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ERA, 34(10): 7175–7200.
DOI: 10.3934/era.2026310
Received: 07 May 2026
Revised: 07 July 2026
Accepted: 12 August 2026
Published: 18 August 2026

Research article

Continuous solutions for a class boundary value problem with mixed ψ-Riemann-Liouville tempered fractional derivatives

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Abstract: We study a class of nonlinear boundary value problems driven by mixed ψ-Riemann-Liouville tempered fractional derivatives of order $\alpha \in \left(\frac{1}{2}, 1\right)$ under homogeneous Dirichlet boundary conditions. The analysis is performed in a suitable fractional Sobolev-type space associated with ψ-tempered operators. We first establish sharp mapping properties for the ψ-Riemann-Liouville tempered fractional integral, proving its boundedness from $L^p((a,b),\psi')$ into Hölder spaces $H^{\alpha-\frac{1}{2}}[a,b]$ with explicit constants depending on the incomplete gamma function. These estimates provide the functional framework required to handle the corresponding nonlocal problem. Using variational methods, we associate the problem with a C^1 energy functional on $\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$ and prove compactness via the Palais–Smale condition under optimal growth assumptions. We then establish a precise variational structure yielding: (i) existence of nontrivial solutions via minimization arguments, (ii) multiplicity results based on a mountain pass geometry, and (iii) existence of infinitely many solutions by means of symmetric critical point theory. Our results extend the current theory of fractional equations with respect to another function by incorporating tempered effects and explicit operator estimates, thus providing a unified and flexible framework for a broad class of nonlocal problems.

Keywords: Riemann-Liouville tempered fractional operators; ψ-tempered fractional operators; variational methods; critical point theory; tempered fractional space of Sobolev type

1. Introduction and main results

Nowadays, fractional calculus has become a well-established area of mathematical analysis, with numerous fractional integral and differential operators being developed for different theoretical and applied purposes. The classical theory of fractional differential equations and its analytical foundations were extensively investigated by Diethelm and Ford [1], while Gorenflo and Mainardi [2] provided a comprehensive overview of fractional calculus and its applications. A systematic treatment of fractional differential equations can also be found in the monograph by Kilbas et al. [3]. Several generalized fractional derivatives have subsequently been introduced. Oliveira and de Oliveira [4] proposed the Hilfer-Katugampola fractional derivative, whereas Sousa et al. [5] developed a variable-order ψ-Hilfer fractional calculus. The fundamental properties of the ψ-Hilfer fractional derivative were studied by Sousa and de Oliveira [6], and further generalizations were considered by Sousa et al. [7] through the ψ-Hilfer pseudo-fractional operator. Zhou et al. [8] provided a systematic account of the basic theory of fractional differential equations. In addition, Sousa et al. [9] established a variational framework for problems involving ψ-Hilfer operators. More recently, Fahad et al. [10] introduced tempered and Hadamard-type fractional operators with respect to functions, extending the scope of generalized fractional calculus.

Motivated by this diversity, it became necessary to introduce fractional operators defined with respect to another function. The first contributions in this direction were made by Erdélyi [11, 12] and later by Osler [13–15]. Subsequently, Kilbas et al. [3] introduced the left ψ-Riemann-Liouville fractional integral defined by

$$\mathbf{I}_{a^+}^{\alpha;\psi} u(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(s)\big(\psi(x) - \psi(s)\big)^{\alpha-1} u(s)\, ds,$$

where $\alpha > 0$ and $\psi \in C^1[a, b]$ is a strictly increasing function. The associated ψ-Riemann-Liouville fractional derivative is given by

$$\mathbf{D}_{a^+}^{\alpha;\psi} u(x) = \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^n \mathbf{I}_{a^+}^{n-\alpha;\psi} u(x),$$

with $\alpha \in [n-1, n)$ and $n \in \mathbb{N}$.

In 2017, Almeida [16] introduced a regularized version of the ψ-Riemann-Liouville fractional derivative, now referred to as the ψ-Caputo fractional derivative. Subsequently, Sousa and Oliveira [6] proposed the ψ-Hilfer fractional derivative, which provides a continuous interpolation between the ψ-Riemann-Liouville and ψ-Caputo operators. More recently, Rodríguez et al. [17] investigated the analytical properties of a novel fractional operator, namely the ψ-Hilfer-Caputo fractional derivative, this operator can be viewed as a regularized counterpart of the ψ-Hilfer fractional derivative and establishes a natural interpolation framework between the ψ-Riemann-Liouville and ψ-Caputo derivatives, while preserving key structural features such as those related to integration by parts.

On the other hand, tempered fractional calculus has emerged as a natural extension of classical fractional calculus, providing a suitable framework for incorporating both nonlocal effects and exponential tempering into fractional models. Tempered fractional differential equations have found applications in a variety of physical settings. In particular, they have been employed to describe anomalous diffusion and transport phenomena in heterogeneous media and geophysical systems [21, 24], as well as diffusion and relaxation processes associated with tempered α-stable dynamics [19, 22]. Related tempered fractional Fokker-Planck models have also been developed to characterize transport processes involving truncated or tempered jump distributions [18, 20]. Furthermore, tempered fractional operators arise naturally in spectral and eigenvalue problems, including tempered fractional Sturm-Liouville problems [23]. These developments demonstrate the broad applicability of tempered fractional calculus in areas such as geophysics, statistical physics, and other branches of mathematical physics. In the context of quantitative finance, tempered fractional derivatives have also been employed to model asset-price dynamics exhibiting semi-heavy-tailed behavior [25].

A natural generalization of the Riemann-Liouville tempered fractional integral with respect to another function can be constructed as follows. Let $\sigma > 0$ and let $\psi \in C^1[a, b]$ be strictly increasing. For $n \in \mathbb{N}$, consider the differential operator

$$D^n_{\psi,\sigma} = \left(\frac{1}{\psi'(x)}\frac{d}{dx} + \sigma\right)^n .$$

Then, for a continuous function u on $[a, b]$, the Cauchy problem

$$\begin{cases} D^n_{\psi,\sigma} z(x) = u(x), & x \in (a, b), \\ \dfrac{d^k}{dx^k} z(a) = 0, & k = 0, 1, \ldots, n-1, \end{cases}$$

admits a unique solution given by

$$z(x) = \frac{1}{(n-1)!}\int_a^x \psi'(s)(\psi(x) - \psi(s))^{n-1} e^{-\sigma(\psi(x)-\psi(s))} u(s)\, ds.$$

Using $(n-1)! = \Gamma(n)$, this representation extends naturally to any $\alpha > 0$, leading to the ψ-Riemann-Liouville tempered fractional integral

$$\mathbb{I}^{\alpha,\sigma}_{\psi,a^+} u(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \psi'(s)(\psi(x) - \psi(s))^{\alpha-1} e^{-\sigma(\psi(x)-\psi(s))} u(s)\, ds.$$

This formulation was used in [10] to establish connections between Hadamard-type fractional calculus and tempered fractional calculus with respect to ψ.

Recently, boundary value problems driven by mixed fractional operators have attracted considerable attention due to their rich analytical structure and broad range of applications. In this direction, Sousa et al. [9] established one of the first variational frameworks for equations involving the ψ-Hilfer fractional derivative. They introduced the fractional Sobolev-type spaces $H^{\alpha,\beta;\psi}_p([0, T], \mathbb{R})$, proved their completeness, reflexivity, and embedding properties, and characterized weak solutions as critical points of the associated energy functional. The existence of nontrivial solutions was obtained by means of the mountain pass theorem. In the same year, Khaliq and Rehman [26] developed an analogous variational framework for the ψ-Caputo fractional derivative. They introduced the Banach space ${}_0E^{\alpha,\psi}_p$, proved its separability and reflexivity, and established existence results by combining critical point arguments with direct minimization techniques. Subsequent works have extended the theory to several classes of ψ-Hilfer fractional problems. Boudjerida et al. [27] studied controllability for coupled systems of impulsive ψ-Hilfer fractional integro-differential inclusions. Ezati and Nyamoradi [29] established existence results for Kirchhoff-type ψ-Hilfer fractional p-Laplacian equations, while in [28] they obtained existence and multiplicity results for a related ψ-Hilfer fractional p-Laplacian problem. Variable-order generalizations were developed by Sousa et al. [30], who introduced a variable-order ψ-Hilfer fractional calculus and considered its applications. Furthermore, Sousa et al. [31] applied the Nehari manifold method to ψ-Hilfer fractional p-Laplacian problems, whereas Sousa et al. [32] investigated the attractivity of solutions for fractional differential equations of ψ-Hilfer type.

Several generalizations of these operators have subsequently been proposed. These include the k-Riemann-Liouville fractional integral [33], the corresponding k-Riemann-Liouville fractional derivative [34], the k-Hilfer fractional derivative [35], and the (k,ψ)-Hilfer fractional derivative introduced by Kucche and Mali [36]. A variational treatment of the latter operator was later developed by Torres Ledesma and Nyamoradi [37].

