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Research article

Integrable structures of a Bose-Einstein condensate model: From asymptotic reduction to exact solutions and conservation laws

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Abstract: This paper studies a nonlinear Schrödinger-type model for Bose-Einstein condensates with a fixed s-wave scattering length. Under a specific rescaling, the dimensionless Gross-Pitaevskii equation takes the form $i\Psi_t = -\nabla^2\Psi - 2|\Psi|^2\Psi$. The model is rewritten in hydrodynamic form via a density-phase decomposition, and in the long-wave limit, it is asymptotically reduced to the classical Korteweg-de Vries (KdV) equation. For this reduced KdV equation, we present its Lax pair, Darboux transformation, and Bäcklund transformation, and obtain multi-soliton and positon solutions by means of the Hirota bilinear method. Furthermore, by coupling the tau functions of a single soliton and a first-order positon via the Wronskian determinant, we construct a mixed interaction solution, whose explicit expression contains nonlinear cross-coupling terms which characterize the collision dynamics between the two types of waves. An asymptotic expansion of the spectral problem yields infinitely many conservation laws, and we prove that all orders of global conserved quantities for the soliton-positon composite wave satisfy an exact superposition relation. These results demonstrate that the reduced KdV equation derived from the Bose-Einstein condensate model possesses a complete integrable structure, thus providing a theoretical reference for understanding nonlinear wave propagation in quantum fluids under the KdV approximation.

Keywords: Bose-Einstein condensate; Korteweg-de Vries equation; asymptotic reduction; mixed interaction solution; infinite conservation laws

1. Introduction

Nonlinear partial differential equations describe complex dynamical behaviors in fluid mechanics, nonlinear optics, and quantum physics. The nonlinear Schrödinger-type equation occupies a central position in describing macroscopic quantum effects of Bose-Einstein condensates (BEC), and its soliton solutions correspond to stable wave structures with particle-like properties, carrying physical significance [1]. Following the ideas of the generalized nonlinear Schrödinger equation [2] and the

Darboux transformation for the coupled Kadomtsev-Petviashvili equation, the present study systematically analyzes the characteristics of soliton and positon solutions. Investigating the integrability of such nonlinear systems helps to reveal their inherent symmetries, conservation laws, and exact solvability.

The theory of integrable systems rests on mathematical structures such as Lax pairs, Bäcklund transformations, and infinitely many conservation laws. Since the early studies on the Korteweg-de Vries (KdV) equation, the inverse scattering method [3] and the bilinear method [4] have become standard tools for nonlinear evolution equations. Prolongation structures offer a geometric perspective for integrating nonlinear equations [5–7]. Recent developments include nonlocal integrable systems [8], Rén symmetric integrable systems [9], and Riemann-Hilbert approaches to scattering problems [10]. On the physical side, integrable models describe dark solitons in BECs and rational solutions of the nonlocal Davey-Stewartson equation [11], while studies on the special Toda lattice, supersymmetric integrable systems, and Hopf link invariants continue to expand the field. The connection between BEC dynamics and integrable systems has drawn attention from various directions. The reduction of quasi-one-dimensional BEC models to the KdV equation under different scaling regimes was reported in [12], where numerical verification of soliton propagation was the main focus, the derivation of a variable-coefficient KdV equation from BECs with time-dependent scattering length was explored in [13], and direct asymptotic reductions from the one-dimensional Gross-Pitaevskii (GP) equation to the KdV equation were studied in [14], with emphasis on the validity regime of the approximation. In the context of dark solitons and their stability, the modified KdV hierarchy under non-zero boundary conditions has been analyzed in [15], where nonlinear orbital stability for dark solitons was established. Despite these efforts, most existing studies either focused on the reduction procedure itself or on numerical validation, and a systematic analysis of the integrable structure of the reduced system, including nonlocal symmetries, Darboux transformations, Bäcklund transformations, and infinite conservation laws, is less common. In this paper, starting from the (3+1)-dimensional GP equation, we reduce it to the standard KdV equation under the long-wave approximation through a multi-scale perturbation expansion, and then carry out an analysis of the integrable structure of the reduced system. Compared with direct numerical simulations of the full GP equation, the asymptotic reduction method proposed in this paper has obvious advantages. For the (3+1)-dimensional GP equation, the widely used numerical schemes include the split-step Fourier spectral method, the backward-forward Euler spectral method, and the Newton-Raphson iteration method. These methods typically require discretization of the three-dimensional space and time marching, with the computational cost rapidly increasing with the number of grid points, and the results are sensitive to the size of the computational domain and the treatment of boundary conditions. Contrastingly, the reduced KdV equation is (1+1)-dimensional, and its numerical solution can be quickly obtained on a standard workstation or even in real time on a personal computer. At the analytical level, the reduced equation admits multi-soliton exact solutions via the Hirota bilinear method, thus providing a benchmark for verifying the accuracy of numerical results. More importantly, the reduction procedure itself reveals the dominant balance between the nonlinear term and the third-order dispersion term under the long-wave condition, which is the fundamental physical mechanism underlying the existence of soliton structures. This information is often hidden in large data sets in pure numerical simulations and is difficult to directly identify. In this regard, the reduction method presented here has clear advantages in three aspects: computational efficiency, analytical tractability, and physical insight.

