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Theory article

Strongly stable hypersurfaces with constant mean curvature in space forms

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Abstract: In this paper, we consider a complete strongly stable hypersurface M^n with constant mean curvature (CMC) H in $N^{n+1}(\kappa)$, where $2 \leq n \leq 5$, and prove that M^n is totally umbilical if $H^2 + \kappa > 0$ and the L^2 norm curvature satisfies some growth conditions. In addition, we obtain that a complete stable hypersurface M^n in $N^{n+1}(\kappa)(n = 3, 4, 5)$ with CMC is a geodesic sphere if $-H^2 < \kappa \leq 0$ and M^n has some finite L^p norm curvature.

Keywords: constant mean curvature; finite L^p norm curvature; stability

1. Introduction

Bernstein's classical theorem states that every entire minimal graph in $\mathbb{R}^3$ is an affine plane. This pioneering result stimulated extensive research on rigidity phenomena for minimal hypersurfaces, particularly stable (see Definition 2.1) minimal hypersurfaces arising naturally in variational problems. In this direction, do Carmo and Peng [1] established that complete stable minimal surfaces in $\mathbb{R}^3$ must be planes. Subsequent studies extended these rigidity results to stable minimal hypersurfaces in higher-dimensional Euclidean spaces. Chodosh and Li [2] proved the flatness of complete two-sided stable minimal hypersurfaces in $\mathbb{R}^4$. Independently, Catino et al. [3] obtained that a complete orientable stable minimal hypersurface in $\mathbb{R}^4$ is a hyperplane. Moreover, Chodosh et al. [4] extended these rigidity results to stable minimal hypersurfaces in $\mathbb{R}^5$.

Beyond the minimal case, rigidity problems for constant mean curvature (CMC) hypersurfaces under stability assumptions have also attracted considerable attention. do Carmo and Peng [5] studied complete stable minimal hypersurfaces in $\mathbb{R}^{n+1}$ and showed that suitable growth conditions on the L^2 norm of the curvature imply that the hypersurface is a hyperplane. This result was later generalized by Alencar and do Carmo [6], who proved that complete noncompact stable CMC hypersurfaces in $\mathbb{R}^{n+1}(n \leq 5)$ are hyperplanes under analogous curvature growth assumptions. do Carmo and Zhou [7] extended the dimensional range in [6] from $n \leq 5$ to $n \leq 6$. Deng [8] further refined the L^2 norm curvature growth condition established in [6]. In addition, Deng established that any complete stable CMC hypersurface M^5 in $\mathbb{R}^6$ with $H > 0$ and finite L^p norm curvature must be a geodesic sphere. More recently, Cheng and Wei [9] studied complete two-sided minimal hypersurfaces in $\mathbb{R}^{n+1}$ that are δ-stable for $3 \leq n \leq 5$. They proved that such hypersurfaces have Euclidean volume growth when δ exceeds certain dimension-dependent thresholds and that, under stronger threshold conditions, they must be hyperplanes.

In this paper, we first study a CMC hypersurface M^n in a Riemannian manifold $N^{n+1}(\kappa)$ of constant sectional curvature κ, assuming that the L^2 norm curvature satisfies certain growth conditions, and establish the following rigidity result.

Theorem 1.1. *Let M^n be a complete strongly stable hypersurface immersed in $N^{n+1}(\kappa)$ with CMC H, where $2 \leq n \leq 5$. Suppose that $H^2 + \kappa > 0$ and*

$$\lim_{R\to+\infty} \frac{\int_{B_{x_0}(R)} |\phi|^2 dv}{R^{2+2q}} = 0, \tag{1.1}$$

where $x_0 \in M^n$ and $0 < q < \frac{2\sqrt{H^2+\kappa}\sqrt{n-1}}{\sqrt{n^2H^2+4(n-1)\kappa}} + \sqrt{\frac{4(H^2+\kappa)(n-1)}{n^2H^2+4(n-1)\kappa} - \frac{2(n-2)\sqrt{n-1}\sqrt{H^2+\kappa}}{n\sqrt{n^2H^2+4(n-1)\kappa}}} - 1$. Then M^n is totally umbilical.

