s,\cdot)\|_\nu^2ds\right]\\&=\mathbb{E}\left[e^{\beta T}|\psi(t)|^2-\int_t^T\beta e^{\beta s}|M_t(s)|^2ds+2\int_t^T e^{\beta s}M_t(s)g(t,s)ds\right].\end{aligned}$$

By the Young inequality (i.e., $2ab \leq \beta a^2 + \frac{1}{\beta}b^2$, $\forall a,b \in \mathbb{R}, \beta > 0$), we obtain

$$\begin{aligned}&\mathbb{E}\left[e^{\beta t}|Y(t)|^2+\int_t^T e^{\beta s}|Z(t,s)|^2ds+\int_t^T e^{\beta s}\|V(t,s,\cdot)\|_\nu^2ds\right]\\&\leq\mathbb{E}\left[e^{\beta T}|\psi(t)|^2-\int_t^T\beta e^{\beta s}|M_t(s)|^2ds+\int_t^T\beta e^{\beta s}|M_t(s)|^2ds+\frac{1}{\beta}\int_t^T e^{\beta s}|g(t,s)|^2ds\right]\\&=\mathbb{E}\left[e^{\beta T}|\psi(t)|^2\right]+\frac{1}{\beta}\left[\int_t^T e^{\beta s}|g(t,s)|^2ds\right],\end{aligned}$$

which means that

$$\mathbb{E}\left[\int_0^T e^{\beta t}|Y(t)|^2dt + \int_0^T\int_t^T e^{\beta s}|Z(t,s)|^2dsdt + \int_0^T\int_t^T e^{\beta s}\|V(t,s,\cdot)\|_\nu^2dsdt\right]$$
$$\leq \mathbb{E}\left[\int_0^T e^{\beta T}|\psi(t)|^2dt\right] + \frac{1}{\beta}\mathbb{E}\left[\int_0^T\int_t^T e^{\beta s}|g(t,s)|^2dsdt\right].$$

This proves (3.7). □

We are now ready to prove our main result of this subsection, Theorem 1, as follows.

Theorem 1. *Let Assumption* (A1) *hold. Then for any* $\psi(\cdot) \in L^2_{\mathcal{F}_T}(0,T;\mathbb{R})$*, MF-BSVIE with jump (3.1) admits a unique adapted M-solution* $(Y(\cdot), Z(\cdot,\cdot), V(\cdot,\cdot,\cdot)) \in \mathcal{M}^2_\beta[0,T]$*. Moreover, there exists a constant* $K > 0$ *such that*

$$\begin{aligned}
&\|(Y(\cdot), Z(\cdot,\cdot), V(\cdot,\cdot,\cdot))\|^2_{\mathcal{M}^2_\beta[0,T]}\\
&:= \mathbb{E}\left[\int_0^T e^{\beta t}|Y(t)|^2dt + \int_0^T\int_t^T e^{\beta s}|Z(t,s)|^2dsdt + \int_0^T\int_t^T e^{\beta s}\|V(t,s,\cdot)\|_\nu^2dsdt\right]\\
&\leq K\mathbb{E}\left[\int_0^T e^{\beta T}|\psi(t)|^2dt + \int_0^T\int_t^T e^{\beta s}|g_0(t,s)|^2dsdt\right],
\end{aligned} \tag{3.9}$$

where $g_0(t,s) := g(t,s,0,0,0,0,0,0,0,0,0,0)$.

Proof. For any given $(y(\cdot), z(\cdot,\cdot), \upsilon(\cdot,\cdot,\cdot)) \in \mathcal{M}^2_\beta[0,T]$, we consider the following equation:

$$\begin{aligned}
Y(t) = \psi(t) + \int_t^T g\,(t,s,y(s),z(t,s),z(s,t),\upsilon(t,s,\cdot),\upsilon(s,t,\cdot),\mathbb{E}[y(s)],\mathbb{E}[z(t,s)],\mathbb{E}[z(s,t)],\\
\mathbb{E}[\upsilon(t,s,\cdot)],\mathbb{E}[\upsilon(s,t,\cdot)])\,ds - \int_t^T Z(t,s)dW(s) - \int_t^T\int_{\mathbb{R}_0} V(t,s,e)\widetilde{N}(ds,de), \quad t\in[0,T].
\end{aligned} \tag{3.10}$$

By Lemma 1, there exists a unique adapted M-solution $(Y(\cdot), Z(\cdot,\cdot), V(\cdot,\cdot,\cdot)) \in \mathcal{M}^2_\beta[0,T]$ of BSVIE with jump (3.10). Under Assumption (A1), by (3.3) and (3.7), and from $|a_1+\cdots+a_n|^2 \leq n(|a_1|^2+\cdots+|a_n|^2)$ and Jensen's inequality, it holds that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{\beta t}|Y(t)|^2dt + \int_0^T\int_t^T e^{\beta s}|Z(t,s)|^2dsdt + \int_0^T\int_t^T e^{\beta s}\|V(t,s,\cdot)\|_\nu^2dsdt\right]\\
&\leq \mathbb{E}\left[\int_0^T e^{\beta T}|\psi(t)|^2dt\right] + \frac{1}{\beta}\mathbb{E}\Bigg[\int_0^T\int_t^T e^{\beta s}\,|g\,(t,s,y(s),z(t,s),z(s,t),\upsilon(t,s,\cdot),\upsilon(s,t,\cdot),\mathbb{E}[y(s)],\mathbb{E}[z(t,s)],\\
&\quad \mathbb{E}[z(s,t)],\mathbb{E}[\upsilon(t,s,\cdot)],\mathbb{E}[\upsilon(s,t,\cdot)])|^2\,dsdt\Bigg]
\end{aligned} \tag{3.11}$$

$$
\begin{aligned}
&\leq \mathbb{E}\left[\int_0^T e^{\beta T}|\psi(t)|^2dt\right] + \frac{2}{\beta}\mathbb{E}\left[\int_0^T\int_t^T e^{\beta s}\,|g\,(t,s,y(s),z(t,s),z(s,t),\upsilon(t,s,\cdot),\upsilon(s,t,\cdot),\mathbb{E}[y(s)],\mathbb{E}[z(t,s)],\right.\\
&\quad\left.\mathbb{E}[z(s,t)],\mathbb{E}[\upsilon(t,s,\cdot)],\mathbb{E}[\upsilon(s,t,\cdot)]) - g_0(t,s)|^2\,dsdt\right] + \frac{2}{\beta}\mathbb{E}\left[\int_0^T\int_t^T e^{\beta s}\,|g_0(t,s)|^2\,dsdt\right]\\
&\leq \mathbb{E}\left[\int_0^T e^{\beta T}|\psi(t)|^2dt\right] + \frac{2}{\beta}\mathbb{E}\left[\int_0^T\int_t^T e^{\beta s}|g_0(t,s)|^2dsdt\right] + \frac{20C^2}{\beta}\mathbb{E}\left[\int_0^T\int_t^T e^{\beta s}\left(|y(s)|^2+|z(t,s)|^2\right.\right.\\
&\quad+|z(s,t)|^2+\|\upsilon(t,s,\cdot)\|_\nu^2+\|\upsilon(s,t,\cdot)\|_\nu^2+|\mathbb{E}[y(s)]|^2+|\mathbb{E}[z(t,s)]|^2+|\mathbb{E}[z(s,t)]|^2+\|\mathbb{E}[\upsilon(t,s,\cdot)]\|_\nu^2\\
&\quad\left.\left.+\|\mathbb{E}[\upsilon(s,t,\cdot)]\|_\nu^2\right)dsdt\right]\\
&\leq \mathbb{E}\left[\int_0^T e^{\beta T}|\psi(t)|^2dt\right] + \frac{2}{\beta}\mathbb{E}\left[\int_0^T\int_t^T e^{\beta s}|g_0(t,s)|^2dsdt\right] + \frac{40C^2}{\beta}\mathbb{E}\left[\int_0^T\int_t^T e^{\beta s}\left(|y(s)|^2+|z(t,s)|^2\right.\right.\\
&\quad\left.\left.+|z(s,t)|^2+\|\upsilon(t,s,\cdot)\|_\nu^2+\|\upsilon(s,t,\cdot)\|_\nu^2\right)dsdt\right]\\
&= \mathbb{E}\left[\int_0^T e^{\beta T}|\psi(t)|^2dt\right] + \frac{2}{\beta}\mathbb{E}\left[\int_0^T\int_t^T e^{\beta s}|g_0(t,s)|^2dsdt\right] + \frac{40C^2}{\beta}\mathbb{E}\left[\int_0^T\int_t^T e^{\beta s}\left(|y(s)|^2+|z(t,s)|^2\right.\right.\\
&\quad\left.\left.+\|\upsilon(t,s,\cdot)\|_\nu^2\right)dsdt\right] + \frac{40C^2}{\beta}\mathbb{E}\left[\int_0^T\int_0^s e^{\beta s}\left(|z(s,t)|^2+\|\upsilon(s,t,\cdot)\|_\nu^2\right)dtds\right]\\
&= \mathbb{E}\left[\int_0^T e^{\beta T}|\psi(t)|^2dt\right] + \frac{2}{\beta}\mathbb{E}\left[\int_0^T\int_t^T e^{\beta s}|g_0(t,s)|^2dsdt\right] + \frac{40C^2}{\beta}\mathbb{E}\left[\int_0^T\int_t^T e^{\beta s}\left(|y(s)|^2+|z(t,s)|^2\right.\right.\\
&\quad\left.\left.+\|\upsilon(t,s,\cdot)\|_\nu^2\right)dsdt\right] + \frac{40C^2}{\beta}\mathbb{E}\left[\int_0^T e^{\beta t}\left(\int_0^t\left(|z(t,s)|^2+\|\upsilon(t,s,\cdot)\|_\nu^2\right)ds\right)dt\right]\\
&\leq \mathbb{E}\left[\int_0^T e^{\beta T}|\psi(t)|^2dt\right] + \frac{2}{\beta}\mathbb{E}\left[\int_0^T\int_t^T e^{\beta s}|g_0(t,s)|^2dsdt\right] + \frac{40(T+1)C^2}{\beta}\mathbb{E}\left[\int_0^T e^{\beta t}|y(t)|^2dt\right]\\
&\quad+\frac{40C^2}{\beta}\mathbb{E}\left[\int_0^T\int_t^T e^{\beta s}\left(|z(t,s)|^2+\|\upsilon(t,s,\cdot)\|_\nu^2\right)dsdt\right] < \infty.
\end{aligned}
$$

Therefore, we can define an operator $\mathcal{L}$ from $\mathcal{M}_\beta^2[0,T]$ to $\mathcal{M}_\beta^2[0,T]$ by

$$
\mathcal{L}(y(\cdot),z(\cdot,\cdot),\upsilon(\cdot,\cdot,\cdot)) = (Y(\cdot),Z(\cdot,\cdot),V(\cdot,\cdot,\cdot)),\ \forall\ (y(\cdot),z(\cdot,\cdot),\upsilon(\cdot,\cdot,\cdot)) \in \mathcal{M}_\beta^2[0,T].
$$

We are going to prove that $\mathcal{L}$ is a contractive map on the Banach space $\mathcal{M}_\beta^2[0,T]$. To this end, for $i=1,2$, take any $(y_i(\cdot),z_i(\cdot,\cdot),\upsilon_i(\cdot,\cdot,\cdot)) \in \mathcal{M}_\beta^2[0,T]$, and let $(Y_i(\cdot),Z_i(\cdot,\cdot),V_i(\cdot,\cdot,\cdot)) = \mathcal{L}(y_i(\cdot),z_i(\cdot,\cdot),\upsilon_i(\cdot,\cdot,\cdot))$. Define

$$
\begin{aligned}
(\Delta y(\cdot),\Delta z(\cdot,\cdot),\Delta\upsilon(\cdot,\cdot,\cdot)) &:= (y_1(\cdot)-y_2(\cdot),z_1(\cdot,\cdot)-z_2(\cdot,\cdot),\upsilon_1(\cdot,\cdot,\cdot)-\upsilon_2(\cdot,\cdot,\cdot)),\\
(\Delta Y(\cdot),\Delta Z(\cdot,\cdot),\Delta V(\cdot,\cdot,\cdot)) &:= (Y_1(\cdot)-Y_2(\cdot),Z_1(\cdot,\cdot)-Z_2(\cdot,\cdot),V_1(\cdot,\cdot,\cdot)-V_2(\cdot,\cdot,\cdot)).
\end{aligned}
$$

Then, thanks to (3.7) and Assumption $(A1)$, we obtain the following:

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{\beta t}|\Delta Y(t)|^2dt+\int_0^T\int_t^T e^{\beta s}|\Delta Z(t,s)|^2dsdt+\int_0^T\int_t^T e^{\beta s}\|\Delta V(t,s,\cdot)\|_\nu^2dsdt\right]\\
&\leq \frac{1}{\beta}\mathbb{E}\left[\int_0^T\int_t^T\left(e^{\beta s}\,|g\,(t,s,y_1(s),z_1(t,s),z_1(s,t),\upsilon_1(t,s,\cdot),\upsilon_1(s,t,\cdot),\mathbb{E}[y_1(s)],\mathbb{E}[z_1(t,s)],\right.\right. \qquad (3.12)\\
&\quad\mathbb{E}[z_1(s,t)],\mathbb{E}[\upsilon_1(t,s,\cdot)],\mathbb{E}[\upsilon_1(s,t,\cdot)]) - g\,(t,s,y_2(s),z_2(t,s),z_2(s,t),\upsilon_2(t,s,\cdot),\upsilon_2(s,t,\cdot),
\end{aligned}
$$

$$\mathbb{E}[y_2(s)],\mathbb{E}[z_2(t,s)],\mathbb{E}[z_2(s,t)],\mathbb{E}[\upsilon_2(t,s,\cdot)],\mathbb{E}[\upsilon_2(s,t,\cdot)])|^2\Big)\,dsdt\Big]$$
$$\leq \frac{10C^2}{\beta}\mathbb{E}\left[\int_0^T\int_t^T e^{\beta s}\Big(|\Delta y(s)|^2+|\Delta z(t,s)|^2+|\Delta z(s,t)|^2+\|\Delta\upsilon(t,s,\cdot)\|_\nu^2+\|\Delta\upsilon(s,t,\cdot)\|_\nu^2\right.$$
$$\left.+|\mathbb{E}[\Delta y(s)]|^2+|\mathbb{E}[\Delta z(t,s)]|^2+|\mathbb{E}[\Delta z(s,t)]|^2+\|\mathbb{E}[\Delta\upsilon(t,s,\cdot)]\|_\nu^2+\|\mathbb{E}[\Delta\upsilon(s,t,\cdot)]\|_\nu^2\Big)\,dsdt\right].$$

From (3.2) and (3.4), it follows that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T\int_t^T e^{\beta s}\Big(|\Delta z(s,t)|^2+\|\Delta\upsilon(s,t,\cdot)\|_\nu^2\Big)\,dsdt\right]\\
&=\mathbb{E}\left[\int_0^T\int_0^s e^{\beta s}\Big(|\Delta z(s,t)|^2+\|\Delta\upsilon(s,t,\cdot)\|_\nu^2\Big)\,dtds\right]\\
&=\mathbb{E}\left[\int_0^T e^{\beta t}\left(\int_0^t\Big(|\Delta z(t,s)|^2+\|\Delta\upsilon(t,s,\cdot)\|_\nu^2\Big)\,ds\right)dt\right]\\
&=\mathbb{E}\left[\int_0^T e^{\beta t}\left(\int_0^t\Delta z(t,s)dW(s)+\int_0^t\int_{\mathbb{R}_0}\Delta\upsilon(t,s,e)\widetilde{N}(ds,de)\right)^2dt\right]\\
&=\mathbb{E}\left[\int_0^T e^{\beta t}\,|\Delta y(t)-\mathbb{E}[\Delta y(t)]|^2\,dt\right]\\
&\leq\mathbb{E}\left[\int_0^T e^{\beta t}|\Delta y(t)|^2dt\right].
\end{aligned}\tag{3.13}$$

By Jensen's inequality and (3.13), one has

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T\int_t^T e^{\beta s}\Big(|\mathbb{E}[\Delta z(s,t)]|^2+\|\mathbb{E}[\Delta\upsilon(s,t,\cdot)]\|_\nu^2\Big)\,dsdt\right]\\
&=\mathbb{E}\left[\int_0^T e^{\beta t}\left(\int_0^t\Big(|\mathbb{E}[\Delta z(t,s)]|^2+\|\mathbb{E}[\Delta\upsilon(t,s,\cdot)]\|_\nu^2\Big)\,ds\right)dt\right]\\
&\leq\mathbb{E}\left[\int_0^T e^{\beta t}\left(\int_0^t\Big(|\Delta z(t,s)|^2+\|\Delta\upsilon(t,s,\cdot)\|_\nu^2\Big)\,ds\right)dt\right]\\
&\leq\mathbb{E}\left[\int_0^T e^{\beta t}|\Delta y(t)|^2dt\right].
\end{aligned}\tag{3.14}$$

Thus, combining (3.13) and (3.14), and by Jensen's inequality, (3.12) implies that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{\beta t}|\Delta Y(t)|^2dt+\int_0^T\int_t^T e^{\beta s}|\Delta Z(t,s)|^2dsdt+\int_0^T\int_t^T e^{\beta s}\|\Delta V(t,s,\cdot)\|_\nu^2dsdt\right]\\
&\leq\frac{10C^2}{\beta}\left\{(T+2)\mathbb{E}\left[\int_0^T e^{\beta t}|\Delta y(t)|^2dt\right]+T\mathbb{E}\left[\int_0^T e^{\beta t}|\mathbb{E}[\Delta y(t)]|^2dt\right]\right.\\
&\quad+\mathbb{E}\left[\int_0^T\int_t^T e^{\beta s}\Big(|\Delta z(t,s)|^2+\|\Delta\upsilon(t,s,\cdot)\|_\nu^2\Big)\,dsdt\right]+\mathbb{E}\left[\int_0^T\int_t^T e^{\beta s}\Big(|\mathbb{E}[\Delta z(t,s)]|^2\right.\\
&\quad\left.\left.+\|\mathbb{E}[\Delta\upsilon(t,s,\cdot)]\|_\nu^2\Big)\,dsdt\right]\right\}\\
&\leq\frac{20C^2(T+1)}{\beta}\mathbb{E}\left[\int_0^T e^{\beta t}|\Delta y(t)|^2dt+\int_0^T\int_t^T e^{\beta s}|\Delta z(t,s)|^2dsdt+\int_0^T\int_t^T e^{\beta s}\|\Delta\upsilon(t,s,\cdot)\|_\nu^2dsdt\right].
\end{aligned}\tag{3.15}$$

Putting $\beta = 40C^2(T+1)+1$ in (3.15), we can show that

$$\begin{aligned}&\mathbb{E}\left[\int_0^T e^{\beta t}|\Delta Y(t)|^2 dt + \int_0^T\int_t^T e^{\beta s}|\Delta Z(t,s)|^2 dsdt + \int_0^T\int_t^T e^{\beta s}\|\Delta V(t,s,\cdot)\|_\nu^2 dsdt\right]\\ &\leq \frac{1}{2}\mathbb{E}\left[\int_0^T e^{\beta t}|\Delta y(t)|^2 dt + \int_0^T\int_t^T e^{\beta s}|\Delta z(t,s)|^2 dsdt + \int_0^T\int_t^T e^{\beta s}\|\Delta \upsilon(t,s,\cdot)\|_\nu^2 dsdt\right],\end{aligned}$$

which implies that

$$\begin{aligned}&\|\mathcal{L}(y_1(\cdot),z_1(\cdot,\cdot),\upsilon_1(\cdot,\cdot,\cdot)) - \mathcal{L}(y_2(\cdot),z_2(\cdot,\cdot),\upsilon_2(\cdot,\cdot,\cdot))\|^2_{\mathcal{M}^2_\beta[0,T]}\\ &\leq \frac{1}{2}\|(y_1(\cdot),z_1(\cdot,\cdot),\upsilon_1(\cdot,\cdot,\cdot)) - (y_2(\cdot),z_2(\cdot,\cdot),\upsilon_2(\cdot,\cdot,\cdot))\|^2_{\mathcal{M}^2_\beta[0,T]}.\end{aligned}$$

Then, $\mathcal{L}$ is a contractive map on the Banach space $\mathcal{M}^2_\beta[0,T]$. From the Banach contraction mapping principle, the map $\mathcal{L}$ admits a unique fixed point $(Y(\cdot),Z(\cdot,\cdot),V(\cdot,\cdot,\cdot))$ such that $\mathcal{L}(Y(\cdot),Z(\cdot,\cdot),V(\cdot,\cdot,\cdot)) = (Y(\cdot),Z(\cdot,\cdot),V(\cdot,\cdot,\cdot))$. We claim that $(Y(\cdot),Z(\cdot,\cdot),V(\cdot,\cdot,\cdot)) \in \mathcal{M}^2_\beta[0,T]$ is the unique adapted M-solution of MF-BSVIE with jump (3.1). Moreover, it follows from (3.11) that

$$\begin{aligned}&\mathbb{E}\left[\int_0^T e^{\beta t}|Y(t)|^2 dt + \int_0^T\int_t^T e^{\beta s}|Z(t,s)|^2 dsdt + \int_0^T\int_t^T e^{\beta s}\|V(t,s,\cdot)\|_\nu^2 dsdt\right]\\ &\leq \mathbb{E}\left[\int_0^T e^{\beta T}|\psi(t)|^2 dt\right] + \frac{2}{\beta}\mathbb{E}\left[\int_0^T\int_t^T e^{\beta s}|g_0(t,s)|^2 dsdt\right] + \frac{40(T+1)C^2}{\beta}\mathbb{E}\left[\int_0^T e^{\beta t}|Y(t)|^2 dt\right]\\ &\quad + \frac{40C^2}{\beta}\mathbb{E}\left[\int_0^T\int_t^T e^{\beta s}\left(|Z(t,s)|^2 + \|V(t,s,\cdot)\|_\nu^2\right) dsdt\right].\end{aligned}$$

Taking $\beta > 40(T+1)C^2$ such that $1 - \frac{40(T+1)C^2}{\beta} > 0$, we can deduce that

$$\begin{aligned}&\mathbb{E}\left[\int_0^T e^{\beta t}|Y(t)|^2 dt + \int_0^T\int_t^T e^{\beta s}|Z(t,s)|^2 dsdt + \int_0^T\int_t^T e^{\beta s}\|V(t,s,\cdot)\|_\nu^2 dsdt\right]\\ &\leq K\mathbb{E}\left[\int_0^T e^{\beta T}|\psi(t)|^2 dt + \int_0^T\int_t^T e^{\beta s}|g_0(t,s)|^2 dsdt\right].\end{aligned}$$

This proves (3.9). □

3.2. Well-posedness of MF-FSVIEs with jumps

In this subsection, we present the well-posedness of adapted solutions to MF-FSVIEs with jumps. These equations have very interesting applications in stochastic optimal control, mathematical finance, and so on. More precisely, we consider the following MF-FSVIE with jump:

$$\begin{aligned}X(t) = \varphi(t) &+ \int_0^t b(t,s,X(s),\mathbb{E}[X(s)])ds + \int_0^t \sigma(t,s,X(s),\mathbb{E}[X(s)])dW(s)\\ &+ \int_0^t\int_{\mathbb{R}_0}\gamma(t,s,X(s),\mathbb{E}[X(s)],e)\widetilde{N}(ds,de),\quad t\in[0,T],\end{aligned} \tag{3.16}$$

where $W(\cdot)$ is a standard Brownian motion, $\widetilde{N}(\cdot,\cdot)$ is an independent compensated Poisson random measure, $T > 0$ is a fixed terminal time, and $\varphi(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R})$. $b,\sigma : \Omega \times \Delta^c \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ are $\mathcal{F}_T \otimes \mathcal{B}(\Delta^c \times \mathbb{R} \times \mathbb{R})$-measurable such that $s \mapsto (b(t,s,x,\overline{x}), \sigma(t,s,x,\overline{x}))$ is $\mathbb{F}$-progressively measurable for all $(t,x,\overline{x}) \in [0,T] \times \mathbb{R} \times \mathbb{R}$. $\gamma : \Omega \times \Delta^c \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}_0 \mapsto \mathbb{R}$ is $\mathcal{F}_T \otimes \mathcal{B}(\Delta^c \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}_0)$-measurable such that $s \mapsto \gamma(t,s,x,\overline{x},\cdot)$ is $\mathbb{F}$-progressively measurable for all $(t,x,\overline{x}) \in [0,T] \times \mathbb{R} \times \mathbb{R}$.