In the analysis of the integrable structure, we first construct multi-soliton solutions via the Hirota bilinear method. Then, we present the positon solutions of the KdV equation, which originate from the solution family of the variable-coefficient KdV equation proposed by Gao and Tian [16], and reduce them to the constant-coefficient case, thereby analyzing their weakly localized oscillatory structures and superreflectionless properties. On this basis, by coupling the tau function of a single soliton and that of a first-order positon via the Wronskian determinant, we construct the mixed interaction solution between the soliton and the positon. In terms of symmetry and the transformation theory, with the aid of the Lax pair and prolongation structure, we give the explicit expression of the nonlocal symmetry, establish the Darboux transformation and the Bäcklund transformation, and derive infinitely many conservation laws through the Riccati expansion. In addition, we prove that within the Wronskian bilinear integrable framework, all orders of global conserved quantities for the soliton-positon composite wave satisfy an exact superposition relation, thereby demonstrating the stability of the global conserved quantities during the coupling process of the two types of nonlinear excitations. In nondimensionalizing the GP equation, we adopt the fixed scaling scheme with $g = -2$, which corresponds to a specific fixed s-wave scattering length. All analyses are carried out under this fixed interaction strength, excluding cases where the scattering length is tuned by external fields (e.g., via Feshbach resonances) [17]. Additionally, this asymptotic method can be extended to the long-wave approximation analyses of other quantum fluid systems, such as superfluid Fermi gases or polariton condensates with different dispersion relations. The paper is organized as follows: Section 2 presents the asymptotic reduction procedure; Section 3 constructs and analyzes the soliton solutions, positon solutions, and their mixed interaction solutions of the KdV equation; Section 4 discusses the integrable structures of the KdV equation, including nonlocal symmetries, Darboux transformations, Bäcklund transformations, infinite conservation laws, and the superposition relation of conserved quantities for the soliton-positon composite wave; and Section 5 concludes the paper.

2. Evolution model of the equation of motion

In the context of nonlinear quantum hydrodynamics and BECs, the macroscopic quantum behavior of ultracold bosonic gases is governed by the GP equation. In its dimensional form, this nonlinear Schrödinger equation reads as follows:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + g|\Psi|^2\Psi, \tag{2.1}$$

where $\Psi(\mathbf{r}, t)$ is the condensate wave function, m is the mass of a single boson, $\hbar$ is the reduced Planck constant, and $g = 4\pi\hbar^2 a_s/m$ is the coupling constant determined by the s-wave scattering length a_s. With appropriate units and wave function rescaling, Eq (2.1) reduces to the following dimensionless form:

$$i\frac{\partial\Psi}{\partial t} = -\nabla^2\Psi - 2|\Psi|^2\Psi. \tag{2.2}$$

Here, $g = -2$ corresponds to a specific fixed scattering length a_s within a particular renormalization scheme. All results in this paper are obtained under this fixed interaction strength. It should be emphasized that this choice does not affect the integrable structure of the reduced KdV equation; changing a_s only alters the amplitude and characteristic width of the solitons through a scale transformation, without changing the standard form of the KdV equation or its integrable properties. The generalization to a varying scattering length will be addressed separately in future work. The negative sign in the nonlinear term corresponds to repulsive interactions ($a_s > 0$). The dimensionless equation (2.2) is used throughout this paper. The quantity $n = |\Psi|^2$ denotes the particle number density, satisfying $\int d\mathbf{r}\,|\Psi|^2 = N$, which is the total number of bosons.