By contrast, the variational theory for problems involving mixed Riemann-Liouville and ψ-Riemann-Liouville tempered fractional derivatives remains comparatively unexplored. For the case $\nu \in (1/2,1)$, Torres Ledesma et al. [38] introduced the tempered Sobolev-type space $H_0^{\nu,\Xi}(\varpi_1,\varpi_2)$, established its Hilbert space structure, proved the Hölder continuity of tempered fractional integrals and a compact embedding into $C(\varpi_1,\varpi_2)$, and obtained nontrivial weak solutions by variational methods. Extending this framework, Nyamoradi and Torres Ledesma [39] incorporated both the ψ-fractional and tempered settings into impulsive problems. They introduced the space $H_{\psi,0}^{\alpha,\sigma}(a,b)$, established an integration-by-parts formula together with Poincaré-type inequalities, and proved the existence of infinitely-many weak solutions via genus theory and the mountain pass theorem.

The complementary regime $\alpha \in (0,1/2)$ was subsequently investigated in [40]. Owing to the loss of compactness in $C(a,b)$, the authors established a compact embedding into $L^p(a,b)$, justified the nonlocal Dirichlet condition on $\mathbb{R}\setminus(a,b)$, and derived existence and multiplicity results using the minimum principle and Morse theory. Nevertheless, the critical case $\alpha = \frac{1}{2}$ remains open. We also refer to the recent works [39, 41] for further developments concerning impulsive equations and related tempered fractional operators.

Motivated by these developments, we investigate the boundary value problem

$$\begin{cases} \mathbb{D}_{\psi,b^-}^{\alpha,\sigma}\left({}^C\mathbb{D}_{\psi,a^+}^{\alpha,\sigma}u(x)\right) = f(x,u(x)), & \text{a.e. } x \in (a,b), \\ u(a) = u(b) = 0, \end{cases} \tag{1.1}$$

where $\alpha \in \left(\frac{1}{2},1\right)$, $\sigma > 0$, and $f \in C((a,b)\times\mathbb{R})$.

We define the space $\mathbb{H}_{\psi,0}^{\alpha,\sigma}(a,b)$ as the closure of $C_0^\infty(a,b)$ with respect to the norm

$$\|u\|_\psi = \left(\int_a^b |u(x)|^2\psi'(x)\,dx + \int_a^b \left|{}^C\mathbb{D}_{\psi,a^+}^{\alpha,\sigma}u(x)\right|^2\psi'(x)\,dx\right)^{1/2}.$$

With this space at hand, we say that a function $u \in \mathbb{H}_{\psi,0}^{\alpha,\sigma}(a,b)$ is a *weak solution* of problem (1.1) if the variational identity

$$\int_a^b {}^C\mathbb{D}_{\psi,a^+}^{\alpha,\sigma}u(x)\,{}^C\mathbb{D}_{\psi,a^+}^{\alpha,\sigma}\varphi(x)\,\psi'(x)\,dx = \int_a^b f(x,u(x))\varphi(x)\,\psi'(x)\,dx$$

holds for every $\varphi \in \mathbb{H}_{\psi,0}^{\alpha,\sigma}(a,b)$. Further details regarding the construction and properties of this space are provided in Section 3.

Throughout the paper, we assume that $\alpha \in (\frac{1}{2},1)$, $\sigma > 0$, $\psi \in C^1[a,b]$ is strictly increasing, and $f : (a,b)\times\mathbb{R} \to \mathbb{R}$ is continuous.

We are now in a position to state our main results.

Theorem 1.1. *Suppose that:*

(G1) *For all $\xi \in \mathbb{R}$, $\mu F(x,\xi) \le \xi f(x,\xi)$, where $\mu > 2$.*

(G1*) *There exist* $\varrho, \vartheta > 0$ *such that*

$$\limsup_{u\to 0} \frac{F(x,u)}{|u|^{\mu}} \leq \varrho \quad \textit{and} \quad \liminf_{u\to\infty} \frac{F(x,u)}{|u|^{\mu}} \geq \vartheta.$$

Then, problem (1.1) *admits at least two weak solutions.*

Theorem 1.2. *Assume that:*

(G2) *There exists* $\iota > 2$ *such that*

$$\liminf_{|u|\to\infty} \frac{\min_{x\in[a,b]} F(x,u) - |u|^{\iota}}{|u|^2} > -\infty.$$

(G3)

$$\limsup_{u\to 0} \frac{\max_{x\in[a,b]} F(x,u)}{|u|^2} < \frac{\sigma^{2\alpha}\Gamma^2(\alpha)}{4(\gamma(\alpha, \sigma(\psi(b) - \psi(a))))^2}.$$

(G4) *There exist constants* $\nu > 2$*,* $\gamma_1 > 0$*,* $\theta_i \geq 0$*,* $0 < \zeta_i < 2$*, and* $0 \leq \theta_F < \frac{(\nu-2)\sigma^{2\alpha}\Gamma^2(\alpha)}{4(\gamma(\alpha,\sigma(\psi(b)-\psi(a))))^2}$ *such that, for* $|u| \geq \gamma_1$*,*

$$\nu F(x,u) - uf(x,u) \leq \sum_{i=1}^{l} \theta_i |u|^{\zeta_i} + \theta_F |u|^2.$$

If $f(x,u)$ *is odd in u, then problem* (1.1) *admits infinitely many weak solutions.*

Theorem 1.3. *If*

(G5)

$$\limsup_{|u|\to\infty} \frac{\max_{x\in[a,b]} F(x,u)}{|u|^2} < \frac{\sigma^{2\alpha}\Gamma^2(\alpha)}{4(\gamma(\alpha, \sigma(\psi(b) - \psi(a))))^2},$$

then problem (1.1) *admits at least one weak solution.*

Finally, observe that if $\sigma = 0$, problem (1.1) reduces to

$$\begin{cases} \mathbb{D}^{\alpha,\psi}_{b^-}({}^{C}\mathbb{D}^{\alpha,\psi}_{a^+} u(x)) = f(x, u(x)), & \text{a.e. } x \in (a,b), \\ u(a) = u(b) = 0. \end{cases} \tag{1.2}$$

This problem was recently studied by Khaliq and Rehman [26], where existence results were obtained via critical point theory. Our results extend and improve those in [26].

Remark 1.4. *It is worth emphasizing that while Theorems 1.1, 1.2, and 1.3 provide the existence of weak solutions in the space* $\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$*, Theorem 3.4 further ensures that such solutions are continuous.*

Finally, before concluding this section, the following table summarizes the notation for the ψ-tempered fractional operators and the associated function spaces used throughout the paper.

Notation	Meaning
$\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}$, $\mathbb{I}^{\alpha,\sigma}_{\psi,b^-}$	Left/right ψ-Riemann-Liouville tempered fractional integral
$\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}$, $\mathbb{D}^{\alpha,\sigma}_{\psi,b^-}$	Left/right ψ-Riemann-Liouville tempered fractional derivative
${}^{C}\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}$, ${}^{C}\mathbb{D}^{\alpha,\sigma}_{\psi,b^-}$	Left/right ψ-Caputo tempered fractional derivative
$L^p((a,b),\psi')$	Weighted Lebesgue space with measure $\psi'(x)dx$
$\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$	Tempered fractional Sobolev-type space
$\|\cdot\|_{\psi}$, $\|\cdot\|$	Norms in $\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$

2. Previous results

For $\alpha > 0$ and $x \geq 0$, the lower incomplete Gamma function is defined by

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1}e^{-t}\,dt.$$

This function satisfies the following sharp bounds:

$$e^{-x}\frac{x^\alpha}{\alpha} \leq \gamma(\alpha, x) \leq \frac{x^\alpha}{\alpha}. \tag{2.1}$$

Moreover, an application of integration by parts yields the recurrence relation

$$\gamma(\alpha+1, x) = \alpha\gamma(\alpha, x) - x^\alpha e^{-x}. \tag{2.2}$$

This identity allows one to extend the definition of $\gamma(\alpha, x)$ to negative non-integer values of α. For further details, see [42].

2.1. ψ-Tempered fractional integrals

Let $\alpha > 0$ and $\sigma > 0$. The left- and right-sided ψ-Riemann-Liouville tempered fractional integrals are defined, respectively, by

$$\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}u(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \psi'(s)\big(\psi(x)-\psi(s)\big)^{\alpha-1}e^{-\sigma(\psi(x)-\psi(s))}u(s)\,ds,$$

and

$$\mathbb{I}^{\alpha,\sigma}_{\psi,b^-}u(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \psi'(s)\big(\psi(s)-\psi(x)\big)^{\alpha-1}e^{-\sigma(\psi(s)-\psi(x))}u(s)\,ds.$$

To investigate the mapping properties of the operator $\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}$, we introduce the weighted Lebesgue space $L^p((a,b),\psi')$, with $1 \leq p < \infty$, defined as the Banach space of measurable functions $u : (a,b) \to \mathbb{R}$ endowed with the norm

$$\|u\|_{p,\psi'} = \left(\int_a^b |u(x)|^p\psi'(x)\,dx\right)^{1/p}.$$

In addition, let $\gamma \in (0,1]$. A function $u : [a,b] \to \mathbb{R}$ is said to be Hölder continuous with exponent γ if there exists a constant $\mathcal{H} > 0$ such that

$$|u(x) - u(y)| \leq \mathcal{H}|x-y|^\gamma, \quad \forall x, y \in [a,b].$$

We denote by

$$H^{\gamma}[a,b] := \{u \in C[a,b] : u \text{ is Hölder continuous with exponent } \gamma\}.$$

Equipped with the norm

$$\|u\|_{H^{\gamma}[a,b]} := \|u\|_{\infty} + [u]_{\gamma}, \quad \text{where} \quad [u]_{\gamma} := \sup_{\substack{x,y\in[a,b]\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\gamma}},$$

the space $(H^{\gamma}[a,b], \|\cdot\|_{H^{\gamma}[a,b]})$ forms a Banach space.