Now, we make the following assumptions for the coefficients of (3.16):

(A2) There exists a constant $C > 0$ such that for almost all $(\omega,t,s) \in \Omega \times \Delta^c$, and for any $x,\overline{x} \in \mathbb{R}$,

$$|b(\omega,t,s,x,\overline{x})| + |\sigma(\omega,t,s,x,\overline{x})| + \|\gamma(\omega,t,s,x,\overline{x},\cdot)\|_\nu \le C\left(1 + |x| + |\overline{x}|\right),$$

and such that for almost all $(\omega,t,s) \in \Omega \times \Delta^c$, and for any $x_1, x_2, \overline{x}_1, \overline{x}_2 \in \mathbb{R}$,

$$\begin{aligned}&|b(\omega,t,s,x_1,\overline{x}_1) - b(\omega,t,s,x_2,\overline{x}_2)| + |\sigma(\omega,t,s,x_1,\overline{x}_1) - \sigma(\omega,t,s,x_2,\overline{x}_2)| \\ &\quad + \|\gamma(\omega,t,s,x_1,\overline{x}_1,\cdot) - \gamma(\omega,t,s,x_2,\overline{x}_2,\cdot)\|_\nu \le C\left(|x_1 - x_2| + |\overline{x}_1 - \overline{x}_2|\right).\end{aligned}$$

Theorem 2. *Let Assumption* (A2) *hold. Then for any* $\varphi(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R})$*, MF-FSVIE with jump (3.16) admits a unique adapted solution* $X(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R})$*. Moreover, for any* $\beta > 0$*, there exists a constant* $K > 0$ *such that*

$$\begin{aligned}&\|X(\cdot)\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R})} := \mathbb{E}\left[\int_0^T e^{-\beta t}|X(t)|^2 dt\right] \\ &\le K\mathbb{E}\left[\int_0^T |\varphi(t)|^2 dt + \int_0^T \int_0^t e^{-\beta s}\left(|b(t,s,0,0)|^2 + |\sigma(t,s,0,0)|^2 + \|\gamma(t,s,0,0,\cdot)\|_\nu^2\right) ds dt\right].\end{aligned} \tag{3.17}$$

Proof. For any fixed $t \in [0,T]$, we discuss the following mean-field stochastic differential equation with jump (MF-SDE with jump, for short):

$$\begin{aligned}\mathcal{X}_t(r) &= \varphi(t) + \int_0^r b(t,s,\mathcal{X}_t(s),\mathbb{E}[\mathcal{X}_t(s)])ds + \int_0^r \sigma(t,s,\mathcal{X}_t(s),\mathbb{E}[\mathcal{X}_t(s)])dW(s) \\ &\quad + \int_0^r \int_{\mathbb{R}_0} \gamma(t,s,\mathcal{X}_t(s),\mathbb{E}[\mathcal{X}_t(s)],e)\widetilde{N}(ds,de), \quad r \in [0,t].\end{aligned} \tag{3.18}$$

By Lemma 4.1 in [22], we know that MF-SDE with jump (3.18) admits a unique adapted solution $\mathcal{X}_t(\cdot) \in L^2_{\mathbb{F}}(0,t;\mathbb{R})$. In the manner analogous to the proof of Lemma 2.1 in [39], we have the following estimate:

$$\mathbb{E}\left[\int_0^t e^{-\beta r}|\mathcal{X}_t(r)|^2 dr\right] \le K\mathbb{E}\left[|\varphi(t)|^2 + \int_0^t e^{-\beta s}\left(|b(t,s,0,0)|^2 + |\sigma(t,s,0,0)|^2 + \|\gamma(t,s,0,0,\cdot)\|_\nu^2\right) ds\right]. \tag{3.19}$$

Define

$$X(t) := \mathcal{X}_t(t), \quad t \in [0,T].$$

For this, it is clear that $X(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R})$ is an adapted solution of (3.16). If (3.19) holds, then we obtain that

$$\mathbb{E}\left[e^{-\beta t}|X(t)|^2\right] \le K\mathbb{E}\left[|\varphi(t)|^2 + \int_0^t e^{-\beta s}\left(|b(t,s,0,0)|^2 + |\sigma(t,s,0,0)|^2 + \|\gamma(t,s,0,0,\cdot)\|_\nu^2\right) ds\right].$$

This means that

$$\|X(\cdot)\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R})} := \mathbb{E}\left[\int_0^T e^{-\beta t}|X(t)|^2 dt\right]$$
$$\leq K\mathbb{E}\left[\int_0^T |\varphi(t)|^2 dt + \int_0^T\int_0^t e^{-\beta s}\left(|b(t,s,0,0)|^2 + |\sigma(t,s,0,0)|^2 + \|\gamma(t,s,0,0,\cdot)\|^2_\nu\right) dsdt\right].$$

Obviously, the uniqueness of (3.16) follows easily from (3.17). □

4. Comparison theorem for MF-BSVIEs with jumps

In this section, we shall use Theorem 1 to establish the comparison theorem for MF-BSVIEs with jumps. For $i = 1, 2$, we focus on the following MF-BSVIE with jump of the form:

$$\begin{aligned} Y_i(t) = \psi_i(t) &+ \int_t^T g_i\left(t,s,Y_i(s),Z_i(t,s),V_i(t,s,\cdot),E[Y_i(s)]\right) ds - \int_t^T Z_i(t,s)dW(s) \\ &- \int_t^T\int_{\mathbb{R}_0} V_i(t,s,e)\widetilde{N}(ds,de), \quad t \in [0,T], \end{aligned} \tag{4.1}$$

where $\psi_i(\cdot) \in L^2_{\mathcal{F}_T}(0,T;\mathbb{R})$ and $g_i : \Omega\times\Delta\times\mathbb{R}\times\mathbb{R}\times L^2_\nu(\mathbb{R})\times\mathbb{R} \mapsto \mathbb{R}$ is $\mathcal{F}_T\otimes\mathcal{B}(\Delta\times\mathbb{R}\times\mathbb{R}\times L^2_\nu(\mathbb{R})\times\mathbb{R})$-measurable such that $s \mapsto g_i(t,s,y,z,\upsilon,\overline{y})$ is $\mathbb{F}$-progressively measurable for all $(t,y,z,\upsilon,\overline{y}) \in [0,T]\times\mathbb{R}\times\mathbb{R}\times L^2_\nu(\mathbb{R})\times\mathbb{R}$. Since the generator g_i is independent of $Z_i(s,t)$, $V_i(s,t,\cdot)$, $\mathbb{E}[Z_i(s,t)]$, and $\mathbb{E}[V_i(s,t,\cdot)]$, we do not need to determine the values of $Z_i(s,t)$, $V_i(s,t,\cdot)$, $\mathbb{E}[Z_i(s,t)]$, and $\mathbb{E}[V_i(s,t,\cdot)]$. Hence, we no longer need to define an adapted M-solution for (4.1). By Theorem 1, we can see that under Assumption $(A1)$, for any $\psi_i(\cdot) \in L^2_{\mathcal{F}_T}(0,T;\mathbb{R})$, there exists a unique adapted solution $(Y_i(\cdot), Z_i(\cdot,\cdot), V_i(\cdot,\cdot,\cdot)) \in L^2_{\mathbb{F}}(0,T;\mathbb{R})\times L^2_{\mathbb{F}}(\Delta;\mathbb{R})\times L^2_N(\Delta;\mathbb{R})$ to MF-BSVIE with jump (4.1).

Theorem 3. *For $i = 1, 2$, let $\psi_i(\cdot) \in L^2_{\mathcal{F}_T}(0,T;\mathbb{R})$, and let g_i satisfy Assumption $(A1)$. Let $(Y_i(\cdot), Z_i(\cdot,\cdot), V_i(\cdot,\cdot,\cdot)) \in L^2_{\mathbb{F}}(0,T;\mathbb{R})\times L^2_{\mathbb{F}}(\Delta;\mathbb{R})\times L^2_N(\Delta;\mathbb{R})$ be the adapted solution of (4.1). Suppose that $\overline{g} : \Omega\times\Delta\times\mathbb{R}\times\mathbb{R}\times L^2_\nu(\mathbb{R})\times\mathbb{R} \mapsto \mathbb{R}$ is $\mathcal{F}_T\otimes\mathcal{B}(\Delta\times\mathbb{R}\times\mathbb{R}\times L^2_\nu(\mathbb{R})\times\mathbb{R})$-measurable such that $s \mapsto \overline{g}(t,s,y,z,\upsilon,\overline{y})$ is $\mathbb{F}$-progressively measurable for all $(t,y,z,\upsilon,\overline{y}) \in [0,T]\times\mathbb{R}\times\mathbb{R}\times L^2_\nu(\mathbb{R})\times\mathbb{R}$. Let $\overline{g}$ satisfy Assumption $(A1)$ and the following conditions:*

(a) For any $(t,s,y,z,\upsilon,\overline{y}) \in \Delta\times\mathbb{R}\times\mathbb{R}\times L^2_\nu(\mathbb{R})\times\mathbb{R}$,

$$g_1(t,s,y,z,\upsilon,\overline{y}) \leq \overline{g}(t,s,y,z,\upsilon,\overline{y}) \leq g_2(t,s,y,z,\upsilon,\overline{y}), \text{ a.s.}$$

(b) For any $(t,s,z,\upsilon,\overline{y}) \in \Delta\times\mathbb{R}\times L^2_\nu(\mathbb{R})\times\mathbb{R}$, $\overline{g}(t,s,\cdot,z,\upsilon,\overline{y})$ is increasing, that is, for any $y_1, y_2 \in \mathbb{R}$, if $y_1 \leq y_2$, then $\overline{g}(t,s,y_1,z,\upsilon,\overline{y}) \leq \overline{g}(t,s,y_2,z,\upsilon,\overline{y})$, a.s.

(c) For any $(t,s,y,z,\upsilon) \in \Delta\times\mathbb{R}\times\mathbb{R}\times L^2_\nu(\mathbb{R})$, $\overline{g}(t,s,y,z,\upsilon,\cdot)$ is increasing, that is, for any $\overline{y}_1, \overline{y}_2 \in \mathbb{R}$, if $\overline{y}_1 \leq \overline{y}_2$, then $\overline{g}(t,s,y,z,\upsilon,\overline{y}_1) \leq \overline{g}(t,s,y,z,\upsilon,\overline{y}_2)$, a.s.

(d) There exists a bounded predictable process $\lambda(t,s,e)$ such that $\lambda(t,s,e) > -1$ and $|\lambda(t,s,e)| \leq \Lambda(e)$, a.s., where $\Lambda \in L^2_\nu(\mathbb{R})$, and such that for any $(t,s,y,z,\overline{y}) \in \Delta\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ and for any $\upsilon_1, \upsilon_2 \in L^2_\nu(\mathbb{R})$,

$$\overline{g}(t,s,y,z,\upsilon_1,\overline{y}) - \overline{g}(t,s,y,z,\upsilon_2,\overline{y}) \leq \int_{\mathbb{R}_0} \lambda(t,s,e)(\upsilon_1(e) - \upsilon_2(e))\nu(de), \text{ a.s.}$$

Therefore, if $\psi_1(t) \leq \psi_2(t)$, $t \in [0,T]$, a.s., then

$$Y_1(t) \leq Y_2(t), \ t \in [0,T], \text{ a.s.}$$

Proof. If $\overline{\psi}(\cdot) \in L^2_{\mathcal{F}_T}(0,T;\mathbb{R})$ such that

$$\psi_1(t) \le \overline{\psi}(t) \le \psi_2(t),\ t \in [0,T],\ a.s.,$$

then from Theorem 1, there exists a unique adapted solution $(\overline{Y}(\cdot), \overline{Z}(\cdot,\cdot), \overline{V}(\cdot,\cdot,\cdot)) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}) \times L^2_{\mathbb{F}}(\Delta;\mathbb{R}) \times L^2_N(\Delta;\mathbb{R})$ to the following equation:

$$\begin{aligned}\overline{Y}(t) = \overline{\psi}(t) &+ \int_t^T \overline{g}\left(t,s,\overline{Y}(s),\overline{Z}(t,s),\overline{V}(t,s,\cdot),\mathbb{E}[\overline{Y}(s)]\right)ds - \int_t^T \overline{Z}(t,s)dW(s)\\ &- \int_t^T\int_{\mathbb{R}_0} \overline{V}(t,s,e)\widetilde{N}(ds,de),\quad t\in[0,T].\end{aligned}$$

Let $\widetilde{Y}_0(\cdot) = Y_2(\cdot)$. Since $\overline{\psi}(\cdot) \in L^2_{\mathcal{F}_T}(0,T;\mathbb{R})$ and $\overline{g}$ satisfies Assumption $(A1)$, it then follows from Theorem 3.2 in [37] that the following equation (4.2) admits a unique adapted solution $(\widetilde{Y}_1(\cdot), \widetilde{Z}_1(\cdot,\cdot), \widetilde{V}_1(\cdot,\cdot,\cdot)) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}) \times L^2_{\mathbb{F}}(\Delta;\mathbb{R}) \times L^2_N(\Delta;\mathbb{R})$:

$$\begin{aligned}\widetilde{Y}_1(t) = \overline{\psi}(t) &+ \int_t^T \overline{g}\left(t,s,\widetilde{Y}_1(s),\widetilde{Z}_1(t,s),\widetilde{V}_1(t,s,\cdot),\mathbb{E}[\widetilde{Y}_0(s)]\right)ds - \int_t^T \widetilde{Z}_1(t,s)dW(s)\\ &- \int_t^T\int_{\mathbb{R}_0} \widetilde{V}_1(t,s,e)\widetilde{N}(ds,de),\quad t\in[0,T].\end{aligned} \tag{4.2}$$

It has been seen that for any $(t,s,y,z,\upsilon) \in \Delta\times\mathbb{R}\times\mathbb{R}\times L^2_\nu(\mathbb{R})$,

$$\overline{g}\left(t,s,y,z,\upsilon,\mathbb{E}[\widetilde{Y}_0(s)]\right) \le g_2\left(t,s,y,z,\upsilon,\mathbb{E}[\widetilde{Y}_0(s)]\right),\ a.s.,$$

and for any $t\in[0,T]$, $\overline{\psi}(t) \le \psi_2(t)$, a.s. Note that $\overline{g}$ satisfies (b) and (d). Thus, by Proposition 2.4 in [40], we can show that

$$\widetilde{Y}_1(t) \le \widetilde{Y}_0(t) = Y_2(t),\ t\in[0,T],\ a.s. \tag{4.3}$$