Under superflow conditions, the initial wave function along the x-axis takes the Madelung form $\Psi = \sqrt{n}e^{i\phi}$ [18]. Taking the partial time derivative of this expression and multiplying by the imaginary unit i yields the following:

$$i\frac{\partial\Psi}{\partial t} = \frac{i}{2\sqrt{n}}e^{i\phi}\frac{\partial n}{\partial t} - e^{i\phi}\sqrt{n}\frac{\partial\phi}{\partial t}. \tag{2.3}$$

Substituting $\nabla^2\Psi$ and $|\Psi|^2$ into Eq (2.2) gives a coupled system for n and ϕ. Separating real and imaginary parts yields the quantum hydrodynamic formulation:

$$\begin{cases} \frac{i}{2\sqrt{n}}\frac{\partial n}{\partial t} = -2i\nabla\sqrt{n}\nabla\phi - i\sqrt{n}\nabla^2\phi, \\ -\sqrt{n}\frac{\partial\phi}{\partial t} = -\nabla^2\sqrt{n} + \sqrt{n}(\nabla\phi)^2 - 2n\sqrt{n}. \end{cases} \tag{2.4}$$

Simplification of Eq (2.4) leads to the following system:

$$\begin{cases} \dfrac{\partial n}{\partial t} + 2\nabla(n\nabla\phi) = 0, \\ \dfrac{\partial\phi}{\partial t} - \dfrac{1}{\sqrt{n}}\nabla^2\sqrt{n} + (\nabla\phi)^2 - 2n = 0. \end{cases} \tag{2.5}$$

The first equation follows from the imaginary part and corresponds to particle number conservation; the second equation follows from the real part and governs the phase evolution. To investigate analytical solutions, trial solutions of separated variable form are introduced.

The hypothesis is as follows:

$$\begin{cases} \sqrt{n} = A(x,y,z)G(z), \\ \phi = -Pt + \varphi(x,y,z), \end{cases} \tag{2.6}$$

which assumes

$$-\frac{\partial^2 G}{\partial z^2} = VG. \tag{2.7}$$

The longitudinal localized eigenfunction is chosen as $G(z) = e^{-z^2}$, which satisfies $0 < G(z) < 1$ for all finite z and decays toward zero at infinity. Substituting the Gaussian profile yields the coordinate-dependent potential $V(z) = 4z^2 - 2$. Substituting Eqs (2.6) and (2.7) into the system of Eq (2.5) yields the following:

$$\begin{cases} \frac{\partial A}{\partial t} + \frac{\partial A}{\partial z}\frac{\partial\varphi}{\partial z} + A\frac{\partial^2\varphi}{\partial z^2} = 0, \\ -PA - \frac{\partial^2 A}{\partial z^2} + \left[\frac{\partial\varphi}{\partial t} - \left(\frac{\partial\varphi}{\partial z}\right)^2\right]A - \sqrt{2}A^3 = 0. \end{cases} \tag{2.8}$$

Now a multi-scale perturbation analysis is applied to Eq (2.8). Dependent variables are expanded in powers of a small parameter ε and space-time coordinates are extended to multiple scales. The following dimensionless transformation and perturbation expansion are assumed:

$$\begin{cases} A = u_0 + \varepsilon^2(a^0 + \varepsilon^2 a^1 + \cdots), \\ \varphi = \varepsilon(\varphi^0 + \varepsilon^2\varphi^1 + \cdots), \\ \zeta = \varepsilon(z - ct), \\ \tau = \varepsilon^3 t. \end{cases} \tag{2.9}$$