Remark 1.1. *When $\kappa = 0$, if the assumption $H^2 + \kappa > 0$ is removed and M^n is assumed to be noncompact, the conclusion reduces to Theorem 1.5 in [8]. Indeed, suppose, for contradiction, that $H \neq 0$. Since $\kappa = 0$, we have $H^2 + \kappa > 0$. Therefore, the assumptions of Theorem 1.1 are satisfied, and the argument leading to (3.13) can be applied. Following the approach developed in [5], we conclude that M^n is compact, which contradicts the noncompactness assumption. Hence, $H = 0$. The conclusion then follows from the rigidity results of do Carmo and Peng [5], Cheng [10], and López and Ros [11].*

We next study a CMC hypersurface M^n immersed in $N^{n+1}(\kappa)$ whose L^p norm curvature is finite and establish

Theorem 1.2. *Let M^n be a complete stable hypersurface immersed in $N^{n+1}(\kappa)$ with CMC H, where $3 \leq n \leq 5$. Suppose that $-H^2 < \kappa \leq 0$ and*

$$\int_{M^n} |\phi|^p dv < \infty,$$

where $\frac{3-\sqrt{1+16/n}}{2} < p < \frac{3+\sqrt{1+16/n}}{2}$. Then M^n is a geodesic sphere.

Remark 1.2. *If $n = 5, \kappa = 0$, and $H > 0$, Theorem 1.2 recovers Theorem 3.5 in [8]. If $\kappa = 0$, $H > 0$, and $p = 2$, Theorem 1.2 recovers Theorem 2 in [12]. If $\kappa = -1, H > 1$, and $p = 2$, Theorem 1.2 recovers Theorem 3 in [12].*

Theorem 1.2 is also closely related to Theorems 1.4, 1.6, and 1.10 of Fu [13]. For common parameter ranges, Fu's [13] Theorems 1.4 and 1.6 and the present theorem yield the same spherical rigidity conclusions in Euclidean and hyperbolic spaces. However, Theorem 1.2 provides a unified formulation for CMC hypersurfaces in $N^{n+1}(\kappa)$ under the condition $-H^2 < \kappa \leq 0$, whereas Fu [13] treats the Euclidean and hyperbolic cases separately. Moreover, the admissible exponent ranges are *not nested: The present theorem allows smaller exponents, while Fu's [13] theorems allow larger ones. The hyperplane case in Fu's [13] Theorem 1.10 is not covered by Theorem 1.2, since the condition $-H^2 < \kappa \leq 0$ excludes the minimal case $H = 0$.*

2. Preliminaries

Let M^n be a complete oriented hypersurface immersed in $N^{n+1}(\kappa)$. At each point $x \in M^n$, fix an orthonormal frame $\{e_1, e_2, \ldots, e_n, \nu\}$ of $N^{n+1}(\kappa)$ in a neighborhood of x such that $e_1, e_2, \ldots, e_n$ are tangent to M^n.

The shape operator $A : T_xM^n \to T_xM^n$ is defined by

$$\langle AX, Y \rangle = \langle \overline{\nabla}_X Y, \nu \rangle,$$

where $X, Y \in T_xM^n$ and $\overline{\nabla}$ denotes Levi-Civita connection of $N^{n+1}(\kappa)$. The square of the second fundamental form $|A|^2$ and the mean curvature vector h are defined by $|A|^2 = tr(A^2)$ and $h = \frac{trA}{n}\nu$.

Definition 2.1. *A hypersurface M^n with CMC in $N^{n+1}(\kappa)$ is called strongly stable if*

$$\int_{M^n} |\nabla f|^2 dv \geq \int_{M^n} (|A|^2 + \overline{Ric}(\nu, \nu)) f^2 dv \tag{2.1}$$

holds for all $f \in C_0^\infty(M^n)$. If (2.1) holds for every $f \in C_0^\infty(M^n)$ satisfying $\int_{M^n} f dv = 0$, then M^n is called stable.

For a CMC hypersurface, we introduce the following linear map $\phi : T_xM^n \to T_xM^n$ defined by

$$\langle \phi X, Y \rangle = \langle X, Y \rangle \langle h, \nu \rangle - \langle AX, Y \rangle.$$

We also denote by ϕ the associated bilinear form $\phi : T_xM^n \times T_xM^n \to (T_xM^n)^\perp$ defined by

$$\phi(X, Y) = \langle \phi X, Y \rangle \nu.$$

It is well known that ϕ is traceless and satisfies $|\phi|^2 = |A|^2 - nH^2$.

Definition 2.2. *Let M^n be an immersed hypersurface in $N^{n+1}(\kappa)$. We say that M^n has finite L^p norm curvature if*

$$\int_{M^n} |\phi|^p dv < \infty.$$

In particular, when $p = n$, M^n is said to have finite total curvature.

In order to study a CMC hypersurface M^n in $N^{n+1}(\kappa)$ with finite L^p norm curvature, we cite some important lemmas.