Next, suppose that $(\widetilde{Y}_2(\cdot), \widetilde{Z}_2(\cdot,\cdot), \widetilde{V}_2(\cdot,\cdot,\cdot)) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}) \times L^2_{\mathbb{F}}(\Delta;\mathbb{R}) \times L^2_N(\Delta;\mathbb{R})$ is the adapted solution to the following equation:

$$\begin{aligned}\widetilde{Y}_2(t) = \overline{\psi}(t) &+ \int_t^T \overline{g}\left(t,s,\widetilde{Y}_2(s),\widetilde{Z}_2(t,s),\widetilde{V}_2(t,s,\cdot),\mathbb{E}[\widetilde{Y}_1(s)]\right)ds - \int_t^T \widetilde{Z}_2(t,s)dW(s)\\ &- \int_t^T\int_{\mathbb{R}_0} \widetilde{V}_2(t,s,e)\widetilde{N}(ds,de),\quad t\in[0,T].\end{aligned}$$

From (c), we know that if $\overline{y}_1 \le \overline{y}_2$, then $\overline{g}(t,s,y,z,\upsilon,\overline{y}_1) \le \overline{g}(t,s,y,z,\upsilon,\overline{y}_2)$, a.s. Thus, (4.3) implies that for any $(t,s,y,z,\upsilon) \in \Delta\times\mathbb{R}\times\mathbb{R}\times L^2_\nu(\mathbb{R})$,

$$\overline{g}\left(t,s,y,z,\upsilon,\mathbb{E}[\widetilde{Y}_1(s)]\right) \le \overline{g}\left(t,s,y,z,\upsilon,\mathbb{E}[\widetilde{Y}_0(s)]\right),\ a.s.$$

Recall that $\overline{g}$ satisfies (b) and (d). Then by Proposition 2.4 in [40] again, we obtain that

$$\widetilde{Y}_2(t) \le \widetilde{Y}_1(t),\ t\in[0,T],\ a.s.$$

Now, we can recursively define a sequence $\{(\widetilde{Y}_n(\cdot), \widetilde{Z}_n(\cdot,\cdot), \widetilde{V}_n(\cdot,\cdot,\cdot))\}_{n=1}^{\infty}$ in $L^2_{\mathbb{F}}(0,T;\mathbb{R}) \times L^2_{\mathbb{F}}(\Delta;\mathbb{R}) \times L^2_N(\Delta;\mathbb{R})$ by

$$\begin{aligned}\widetilde{Y}_n(t) = \overline{\psi}(t) &+ \int_t^T \overline{g}\left(t,s,\widetilde{Y}_n(s),\widetilde{Z}_n(t,s),\widetilde{V}_n(t,s,\cdot),\mathbb{E}[\widetilde{Y}_{n-1}(s)]\right)ds - \int_t^T \widetilde{Z}_n(t,s)dW(s)\\ &- \int_t^T \int_{\mathbb{R}_0} \widetilde{V}_n(t,s,e)\widetilde{N}(ds,de), \quad t \in [0,T].\end{aligned} \tag{4.4}$$

By induction, we can then show that

$$Y_2(t) = \widetilde{Y}_0(t) \geq \widetilde{Y}_1(t) \geq \widetilde{Y}_2(t) \geq \cdots \geq \widetilde{Y}_n(t) \geq \cdots, \ t \in [0,T], \ a.s.$$

In what follows, let us proceed to prove that $\{(\widetilde{Y}_n(\cdot), \widetilde{Z}_n(\cdot,\cdot), \widetilde{V}_n(\cdot,\cdot,\cdot))\}_{n=1}^{\infty}$ is a Cauchy sequence in the Banach space $L^2_{\mathbb{F}}(0,T;\mathbb{R}) \times L^2_{\mathbb{F}}(\Delta;\mathbb{R}) \times L^2_N(\Delta;\mathbb{R})$. To this end, we introduce a norm in the Banach space $L^2_{\mathbb{F}}(0,T;\mathbb{R}) \times L^2_{\mathbb{F}}(\Delta;\mathbb{R}) \times L^2_N(\Delta;\mathbb{R})$ as follows:

$$\begin{aligned}&\|(\widetilde{Y}(\cdot), \widetilde{Z}(\cdot,\cdot), \widetilde{V}(\cdot,\cdot,\cdot))\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R})\times L^2_{\mathbb{F}}(\Delta;\mathbb{R})\times L^2_N(\Delta;\mathbb{R})}\\ &:= \mathbb{E}\left[\int_0^T e^{\beta t}|\widetilde{Y}(t)|^2 dt + \int_0^T \int_t^T e^{\beta s}|\widetilde{Z}(t,s)|^2 dsdt + \int_0^T \int_t^T e^{\beta s}\|\widetilde{V}(t,s,\cdot)\|^2_\nu dsdt\right].\end{aligned}$$

Indeed, it follows from the estimate (3.7) that

$$\begin{aligned}&\mathbb{E}\left[\int_0^T e^{\beta t}|\widetilde{Y}_n(t) - \widetilde{Y}_{n-1}(t)|^2 dt + \int_0^T \int_t^T e^{\beta s}|\widetilde{Z}_n(t,s) - \widetilde{Z}_{n-1}(t,s)|^2 dsdt\right.\\ &\left.+ \int_0^T \int_t^T e^{\beta s}\|\widetilde{V}_n(t,s,\cdot) - \widetilde{V}_{n-1}(t,s,\cdot)\|^2_\nu dsdt\right]\\ &\leq \frac{1}{\beta}\mathbb{E}\left[\int_0^T \int_t^T e^{\beta s}\left|\overline{g}\left(t,s,\widetilde{Y}_n(s),\widetilde{Z}_n(t,s),\widetilde{V}_n(t,s,\cdot),\mathbb{E}[\widetilde{Y}_{n-1}(s)]\right)\right.\right.\\ &\left.\left.-\overline{g}\left(t,s,\widetilde{Y}_{n-1}(s),\widetilde{Z}_{n-1}(t,s),\widetilde{V}_{n-1}(t,s,\cdot),\mathbb{E}[\widetilde{Y}_{n-2}(s)]\right)\right|^2 dsdt\right]\\ &\leq \frac{4C^2(T+1)}{\beta}\mathbb{E}\left[\int_0^T e^{\beta t}|\widetilde{Y}_n(t) - \widetilde{Y}_{n-1}(t)|^2 dt + \int_0^T e^{\beta t}|\widetilde{Y}_{n-1}(t) - \widetilde{Y}_{n-2}(t)|^2 dt\right.\\ &\left.+ \int_0^T \int_t^T e^{\beta s}|\widetilde{Z}_n(t,s) - \widetilde{Z}_{n-1}(t,s)|^2 dsdt + \int_0^T \int_t^T e^{\beta s}\|\widetilde{V}_n(t,s,\cdot) - \widetilde{V}_{n-1}(t,s,\cdot)\|^2_\nu dsdt\right].\end{aligned} \tag{4.5}$$

Putting $\beta = 12C^2(T+1) > 0$ in (4.5), we arrive at

$$\begin{aligned}&\mathbb{E}\left[\int_0^T e^{\beta t}|\widetilde{Y}_n(t) - \widetilde{Y}_{n-1}(t)|^2 dt + \int_0^T \int_t^T e^{\beta s}|\widetilde{Z}_n(t,s) - \widetilde{Z}_{n-1}(t,s)|^2 dsdt\right.\\ &\left.+ \int_0^T \int_t^T e^{\beta s}\|\widetilde{V}_n(t,s,\cdot) - \widetilde{V}_{n-1}(t,s,\cdot)\|^2_\nu dsdt\right]\\ &\leq \frac{1}{2}\mathbb{E}\left[\int_0^T e^{\beta t}|\widetilde{Y}_{n-1}(t) - \widetilde{Y}_{n-2}(t)|^2 dt\right]\\ &\leq \frac{1}{2}\mathbb{E}\left[\int_0^T e^{\beta t}|\widetilde{Y}_{n-1}(t) - \widetilde{Y}_{n-2}(t)|^2 dt + \int_0^T \int_t^T e^{\beta s}|\widetilde{Z}_{n-1}(t,s) - \widetilde{Z}_{n-2}(t,s)|^2 dsdt\right.\end{aligned}$$

$$
+\int_0^T\int_t^T e^{\beta s}\|\widetilde{V}_{n-1}(t,s,\cdot)-\widetilde{V}_{n-2}(t,s,\cdot)\|_\nu^2 dsdt\Big]
$$

$$
\le\left(\frac{1}{2}\right)^{n-2}\mathbb{E}\Big[\int_0^T e^{\beta t}|\widetilde{Y}_2(t)-\widetilde{Y}_1(t)|^2dt+\int_0^T\int_t^T e^{\beta s}|\widetilde{Z}_2(t,s)-\widetilde{Z}_1(t,s)|^2dsdt
$$

$$
+\int_0^T\int_t^T e^{\beta s}\|\widetilde{V}_2(t,s,\cdot)-\widetilde{V}_1(t,s,\cdot)\|_\nu^2 dsdt\Big].
$$

It is directly concluded that $\{(\widetilde{Y}_n(\cdot),\widetilde{Z}_n(\cdot,\cdot),\widetilde{V}_n(\cdot,\cdot,\cdot))\}_{n=1}^\infty$ is a Cauchy sequence in $L^2_{\mathbb{F}}(0,T;\mathbb{R})\times L^2_{\mathbb{F}}(\Delta;\mathbb{R})\times L^2_N(\Delta;\mathbb{R})$, and its limit $(\widetilde{Y}(\cdot),\widetilde{Z}(\cdot,\cdot),\widetilde{V}(\cdot,\cdot,\cdot))\in L^2_{\mathbb{F}}(0,T;\mathbb{R})\times L^2_{\mathbb{F}}(\Delta;\mathbb{R})\times L^2_N(\Delta;\mathbb{R})$ satisfies

$$
\lim_{n\to\infty}\mathbb{E}\Big[\int_0^T e^{\beta t}|\widetilde{Y}_n(t)-\widetilde{Y}(t)|^2dt+\int_0^T\int_t^T e^{\beta s}|\widetilde{Z}_n(t,s)-\widetilde{Z}(t,s)|^2dsdt
$$

$$
+\int_0^T\int_t^T e^{\beta s}\|\widetilde{V}_n(t,s,\cdot)-\widetilde{V}(t,s,\cdot)\|_\nu^2 dsdt\Big]=0.
$$

Finally, since $\overline{g}$ satisfies Assumption $(A1)$, we can obtain that

$$
\mathbb{E}\Big[\int_0^T e^{\beta t}\Big|\int_t^T\overline{g}\left(t,s,\widetilde{Y}_n(s),\widetilde{Z}_n(t,s),\widetilde{V}_n(t,s,\cdot),\mathbb{E}[\widetilde{Y}_{n-1}(s)]\right)ds
$$

$$
-\int_t^T\overline{g}\left(t,s,\widetilde{Y}(s),\widetilde{Z}(t,s),\widetilde{V}(t,s,\cdot),\mathbb{E}[\widetilde{Y}(s)]\right)ds\Big|^2dt\Big]
$$

$$
\le 4TC^2(T+1)\mathbb{E}\Big[\int_0^T e^{\beta t}|\widetilde{Y}_n(t)-\widetilde{Y}(t)|^2dt+\int_0^T\int_t^T e^{\beta s}|\widetilde{Z}_n(t,s)-\widetilde{Z}(t,s)|^2dsdt
$$

$$
+\int_0^T\int_t^T e^{\beta s}\|\widetilde{V}_n(t,s,\cdot)-\widetilde{V}(t,s,\cdot)\|_\nu^2dsdt+\int_0^T e^{\beta t}|\widetilde{Y}_{n-1}(t)-\widetilde{Y}(t)|^2dt\Big]\to 0,\quad as\ n\mapsto\infty.
$$

Therefore, we can let $n\mapsto\infty$ in (4.4) to obtain that

$$
\widetilde{Y}(t)=\overline{\psi}(t)+\int_t^T\overline{g}\left(t,s,\widetilde{Y}(s),\widetilde{Z}(t,s),\widetilde{V}(t,s,\cdot),\mathbb{E}[\widetilde{Y}(s)]\right)ds-\int_t^T\widetilde{Z}(t,s)dW(s)
$$

$$
-\int_t^T\int_{\mathbb{R}_0}\widetilde{V}(t,s,e)\widetilde{N}(ds,de),\quad t\in[0,T].
$$

It then follows from Theorem 1 that

$$
Y_2(t)=\widetilde{Y}_0(t)\ge\widetilde{Y}(t)=\overline{Y}(t),\ t\in[0,T],\ a.s.
$$