Following the approach developed in [43], we establish the boundedness results stated below. For the reader's convenience, detailed proofs are provided in the Appendix.

Theorem 2.1. *Let $\alpha \in (0,1)$, $\sigma > 0$, and $p \in [1,\infty)$. Then, the operator $\mathbb{I}^{\alpha,\sigma}_{\psi,a^+} : L^p((a,b),\psi') \to L^p((a,b),\psi')$ is bounded. Moreover,*

$$\|\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}u\|_{p,\psi'} \leq \frac{1}{\sigma^{\alpha}\Gamma(\alpha)}\gamma(\alpha,\sigma(\psi(b)-\psi(a)))\,\|u\|_{p,\psi'}.$$

As a direct consequence, we obtain the semigroup (index) property.

Theorem 2.2. *Let $\alpha,\beta \in (0,1)$, $\sigma > 0$, and $u \in L^1((a,b),\psi')$. Then,*

$$\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}\mathbb{I}^{\beta,\sigma}_{\psi,a^+}u(x) = \mathbb{I}^{\alpha+\beta,\sigma}_{\psi,a^+}u(x). \tag{2.3}$$

Theorem 2.3. *Let $\sigma > 0$ and $\alpha \in (\frac{1}{2},\frac{3}{2})$. Let $\psi \in C^1[a,b]$ be strictly increasing. Then, the operator*

$$\mathbb{I}^{\alpha,\sigma}_{\psi,a^+} : L^2((a,b),\psi') \to H^{\alpha-\frac{1}{2}}[a,b]$$

is bounded. Moreover,

$$\lim_{x\to a^+}\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}u(x) = 0.$$

Theorem 2.4. *[39] If $u,v \in L^2((a,b),\psi')$, then the following integration by parts formula holds:*

$$\int_a^b \mathbb{I}^{\alpha,\sigma}_{\psi,a^+}u(x)\,v(x)\,\psi'(x)\,dx = \int_a^b u(x)\,\mathbb{I}^{\alpha,\sigma}_{\psi,b^-}v(x)\,\psi'(x)\,dx. \tag{2.4}$$

2.2. *ψ-Tempered fractional derivatives*

Let $\alpha \in (0,1)$ and $\sigma > 0$. For a suitable function u, the left- and right-sided ψ-Riemann-Liouville tempered fractional derivatives are defined by

$$\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}u(x) = \left(\frac{1}{\psi'(x)}\frac{d}{dx} + \sigma\right)\mathbb{I}^{1-\alpha,\sigma}_{\psi,a^+}u(x), \tag{2.5}$$

and

$$\mathbb{D}^{\alpha,\sigma}_{\psi,b^-}u(x) = -\left(\frac{1}{\psi'(x)}\frac{d}{dx} - \sigma\right)\mathbb{I}^{1-\alpha,\sigma}_{\psi,b^-}u(x), \tag{2.6}$$

respectively. The corresponding ψ-Caputo tempered fractional derivatives are given by

$$ {}^{C}\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}u(x) = \mathbb{I}^{1-\alpha,\sigma}_{\psi,a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx} + \sigma\right)u(x), \tag{2.7} $$

and

$$ {}^{C}\mathbb{D}^{\alpha,\sigma}_{\psi,b^-}u(x) = -\mathbb{I}^{1-\alpha,\sigma}_{\psi,b^-}\left(\frac{1}{\psi'(x)}\frac{d}{dx} - \sigma\right)u(x). \tag{2.8} $$

The following result summarizes several fundamental structural properties of the ψ-tempered fractional operators. Some of these properties were previously established in [44] for the special case $\psi(x) = x$, while here they are presented in the more general setting.

Theorem 2.5. *[39] Let $\alpha \in (0, 1)$ and $\sigma > 0$.*

(i) If $u \in C^1[a, b]$, then the ψ-Riemann–Liouville and ψ-Caputo tempered fractional derivatives are related by

$$ \mathbb{D}^{\alpha,\sigma}_{\psi,a^+}u(x) = \frac{u(a)}{\Gamma(1-\alpha)}(\psi(x) - \psi(a))^{-\alpha}e^{-\sigma(\psi(x)-\psi(a))} + {}^{C}\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}u(x). \tag{2.9} $$

(ii) If $u \in C[a, b]$, then the left inverse property holds:

$$ \mathbb{D}^{\alpha,\sigma}_{\psi,a^+}\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}u(x) = u(x). \tag{2.10} $$

Moreover, if $u \in C^1[a, b]$, then

$$ {}^{C}\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}u(x) = u(x). \tag{2.11} $$

(iii) If $u \in C^1[a, b]$, then the right inverse property is given by

$$ \mathbb{I}^{\alpha,\sigma}_{\psi,a^+}\,{}^{C}\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}u(x) = u(x) - e^{-\sigma(\psi(x)-\psi(a))}u(a). \tag{2.12} $$

Finally, we recall the integration by parts formula for tempered fractional derivatives.

Theorem 2.6. *[39] If $u, v \in C^1[a, b]$, then*

$$ \begin{aligned} \int_a^b u(x)\,\mathbb{D}^{\alpha,\sigma}_{\psi,b^-}v(x)\,\psi'(x)\,dx &= \lim_{x\to a^+} u(x)\,\mathbb{I}^{1-\alpha,\sigma}_{\psi,b^-}v(x) - \lim_{x\to b^-} u(x)\,\mathbb{I}^{1-\alpha,\sigma}_{\psi,b^-}v(x) \\ &\quad + \int_a^b {}^{C}\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}u(x)\,v(x)\,\psi'(x)\,dx. \end{aligned} $$

3. Tempered fractional space of Sobolev type

In this section, we study some properties of the tempered fractional space of Sobolev type $\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a, b)$. Some of these results were considered in Nemat and Torres Ledesma [39], but for better understanding and for the convenience of the reader we consider the proof of these properties.

We begin by introducing the notions of classical and weak solutions for problem (1.1). The latter motivates the definition of the spaces $\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a, b)$.

Definition 3.1. *A function $u : [a,b] \to \mathbb{R}$ is a* classical solution *of* (1.1) *if $u \in C^1(a,b) \cap C[a,b]$, $\mathbb{I}^{1-\alpha,\sigma}_{\psi,a^+}\left(\frac{1}{\psi'}\frac{d}{dx} + \sigma\right)u \in C(a,b)$, $\mathbb{D}^{\alpha,\sigma}_{\psi,b^-}({}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}u) \in C(a,b)$, and the following pointwise identity holds:*

$$\mathbb{D}^{\alpha,\sigma}_{\psi,b^-}\big({}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}u(x)\big) = f(x,u(x)) \qquad \forall x \in (a,b),$$

together with the Dirichlet boundary conditions $u(a) = u(b) = 0$.

Now, let $u \in C^1(a,b) \cap C([a,b])$ be a classical solution of (1.1), and let $\varphi \in C_0^\infty(a,b)$ be an arbitrary test function. Multiplying the identity in (1.1) by φ and integrating with respect to the measure $\psi'(x)\,dx$ over (a,b), we obtain

$$\int_a^b \mathbb{D}^{\alpha,\sigma}_{\psi,b^-}\big({}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}u(x)\big)\varphi(x)\psi'(x)\,dx = \int_a^b f(x,u(x))\varphi(x)\psi'(x)\,dx.$$

Applying the integration by parts formula established in Theorem 2.6 and taking into account the boundary conditions $u(a) = u(b) = 0$, it follows that

$$\int_a^b {}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}u(x)\,{}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}\varphi(x)\,\psi'(x)\,dx = \int_a^b f(x,u(x))\varphi(x)\,\psi'(x)\,dx.$$

Since $C_0^\infty(a,b)$ is dense in $\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$ by definition, the above identity extends by continuity to all $\varphi \in \mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$. This motivates the following definition.

Definition 3.2. *A function $u \in \mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$ is said to be a* weak solution *of problem* (1.1) *if*

$$\int_a^b {}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}u(x)\,{}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}\varphi(x)\,\psi'(x)\,dx = \int_a^b f(x,u(x))\varphi(x)\,\psi'(x)\,dx$$

for every $\varphi \in \mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$.

We begin by proving that $\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$ is a Hilbert space.