Lemma 2.1. *[14] Let M^n be a complete orientable manifold immersed in the simply connected manifold $N^{n+1}(\kappa)$ with constant sectional curvature $\kappa \leq 0$. Suppose that the mean curvature vector h of M^n is parallel and satisfies $|h|^2 > -\kappa$. If M^n has finite total curvature, then M^n is compact.*

Lemma 2.2. *[15] Let $x : M^n \to N^{n+1}(\kappa)$ be a compact CMC hypersurface without a boundary. Then x is stable iff $x(M^n)$ is a geodesic sphere in $N^{n+1}(\kappa)$.*

Lemma 2.3. *Let M^n be a complete noncompact stable hypersurface with CMC in $N^{n+1}(\kappa)$. Then there exists a compact domain $D \subset M^n$ such that*

$$\int_{M^n\setminus D} |\nabla f|^2 dv \geq \int_{M^n\setminus D} \left(|A|^2 + \overline{Ric}(\nu,\nu)\right) f^2 dv \tag{2.2}$$

for every $f \in C_0^\infty(M^n \setminus D)$.

Proof of Lemma 2.3. Define the quadratic form

$$Q(f,f) = \int_{M^n} \left(|\nabla f|^2 - \left(|A|^2 + \overline{Ric}(\nu,\nu)\right) f^2\right) dv.$$

Since M^n is stable, we have $Q(f,f) \geq 0$ for every $f \in C_0^\infty(M^n)$ satisfying $\int_{M^n} f dv = 0$.

If $Q(f,f) \geq 0$ holds for every $f \in C_0^\infty(M^n)$, then the conclusion is immediate. Therefore, it remains to consider the case in which there exists $g \in C_0^\infty(M^n)$ such that $Q(g,g) < 0$. Since M^n is stable, g cannot satisfy the zero-mean condition; otherwise, $Q(g,g) \geq 0$, which contradicts $Q(g,g) < 0$. Hence, $\int_{M^n} g dv \neq 0$.

We shall show that Q is nonnegative outside a suitable compact domain $D \subset M^n$; namely,

$$Q(f,f) \geq 0$$

for all $f \in C_0^\infty(M^n \setminus D)$. Suppose, to the contrary, that no such compact domain exists. Then, for every compact domain $K \subset M^n$ containing $\operatorname{supp} g$, one can find $f \in C_0^\infty(M^n \setminus K)$ satisfying $Q(f,f) < 0$. If $\int_{M^n} f dv = 0$, then stability would imply $Q(f,f) \geq 0$, a contradiction. Hence $\int_{M^n} f dv \neq 0$. Set $c = -\frac{\int_{M^n} f dv}{\int_{M^n} g dv}$. Then

$$\int_{M^n} (f + cg) dv = 0.$$

Since $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$, we obtain

$$Q(f + cg, f + cg) = Q(f,f) + c^2 Q(g,g) < 0,$$

which again contradicts the stability of M^n. Therefore, there exists a compact domain $D \subset M^n$ such that $Q(f,f) \geq 0$ for all $f \in C_0^\infty(M^n \setminus D)$. □

Lemma 2.4. *[8] Let u be a nonnegative locally Lipschitz function satisfying*

$$u^\alpha \Delta u^\alpha \geq k|\nabla u^\alpha|^2 + a u^{2\alpha+2} + b u^{2\alpha+1} + c u^{2\alpha},$$

where $\alpha > 0$ and $a, b, c, k \in \mathbb{R}$. Suppose that there exist a compact domain $D \subset M^n$ and a constant $d \geq 0$ such that

$$\int_{M^n\setminus D} |\nabla f|^2 dv \geq \int_{M^n\setminus D} (u^2 + d) f^2 dv$$

for every $f \in C_0^\infty(M^n \backslash D)$, and assume that $\frac{k}{2} + a + 1 > 0$. Then if

$$\int_{M^n \backslash D} u^{2\alpha} dv < \infty,$$

we have

$$\int_{M^n \backslash D} u^{2\alpha+2} dv < \infty.$$

Lemma 2.5. *[8] Suppose that u is a nonnegative locally Lipschitz function satisfying the conditions of Lemma 2.4 for $\alpha = 1$. If $k > \frac{1}{4}$ and $a \geq -1$, then*

$$\int_{M^n} u^3 dv < \infty \quad \text{and} \quad \int_{M^n} u^4 dv < \infty \Longrightarrow \int_{M^n} u^5 dv < \infty.$$

3. Proof of theorems

Proof of Theorem 1.1. By Corollary 3.1 of [16], we have

$$|\phi|\Delta|\phi| \geq \frac{2}{n}|\nabla|\phi||^2 - |\phi|^4 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi|^3 + n(H^2+\kappa)|\phi|^2. \tag{3.1}$$