In the same way as in the above proof, we can show that

$$
\overline{Y}(t)\ge Y_1(t),\ t\in[0,T],\ a.s.
$$

This implies that

$$
Y_2(t)\ge Y_1(t),\ t\in[0,T],\ a.s.
$$

□

Remark 1. *It should be pointed out that there are many assumptions in the comparison theorem for MF-BSVIEs with jumps, and these assumptions are relatively strict. We will relax these assumptions in our future works, such as we might be able to apply Malliavin calculus to weaken the condition (d) in the comparison theorem.*

5. Stochastic optimal control problems

In this section, we shall apply the theory of MF-BSVIEs with jumps to a stochastic optimal control problem of MF-FSVIEs with jumps. More precisely, the state process is described by the following MF-FSVIE with jump:

$$
\begin{aligned}
X(t) = \varphi(t) &+ \int_0^t b\left(t, s, X(s), E[X(s)], u(s), E[u(s)]\right) ds + \int_0^t \sigma\left(t, s, X(s), E[X(s)], u(s), E[u(s)]\right) dW(s) \\
&+ \int_0^t \int_{\mathbb{R}_0} \gamma(t, s, X(s), E[X(s)], u(s), E[u(s)], e)\widetilde{N}(ds, de), \quad t \in [0, T],
\end{aligned}
\tag{5.1}
$$

where $\varphi(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R})$. $b, \sigma : \Omega \times \Delta^c \times \mathbb{R} \times \mathbb{R} \times \mathbf{U} \times \mathbf{U} \mapsto \mathbb{R}$ are $\mathcal{F}_T \otimes \mathcal{B}(\Delta^c \times \mathbb{R} \times \mathbb{R} \times \mathbf{U} \times \mathbf{U})$-measurable such that $s \mapsto (b(t, s, x, \overline{x}, u, \overline{u}), \sigma(t, s, x, \overline{x}, u, \overline{u}))$ is $\mathbb{F}$-progressively measurable for all $(t, x, \overline{x}, u, \overline{u}) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbf{U} \times \mathbf{U}$. $\gamma : \Omega \times \Delta^c \times \mathbb{R} \times \mathbb{R} \times \mathbf{U} \times \mathbf{U} \times \mathbb{R}_0 \mapsto \mathbb{R}$ is $\mathcal{F}_T \otimes \mathcal{B}(\Delta^c \times \mathbb{R} \times \mathbb{R} \times \mathbf{U} \times \mathbf{U} \times \mathbb{R}_0)$-measurable such that $s \mapsto \gamma(t, s, x, \overline{x}, u, \overline{u}, \cdot)$ is $\mathbb{F}$-progressively measurable for all $(t, x, \overline{x}, u, \overline{u}) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbf{U} \times \mathbf{U}$. Here, $\mathbf{U}$ is a non-empty convex subset of $\mathbb{R}$, and $u(\cdot)$ is a càdlàg and $\mathbb{F}$-predictable control process of the controller.

Definition 2. *A control process $u(\cdot)$ is said to be admissible if $u(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R})$ and $u(t) \in \mathbf{U}$, $t \in [0, T]$, a.s. Denote by $\mathcal{A}$ the set of all admissible control processes.*

The cost functional is defined by

$$
J(u(\cdot)) = \mathbb{E}\left[\int_0^T h(t, X(t), \mathbb{E}[X(t)], u(t), \mathbb{E}[u(t)])dt\right],
\tag{5.2}
$$

where

$$
h : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbf{U} \times \mathbf{U} \mapsto \mathbb{R}
$$

is an $\mathcal{F}_T \otimes \mathcal{B}([0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbf{U} \times \mathbf{U})$-measurable map such that

$$
\mathbb{E}\left[\int_0^T |h(t, X(t), \mathbb{E}[X(t)], u(t), \mathbb{E}[u(t)])|\, dt\right] < \infty, \quad \forall\, u(\cdot) \in \mathcal{A}.
$$

Let us make the following assumptions:

(A3) $b(\cdot, \cdot, 0, 0, 0, 0), \sigma(\cdot, \cdot, 0, 0, 0, 0) \in L^2_{\mathbb{F}}(\Delta^c; \mathbb{R})$ and $\gamma(\cdot, \cdot, 0, 0, 0, 0, \cdot) \in L^2_N(\Delta^c; \mathbb{R})$. For almost all $(\omega, t, s) \in \Omega \times \Delta^c$, the maps $(x, \overline{x}, u, \overline{u}) \mapsto b(\omega, t, s, x, \overline{x}, u, \overline{u})$, $(x, \overline{x}, u, \overline{u}) \mapsto \sigma(\omega, t, s, x, \overline{x}, u, \overline{u})$, and $(x, \overline{x}, u, \overline{u}) \mapsto \gamma(\omega, t, s, x, \overline{x}, u, \overline{u}, \cdot)$ are continuously differentiable, and their partial derivatives b_x, $b_{\overline{x}}$, b_u, $b_{\overline{u}}$, σ_x, $\sigma_{\overline{x}}$, σ_u, $\sigma_{\overline{u}}$, γ_x, $\gamma_{\overline{x}}$, γ_u, $\gamma_{\overline{u}}$ are uniformly bounded.

(A4) For almost all $(\omega, t) \in \Omega \times [0, T]$, the map $(x, \overline{x}, u, \overline{u}) \mapsto h(\omega, t, x, \overline{x}, u, \overline{u})$ is continuously differentiable, and there exists a constant $C > 0$ such that

$$
\begin{aligned}
&|h_\rho(\omega, t, x, \overline{x}, u, \overline{u})| \le C(1 + |x| + |\overline{x}| + |u| + |\overline{u}|), \\
&\forall\, (\omega, t, x, \overline{x}, u, \overline{u}) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbf{U} \times \mathbf{U},
\end{aligned}
$$

where $\rho = x, \overline{x}, u, \overline{u}$.

By Theorem 2, we can obtain that under Assumption $(A3)$, for any $\varphi(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R})$ and $u(\cdot) \in \mathcal{A}$, MF-FSVIE with jump (5.1) has a unique adapted solution $X(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R})$. For this, we state the stochastic optimal control problem as follows:

Problem 1. *Find an admissible control $u^*(\cdot) \in \mathcal{A}$ such that*

$$J(u^*(\cdot)) = \inf_{u(\cdot)\in\mathcal{A}} J(u(\cdot)). \tag{5.3}$$

The admissible control $u^*(\cdot)$ satisfying (5.3) is called an optimal control, and the corresponding $X^*(\cdot)$ an optimal state process. Thus, $(X^*(\cdot), u^*(\cdot))$ is called an optimal pair of Problem 1.

Let us now turn to establish the duality principle between linear MF-FSVIEs with jumps and linear MF-BSVIEs with jumps.

Theorem 4. *For $i = 1, 2$, let $A_i(\cdot,\cdot), B_i(\cdot,\cdot) \in L^\infty_{\mathbb{F}}([0,T]^2;\mathbb{R})$, $C_i(\cdot,\cdot,\cdot) \in L^\infty_N([0,T]^2;\mathbb{R})$, $\varphi(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R})$, and $\psi(\cdot) \in L^2_{\mathcal{F}_T}(0,T;\mathbb{R})$. Let $X(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R})$ be the adapted solution to the linear MF-FSVIE with jump:*

$$\begin{aligned} X(t) = \varphi(t) &+ \int_0^t (A_1(t,s)X(s) + A_2(t,s)\mathbb{E}[X(s)])\,ds + \int_0^t (B_1(t,s)X(s) + B_2(t,s)\mathbb{E}[X(s)])\,dW(s) \\ &+ \int_0^t \int_{\mathbb{R}_0} (C_1(t,s,e)X(s) + C_2(t,s,e)\mathbb{E}[X(s)])\,\widetilde{N}(ds,de), \quad t \in [0,T], \end{aligned} \tag{5.4}$$

and let $(Y(\cdot), Z(\cdot,\cdot), V(\cdot,\cdot,\cdot)) \in \mathcal{M}^2_\beta[0,T]$ be the adapted M-solution to the linear MF-BSVIE with jump:

$$\begin{aligned} Y(t) = \psi(t) &+ \int_t^T \Big[A_1(s,t)Y(s) + \mathbb{E}[A_2(s,t)Y(s)] + B_1(s,t)Z(s,t) + \mathbb{E}[B_2(s,t)Z(s,t)] \\ &+ \int_{\mathbb{R}_0} (C_1(s,t,e)V(s,t,e) + \mathbb{E}[C_2(s,t,e)V(s,t,e)])\,\nu(de)\Big] ds - \int_t^T Z(t,s)dW(s) \\ &- \int_t^T \int_{\mathbb{R}_0} V(t,s,e)\widetilde{N}(ds,de), \quad t \in [0,T]. \end{aligned} \tag{5.5}$$

Then

$$\mathbb{E}\left[\int_0^T \psi(t)X(t)dt\right] = \mathbb{E}\left[\int_0^T \varphi(t)Y(t)dt\right]. \tag{5.6}$$

Proof. It follows from (3.2), (5.4), and (5.5) that

$$\begin{aligned} &\mathbb{E}\left[\int_0^T \varphi(t)Y(t)dt\right] \\ &= \mathbb{E}\left[\int_0^T X(t)Y(t)dt\right] - \mathbb{E}\left\{\int_0^T Y(t)\left[\int_0^t (A_1(t,s)X(s) + A_2(t,s)\mathbb{E}[X(s)])\,ds\right]dt\right\} \\ &\quad - \mathbb{E}\left\{\int_0^T Y(t)\left[\int_0^t (B_1(t,s)X(s) + B_2(t,s)\mathbb{E}[X(s)])\,dW(s)\right]dt\right\} \\ &\quad - \mathbb{E}\left\{\int_0^T Y(t)\left[\int_0^t \int_{\mathbb{R}_0} (C_1(t,s,e)X(s) + C_2(t,s,e)\mathbb{E}[X(s)])\,\widetilde{N}(ds,de)\right]dt\right\} \\ &= \mathbb{E}\left[\int_0^T X(t)Y(t)dt\right] - \mathbb{E}\left[\int_0^T \int_s^T Y(t)\,(A_1(t,s)X(s) + A_2(t,s)\mathbb{E}[X(s)])\,dtds\right] \end{aligned}$$

$$
\begin{aligned}
&-\mathbb{E}\left\{\int_0^T\left[\int_0^t(B_1(t,s)X(s)+B_2(t,s)\mathbb{E}[X(s)])\,dW(s)+\int_0^t\int_{\mathbb{R}_0}(C_1(t,s,e)X(s)\right.\right.\\
&\left.\left.+C_2(t,s,e)\mathbb{E}[X(s)])\,\widetilde{N}(ds,de)\right]\left(\mathbb{E}[Y(t)]+\int_0^t Z(t,s)dW(s)+\int_0^t\int_{\mathbb{R}_0}V(t,s,e)\widetilde{N}(ds,de)\right)dt\right\}\\
&=\mathbb{E}\left[\int_0^T X(t)Y(t)dt\right]-\mathbb{E}\left[\int_0^T\int_t^T Y(s)\,(A_1(s,t)X(t)+A_2(s,t)\mathbb{E}[X(t)])\,dsdt\right]\\
&-\mathbb{E}\left[\int_0^T\int_0^t Z(t,s)\,(B_1(t,s)X(s)+B_2(t,s)\mathbb{E}[X(s)])\,dsdt\right]\\
&-\mathbb{E}\left[\int_0^T\int_0^t\int_{\mathbb{R}_0}V(t,s,e)\,(C_1(t,s,e)X(s)+C_2(t,s,e)\mathbb{E}[X(s)])\,\nu(de)dsdt\right]\\
&=\mathbb{E}\left[\int_0^T X(t)Y(t)dt\right]-\mathbb{E}\left[\int_0^T\int_t^T Y(s)\,(A_1(s,t)X(t)+A_2(s,t)\mathbb{E}[X(t)])\,dsdt\right]\\
&-\mathbb{E}\left[\int_0^T\int_t^T Z(s,t)\,(B_1(s,t)X(t)+B_2(s,t)\mathbb{E}[X(t)])\,dsdt\right]\\
&-\mathbb{E}\left[\int_0^T\int_t^T\int_{\mathbb{R}_0}V(s,t,e)\,(C_1(s,t,e)X(t)+C_2(s,t,e)\mathbb{E}[X(t)])\,\nu(de)dsdt\right]\\
&=\mathbb{E}\left[\int_0^T X(t)Y(t)dt\right]-\mathbb{E}\left[\int_0^T\int_t^T X(t)\,(A_1(s,t)Y(s)+\mathbb{E}[A_2(s,t)Y(s)])\,dsdt\right]\\
&-\mathbb{E}\left[\int_0^T\int_t^T X(t)\,(B_1(s,t)Z(s,t)+\mathbb{E}[B_2(s,t)Z(s,t)])\,dsdt\right]\\
&-\mathbb{E}\left[\int_0^T\int_t^T\int_{\mathbb{R}_0}X(t)\,(C_1(s,t,e)V(s,t,e)+\mathbb{E}[C_2(s,t,e)V(s,t,e)])\,\nu(de)dsdt\right]\\
&=\mathbb{E}\left[\int_0^T X(t)\left(\psi(t)-\int_t^T Z(t,s)dW(s)-\int_t^T\int_{\mathbb{R}_0}V(t,s,e)\widetilde{N}(ds,de)\right)dt\right]\\
&=\mathbb{E}\left[\int_0^T\psi(t)X(t)dt\right],
\end{aligned}
$$

and the required (5.6) follows. □

To prove Pontryagin's type maximum principle for Problem 1, we write

$$
\begin{aligned}
&\chi_p^*(t,s)=\chi_p(t,s,X^*(s),\mathbb{E}[X^*(s)],u^*(s),\mathbb{E}[u^*(s)]),\quad \chi_p^*(s,t)=\chi_p(s,t,X^*(t),\mathbb{E}[X^*(t)],u^*(t),\mathbb{E}[u^*(t)]),\\
&\gamma_p^*(t,s,e)=\gamma_p(t,s,X^*(s),\mathbb{E}[X^*(s)],u^*(s),\mathbb{E}[u^*(s)],e),\\
&\gamma_p^*(s,t,e)=\gamma_p(s,t,X^*(t),\mathbb{E}[X^*(t)],u^*(t),\mathbb{E}[u^*(t)],e),\quad h_p^*(t)=h_p(t,X^*(t),\mathbb{E}[X^*(t)],u^*(t),\mathbb{E}[u^*(t)]),
\end{aligned}
$$

where $\chi=b,\sigma$, $p=x,\overline{x},u,\overline{u}$ and $t,s\in[0,T]$.