Proposition 3.1. *The space $\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$ is a Hilbert space.*

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset \mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$ be a Cauchy sequence. Then, (u_n) and $({}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}u_n)$ are Cauchy in $L^2((a,b),\psi')$. Hence, there exist $u, v \in L^2((a,b),\psi')$ such that

$$u_n \to u, \qquad {}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}u_n \to v \quad \text{in } L^2((a,b),\psi').$$

For any $\varphi \in C_0^\infty(a,b)$, using the integration by parts formula, we have

$$\int_a^b {}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}u_n(x)\,\varphi(x)\psi'(x)\,dx = \int_a^b u_n(x)\,\mathbb{D}^{\alpha,\sigma}_{\psi,b^-}\varphi(x)\psi'(x)\,dx.$$

Passing to the limit yields

$$\int_a^b v(x)\,\varphi(x)\psi'(x)\,dx = \int_a^b u(x)\,\mathbb{D}^{\alpha,\sigma}_{\psi,b^-}\varphi(x)\psi'(x)\,dx.$$

Applying integration by parts once more, we obtain

$$\int_a^b v(x)\,\varphi(x)\psi'(x)\,dx = \int_a^b {}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}u(x)\,\varphi(x)\psi'(x)\,dx,$$

for every $\varphi \in C_0^\infty(a,b)$. By density, $v = {}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}u$ a.e. in (a,b). Thus, $u \in \mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$ and $\|u_n - u\|_\psi \to 0$. Therefore, $\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$ is complete. Since the norm is induced by the inner product defined above, $\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$ is a Hilbert space. □

The following result establishes the fundamental theorem of calculus for functions in $\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$. This result is fundamental to the subsequent development of the theory and provides an essential tool for investigating the properties of the space $\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$.

Lemma 3.2.

$$\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}\,{}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}u(x) = u(x), \quad \textit{a.e. in } (a,b) \textit{ and for all } u \in \mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b).$$

Proof. Let $u \in \mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$. Then, there is $(\varphi_n)_{n\in\mathbb{N}} \subset C_0^\infty(a,b)$ such that

$$\lim_{n\to\infty} \|u - \varphi_n\|_\psi = 0.$$

Consequently,

$$\lim_{n\to\infty} \|u - \varphi_n\|_{2,\psi'} = 0 \quad \text{and} \quad \lim_{n\to\infty} \|{}^C\mathbb{D}^{\alpha,\sigma}_{a^+}(u - \varphi_n)\|_{2,\psi'} = 0. \tag{3.1}$$

Fatou's lemma yields that

$$\begin{aligned} &\int_a^b |u(x)|^2\psi'(x)dx \le \liminf_{n\to\infty} \int_a^b |\varphi_n(x)|^2\psi'(x)dx < \infty \quad \text{and} \\ &\int_a^b |{}^C\mathbb{D}^{\alpha,\sigma}_{a^+}u(x)|^2\psi'(x)dx \le \liminf_{n\to\infty} \int_a^b |{}^C\mathbb{D}^{\alpha,\sigma}_{a^+}\varphi_n(x)|^2\psi'(x)dx < +\infty. \end{aligned} \tag{3.2}$$

As $\varphi_n(a) = 0$ for any $n \in \mathbb{N}$, Theorem 2.5-(i) implies that

$$\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}\,{}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}\varphi_n(x) = \varphi_n(x) \quad \forall n \in \mathbb{N},$$

hence

$$\|\mathbb{I}^{\alpha,\sigma}_{a^+}{}^C\mathbb{D}^{\alpha,\sigma}_{a^+}\varphi_n - \varphi_n\|_{2,\psi'} = 0 \quad \forall n \in \mathbb{N}. \tag{3.3}$$

Moreover, by Theorem 2.1 we obtain

$$\|\mathbb{I}^{\alpha,\sigma}_{\psi,,a^+}\,{}^C\mathbb{D}^{\alpha,\sigma}_{\psi,,a^+}(u - \varphi_n)\|_{2,\psi'} \le \frac{\gamma(\alpha, \sigma(\psi(b) - \psi(a)))}{\sigma^\alpha\Gamma(\alpha)}\|{}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}(u - \varphi_n)\|_{2,\psi'}. \tag{3.4}$$

Therefore, the previous estimates combined with

$$\begin{aligned} &\|\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}\,{}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}u - u\|_{2,\psi'} \\ &\le \|\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}\,{}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}(u - \varphi_n)\|_{2,\psi'} + \|\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}\,{}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}\varphi_n - \varphi_n\|_{2,\psi'} + \|\varphi_n - u\|_{2,\psi'}. \end{aligned}$$

yield that

$$\|\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}\,{}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}u - u\|_{2,\psi'} = 0,$$

which implies the desired result. □

Remark 3.3. *By Lemma 3.2, the following tempered Poincaré type inequality*

$$\|u\|_{2,\psi'} \leq \frac{\gamma(\alpha, \sigma(\psi(b) - \psi(a)))}{\sigma^{\alpha}\Gamma(\alpha)} \|{}^{C}\mathbb{D}^{\alpha,\sigma}_{a^+} u\|_{2,\psi'}, \tag{3.5}$$

holds for any $u \in \mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$. Hence, we can endowed $\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$ with the norm

$$\|u\| = \left(\int_a^b |{}^{C}\mathbb{D}^{\alpha,\sigma}_{\psi,a^+} u(x)|^2 \psi'(x)dx\right)^{1/2},$$

which is equivalent to $\|\cdot\|_{\psi}$.

Next we consider some embedding results of $\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$ into $C\overline{(a,b)}$.

Theorem 3.4. *If $\alpha \in \left(\frac{1}{2}, 1\right)$ and $\sigma > 0$, then the embedding $\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b) \hookrightarrow C\overline{(a,b)}$ is continuous.*

Proof. If $u \in \mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$, by Lemma 3.2 we obtain that $u, {}^{C}\mathbb{D}^{\alpha,\sigma}_{\psi,a^+} u \in L^2((a,b),\psi')$ and

$$u(x) = \mathbb{I}^{\alpha,\sigma}_{\psi,a^+}\, {}^{C}\mathbb{D}^{\alpha,\sigma}_{\psi,a^+} u(x) \ \text{ a.e. } x \in (a,b).$$

So, Theorem 2.3 implies that

$$\begin{aligned}
\|u\|_{\infty} &= \|\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}\, {}^{C}\mathbb{D}^{\alpha,\sigma}_{\psi,a^+} u\|_{\infty} \\
&\leq \frac{1}{(2\sigma)^{\alpha-\frac{1}{2}}\Gamma(\alpha)} [\gamma(2\alpha-1, 2\sigma(\psi(b)-\psi(a)))]^{1/2} \|{}^{C}\mathbb{D}^{\alpha,\sigma}_{\psi,a^+} u\|_{2,\psi'} \\
&= \frac{1}{(2\sigma)^{\alpha-\frac{1}{2}}\Gamma(\alpha)} [\gamma(2\alpha-1, 2\sigma(\psi(b)-\psi(a)))]^{1/2} \|u\|,
\end{aligned}$$

and the proof is completed. □

The following compactness result will be crucial for our purpose.

Theorem 3.5. *Let $\alpha \in \left(\frac{1}{2}, 1\right)$ and $\sigma > 0$. Then, the embedding*

$$\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b) \hookrightarrow C\overline{(a,b)}$$

is compact.

Proof. Let B be a bounded subset of $\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$. We prove that B is uniformly bounded and equicontinuous in $C(\overline{(a,b)})$. Since B is bounded, there exists a constant $M > 0$ such that

$$\|u\| \leq M, \quad \forall\, u \in B. \tag{3.6}$$

Moreover, by the proof of Theorem 2.3 (see the Appendix),

$$\|u\|_{\infty} \leq \frac{[\gamma(2\alpha-1, 2\sigma(\psi(b)-\psi(a)))]^{1/2}}{(2\sigma)^{\alpha-\frac{1}{2}}\Gamma(\alpha)} \|u\|, \tag{3.7}$$

for every $u \in \mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$. Combining (3.6) and (3.7), we obtain

$$\|u\|_\infty \leq \frac{\left[\gamma(2\alpha-1, 2\sigma(\psi(b)-\psi(a)))\right]^{1/2}}{(2\sigma)^{\alpha-\frac{1}{2}}\Gamma(\alpha)} M, \quad \forall\, u \in B,$$

which proves that B is uniformly bounded in $C(\overline{(a,b)})$. Next, let $x, y \in [a,b]$ with $x > y$. By Lemma 3.2,

$$|u(x)-u(y)| = \left|\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}\,{}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}u(x) - \mathbb{I}^{\alpha,\sigma}_{\psi,a^+}\,{}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}u(y)\right|.$$

Applying the Hölder continuity estimate established in Theorem 2.3, there exists a constant $\mathcal{K} > 0$, independent of u, such that

$$|u(x)-u(y)| \leq \mathcal{K}\|{}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}u\|_{2,\psi'}|x-y|^{\alpha-\frac{1}{2}}.$$

Hence, from (3.6) it follows that

$$|u(x)-u(y)| \leq \mathcal{K}M|x-y|^{\alpha-\frac{1}{2}}, \quad \forall\, u \in B.$$

Therefore, B is equicontinuous in $C(\overline{(a,b)})$. Since B is uniformly bounded and equicontinuous, the Arzelà–Ascoli theorem implies that every sequence in B possesses a subsequence converging in $C(\overline{(a,b)})$. Consequently, the embedding

$$\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b) \hookrightarrow C(\overline{(a,b)})$$

is compact. □

Remark 3.6. *If $\alpha \in \left(\frac{1}{2}, 1\right)$ and $\sigma > 0$, then for every $u \in \mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$, there exists $(\phi_n)_{n\in\mathbb{N}} \subset C_0^\infty(a,b)$ such that*

$$\lim_{n\to\infty} \|u-\phi_n\| = 0.$$

Combining this limit with Theorem 3.4 we arrive to

$$0 \leq |u(a)| = |u(a)-\phi_n(a)| \leq \frac{1}{(2\sigma)^{\alpha-\frac{1}{2}}\Gamma(\alpha)}\left[\gamma(2\alpha-1, 2\sigma(\psi(b)-\psi(a)))\right]^{1/2}\|u-\phi_n\| \to 0 \ \text{ as } \ n \to \infty.$$

Consequently $u(a) = 0$. Similarly, we can obtain that $u(b) = 0$.