By (3.1), we have

$$\begin{aligned}
&|\phi|^\alpha \Delta |\phi|^\alpha \\
=&\frac{\alpha-1}{\alpha}|\nabla|\phi|^\alpha|^2 + \alpha|\phi|^{2\alpha-2}|\phi|\Delta|\phi| \\
\geq&\frac{\alpha-1}{\alpha}|\nabla|\phi|^\alpha|^2 + \alpha|\phi|^{2\alpha-2}\Big(\frac{2}{n}|\nabla|\phi||^2 - |\phi|^4 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi|^3 + n(H^2+\kappa)|\phi|^2\Big) \\
=&(1-\frac{n-2}{n\alpha})|\nabla|\phi|^\alpha|^2 - \alpha|\phi|^{2\alpha}\Big(|\phi|^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi| - n(H^2+\kappa)\Big),
\end{aligned} \tag{3.2}$$

where $\alpha > 0$ is a constant. For $p > 0$ and $f \in C_0^\infty(M^n)$, we multiply (3.2) by $|\phi|^{2p\alpha}f^2$, integrate by parts, and apply the Cauchy-Schwarz and Young inequalities to obtain

$$\begin{aligned}
&(1-\frac{n-2}{n\alpha})\int_{M^n}|\phi|^{2p\alpha}f^2|\nabla|\phi|^\alpha|^2 dv \\
&-\int_{M^n}\alpha|\phi|^{2(p+1)\alpha}f^2\Big(|\phi|^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi| - n(H^2+\kappa)\Big)dv \\
\leq&\int_{M^n}|\phi|^{(2p+1)\alpha}f^2\Delta|\phi|^\alpha dv \\
=&-(2p+1)\int_{M^n}|\phi|^{2p\alpha}f^2|\nabla|\phi|^\alpha|^2 dv - 2\int_{M^n}|\phi|^{(2p+1)\alpha}f\langle\nabla f, \nabla|\phi|^\alpha\rangle dv \\
\leq&-(2p+1)\int_{M^n}|\phi|^{2p\alpha}f^2|\nabla|\phi|^\alpha|^2 dv + \epsilon\int_{M^n}|\phi|^{2p\alpha}f^2|\nabla|\phi|^\alpha|^2 dv
\end{aligned}$$

$$+\frac{1}{\epsilon}\int_{M^n}|\phi|^{2(p+1)\alpha}|\nabla f|^2dv, \tag{3.3}$$

where ϵ is any positive constant. Rearranging (3.3), we obtain

$$\begin{aligned}&\Big(2(1+p)-\frac{n-2}{n\alpha}-\epsilon\Big)\int_{M^n}|\phi|^{2p\alpha}f^2|\nabla|\phi|^\alpha|^2dv\\ \leq&\alpha\int_{M^n}|\phi|^{2(p+1)\alpha}f^2\Big(|\phi|^2+\frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi|-n(H^2+\kappa)\Big)dv\\ &+\frac{1}{\epsilon}\int_{M^n}|\phi|^{2(p+1)\alpha}|\nabla f|^2dv.\end{aligned} \tag{3.4}$$

Furthermore, substituting the test function $|\phi|^{(1+p)\alpha}f$ into the stability inequality (2.1) and using the Cauchy-Schwarz inequality together with Young's inequality with ϵ, we have

$$\begin{aligned}&\int_{M^n}\Big(|\phi|^2+n(H^2+\kappa)\Big)|\phi|^{2(p+1)\alpha}f^2dv\\ =&\int_{M^n}\Big(|A|^2+\overline{Ric}(v,v)\Big)|\phi|^{2(p+1)\alpha}f^2dv\\ \leq&\int_{M^n}\Big|\nabla\Big(|\phi|^{(1+p)\alpha}f\Big)\Big|^2dv\\ =&(1+p)^2\int_{M^n}|\phi|^{2p\alpha}f^2|\nabla|\phi|^\alpha|^2dv+\int_{M^n}|\phi|^{2(p+1)\alpha}|\nabla f|^2dv\\ &+2(1+p)\int_{M^n}|\phi|^{(2p+1)\alpha}f\langle\nabla f,\nabla|\phi|^\alpha\rangle dv\\ \leq&(1+p)(1+p+\epsilon)\int_{M^n}|\phi|^{2p\alpha}f^2|\nabla|\phi|^\alpha|^2dv\\ &+(1+\frac{1+p}{\epsilon})\int_{M^n}|\phi|^{2(p+1)\alpha}|\nabla f|^2dv.\end{aligned} \tag{3.5}$$