Theorem 5. *Let Assumptions* (A3) *and* (A4) *hold. Let* $\mathbf{U}$ *be a non-empty convex subset of* $\mathbb{R}$*, and* $(X^*(\cdot),u^*(\cdot))$ *be an optimal pair of Problem 1. Then there exists a unique adapted M-solution*

$(Y(\cdot), Y_0(\cdot); Z(\cdot,\cdot), Z_0(\cdot,\cdot); V(\cdot,\cdot,\cdot), V_0(\cdot,\cdot,\cdot))$ *to the following MF-BSVIE with jump:*

$$\begin{cases} Y(t) = h_x^*(t) + \mathbb{E}\left[h_{\overline{x}}^*(t)\right] + \int_t^T \Big[b_x^*(s,t)Y(s) + \mathbb{E}\left[b_{\overline{x}}^*(s,t)Y(s)\right] + \sigma_x^*(s,t)Z(s,t) + \mathbb{E}\left[\sigma_{\overline{x}}^*(s,t)Z(s,t)\right] \\ \qquad + \int_{\mathbb{R}_0}\Big(\gamma_x^*(s,t,e)V(s,t,e) + \mathbb{E}\left[\gamma_{\overline{x}}^*(s,t,e)V(s,t,e)\right]\Big)\nu(de)\Big]ds - \int_t^T Z(t,s)dW(s) \\ \qquad - \int_t^T\int_{\mathbb{R}_0} V(t,s,e)\widetilde{N}(ds,de), \quad t\in[0,T], \\ Y_0(t) = \int_t^T \Big[b_u^*(s,t)Y(s) + \mathbb{E}\left[b_{\overline{u}}^*(s,t)Y(s)\right] + \sigma_u^*(s,t)Z(s,t) + \mathbb{E}\left[\sigma_{\overline{u}}^*(s,t)Z(s,t)\right] \\ \qquad + \int_{\mathbb{R}_0}\Big(\gamma_u^*(s,t,e)V(s,t,e) + \mathbb{E}\left[\gamma_{\overline{u}}^*(s,t,e)V(s,t,e)\right]\Big)\nu(de)\Big]ds - \int_t^T Z_0(t,s)dW(s) \\ \qquad - \int_t^T\int_{\mathbb{R}_0} V_0(t,s,e)\widetilde{N}(ds,de), \quad t\in[0,T], \end{cases} \tag{5.7}$$

such that

$$\left(Y_0(t) + h_u^*(t) + \mathbb{E}\left[h_{\overline{u}}^*(t)\right]\right)(u - u^*(t)) \geq 0, \tag{5.8}$$

for any $u \in \mathbf{U}$, $t\in[0,T]$, *a.s.*

Proof. Take any $u(\cdot)\in\mathcal{A}$. Since $\mathbf{U}$ is convex, for any $\varepsilon\in[0,1]$, we have

$$u^\varepsilon(\cdot) := u^*(\cdot) + \varepsilon[u(\cdot) - u^*(\cdot)] \in \mathcal{A}.$$

Let $X^\varepsilon(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R})$ be the adapted solution of MF-FSVIE with jump (5.1) along with $u^\varepsilon(\cdot)$. Set

$$\widehat{X}^\varepsilon(t) := \frac{X^\varepsilon(t) - X^*(t)}{\varepsilon}, \quad t\in[0,T]. \tag{5.9}$$

Now, letting $\varepsilon\mapsto 0$ in (5.9), we shall prove that $\widehat{X}^\varepsilon(\cdot)\mapsto\widehat{X}(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R})$ and $\widehat{X}(\cdot)$ satisfies the following equation:

$$\begin{aligned} \widehat{X}(t) &= \int_0^t \Big(b_x^*(t,s)\widehat{X}(s) + b_{\overline{x}}^*(t,s)\mathbb{E}[\widehat{X}(s)] + b_u^*(t,s)(u(s)-u^*(s)) + b_{\overline{u}}^*(t,s)\mathbb{E}[u(s)-u^*(s)]\Big)ds \\ &\quad + \int_0^t \Big(\sigma_x^*(t,s)\widehat{X}(s) + \sigma_{\overline{x}}^*(t,s)\mathbb{E}[\widehat{X}(s)] + \sigma_u^*(t,s)(u(s)-u^*(s)) \\ &\quad + \sigma_{\overline{u}}^*(t,s)\mathbb{E}[u(s)-u^*(s)]\Big)dW(s) + \int_0^t\int_{\mathbb{R}_0}\Big(\gamma_x^*(t,s,e)\widehat{X}(s) + \gamma_{\overline{x}}^*(t,s,e)\mathbb{E}[\widehat{X}(s)] \\ &\quad + \gamma_u^*(t,s,e)(u(s)-u^*(s)) + \gamma_{\overline{u}}^*(t,s,e)\mathbb{E}[u(s)-u^*(s)]\Big)\widetilde{N}(ds,de) \\ &= \varphi^*(t) + \int_0^t \Big(b_x^*(t,s)\widehat{X}(s) + b_{\overline{x}}^*(t,s)\mathbb{E}[\widehat{X}(s)]\Big)ds + \int_0^t\Big(\sigma_x^*(t,s)\widehat{X}(s) + \sigma_{\overline{x}}^*(t,s)\mathbb{E}[\widehat{X}(s)]\Big)dW(s) \\ &\quad + \int_0^t\int_{\mathbb{R}_0}\Big(\gamma_x^*(t,s,e)\widehat{X}(s) + \gamma_{\overline{x}}^*(t,s,e)\mathbb{E}[\widehat{X}(s)]\Big)\widetilde{N}(ds,de), \quad t\in[0,T], \end{aligned} \tag{5.10}$$

where

$$\begin{aligned} \varphi^*(t) &= \int_0^t \Big(b_u^*(t,s)(u(s)-u^*(s)) + b_{\overline{u}}^*(t,s)\mathbb{E}[u(s)-u^*(s)]\Big)ds \\ &\quad + \int_0^t\Big(\sigma_u^*(t,s)(u(s)-u^*(s)) + \sigma_{\overline{u}}^*(t,s)\mathbb{E}[u(s)-u^*(s)]\Big)dW(s) \\ &\quad + \int_0^t\int_{\mathbb{R}_0}\Big(\gamma_u^*(t,s,e)(u(s)-u^*(s)) + \gamma_{\overline{u}}^*(t,s,e)\mathbb{E}[u(s)-u^*(s)]\Big)\widetilde{N}(ds,de), \quad t\in[0,T]. \end{aligned}$$

In fact, by the Taylor expansion, we have

$$
\begin{aligned}
&X^{\varepsilon}(t)-X^{*}(t)\\
&=\int_0^t\Big(\tilde{b}_x^{\varepsilon}(t,s)(X^{\varepsilon}(s)-X^{*}(s))+\tilde{b}_{\overline{x}}^{\varepsilon}(t,s)\mathbb{E}[X^{\varepsilon}(s)-X^{*}(s)]+\tilde{b}_u^{\varepsilon}(t,s)\varepsilon(u(s)-u^{*}(s))\\
&+\tilde{b}_{\overline{u}}^{\varepsilon}(t,s)\varepsilon\mathbb{E}[u(s)-u^{*}(s)]\Big)ds\\
&+\int_0^t\Big(\tilde{\sigma}_x^{\varepsilon}(t,s)(X^{\varepsilon}(s)-X^{*}(s))+\tilde{\sigma}_{\overline{x}}^{\varepsilon}(t,s)\mathbb{E}[X^{\varepsilon}(s)-X^{*}(s)]+\tilde{\sigma}_u^{\varepsilon}(t,s)\varepsilon(u(s)-u^{*}(s))\\
&+\tilde{\sigma}_{\overline{u}}^{\varepsilon}(t,s)\varepsilon\mathbb{E}[u(s)-u^{*}(s)]\Big)dW(s)\\
&+\int_0^t\int_{\mathbb{R}_0}\Big(\tilde{\gamma}_x^{\varepsilon}(t,s,e)(X^{\varepsilon}(s)-X^{*}(s))+\tilde{\gamma}_{\overline{x}}^{\varepsilon}(t,s,e)\mathbb{E}[X^{\varepsilon}(s)-X^{*}(s)]+\tilde{\gamma}_u^{\varepsilon}(t,s,e)\varepsilon(u(s)-u^{*}(s))\\
&+\tilde{\gamma}_{\overline{u}}^{\varepsilon}(t,s,e)\varepsilon\mathbb{E}[u(s)-u^{*}(s)]\Big)\widetilde{N}(ds,de),\quad\forall t\in[0,T],
\end{aligned}
\tag{5.11}
$$

where

$$
\begin{aligned}
\tilde{b}_x^{\varepsilon}(t,s)&=\int_0^1 b_x(t,s,X^{*}(s)+\lambda(X^{\varepsilon}(s)-X^{*}(s)),\mathbb{E}[X^{\varepsilon}(s)],u^{\varepsilon}(s),\mathbb{E}[u^{\varepsilon}(s)])d\lambda,\\
\tilde{b}_{\overline{x}}^{\varepsilon}(t,s)&=\int_0^1 b_{\overline{x}}(t,s,X^{*}(s),\mathbb{E}[X^{*}(s)+\lambda(X^{\varepsilon}(s)-X^{*}(s))],u^{\varepsilon}(s),\mathbb{E}[u^{\varepsilon}(s)])d\lambda,\\
\tilde{b}_u^{\varepsilon}(t,s)&=\int_0^1 b_u(t,s,X^{*}(s),\mathbb{E}[X^{*}(s)],u^{*}(s)+\lambda\varepsilon(u(s)-u^{*}(s)),\mathbb{E}[u^{\varepsilon}(s)])d\lambda,\\
\tilde{b}_{\overline{u}}^{\varepsilon}(t,s)&=\int_0^1 b_{\overline{u}}(t,s,X^{*}(s),\mathbb{E}[X^{*}(s)],u^{*}(s),\mathbb{E}[u^{*}(s)+\lambda\varepsilon(u(s)-u^{*}(s))])d\lambda,
\end{aligned}
$$

and $\tilde{\sigma}_x^{\varepsilon}(t,s)$, $\tilde{\sigma}_{\overline{x}}^{\varepsilon}(t,s)$, $\tilde{\sigma}_u^{\varepsilon}(t,s)$, $\tilde{\sigma}_{\overline{u}}^{\varepsilon}(t,s)$, $\tilde{\gamma}_x^{\varepsilon}(t,s,e)$, $\tilde{\gamma}_{\overline{x}}^{\varepsilon}(t,s,e)$, $\tilde{\gamma}_u^{\varepsilon}(t,s,e)$, and $\tilde{\gamma}_{\overline{u}}^{\varepsilon}(t,s,e)$ can be derived in a similar way. We do not repeat them here.