On the other hand, Lemma 3.2 yields that

$${}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}u \in L^2((a,b),\psi').$$

Therefore, $\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$ can be rewritten as

$$\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b) = \{u \in L^2((a,b),\psi') :\ {}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}u \in L^2((a,b),\psi') \ \text{ and } \ u(a) = u(b) = 0\}.$$

4. Proof of Theorem 1.1, 1.2, 1.3

In this section, we establish Theorems 1.1, 1.2, and 1.3. We begin with the proof of Theorem 1.1. To this end, we first recall some auxiliary results.

Definition 4.1. *An operator $\Upsilon : E \to E^*$ is said to be of type $(S)_+$ if, for any sequence $\{\xi_n\} \subset E$ such that $\xi_n \rightharpoonup \xi$ in E and*

$$\limsup_{n\to+\infty} \langle \Upsilon(\xi_n), \xi_n - \xi \rangle \leq 0,$$

it follows that $\xi_n \to \xi$ strongly in E.

Lemma 4.1. *[45] Let $\mathcal{F} \in C^1(E, \mathbb{R})$. Assume that there exist $\xi_0, \xi_1 \in E$ and a bounded neighborhood Λ of ξ_0 such that $\xi_1 \notin \Lambda$ and*

$$\inf_{Z\in\partial\Lambda} \mathcal{F}(Z) > \max\{\mathcal{F}(\xi_0), \mathcal{F}(\xi_1)\}.$$

Then, $\mathcal{F}$ admits a critical point $\xi \in E$, that is, $\mathcal{F}'(\xi) = 0$ such that

$$\mathcal{F}(\xi) > \max\{\mathcal{F}(\xi_0), \mathcal{F}(\xi_1)\}.$$

Moreover, if either ξ_0 or ξ_1 is already a critical point of $\mathcal{F}$, then $\mathcal{F}$ possesses at least two distinct critical points.

Lemma 4.2. *[50, Theorem 38.A] Let $\mathcal{F} : \Omega \subseteq E \to \mathbb{R}$ be a functional defined on a nonempty set Ω. The minimization problem*

$$\min_{\xi\in\Omega} \mathcal{F}(\xi) = \delta$$

admits a solution provided that the following conditions hold:

(i) E is a real reflexive Banach space;
(ii) Ω is weakly sequentially closed and bounded;
(iii) $\mathcal{F}$ is weakly sequentially lower semicontinuous on Ω.

Associated with problem (1.1), we consider the functional $\mathcal{F} : \mathbb{H}^{\alpha,\sigma}_{\psi,0}(a, b) \to \mathbb{R}$ defined by

$$\mathcal{F}(u) = \frac{1}{2}\int_a^b \left| {}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+} u(x) \right|^2 \psi'(x)\, dx - \int_a^b F(x, u(x))\, \psi'(x)\, dx, \quad \forall\, u \in \mathbb{H}^{\alpha,\sigma}_{\psi,0}(a, b), \tag{4.1}$$

where $F(x, u) = \int_0^u f(x, \xi)\, d\xi$.

It is straightforward to verify that $\mathcal{F} \in C^1(\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a, b), \mathbb{R})$. Moreover, for every $u, \varphi \in \mathbb{H}^{\alpha,\sigma}_{\psi,0}(a, b)$, its Fréchet derivative is given by

$$\langle \mathcal{F}'(u), \varphi \rangle = \int_a^b {}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+} u(x)\, {}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+} \varphi(x)\, \psi'(x)\, dx - \int_a^b f(x, u(x))\varphi(x)\, \psi'(x)\, dx. \tag{4.2}$$

Therefore, the critical points of $\mathcal{F}$ correspond to weak solutions of problem (1.1).

Finally, we define the operator $\Upsilon : \mathbb{H}^{\alpha,\sigma}_{\psi,0}(a, b) \to (\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a, b))^*$ by

$$\langle \Upsilon u, \varphi \rangle := \int_a^b {}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+} u(x)\, {}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+} \varphi(x)\, \psi'(x)\, dx.$$

Lemma 4.3. *The functional $\mathcal{F}$ is continuous and weakly lower semicontinuous. Moreover, if* (**G1**) *holds, then $\mathcal{F}$ satisfies the Palais-Smale* (PS) *condition.*

Proof. Since f is continuous, both $\mathcal{F}$ and $\mathcal{F}'$ are continuous, and hence $\mathcal{F} \in C^1(\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b), \mathbb{R})$.

We first show that $\mathcal{F}$ is weakly lower semicontinuous. Let $\xi_n \rightharpoonup \xi$ in $\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$. Then,

$$\|\xi\| \leq \liminf_{n\to\infty} \|\xi_n\|,$$

and, by Theorem 3.5, $\xi_n \to \xi$ uniformly in $C\overline{(a,b)}$. Hence,

$$\int_a^b F(x, \xi_n(x))\psi'(x)\,dx \to \int_a^b F(x, \xi(x))\psi'(x)\,dx,$$

which implies

$$\mathcal{F}(\xi) \leq \liminf_{n\to\infty} \mathcal{F}(\xi_n).$$

Thus, $\mathcal{F}$ is weakly lower semicontinuous.

We now verify the (PS) condition. Let $\{\xi_n\}$ be such that $\{\mathcal{F}(\xi_n)\}$ is bounded and $\mathcal{F}'(\xi_n) \to 0$. From (4.2), we have

$$\int_a^b f(x, \xi_n(x))\xi_n(x)\psi'(x)\,dx = \|\xi_n\|^2 - \langle \mathcal{F}'(\xi_n), \xi_n\rangle. \tag{4.3}$$

Using (4.3) and (**G1**), we obtain

$$\begin{aligned} \mathcal{F}(\xi_n) &\geq \frac{1}{2}\|\xi_n\|^2 - \frac{1}{\mu}\int_a^b f(x, \xi_n(x))\xi_n(x)\psi'(x)\,dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\|\xi_n\|^2 - \frac{1}{\mu}\|\mathcal{F}'(\xi_n)\|_{(\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b))^*}\|\xi_n\|. \end{aligned} \tag{4.4}$$

Hence, $\{\xi_n\}$ is bounded. By reflexivity, up to a subsequence,

$$\begin{cases} \xi_n \rightharpoonup \xi & \text{in } \mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b), \\ \xi_n \to \xi & \text{a.e. in } (a,b), \end{cases} \tag{4.5}$$

and also uniformly by Theorem 3.5. Therefore,

$$\langle \mathcal{F}'(\xi_n), \xi_n - \xi\rangle \to 0, \quad \int_a^b f(x, \xi_n(x))(\xi_n(x) - \xi(x))\psi'(x)\,dx \to 0.$$

Thus,

$$\limsup_{n\to\infty} \langle \Upsilon\xi_n, \xi_n - \xi\rangle \leq 0,$$

and arguing as in [48, Theorem 5.1], Υ is of type $(S)_+$, which yields $\xi_n \to \xi$ strongly. Hence, $\mathcal{F}$ satisfies the (PS) condition. □

Proof of Theorem 1.1. We apply Lemma 4.2 in the space $E := \mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$. For $\varpi_0 > 0$, define

$$\Omega_{\varpi_0} := \{\xi \in \mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b) : \|\xi\| < \varpi_0\}.$$

Since E is a Hilbert space, $\overline{\Omega}_{\varpi_0}$ is weakly sequentially closed and bounded. By Lemma 4.3, $\mathcal{F}$ is weakly lower semicontinuous, hence there exists $\xi_0 \in \overline{\Omega}_{\varpi_0}$ such that

$$\mathcal{F}(\xi_0) = \min_{\xi \in \overline{\Omega}_{\varpi_0}} \mathcal{F}(\xi).$$

Next, we show that ξ_0 is a local minimizer. From (**G2**), there exist $\varpi_1 > 0$ and $\varepsilon > 0$ such that

$$F(x,\xi) \leq \varrho|\xi|^{\mu}, \quad \|\xi\| \leq \varpi_1,$$

and

$$\frac{1}{2}\varpi_1^2 - \varrho\left(\frac{\varpi_1}{\sigma^{\alpha}\Gamma(\alpha)}\gamma(\alpha, \sigma(\psi(b) - \psi(a)))\right)^{\mu} > \varepsilon.$$

For any $u \in \partial\Omega_{\varpi_1}$, $\|u\| = \varpi_1$, Theorem 2.1 yields

$$\mathcal{F}(u) \geq \frac{1}{2}\varpi_1^2 - \varrho\left(\frac{\varpi_1}{\sigma^{\alpha}\Gamma(\alpha)}\gamma(\alpha, \sigma(\psi(b) - \psi(a)))\right)^{\mu} > \varepsilon.$$

Since $\mathcal{F}(0) = 0$, it follows that

$$\mathcal{F}(\xi_0) \leq 0 < \varepsilon \leq \inf_{u \in \partial\Omega_{\varpi_1}} \mathcal{F}(u),$$

so ξ_0 is a local minimizer.