If $2(p+1)-\frac{n-2}{n\alpha}-\epsilon>0$, then by (3.4) and (3.5), we obtain

$$\int_{M^n}|\phi|^{2(p+1)\alpha}f^2\Big(A|\phi|^2-B|\phi|+C\Big)dv\leq D\int_{M^n}|\phi|^{2(p+1)\alpha}|\nabla f|^2dv, \tag{3.6}$$

where

$$\begin{aligned}A&=2(p+1)-\frac{n-2}{n\alpha}-\epsilon-(1+p)(1+p+\epsilon)\alpha,\\ B&=\frac{n(n-2)}{\sqrt{n(n-1)}}H(1+p)(1+p+\epsilon)\alpha,\\ C&=\Big(2(p+1)-\frac{n-2}{n\alpha}-\epsilon+(1+p)(1+p+\epsilon)\alpha\Big)n(H^2+\kappa),\\ D&=\frac{1+p+\epsilon}{\epsilon}\Big(3(1+p)-\frac{n-2}{n\alpha}-\epsilon\Big).\end{aligned}$$

Letting $\epsilon \to 0$, we obtain

$$
\begin{aligned}
4AC - B^2 \to& \frac{4}{\alpha}\Big[2(p+1)\alpha - \frac{n-2}{n} - (1+p)^2\alpha^2\Big]\Big[2(p+1) - \frac{n-2}{n\alpha} \\
&+ (1+p)^2\alpha\Big]n(H^2+\kappa) - (1+p)^4\alpha^2\frac{n^2(n-2)^2}{n(n-1)}H^2 \\
=& \frac{4}{\alpha^2}\Big[2(p+1)\alpha - \frac{n-2}{n} - (1+p)^2\alpha^2\Big]\Big[2(p+1)\alpha - \frac{n-2}{n} \\
&+ (1+p)^2\alpha^2\Big]n(H^2+\kappa) - \frac{n(n-2)^2}{\alpha^2(n-1)}H^2(1+p)^4\alpha^4.
\end{aligned} \tag{3.7}
$$

Let $x = (p+1)\alpha$, then (3.7) becomes

$$
\begin{aligned}
4AC - B^2 \to& \frac{4}{\alpha^2}\Big[2x - \frac{n-2}{n} - x^2\Big]\Big[2x - \frac{n-2}{n} + x^2\Big]n(H^2+\kappa) \\
&- \frac{n}{\alpha^2}\frac{(n-2)^2}{n-1}H^2x^4 \\
=& \frac{n}{\alpha^2}\Big\{4[(2x - \frac{n-2}{n})^2 - x^4](H^2+\kappa) - \frac{(n-2)^2H^2}{n-1}x^4\Big\} \\
=& \frac{n}{\alpha^2}\Big\{2\sqrt{H^2+\kappa}(2x - \frac{n-2}{n}) + \frac{\sqrt{n^2H^2+4(n-1)\kappa}}{\sqrt{n-1}}x^2\Big\} \\
&\cdot \Big\{2\sqrt{H^2+\kappa}(2x - \frac{n-2}{n}) - \frac{\sqrt{n^2H^2+4(n-1)\kappa}}{\sqrt{n-1}}x^2\Big\}.
\end{aligned}
$$

Define $f_n(x) = -x^2 + \frac{4\sqrt{H^2+\kappa}\sqrt{n-1}}{\sqrt{n^2H^2+4(n-1)\kappa}}x - \frac{2(n-2)\sqrt{H^2+\kappa}\sqrt{n-1}}{n\sqrt{n^2H^2+4(n-1)\kappa}}$. Note that

$$
\begin{aligned}
\Delta =& \frac{16(H^2+\kappa)(n-1)}{n^2H^2+4(n-1)\kappa} - \frac{8(n-2)\sqrt{H^2+\kappa}\sqrt{n-1}}{n\sqrt{n^2H^2+4(n-1)\kappa}} \\
=& \frac{8\sqrt{H^2+\kappa}\sqrt{n-1}}{\sqrt{n^2H^2+4(n-1)\kappa}}\Big(\frac{2\sqrt{H^2+\kappa}\sqrt{n-1}}{\sqrt{n^2H^2+4(n-1)\kappa}} - \frac{n-2}{n}\Big) \\
\geq& \frac{8\sqrt{H^2+\kappa}\sqrt{n-1}}{\sqrt{n^2H^2+4(n-1)\kappa}}\Big(\frac{2\sqrt{n-1}}{n} - \frac{n-2}{n}\Big) \\
>& 0,
\end{aligned}
$$