Under Assumption (A3), by the estimate (3.17), we derive that

$$
\begin{aligned}
&\|X^{\varepsilon}(\cdot)-X^{*}(\cdot)\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R})}:=\mathbb{E}\left[\int_0^T e^{-\beta t}|X^{\varepsilon}(t)-X^{*}(\cdot)|^2dt\right]\\
&\le K\mathbb{E}\Bigg[\int_0^T\int_0^t e^{-\beta s}\Big(\left|\tilde{b}_u^{\varepsilon}(t,s)\varepsilon(u(s)-u^{*}(s))\right|^2+\left|\tilde{b}_{\overline{u}}^{\varepsilon}(t,s)\varepsilon\mathbb{E}[u(s)-u^{*}(s)]\right|^2+\left|\tilde{\sigma}_u^{\varepsilon}(t,s)\varepsilon(u(s)-u^{*}(s))\right|^2\\
&\quad+\left|\tilde{\sigma}_{\overline{u}}^{\varepsilon}(t,s)\varepsilon\mathbb{E}[u(s)-u^{*}(s)]\right|^2+\left\|\tilde{\gamma}_u^{\varepsilon}(t,s,\cdot)\varepsilon(u(s)-u^{*}(s))\right\|_{\nu}^2+\left\|\tilde{\gamma}_{\overline{u}}^{\varepsilon}(t,s,\cdot)\varepsilon\mathbb{E}[u(s)-u^{*}(s)]\right\|_{\nu}^2\Big)dsdt\Bigg]\\
&\le K\varepsilon^2\to 0,\quad as\ \varepsilon\mapsto 0,
\end{aligned}
$$

which means that

$$
X^{\varepsilon}(\cdot)\mapsto X^{*}(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}),\quad as\ \varepsilon\mapsto 0. \tag{5.12}
$$

Then, let

$$
\eta^{\varepsilon}(t):=\frac{X^{\varepsilon}(t)-X^{*}(t)}{\varepsilon}-\widehat{X}(t),\quad t\in[0,T].
$$

It follows from (5.10) and (5.11) that

$$
\eta^{\varepsilon}(t)=\int_0^t\Big(\tilde{b}_x^{\varepsilon}(t,s)\eta^{\varepsilon}(s)+\tilde{b}_{\overline{x}}^{\varepsilon}(t,s)\mathbb{E}[\eta^{\varepsilon}(s)]\Big)ds+\int_0^t\Big(\tilde{\sigma}_x^{\varepsilon}(t,s)\eta^{\varepsilon}(s)+\tilde{\sigma}_{\overline{x}}^{\varepsilon}(t,s)\mathbb{E}[\eta^{\varepsilon}(s)]\Big)dW(s)
$$

$$+\int_0^t \int_{\mathbb{R}_0} \left(\tilde{\gamma}_x^{\varepsilon}(t,s,e)\eta^{\varepsilon}(s) + \tilde{\gamma}_{\bar{x}}^{\varepsilon}(t,s,e)\mathbb{E}[\eta^{\varepsilon}(s)]\right)\widetilde{N}(ds,de) + \int_0^t \Pi_1^{\varepsilon}(t,s)ds + \int_0^t \Pi_2^{\varepsilon}(t,s)dW(s)$$
$$+\int_0^t \int_{\mathbb{R}_0} \Pi_3^{\varepsilon}(t,s,e)\widetilde{N}(ds,de), \quad t \in [0,T],$$

where

$$\begin{aligned}
\Pi_1^{\varepsilon}(t,s) &= (\tilde{b}_x^{\varepsilon}(t,s) - b_x^*(t,s))\widehat{X}(s) + (\tilde{b}_{\bar{x}}^{\varepsilon}(t,s) - b_{\bar{x}}^*(t,s))\mathbb{E}[\widehat{X}(s)] + (\tilde{b}_u^{\varepsilon}(t,s) - b_u^*(t,s))(u(s) - u^*(s)) \\
&\quad + (\tilde{b}_{\bar{u}}^{\varepsilon}(t,s) - b_{\bar{u}}^*(t,s))\mathbb{E}[u(s) - u^*(s)], \\
\Pi_2^{\varepsilon}(t,s) &= (\tilde{\sigma}_x^{\varepsilon}(t,s) - \sigma_x^*(t,s))\widehat{X}(s) + (\tilde{\sigma}_{\bar{x}}^{\varepsilon}(t,s) - \sigma_{\bar{x}}^*(t,s))\mathbb{E}[\widehat{X}(s)] + (\tilde{\sigma}_u^{\varepsilon}(t,s) - \sigma_u^*(t,s))(u(s) - u^*(s)) \\
&\quad + (\tilde{\sigma}_{\bar{u}}^{\varepsilon}(t,s) - \sigma_{\bar{u}}^*(t,s))\mathbb{E}[u(s) - u^*(s)], \\
\Pi_3^{\varepsilon}(t,s,e) &= (\tilde{\gamma}_x^{\varepsilon}(t,s,e) - \gamma_x^*(t,s,e))\widehat{X}(s) + (\tilde{\gamma}_{\bar{x}}^{\varepsilon}(t,s,e) - \gamma_{\bar{x}}^*(t,s,e))\mathbb{E}[\widehat{X}(s)] \\
&\quad + (\tilde{\gamma}_u^{\varepsilon}(t,s,e) - \gamma_u^*(t,s,e))(u(s) - u^*(s)) + (\tilde{\gamma}_{\bar{u}}^{\varepsilon}(t,s,e) - \gamma_{\bar{u}}^*(t,s,e))\mathbb{E}[u(s) - u^*(s)].
\end{aligned}$$

Under Assumption (A3), by the estimates (3.17) and (5.12), it then follows from the dominated convergence theorem that

$$\begin{aligned}
&\|\eta^{\varepsilon}(\cdot)\|^2_{L^2_{\mathbb{F}}(0,T;\mathbb{R})} := \mathbb{E}\left[\int_0^T e^{-\beta t}|\eta^{\varepsilon}(t)|^2 dt\right] \\
&\leq K\mathbb{E}\left[\int_0^T \int_0^t e^{-\beta s}\left(\left|\Pi_1^{\varepsilon}(t,s)\right|^2 + \left|\Pi_2^{\varepsilon}(t,s)\right|^2 + \left\|\Pi_3^{\varepsilon}(t,s,\cdot)\right\|_{\nu}^2\right)dsdt\right] \\
&\to 0, \quad as\ \varepsilon \mapsto 0.
\end{aligned}$$

Hence,

$$\widehat{X}^{\varepsilon}(\cdot) := \frac{X^{\varepsilon}(\cdot) - X^*(\cdot)}{\varepsilon} \mapsto \widehat{X}(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}), \quad as\ \varepsilon \mapsto 0. \tag{5.13}$$

In what follows, it should be pointed out that Assumptions (A3) and (A4) are stronger than Assumption (A1), and $h_x^*(\cdot) + \mathbb{E}\left[h_{\bar{x}}^*(\cdot)\right] \in L^2_{\mathcal{F}_T}(0,T;\mathbb{R})$ and $u^*(\cdot) \in \mathcal{A}$. Thus, by Theorem 1, we know that $(Y(\cdot), Y_0(\cdot); Z(\cdot,\cdot), Z_0(\cdot,\cdot); V(\cdot,\cdot,\cdot), V_0(\cdot,\cdot,\cdot))$ is the unique adapted M-solution of (5.7). Under Assumption (A4), for each optimal pair $(X^*(\cdot), u^*(\cdot))$ of Problem 1, from (5.2), (5.12), and (5.13), and by the dominated convergence theorem, we have the following:

$$\begin{aligned}
0 &\leq \frac{J(u^{\varepsilon}(\cdot)) - J(u^*(\cdot))}{\varepsilon} \\
&= \frac{1}{\varepsilon}\mathbb{E}\left[\int_0^T (h(t, X^{\varepsilon}(t), \mathbb{E}[X^{\varepsilon}(t)], u^{\varepsilon}(t), \mathbb{E}[u^{\varepsilon}(t)]) - h(t, X^*(t), \mathbb{E}[X^*(t)], u^*(t), \mathbb{E}[u^*(t)]))\, dt\right] \\
&= \mathbb{E}\Bigg[\int_0^T \Bigg(\int_0^1 h_x(t, X^*(t) + \lambda(X^{\varepsilon}(t) - X^*(t)), \mathbb{E}[X^{\varepsilon}(t)], u^{\varepsilon}(t), \mathbb{E}[u^{\varepsilon}(t)])d\lambda \cdot \frac{X^{\varepsilon}(t) - X^*(t)}{\varepsilon} \\
&\quad + \int_0^1 h_{\bar{x}}(t, X^*(t), \mathbb{E}[X^*(t) + \lambda(X^{\varepsilon}(t) - X^*(t))], u^{\varepsilon}(t), \mathbb{E}[u^{\varepsilon}(t)])d\lambda \cdot \mathbb{E}\left[\frac{X^{\varepsilon}(t) - X^*(t)}{\varepsilon}\right] \\
&\quad + \int_0^1 h_u(t, X^*(t), \mathbb{E}[X^*(t)], u^*(t) + \lambda\varepsilon(u(t) - u^*(t)), \mathbb{E}[u^{\varepsilon}(t)])d\lambda \cdot (u(t) - u^*(t)) \\
&\quad + \int_0^1 h_{\bar{u}}(t, X^*(t), \mathbb{E}[X^*(t)], u^*(t), \mathbb{E}[u^*(t) + \lambda\varepsilon(u(t) - u^*(t))])d\lambda \cdot \mathbb{E}[u(t) - u^*(t)]\Bigg) dt\Bigg]
\end{aligned} \tag{5.14}$$

$$\to \mathbb{E}\left[\int_0^T \left(h_x^*(t)\widehat{X}(t) + h_{\overline{x}}^*(t)\mathbb{E}[\widehat{X}(t)] + h_u^*(t)(u(t)-u^*(t)) + h_{\overline{u}}^*(t)\mathbb{E}[u(t)-u^*(t)]\right)dt\right].$$

By Theorem 4, it follows from (3.2) and (5.14) that

$$\begin{aligned}
0 &\le \frac{J(u^\varepsilon(\cdot)) - J(u^*(\cdot))}{\varepsilon} \\
&= \mathbb{E}\left[\int_0^T \left(h_x^*(t) + \mathbb{E}[h_{\overline{x}}^*(t)]\right)\widehat{X}(t)dt\right] + \mathbb{E}\left[\int_0^T \left(h_u^*(t) + \mathbb{E}[h_{\overline{u}}^*(t)]\right)(u(t)-u^*(t))dt\right] \\
&= \mathbb{E}\left[\int_0^T \varphi^*(t)Y(t)dt\right] + \mathbb{E}\left[\int_0^T \left(h_u^*(t) + \mathbb{E}[h_{\overline{u}}^*(t)]\right)(u(t)-u^*(t))dt\right] \\
&= \mathbb{E}\left\{\int_0^T \left[\int_0^t \left(b_u^*(t,s)(u(s)-u^*(s)) + b_{\overline{u}}^*(t,s)\mathbb{E}[u(s)-u^*(s)]\right)ds\right]Y(t)dt\right\} \\
&\quad + \mathbb{E}\left\{\int_0^T \left[\int_0^t \left(\sigma_u^*(t,s)(u(s)-u^*(s)) + \sigma_{\overline{u}}^*(t,s)\mathbb{E}[u(s)-u^*(s)]\right)dW(s)\right]Y(t)dt\right\} \\
&\quad + \mathbb{E}\left\{\int_0^T \left[\int_0^t \int_{\mathbb{R}_0} \left(\gamma_u^*(t,s,e)(u(s)-u^*(s)) + \gamma_{\overline{u}}^*(t,s,e)\mathbb{E}[u(s)-u^*(s)]\right)\widetilde{N}(ds,de)\right]Y(t)dt\right\} \\
&\quad + \mathbb{E}\left[\int_0^T \left(h_u^*(t) + \mathbb{E}[h_{\overline{u}}^*(t)]\right)(u(t)-u^*(t))dt\right] \\
&= \mathbb{E}\left\{\int_0^T \left[\int_s^T \left(b_u^*(t,s)Y(t) + \mathbb{E}[b_{\overline{u}}^*(t,s)Y(t)]\right)dt\right](u(s)-u^*(s))ds\right\} \\
&\quad + \mathbb{E}\left\{\int_0^T \left[\int_0^t \left(\sigma_u^*(t,s)Z(t,s) + \mathbb{E}[\sigma_{\overline{u}}^*(t,s)Z(t,s)]\right)(u(s)-u^*(s))ds\right]dt\right\} \\
&\quad + \mathbb{E}\left\{\int_0^T \left[\int_0^t \int_{\mathbb{R}_0} \left(\gamma_u^*(t,s,e)V(t,s,e) + \mathbb{E}[\gamma_{\overline{u}}^*(t,s,e)V(t,s,e)]\right)(u(s)-u^*(s))\nu(de)ds\right]dt\right\} \\
&\quad + \mathbb{E}\left[\int_0^T \left(h_u^*(t) + \mathbb{E}[h_{\overline{u}}^*(t)]\right)(u(t)-u^*(t))dt\right] \\
&= \mathbb{E}\left\{\int_0^T \left[\int_t^T \left(b_u^*(s,t)Y(s) + \mathbb{E}[b_{\overline{u}}^*(s,t)Y(s)]\right)ds\right](u(t)-u^*(t))dt\right\} \\
&\quad + \mathbb{E}\left\{\int_0^T \left[\int_t^T \left(\sigma_u^*(s,t)Z(s,t) + \mathbb{E}[\sigma_{\overline{u}}^*(s,t)Z(s,t)]\right)ds\right](u(t)-u^*(t))dt\right\} \\
&\quad + \mathbb{E}\left\{\int_0^T \left[\int_t^T \int_{\mathbb{R}_0} \left(\gamma_u^*(s,t,e)V(s,t,e) + \mathbb{E}[\gamma_{\overline{u}}^*(s,t,e)V(s,t,e)]\right)\nu(de)ds\right](u(t)-u^*(t))dt\right\} \\
&\quad + \mathbb{E}\left[\int_0^T \left(h_u^*(t) + \mathbb{E}[h_{\overline{u}}^*(t)]\right)(u(t)-u^*(t))dt\right] \\
&= \mathbb{E}\left[\int_0^T \left(Y_0(t) + h_u^*(t) + \mathbb{E}[h_{\overline{u}}^*(t)]\right)(u(t)-u^*(t))dt\right].
\end{aligned} \tag{5.15}$$