We now construct ξ_1 such that $\|\xi_1\| > \varpi_1$ and $\mathcal{F}(\xi_1) < 0$. By (**G1***), there exist $\vartheta > 0$ and $\varpi_2 > \varpi_1$ such that

$$F(x,u) \geq \vartheta|u|^{\mu}, \quad \|u\| \geq \varpi_2.$$

Then, for $r > 0$,

$$\mathcal{F}(ru) \leq \frac{1}{2}r^2\|u\|^2 - \vartheta r^{\mu}\|u\|^{\mu}_{\mu,\psi'} \to -\infty \quad \text{as } r \to \infty.$$

Thus, there exists $r_0 > 0$ such that $\xi_1 := r_0 u$ satisfies $\mathcal{F}(\xi_1) < 0$ and $\|\xi_1\| > \varpi_2$.

Therefore,

$$\max\{\mathcal{F}(\xi_0), \mathcal{F}(\xi_1)\} < \inf_{u \in \partial\Omega_{\varpi_1}} \mathcal{F}(u).$$

By Lemmas 4.1 and 4.3, $\mathcal{F}$ admits another critical point ξ^*. Hence, ξ_0 and ξ^* are two distinct critical points, i.e., two weak solutions of (1.1). □

In order to establish Theorems 1.2 and 1.3, we first recall the following abstract results.

Theorem 4.4. *[45, Theorem 1.1] Let E be a reflexive Banach space. If $\mathcal{F} : E \to \mathbb{R}$ is weakly lower semicontinuous and admits a bounded minimizing sequence, then $\mathcal{F}$ attains its minimum on E.*

Theorem 4.5. *[47, Theorem 9.12] Let E be a real Banach space and $\mathcal{F} \in C^1(E,\mathbb{R})$ be an even functional satisfying the Palais-Smale (PS) condition and $\mathcal{F}(0) = 0$. Assume that $E = W \oplus X$, where W is infinite-dimensional. Suppose that:*

(i) *there exist $m_0 > 0$ and $\tau_0 > 0$ such that*

$$\mathcal{F}(\xi) \geq m_0, \quad \forall\, \xi \in \partial B_{\tau_0} \cap X,$$

where $B_{\tau_0} := \{\xi \in E : \|\xi\| < \tau_0\}$;

(ii) *for every finite-dimensional subspace $V \subset E$, there exists $R = R(V) > 0$ such that*

$$\mathcal{F}(\xi) \leq 0, \quad \forall\, \xi \in V \setminus B_R.$$

Then, $\mathcal{F}$ possesses an unbounded sequence of critical values.

We now prove Theorem 1.2.

Proof of Theorem 1.2. It is immediate that $\mathcal{F}$ is even and $\mathcal{F}(0) = 0$. We first verify that $\mathcal{F}$ satisfies the (PS) condition.

Let $\{\xi_n\} \subset \mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$ be such that $\{\mathcal{F}(\xi_n)\}$ is bounded and $\mathcal{F}'(\xi_n) \to 0$. We claim that $\{\xi_n\}$ is bounded. Define

$$\Lambda_n := \{x \in [a,b] : |\xi_n(x)| \geq \gamma_1\},$$

where $\gamma_1 > 0$ is given by (**G4**). Using (4.1), (4.2), (**G4**), and Theorem 2.1, we obtain

$$\begin{aligned}
\nu\mathcal{F}(\xi_n) - \langle \mathcal{F}'(\xi_n), \xi_n\rangle &= \left(\frac{\nu}{2} - 1\right)\|\xi_n\|^2 \\
&\quad + \int_a^b \left(f(x,\xi_n(x))\xi_n(x) - \nu F(x,\xi_n(x))\right)\psi'(x)\,dx \\
&\geq \left(\frac{\nu}{2} - 1\right)\|\xi_n\|^2 \\
&\quad - \int_{\Lambda_n} \left(\sum_{i=1}^{l} \theta_i |\xi_n(x)|^{\zeta_i} + \theta_F |\xi_n(x)|^2\right)\psi'(x)\,dx \\
&\quad - C_0 \int_a^b \psi'(x)\,dx,
\end{aligned}$$

where

$$C_0 := \max_{\substack{x\in[a,b] \\ |s|\leq L_1}} |f(x,s)s - \nu F(x,s)|.$$

Applying Theorem 2.1, we deduce

$$\begin{aligned}
\nu\mathcal{F}(\xi_n) - \langle \mathcal{F}'(\xi_n), \xi_n\rangle &\geq \left(\frac{\nu-2}{2} - \theta_F\left(\frac{1}{\sigma^\alpha\Gamma(\alpha)}\gamma(\alpha,\sigma(\psi(b)-\psi(a)))\right)^2\right)\|\xi_n\|^2 \\
&\quad - \sum_{i=1}^{l} \theta_i \left(\frac{1}{\sigma^\alpha\Gamma(\alpha)}\gamma(\alpha,\sigma(\psi(b)-\psi(a)))\right)^{\zeta_i} \|\xi_n\|^{\zeta_i} \\
&\quad - C_0 \int_a^b \psi'(x)\,dx.
\end{aligned}$$

Since $\nu > 2$, $0 < \zeta_i < 2$ for all i, and

$$0 \leq \theta_F < \frac{(\nu-2)\sigma^{2\alpha}\Gamma^2(\alpha)}{4\big(\gamma(\alpha,\sigma(\psi(b)-\psi(a)))\big)^2},$$

it follows that $\{\xi_n\}$ is bounded in $\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$. By Theorem 3.5, $\{\xi_n\}$ admits a strongly convergent subsequence. Hence, $\mathcal{F}$ satisfies the (PS) condition.

Next, we verify condition (i) of Theorem 4.5. By (**G3**), choose ϑ such that

$$\limsup_{u\to 0} \frac{\max_{x\in[a,b]} F(x,u)}{|u|^2} < \vartheta < \frac{\sigma^{2\alpha}\Gamma^2(\alpha)}{4(\gamma(\alpha,\sigma(\psi(b)-\psi(a))))^2}. \tag{4.6}$$

Then, there exists $\delta > 0$ such that

$$F(x,u) \le \vartheta|u|^2, \quad \forall x \in [a,b],\ |u| \le \delta. \tag{4.7}$$

Using (4.7) and Theorem 2.1, we obtain

$$\mathcal{F}(u) \ge \left(\frac{1}{2} - \vartheta\left(\frac{1}{\sigma^\alpha\Gamma(\alpha)}\gamma(\alpha,\sigma(\psi(b)-\psi(a)))\right)^2\right)\|u\|^2.$$

Thus, choosing $\tau_0 > 0$ sufficiently small, there exists $m_0 > 0$ such that

$$\mathcal{F}(u) \ge m_0 \quad \text{for all } \|u\| = \tau_0.$$

We now verify condition (ii). Let $V \subset \mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$ be finite dimensional. By (**G2**), there exists $\Theta_0 < 0$ such that

$$\liminf_{|u|\to\infty} \frac{\min_{x\in[a,b]} F(x,u) - |u|^\iota}{|u|^2} > \Theta_0.$$

Hence, there exist constants $\tilde{B} > 0$ and $\rho_0 > 0$ such that

$$F(x,u) \ge \Theta_0|u|^2 + |u|^\iota - \rho_0, \quad \forall (x,u) \in [a,b]\times\mathbb{R}. \tag{4.8}$$

Let $u \in V$ with $\|u\| = 1$ and $t > 0$. Using (4.8) and Theorem 2.1, we obtain

$$\begin{aligned}\mathcal{F}(tu) \le{}& \frac{1}{2}t^2 - \Theta_0\left(\frac{1}{\sigma^\alpha\Gamma(\alpha)}\gamma(\alpha,\sigma(\psi(b)-\psi(a)))\right)^2 t^2 \\ &- t^\iota\int_a^b |u(x)|^\iota\psi'(x)\,dx + \rho_0\int_a^b \psi'(x)\,dx.\end{aligned}$$

Since $\iota > 2$, it follows that $\mathcal{F}(tu) \to -\infty$ as $t \to \infty$. By the equivalence of norms in V, there exists $R > 0$ such that

$$\mathcal{F}(u) \le 0, \quad \forall u \in V \setminus B_R.$$

Therefore, all the assumptions of Theorem 4.5 are satisfied. Consequently, $\mathcal{F}$ possesses an unbounded sequence of critical values, which correspond to infinitely many weak solutions of (1.1). ∎

Proof of Theorem 1.3. We first prove that $\mathcal{F}$ is weakly lower semicontinuous. Let $\{u_n\} \subset \mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$ be such that

$$u_n \rightharpoonup u \quad \text{in } \mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b).$$

By Theorem 3.5, $u_n \to u$ uniformly in $C\overline{(a,b)}$, and

$$\|u\| \le \liminf_{n\to\infty} \|u_n\|.$$

Using the continuity of f, we obtain

$$\begin{aligned}\liminf_{n\to\infty}\mathcal{F}(u_n) &= \liminf_{n\to\infty}\left(\frac{1}{2}\int_a^b \left|{}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}u_n(x)\right|^2\psi'(x)\,dx - \int_a^b F(x,u_n(x))\psi'(x)\,dx\right)\\ &\geq \frac{1}{2}\int_a^b \left|{}^C\mathbb{D}^{\alpha,\sigma}_{\psi,a^+}u(x)\right|^2\psi'(x)\,dx - \int_a^b F(x,u(x))\psi'(x)\,dx\\ &= \mathcal{F}(u).\end{aligned}$$

Hence, $\mathcal{F}$ is weakly lower semicontinuous.