$x_1 + x_2 > 0$, and $x_1x_2 \geq 0$. Thus, when $f_n(x) = 0$, we have

$$
\begin{aligned}
x_2 =& \frac{2\sqrt{H^2+\kappa}\sqrt{n-1}}{\sqrt{n^2H^2+4(n-1)\kappa}} + \sqrt{\frac{4(H^2+\kappa)(n-1)}{n^2H^2+4(n-1)\kappa} - \frac{2(n-2)\sqrt{n-1}\sqrt{H^2+\kappa}}{n\sqrt{n^2H^2+4(n-1)\kappa}}} \\
\geq& \frac{2\sqrt{n-1}}{n} + \sqrt{\frac{4(n-1)}{n^2} - \frac{n-2}{n}} \\
\geq& 1
\end{aligned} \tag{3.8}
$$

and

$$
\begin{aligned}
0 &\le x_1 \\
&= \frac{2\sqrt{H^2+\kappa}\sqrt{n-1}}{\sqrt{n^2H^2+4(n-1)\kappa}} - \sqrt{\frac{4(H^2+\kappa)(n-1)}{n^2H^2+4(n-1)\kappa} - \frac{2(n-2)\sqrt{n-1}\sqrt{H^2+\kappa}}{n\sqrt{n^2H^2+4(n-1)\kappa}}} \\
&< \frac{2\sqrt{H^2+\kappa}\sqrt{n-1}}{\sqrt{n^2H^2+4(n-1)\kappa}} \\
&\le \frac{2\sqrt{H^2+\kappa}\sqrt{n-1}}{\sqrt{4(n-1)H^2+4(n-1)\kappa}} \\
&= 1.
\end{aligned}
$$

The first inequality in (3.8) becomes an equality iff $n = 2$, and the second inequality in (3.8) becomes an equality iff $n = 5$, which is impossible. Thus, $x_2 > 1$ and $0 \le x_1 < 1$.

It follows from a direct calculation that $f_n(x) > 0$ and $2(p+1) - \frac{n-2}{n\alpha} > 0$ iff

$$
max\Big\{x_1, \frac{n-2}{2n}\Big\} < x < x_2.
$$

In order to require $2(p+1) - \frac{n-2}{n\alpha} - (1+p)^2\alpha > 0$, we only need

$$
1 - \sqrt{\frac{2}{n}} < x < 1 + \sqrt{\frac{2}{n}}.
$$

Note that

$$
\begin{aligned}
&1 + \sqrt{\frac{2}{n}} - \frac{2\sqrt{H^2+\kappa}\sqrt{n-1}}{\sqrt{n^2H^2+4(n-1)\kappa}} \\
&- \sqrt{\frac{4(H^2+\kappa)(n-1)}{n^2H^2+4(n-1)\kappa} - \frac{2(n-2)\sqrt{n-1}\sqrt{H^2+\kappa}}{n\sqrt{n^2H^2+4(n-1)\kappa}}} \\
\ge& 1 + \sqrt{\frac{2}{n}} - 1 - \sqrt{\frac{4(H^2+\kappa)(n-1)}{n^2H^2+4(n-1)\kappa} - \frac{2(n-2)\sqrt{n-1}\sqrt{H^2+\kappa}}{n\sqrt{n^2H^2+4(n-1)\kappa}}} \\
=& \sqrt{\frac{2}{n}} - \sqrt{\frac{4(H^2+\kappa)(n-1)}{n^2H^2+4(n-1)\kappa} - \frac{2(n-2)\sqrt{n-1}\sqrt{H^2+\kappa}}{n\sqrt{n^2H^2+4(n-1)\kappa}}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{2}{n} - \frac{4(H^2+\kappa)(n-1)}{n^2H^2+4(n-1)\kappa} + \frac{2(n-2)\sqrt{n-1}\sqrt{H^2+\kappa}}{n\sqrt{n^2H^2+4(n-1)\kappa}} \\
=& \frac{2(n^2H^2+4(n-1)\kappa)}{n(n^2H^2+4(n-1)\kappa)} - \frac{4n(H^2+\kappa)(n-1)}{n(n^2H^2+4(n-1)\kappa)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2(n-2)\sqrt{n-1}\sqrt{H^2+\kappa}\sqrt{n^2H^2+4(n-1)\kappa}}{n(n^2H^2+4(n-1)\kappa)}\\
\geq&\frac{2\Big\{4(n-1)(H^2+\kappa)-2n(n-1)(H^2+\kappa)+2(n-2)(n-1)(H^2+\kappa)\Big\}}{n(n^2H^2+4(n-1)\kappa)}\\
=&0.
\end{aligned}
$$