Since $u(\cdot) \in \mathcal{A}$ and $\mathbf{U}$ is convex, we suppose that

$$u(s) = \begin{cases} u, & s \in [t, t+\varepsilon], \quad \omega \in A, \\ u^*(s), & \text{otherwise}, \end{cases}$$

where $0 \leq t < t + \varepsilon \leq T$, $u \in \mathbf{U}$, and $A \in \mathcal{F}_t$. Then (5.15) leads to

$$\frac{1}{\varepsilon}\mathbb{E}\left[\int_t^{t+\varepsilon}\left(Y_0(s) + h_u^*(s) + \mathbb{E}[h_{\bar{u}}^*(s)]\right)(u - u^*(s))dsI_A\right] \geq 0.$$

Letting $\varepsilon \mapsto 0$ in the above inequality, we obtain that

$$\mathbb{E}\left[\left(Y_0(t) + h_u^*(t) + \mathbb{E}[h_{\bar{u}}^*(t)]\right)(u - u^*(t))I_A\right] \geq 0$$

for any $u \in \mathbf{U}$, $A \in \mathcal{F}_t$. This means that

$$\mathbb{E}\left[\left(Y_0(t) + h_u^*(t) + \mathbb{E}[h_{\bar{u}}^*(t)]\right)(u - u^*(t))\middle|\mathcal{F}_t\right] = \left(Y_0(t) + h_u^*(t) + \mathbb{E}[h_{\bar{u}}^*(t)]\right)(u - u^*(t)) \geq 0,$$

for any $u \in \mathbf{U}$, $t \in [0, T]$, a.s., which is the required (5.8). □

Remark 2. *Note that the Assumptions* (A3) *and* (A4) *are more than sufficient, and many of these conditions can be relaxed. In order to satisfy the Lipschitz conditions of (5.7) and (5.10), we usually adopt these stronger conditions in stochastic optimal control theory. In the future research, we will relax these assumptions so that $b(\cdot)$, $\sigma(\cdot)$, $\gamma(\cdot)$, and $h(\cdot)$ can satisfy the controlled growth.*

By Theorem 5, we define the Hamiltonian function H by

$$\begin{aligned}
&H(t, X^*(t), \mathbb{E}[X^*(t)], u^*(t), \mathbb{E}[u^*(t)], Y(\cdot), Z(\cdot, t), V(\cdot, t, e); u)\\
&:= -\left(Y_0(t) + h_u^*(t) + \mathbb{E}[h_{\bar{u}}^*(t)]\right)u\\
&= -\mathbb{E}\Bigg[h_u^*(t) + \mathbb{E}[h_{\bar{u}}^*(t)] + \int_t^T \Big[b_u^*(s,t)Y(s) + \mathbb{E}\left[b_{\bar{u}}^*(s,t)Y(s)\right] + \sigma_u^*(s,t)Z(s,t) + \mathbb{E}\left[\sigma_{\bar{u}}^*(s,t)Z(s,t)\right]\\
&\quad + \int_{\mathbb{R}_0}\left(\gamma_u^*(s,t,e)V(s,t,e) + \mathbb{E}\left[\gamma_{\bar{u}}^*(s,t,e)V(s,t,e)\right]\right)\nu(de)\Big]ds\Bigg|\mathcal{F}_t\Bigg]u, \quad \forall\, t \in [0, T],\ e \in \mathbb{R}_0.
\end{aligned}$$

Thus, (5.8) means that

$$\begin{aligned}
&H(t, X^*(t), \mathbb{E}[X^*(t)], u^*(t), \mathbb{E}[u^*(t)], Y(\cdot), Z(\cdot, t), V(\cdot, t, e); u^*(t))\\
&= \max_{u \in \mathcal{A}} H(t, X^*(t), \mathbb{E}[X^*(t)], u^*(t), \mathbb{E}[u^*(t)], Y(\cdot), Z(\cdot, t), V(\cdot, t, e); u),
\end{aligned}$$

where $(Y(\cdot), Z(\cdot, \cdot), V(\cdot, \cdot, \cdot))$ satisfies the following adjoint equation with respect to the optimal pair $(X^*(\cdot), u^*(\cdot))$:

$$\begin{aligned}
Y(t) &= h_x^*(t) + \mathbb{E}\left[h_{\bar{x}}^*(t)\right] + \int_t^T \Big[b_x^*(s,t)Y(s) + \mathbb{E}\left[b_{\bar{x}}^*(s,t)Y(s)\right] + \sigma_x^*(s,t)Z(s,t) + \mathbb{E}\left[\sigma_{\bar{x}}^*(s,t)Z(s,t)\right]\\
&\quad + \int_{\mathbb{R}_0}\left(\gamma_x^*(s,t,e)V(s,t,e) + \mathbb{E}\left[\gamma_{\bar{x}}^*(s,t,e)V(s,t,e)\right]\right)\nu(de)\Big]ds - \int_t^T Z(t,s)dW(s)\\
&\quad - \int_t^T\int_{\mathbb{R}_0} V(t,s,e)\widetilde{N}(ds,de), \quad t \in [0, T].
\end{aligned}$$

Now, we are devoted to applying the maximum principle (i.e., Theorem 5) to study a linear-quadratic optimal control problem of MF-FSVIEs with jumps.

Example 1. *Let* $\mathbf{U} := \mathbb{R}$. *We consider the controlled linear MF-FSVIE with jump as follows:*

$$\begin{aligned} X(t) = \varphi(t) &+ \int_0^t (a_1(t,s)X(s) + a_2(t,s)\mathbb{E}[X(s)] + a_3(t,s)u(s) + a_4(t,s)\mathbb{E}[u(s)])\, ds \\ &+ \int_0^t (b_1(t,s)X(s) + b_2(t,s)\mathbb{E}[X(s)] + b_3(t,s)u(s) + b_4(t,s)\mathbb{E}[u(s)])\, dW(s) \\ &+ \int_0^t \int_{\mathbb{R}_0} (c_1(t,s,e)X(s) + c_2(t,s,e)E[X(s)] + c_3(t,s,e)u(s) + c_4(t,s,e)E[u(s)])\, \widetilde{N}(ds,de), \\ & t \in [0,T], \end{aligned} \tag{5.16}$$

where $\varphi(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R})$. $a_i(\cdot,\cdot), b_i(\cdot,\cdot) : \Delta^c \mapsto \mathbb{R}$ *are given bounded measurable functions, and* $c_i(\cdot,\cdot,\cdot) : \Delta^c \times \mathbb{R}_0 \mapsto \mathbb{R}$ *is a given bounded measurable function,* $i = 1,2,3,4$. $u(\cdot)$ *is a càdlàg and* $\mathbb{F}$*-predictable control process of the controller, and the admissible control set is denoted by* $\mathcal{A} := L^2_{\mathbb{F}}(0,T;\mathbb{R})$.

The cost functional is defined by

$$J(u(\cdot)) = \frac{1}{2}\mathbb{E}\left[\int_0^T \left(q_1(t)X^2(t) + q_2(t)(\mathbb{E}[X(t)])^2 + r_1(t)u^2(t) + r_2(t)(\mathbb{E}[u(t)])^2\right) dt\right], \tag{5.17}$$

where $q_i(\cdot)$ *is a nonnegative bounded measurable function, and* $r_i(\cdot)$ *is a positive bounded measurable function,* $i = 1,2$.

By Theorem 2, we can obtain that for any $\varphi(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R})$ *and* $u(\cdot) \in \mathcal{A}$*, the linear MF-FSVIE with jump (5.16) admits a unique adapted solution* $X(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R})$. *Then the linear-quadratic optimal control problem driven by MF-FSVIEs with jumps is given below.*

Problem 2. *Find an admissible control* $u^*(\cdot) \in \mathcal{A}$ *such that*

$$J(u^*(\cdot)) = \inf_{u(\cdot)\in\mathcal{A}} J(u(\cdot)). \tag{5.18}$$

Next, we define the Hamiltonian function H of Problem 2 by

$$\begin{aligned} & H(t, X^*(t), \mathbb{E}[X^*(t)], u^*(t), \mathbb{E}[u^*(t)], Y(\cdot), Z(\cdot,t), V(\cdot,t,e); u) \\ := & -(Y_0(t) + r_1(t)u^*(t) + r_2(t)\mathbb{E}[u^*(t)])\, u, \end{aligned}$$

and the corresponding adjoint equation has the following form:

$$\begin{aligned} Y(t) = q_1(t)X^*(t) + q_2(t)\mathbb{E}[X^*(t)] &+ \int_t^T \Big[a_1(s,t)Y(s) + a_2(s,t)\mathbb{E}[Y(s)] + b_1(s,t)Z(s,t) + b_2(s,t)\mathbb{E}[Z(s,t)] \\ & \int_{\mathbb{R}_0} (c_1(s,t,e)V(s,t,e) + c_2(s,t,e)\mathbb{E}[V(s,t,e)])\, \nu(de)\Big] ds - \int_t^T Z(t,s)dW(s) \\ & - \int_t^T \int_{\mathbb{R}_0} V(t,s,e)\widetilde{N}(ds,de), \quad t \in [0,T], \end{aligned}$$

which has a unique adapted M-solution $(Y(\cdot), Z(\cdot,\cdot), V(\cdot,\cdot,\cdot)) \in \mathcal{M}^2_\beta[0,T]$ *according to Theorem 1.*

From Theorem 5, the optimal control $u^*(\cdot) \in \mathcal{A}$ *of Problem 2 satisfies the following equation:*

$$Y_0(t) + r_1(t)u^*(t) + r_2(t)\mathbb{E}[u^*(t)] = 0, \quad t \in [0,T].$$

Therefore, the optimal control $u^*(\cdot) \in \mathcal{A}$ of Problem 2 is given by

$$u^*(t) = [r_1(t)(r_1(t) + r_2(t))]^{-1} r_2(t)\mathbb{E}[Y_0(t)] - r_1(t)^{-1} Y_0(t), \quad t \in [0, T], \tag{5.19}$$

where $(Y_0(\cdot), Z_0(\cdot,\cdot), V_0(\cdot,\cdot,\cdot)) \in \mathcal{M}^2_\beta[0,T]$ is the unique adapted M-solution of the following equation:

$$\begin{aligned} Y_0(t) = \int_t^T \Big[& a_3(s,t)Y(s) + a_4(s,t)\mathbb{E}[Y(s)] + b_3(s,t)Z(s,t) + b_4(s,t)\mathbb{E}[Z(s,t)] + \int_{\mathbb{R}_0} (c_3(s,t,e)V(s,t,e) \\ & + c_4(s,t,e)\mathbb{E}[V(s,t,e)])\, \nu(de)\Big]\, ds - \int_t^T Z_0(t,s)dW(s) - \int_t^T \int_{\mathbb{R}_0} V_0(t,s,e)\widetilde{N}(ds,de), \quad t \in [0,T]. \end{aligned}$$

Proposition 1. *For the linear-quadratic optimal control problem of MF-FSVIE with jump (5.16)–(5.18), an optimal control $u^*(\cdot)$ is given by (5.19).*

6. Conclusions

This paper mainly focuses on the research for a new class of MF-BSVIEs with jumps, including the well-posedness and the comparison theorem of MF-BSVIEs with jumps. Moreover, by virtue of the duality principle, we present Pontryagin's type maximum principle for optimal control problems of MF-FSVIEs with jumps. As one of the applications of the principle, we get an optimal control for the linear-quadratic optimal control problem of MF-FSVIEs with jumps. We point out that there are many assumptions in the comparison theorem for MF-BSVIEs with jumps, and these assumptions are relatively strict. We will relax these assumptions in our future works, such as we might be able to apply Malliavin calculus to weaken the condition (d) in the comparison theorem. In years to come, we could study the relationship between MF-BSVIEs with jumps and quasilinear stochastic partial differential equations. We would also discuss the BSVIE driven by a fractional Brownian motion, and study its application to the stochastic optimal control. Motivated by [41–43], we could study the stability analysis for the BSVIEs. In the future research, we could explore some other directions, such as multidimensional systems, stochastic delay differential equations, infinite-dimensional settings, and numerical approximations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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