We now show that $\mathcal{F}$ is coercive. By assumption (**G5**), there exists $\Theta_0 > 0$ such that

$$\limsup_{|u|\to\infty}\frac{\max_{x\in[a,b]}F(x,u)}{|u|^2} < \Theta_0 < \frac{\sigma^{2\alpha}\Gamma^2(\alpha)}{4\big(\gamma(\alpha,\sigma(\psi(b)-\psi(a)))\big)^2}.$$

Therefore, there exists $\tilde{A} > 0$ such that

$$F(x,u) \leq \Theta_0|u|^2, \quad \forall x\in[a,b],\ |u|\geq \tilde{A}.$$

Since

$$\max_{\substack{x\in[a,b]\\ |u|\leq\tilde{A}}} F(x,u) < \infty,$$

there exists a constant $\rho_1 > 0$ such that

$$F(x,u) \leq \Theta_0|u|^2 + \rho_1, \quad \forall (x,u)\in[a,b]\times\mathbb{R}. \tag{4.9}$$

Using (4.1), (4.9), and Theorem 2.1, we deduce

$$\mathcal{F}(u) \geq \frac{1}{2}\|u\|^2 - \Theta_0\left(\frac{1}{\sigma^\alpha\Gamma(\alpha)}\gamma(\alpha,\sigma(\psi(b)-\psi(a)))\right)^2\|u\|^2 - \rho_1\int_a^b\psi'(x)\,dx,$$

for every $u\in\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$.

Since

$$\Theta_0 < \frac{\sigma^{2\alpha}\Gamma^2(\alpha)}{4\big(\gamma(\alpha,\sigma(\psi(b)-\psi(a)))\big)^2},$$

it follows that the coefficient of $\|u\|^2$ is strictly positive, and hence

$$\lim_{\|u\|\to\infty}\mathcal{F}(u) = +\infty,$$

that is, $\mathcal{F}$ is coercive.

Therefore, by Theorem 4.4, $\mathcal{F}$ attains its minimum on $\mathbb{H}^{\alpha,\sigma}_{\psi,0}(a,b)$. This minimizer is a critical point of $\mathcal{F}$ and thus corresponds to a weak solution of (1.1).

■

5. Conclusions

In this work, we developed a variational framework for a class of nonlinear boundary value problems involving mixed ψ-Riemann–Liouville tempered fractional derivatives for $\alpha \in (1/2, 1)$ and $\sigma > 0$. We established boundedness properties of the associated ψ-Riemann-Liouville tempered fractional integral, with explicit estimates involving the incomplete gamma function.

We also proved that the tempered fractional Sobolev-type space $H^{\alpha,\sigma}_{\psi,0}(a, b)$ is a Hilbert space and that its embedding into $C([a, b])$ is continuous and compact. These properties provide the appropriate functional setting for the variational formulation and ensure the continuity of weak solutions.

Finally, by applying critical point theory, we established existence and multiplicity results. Depending on the assumptions on the nonlinear term, the problem admits at least one, at least two, or infinitely many weak solutions. In particular, the results extend the corresponding non-tempered case $\sigma = 0$ and provide a unified framework for a broader class of nonlocal fractional problems.

Author contributions

All authors contributed equally to this work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflicts of interest.

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Appendix

Proof of Theorem 2.1. Take $p = 1$ and $u \in L^1((a,b),\psi')$. Then, using the change of variable $t = \sigma(\psi(x) - \psi(s))$, we get

$$\begin{aligned}
\int_a^b |\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}u(x)|\psi'(x)dx &\le \frac{1}{\Gamma(\alpha)}\int_a^b\int_a^x \psi'(s)(\psi(x)-\psi(s))^{\alpha-1}e^{-\sigma(\psi(x)-\psi(s))}|u(s)|ds\psi'(x)dx \\
&= \frac{1}{\Gamma(\alpha)}\int_a^b\int_s^b \psi'(s)(\psi(x)-\psi(s))^{\alpha-1}e^{-\sigma(\psi(x)-\psi(s))}|u(s)|\psi'(x)dxds \\
&= \frac{1}{\Gamma(\alpha)}\int_a^b |u(s)|\psi'(s)\int_s^b (\psi(x)-\psi(s))^{\alpha-1}e^{-\sigma(\psi(x)-\psi(s))}\psi'(x)dxds \\
&= \frac{1}{\sigma^\alpha\Gamma(\alpha)}\int_a^b |u(s)|\psi'(s)\gamma(\alpha,\sigma(\psi(b)-\psi(s)))ds \\
&\le \frac{\gamma(\alpha,\sigma(\psi(b)-\psi(a)))}{\sigma^\alpha\Gamma(\alpha)}\int_a^b |u(x)|\psi'(x)dx.
\end{aligned}$$

Therefore,

$$||\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}u||_{p,\psi'} \le \frac{\gamma(\alpha,\sigma(\psi(b)-\psi(a)))}{\sigma^\alpha\Gamma(\alpha)}||u||_{p,\psi'}. \tag{A.1}$$

Let $1 < p < \infty$, $u \in L^p((a,b),\psi')$ and q be a real number such that $\frac{1}{p}+\frac{1}{q}=1$, we denote $K_\psi(x,s) = (\psi(x)-\psi(s))^{\alpha-1}e^{-\sigma(\psi(x)-\psi(s))}$. By the Hölder inequality, we can then derive

$$\begin{aligned}
|\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}u(x)| &\le \frac{1}{\Gamma(\alpha)}\int_a^x \psi'(s)K_\psi(x,s)|u(s)|ds \\
&\le \frac{1}{\Gamma(\alpha)}\left(\int_a^x \psi'(s)K_\psi(x,s)ds\right)^{1/q}\left(\int_a^x \psi'(s)K_\psi(x,s)|u(s)|^p ds\right)^{1/p}.
\end{aligned}$$

Note that, by using the change of variable $t = \sigma(\psi(x)-\psi(s))$, we derive

$$\int_a^x \psi'(s)K_\psi(x,s)ds = \frac{1}{\sigma^\alpha}\gamma(\alpha,\sigma(\psi(x)-\psi(s))).$$

Hence,

$$\begin{aligned}
&\int_a^b |\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}u(x)|^p\psi'(x)dx \\
&\le \frac{1}{\Gamma^p(\alpha)\sigma^{\frac{\alpha p}{q}}}\int_a^b \gamma(\alpha,\sigma(\psi(x)-\psi(a)))^{\frac{p}{q}}\int_a^x \psi'(s)K_\psi(x,s)|u(s)|^p\psi'(x)dx \\
&\le \frac{1}{\Gamma^p(\alpha)\sigma^{\frac{\alpha p}{q}}}\gamma(\alpha,\sigma(\psi(b)-\psi(a)))^{\frac{p}{q}}\int_a^b\int_a^x \psi'(s)K_\psi(x,s)|u(s)|^p\psi'(x)dsdx \\
&= \frac{1}{\Gamma^p(\alpha)\sigma^{\frac{\alpha p}{q}}}\gamma(\alpha,\sigma(\psi(b)-\psi(a)))^{\frac{p}{q}}\int_a^b |u(s)|^p\psi'(s)\int_s^b \psi'(x)K_\psi(x,s)dxds \\
&\le \frac{1}{\Gamma^p(\alpha)\sigma^{\frac{\alpha p}{q}}}\gamma(\alpha,\sigma(\psi(b)-\psi(a)))^{\frac{p}{q}}\int_a^b |u(s)|^p\psi'(s)\frac{1}{\sigma^\alpha}\gamma(\alpha,\sigma(\psi(b)-\psi(s)))ds \\
&\le \frac{1}{\Gamma^p(\alpha)\sigma^{\alpha p}}\gamma(\alpha,\sigma(\psi(b)-\psi(a)))^p||u||^p_{p,\psi'}.
\end{aligned}$$

Therefore,

$$||\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}u||_{p,\psi'} \le \frac{1}{\sigma^\alpha\Gamma(\alpha)}\gamma(\alpha,\sigma(\psi(b)-\psi(a)))||u||_{p,\psi'}.$$

Proof of Theorem 2.3. Since $\psi \in C^1[a,b]$, its derivative ψ' is continuous on the compact interval $[a,b]$, and therefore bounded. By the mean value theorem, there exists a constant $L > 0$ such that

$$|\psi(x_1) - \psi(x_2)| \leq L|x_1 - x_2|, \quad \forall x_1, x_2 \in [a,b]. \tag{A.2}$$