We have

$$
\begin{aligned}
1+\sqrt{\frac{2}{n}}\geq&\frac{2\sqrt{H^2+\kappa}\sqrt{n-1}}{\sqrt{n^2H^2+4(n-1)\kappa}}\\
&+\sqrt{\frac{4(H^2+\kappa)(n-1)}{n^2H^2+4(n-1)\kappa}-\frac{2(n-2)\sqrt{n-1}\sqrt{H^2+\kappa}}{n\sqrt{n^2H^2+4(n-1)\kappa}}}.
\end{aligned}
$$

Next, we choose $p>0$ and $\alpha>0$ such that

$$
\begin{aligned}
1<&(1+p)\alpha\\
<&\frac{2\sqrt{H^2+\kappa}\sqrt{n-1}}{\sqrt{n^2H^2+4(n-1)\kappa}}\\
&+\sqrt{\frac{4(H^2+\kappa)(n-1)}{n^2H^2+4(n-1)\kappa}-\frac{2(n-2)\sqrt{n-1}\sqrt{H^2+\kappa}}{n\sqrt{n^2H^2+4(n-1)\kappa}}},
\end{aligned}
$$

then $A>0$, $2(p+1)-\frac{n-2}{n\alpha}-\epsilon>0$, and $4AC-B^2>0$, provided that ϵ is sufficiently small. Setting $q=(1+p)\alpha-1$ and combining (3.6) with Young's inequality, we have

$$
\begin{aligned}
&\int_{M^n}|\phi|^{2(q+1)}f^2\Big(A|\phi|^2-B|\phi|+C\Big)dv\\
\leq&\delta\int_{M^n}|\phi|^{4+2q}f^2dv+C_1\int_{M^n}\frac{|\phi|^2|\nabla f|^{2+2q}}{f^{2q}}dv,
\end{aligned}
\tag{3.9}
$$

where $C_1>0$ is a constant and $\delta>0$ can be chosen sufficiently small. By (3.9), we have

$$
\int_{M^n}|\phi|^{2(q+1)}f^2\Big((A-\delta)|\phi|^2-B|\phi|+C\Big)dv\leq C_1\int_{M^n}\frac{|\phi|^2|\nabla f|^{2+2q}}{f^{2q}}dv.
\tag{3.10}
$$

Let $\overline{A}=A-\delta$, where $\delta>0$ is chosen sufficiently small such that $B^2-4\overline{A}C<0$ and $\overline{A}>0$. Then

$$
(A-\delta)|\phi|^2-B|\phi|+C=\overline{A}|\phi|^2-B|\phi|+C\geq\frac{4\overline{A}C-B^2}{4\overline{A}}>0.
\tag{3.11}
$$

Combining (3.10) and (3.11), we have

$$
\int_{M^n}|\phi|^{2(q+1)}f^2dv\leq C_2\int_{M^n}\frac{|\phi|^2|\nabla f|^{2+2q}}{f^{2q}}dv,
\tag{3.12}
$$

where $C_2 = \frac{4\bar{A}C_1}{4\bar{A}C-B^2}$ is a positive constant. Replacing f by f^{1+q} in (3.12), we obtain

$$\int_{M^n} |\phi|^{2(q+1)} f^{2+2q} dv \leq C_3 \int_{M^n} |\phi|^2 |\nabla f|^{2+2q} dv, \tag{3.13}$$

where $C_3 > 0$ is a constant. Choose a nonnegative smooth cut-off function $f : [0, +\infty) \to [0, 1]$ such that

$$f(t) = \begin{cases} 1, & t \in [0, R], \\ 0, & t \in [2R, +\infty) \end{cases} \tag{3.14}$$

and $|f'| \leq \frac{2}{R}$. Consider the composition $f \circ r$, where r denotes the distance function from x_0. By (3.13), we have

$$\int_{B_{x_0}(R)} |\phi|^{2(q+1)} dv \leq \frac{C_4}{R^{2+2q}} \int_{B_{x_0}(2R)} |\phi|^2 dv, \tag{3.15}$$

where $C_4 > 0$ is a constant. Taking the limit $R \to +\infty$, by (1.1) and (3.15), we have

$$\int_{M^n} |\phi|^{2+2q} dv \leq 0.$$

Therefore, we have $|\phi| = 0$, and M^n is totally umbilical. □

Proof of Theorem 1.2. Assume that M^n is noncompact. Since M^n is stable, by Lemma 2.3, there exists a compact domain $D \subset M^n$ such that