Define the kernel

$$K_\psi(x,s) = (\psi(x) - \psi(s))^{\alpha-1} e^{-\sigma(\psi(x)-\psi(s))}.$$

Then, the fractional integral operator can be written as

$$\mathbb{I}^{\alpha,\sigma}_{\psi,a^+} u(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(s) K_\psi(x,s) u(s)\, ds.$$

Applying Hölder's inequality, we obtain

$$\left|\mathbb{I}^{\alpha,\sigma}_{\psi,a^+} u(x)\right| \leq \frac{1}{\Gamma(\alpha)} \left(\int_a^x \psi'(s) K_\psi^2(x,s)\, ds \right)^{1/2} \|u\|_{2,\psi'}.$$

Since ψ is strictly increasing, it follows that

$$e^{-2\sigma(\psi(x)-\psi(s))} \leq 1,$$

and therefore

$$\int_a^x \psi'(s) K_\psi^2(x,s)\, ds \leq \frac{(\psi(x) - \psi(a))^{2\alpha-1}}{2\alpha - 1}.$$

Consequently,

$$\left|\mathbb{I}^{\alpha,\sigma}_{\psi,a^+} u(x)\right| \leq \frac{(\psi(x) - \psi(a))^{\alpha-\frac{1}{2}}}{\sqrt{2\alpha-1}\,\Gamma(\alpha)} \|u\|_{2,\psi'}. \tag{A.3}$$

In particular, $\mathbb{I}^{\alpha,\sigma}_{\psi,a^+} u$ is well-defined and satisfies

$$\lim_{x \to a^+} \mathbb{I}^{\alpha,\sigma}_{\psi,a^+} u(x) = 0. \tag{A.4}$$

Let $a < x_1 < x_2 \leq b$. Then,

$$\left|\mathbb{I}^{\alpha,\sigma}_{\psi,a^+} u(x_2) - \mathbb{I}^{\alpha,\sigma}_{\psi,a^+} u(x_1)\right| \leq \frac{1}{\Gamma(\alpha)} \Big(\Sigma_1 + \Sigma_2 + \Sigma_3 \Big),$$

where

$$\begin{aligned}
\Sigma_1 &= \int_a^{x_1} \psi'(s) \Big| (\psi(x_2) - \psi(s))^{\alpha-1} - (\psi(x_1) - \psi(s))^{\alpha-1} \Big| e^{-\sigma(\psi(x_2)-\psi(s))} |u(s)|\, ds, \\
\Sigma_2 &= \int_a^{x_1} \psi'(s) (\psi(x_1) - \psi(s))^{\alpha-1} \Big| e^{-\sigma(\psi(x_2)-\psi(s))} - e^{-\sigma(\psi(x_1)-\psi(s))} \Big| |u(s)|\, ds, \\
\Sigma_3 &= \int_{x_1}^{x_2} \psi'(s) (\psi(x_2) - \psi(s))^{\alpha-1} e^{-\sigma(\psi(x_2)-\psi(s))} |u(s)|\, ds.
\end{aligned}$$

To estimate Σ_1, we use the fundamental theorem of calculus, Hölder's inequality, and the generalized Minkowski inequality:

$$\begin{aligned}
\Sigma_1 &\leq \int_a^{x_1} \left| \int_{x_1}^{x_2} \frac{d}{dt}(\psi(t) - \psi(s))^{\alpha-1} dt \right| |u(s)| ds \\
&\leq |\alpha - 1| \int_a^{x_1} \left| \int_{x_1}^{x_2} (\psi(t) - \psi(s))^{\alpha-2} \psi'(t) dt \right| |u(s)| \psi'(s) ds \\
&\leq |\alpha - 1| \left(\int_a^{x_1} \left| \int_{x_1}^{x_2} (\psi(t) - \psi(s))^{\alpha-2} \psi'(t) dt \right|^2 \psi'(s) ds \right)^{1/2} \|u\|_{2,\psi'} \\
&\leq |\alpha - 1| \int_{x_1}^{x_2} \left| \int_a^{x_1} (\psi(t) - \psi(s))^{2(\alpha-2)} \psi'(s) ds \right|^{1/2} \psi'(t) dt \, \|u\|_{2,\psi'}.
\end{aligned}$$

Using the change of variables $y = \psi(t) - \psi(s)$, we obtain

$$\int_a^{x_1} (\psi(t) - \psi(s))^{2(\alpha-2)} \psi'(s) \, ds = \frac{(\psi(t) - \psi(x_1))^{2\alpha-3} - (\psi(t) - \psi(a))^{2\alpha-3}}{2\alpha - 3}.$$

Since $2\alpha - 3 < 0$, it follows that

$$\left(\int_a^{x_1} (\psi(t) - \psi(s))^{2(\alpha-2)} \psi'(s) \, ds \right)^{1/2} \leq \frac{(\psi(t) - \psi(x_1))^{\alpha - \frac{3}{2}}}{|2\alpha - 3|^{1/2}}.$$

Thus,

$$\Sigma_1 \leq \frac{|\alpha - 1|}{|2\alpha - 3|^{1/2}} \|u\|_{2,\psi'} \int_{x_1}^{x_2} (\psi(t) - \psi(x_1))^{\alpha - \frac{3}{2}} \psi'(t) \, dt,$$

and by (A.2),

$$\Sigma_1 \leq \frac{L^{\alpha - \frac{1}{2}} |\alpha - 1|}{|2\alpha - 3|^{1/2} \left(\alpha - \frac{1}{2}\right)} |x_2 - x_1|^{\alpha - \frac{1}{2}} \|u\|_{2,\psi'}. \tag{A.5}$$

To estimate Σ_2, we use the inequality

$$|e^{-x} - e^{-y}| \leq 2|x - y|^\beta, \quad \beta \in (0, 1),$$

with $\beta = \alpha - \frac{1}{2}$, and (A.2), to obtain

$$\left| e^{-\sigma(\psi(x_2) - \psi(s))} - e^{-\sigma(\psi(x_1) - \psi(s))} \right| \leq 2\sigma L^{\alpha - \frac{1}{2}} |x_2 - x_1|^{\alpha - \frac{1}{2}}.$$

Hence,

$$\Sigma_2 \leq \frac{2\sigma L^{\alpha - \frac{1}{2}} (\psi(b) - \psi(a))^{\alpha - \frac{1}{2}}}{(2\alpha - 1)^{1/2}} |x_2 - x_1|^{\alpha - \frac{1}{2}} \|u\|_{2,\psi'}.$$

Finally, for Σ_3, applying Hölder's inequality and (A.2), we obtain

$$\Sigma_3 \leq \frac{L^{\alpha - \frac{1}{2}}}{(2\alpha - 1)^{1/2}} |x_2 - x_1|^{\alpha - \frac{1}{2}} \|u\|_{2,\psi'}.$$

Collecting the above estimates, we conclude that

$$\left|\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}u(x_2) - \mathbb{I}^{\alpha,\sigma}_{\psi,a^+}u(x_1)\right| \leq \mathcal{K}|x_2 - x_1|^{\alpha-\frac{1}{2}}\|u\|_{2,\psi'}, \tag{A.6}$$

where

$$\mathcal{K} = \frac{1}{\Gamma(\alpha)}\left(\frac{L^{\alpha-\frac{1}{2}}|\alpha-1|}{|2\alpha-3|^{1/2}\left(\alpha-\frac{1}{2}\right)} + \frac{L^{\alpha-\frac{1}{2}}(2\sigma(\psi(b)-\psi(a))^{\alpha-\frac{1}{2}}+1)}{(2\alpha-1)^{1/2}}\right).$$

If $x_1 = a$, then using (A.3) and (A.2), we obtain

$$\left|\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}u(x_2)\right| \leq \frac{L^{\alpha-\frac{1}{2}}}{[2(\alpha-\frac{1}{2})]^{1/2}}|x_2 - a|^{\alpha-\frac{1}{2}}\|u\|_{2,\psi'}. \tag{A.7}$$

Consequently,

$$\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}u \in H^{\alpha-\frac{1}{2}}[a,b].$$

Moreover,

$$\left[\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}u\right]_{\alpha-\frac{1}{2}} \leq \mathcal{K}\|u\|_{2,\psi'}.$$

Finally, estimating the supremum norm and performing the change of variables $\lambda = 2\sigma(\psi(x)-\psi(s))$, we obtain

$$\left\|\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}u\right\|_{\infty} \leq \frac{\gamma(2\alpha-1,\, 2\sigma(\psi(b)-\psi(a)))^{1/2}}{\Gamma(\alpha)\,(2\sigma)^{\alpha-\frac{1}{2}}}\|u\|_{2,\psi'}.$$

Therefore,

$$\left\|\mathbb{I}^{\alpha,\sigma}_{\psi,a^+}u\right\|_{H^{\alpha-\frac{1}{2}}[a,b]} \leq C\|u\|_{2,\psi'},$$

where

$$C = \frac{\gamma(2\alpha-1,\, 2\sigma(\psi(b)-\psi(a)))^{1/2}}{\Gamma(\alpha)\,(2\sigma)^{\alpha-\frac{1}{2}}} + \mathcal{K}.$$

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