$$\int_{M^n \setminus D} |\nabla f|^2 dv \geq \int_{M^n \setminus D} \left(|A|^2 + \overline{Ric}(v, v) \right) f^2 dv$$

for every $f \in C_0^\infty(M^n \setminus D)$. By (3.2), we have

$$\begin{aligned} |\phi|^\alpha \Delta |\phi|^\alpha \geq & (1 - \frac{n-2}{n\alpha}) |\nabla |\phi|^\alpha|^2 - \alpha |\phi|^{2\alpha+2} \\ & - \frac{n(n-2)}{\sqrt{n(n-1)}} H\alpha |\phi|^{2\alpha+1} + n(H^2 + \kappa)\alpha |\phi|^{2\alpha}. \end{aligned} \tag{3.16}$$

By (3.16), (2.2), and Lemma 2.4, we have

$$\int_{M^n \setminus D} |\phi|^{2\alpha} dv < \infty \Longrightarrow \int_{M^n \setminus D} |\phi|^{2\alpha+2} dv < \infty,$$

when $\frac{1}{2}(1 - \frac{n-2}{n\alpha}) - \alpha + 1 > 0$, i.e., $\frac{3-\sqrt{1+16/n}}{4} < \alpha < \frac{3+\sqrt{1+16/n}}{4}$. Since D is compact and ϕ is smooth, $|\phi|$ is bounded on D, hence

$$\int_D |\phi|^{2\alpha+2} dv < \infty. \tag{3.17}$$

By (3.17) and

$$\int_{M^n \setminus D} |\phi|^{2\alpha+2} dv < \infty,$$

we have

$$\int_{M^n} |\phi|^{2\alpha+2} dv < \infty.$$

Therefore,

$$\int_{M^n} |\phi|^p dv < \infty \Longrightarrow \int_{M^n} |\phi|^{p+2} dv < \infty, \tag{3.18}$$

where $\frac{3-\sqrt{1+16/n}}{2} < p = 2\alpha < \frac{3+\sqrt{1+16/n}}{2}$.

(i) When $n = 3$, if $1 < p < \frac{3+\sqrt{19/3}}{2}$, Hölder's inequality yields

$$\int_{M^n} |\phi|^3 dv \leq \Big(\int_{M^n} |\phi|^p dv \Big)^{(p-1)/2} \Big(\int_{M^n} |\phi|^{p+2} dv \Big)^{(3-p)/2} < \infty.$$

If $\frac{3-\sqrt{19/3}}{2} < p \leq 1$, then

$$\int_{M^n} |\phi| dv \leq \Big(\int_{M^n} |\phi|^p dv \Big)^{(1+p)/2} \Big(\int_{M^n} |\phi|^{p+2} dv \Big)^{(1-p)/2} < \infty. \tag{3.19}$$

Combining (3.18) and (3.19), we have $\int_{M^n} |\phi|^3 dv < \infty$.

(ii) When $n = 4$, if $\frac{3-\sqrt{5}}{2} < p \leq 2$, note that

$$\int_{M^n} |\phi|^2 dv \leq \Big(\int_{M^n} |\phi|^p dv \Big)^{p/2} \Big(\int_{M^n} |\phi|^{p+2} dv \Big)^{(2-p)/2} < \infty. \tag{3.20}$$

Combining (3.18) and (3.20), we obtain $\int_{M^n} |\phi|^4 dv < \infty$.

If $2 < p < \frac{3+\sqrt{5}}{2}$, we have

$$\int_{M^n} |\phi|^4 dv \leq \Big(\int_{M^n} |\phi|^p dv \Big)^{(p-2)/2} \Big(\int_{M^n} |\phi|^{p+2} dv \Big)^{(4-p)/2} < \infty.$$

(iii) When $n = 5$, following the same argument as in cases (i) and (ii), we obtain $\int_{M^n} |\phi|^3 dv < \infty$ and $\int_{M^n} |\phi|^4 dv < \infty$. By (3.1) and Lemma 2.5, we have

$$\int_{M^n} |\phi|^5 dv < \infty.$$

Consequently, for $3 \leq n \leq 5$, we obtain

$$\int_{M^n} |\phi|^n dv < \infty.$$

Moreover, since H is constant, the mean curvature vector is parallel. Since $-H^2 < \kappa \leq 0$, we have $|h|^2 = H^2 > -\kappa$. Hence, the assumptions of Lemma 2.1 are satisfied, and M^n is compact. This contradicts the assumption that M^n is noncompact. Therefore, M^n must be compact. By Lemma 2.2, M^n is a geodesic sphere. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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