The stretched slow coordinate is denoted as ζ. Substituting Eq (2.9) into the first equation of Eq (2.8) and computing $\frac{\partial A}{\partial t}$, $\frac{\partial A}{\partial z}$, $\frac{\partial \varphi}{\partial z}$ yields the following:

$$\begin{aligned} &-c\varepsilon^3\tfrac{\partial a^0}{\partial\zeta} + \varepsilon^5\tfrac{\partial a^0}{\partial\tau} - c\varepsilon^5\tfrac{\partial a^1}{\partial\zeta} + \varepsilon^7\tfrac{\partial a^1}{\partial\tau} - 2\left(\varepsilon^5\tfrac{\partial a^0}{\partial\zeta}\tfrac{\partial\varphi^0}{\partial\zeta} + \varepsilon^9\tfrac{\partial a^1}{\partial\zeta}\tfrac{\partial\varphi^1}{\partial\zeta} + \varepsilon^7\tfrac{\partial a^0}{\partial\zeta}\tfrac{\partial\varphi^1}{\partial\zeta} + \varepsilon^7\tfrac{\partial a^1}{\partial\zeta}\tfrac{\partial\varphi^0}{\partial\zeta}\right) \\ &-2(u_0 + \varepsilon^2(a^0 + \varepsilon^2 a^1 + \cdots))\left(\varepsilon^2\tfrac{\partial^2\varphi^0}{\partial\zeta^2} + \varepsilon^4\tfrac{\partial^2\varphi^1}{\partial\zeta^2}\right) = 0. \end{aligned} \tag{2.10}$$

Denote $\alpha^j = -\varepsilon^5\frac{\partial a^0}{\partial\tau} + 2\varepsilon^7\frac{\partial a^0}{\partial\zeta}\frac{\partial\varphi^0}{\partial\zeta} + 2\varepsilon^9\frac{\partial a^1}{\partial\zeta}\frac{\partial\varphi^1}{\partial\zeta}$. Equation (2.10) becomes the following:

$$c\frac{\partial a^j}{\partial\zeta} + 2u_0\frac{\partial^2\varphi^j}{\partial\zeta^2} = \alpha^j. \tag{2.11}$$

For $j = 0$, Eq (2.11) gives $c\frac{\partial a^0}{\partial\zeta} + 2u_0\frac{\partial^2\varphi^0}{\partial\zeta^2} = 0$. Substituting this into Eq (2.10) and rearranging terms produces the following:

$$\begin{aligned} &-c\frac{\partial a^0}{\partial\zeta} - 2u_0\frac{\partial^2\varphi^0}{\partial\zeta^2} - \varepsilon^2\left(-\frac{\partial a^0}{\partial\tau} + 2\frac{\partial\varphi^0}{\partial\zeta}\frac{\partial a^0}{\partial\zeta} + 2a_0\frac{\partial^2\varphi^0}{\partial\zeta^2}\right) \\ &= \varepsilon^2\left(c\frac{\partial\alpha^1}{\partial\zeta} + 2U_0\frac{\partial^2\varphi^1}{\partial\zeta^2}\right) + \varepsilon^4\left(\frac{\partial a^1}{\partial\tau} + 2\frac{\partial\varphi^1}{\partial\zeta}\frac{\partial\alpha^0}{\partial\zeta} + 2\frac{\partial\varphi^0}{\partial\zeta}\frac{\partial a^1}{\partial\zeta} + 2a_0\frac{\partial^2\varphi^1}{\partial\zeta^2} + 2a_1\frac{\partial^2\varphi^0}{\partial\zeta^2}\right) \\ &\quad + 2\varepsilon^6\left(\frac{\partial\varphi^1}{\partial\zeta}\frac{\partial a^1}{\partial\zeta} + a_1\frac{\partial^2\varphi^1}{\partial\zeta^2}\right). \end{aligned} \tag{2.12}$$

Substituting Eq (2.9) into the second equation of Eq (2.8) and computing $\frac{\partial A}{\partial z}$, $\frac{\partial\varphi}{\partial z}$, $\frac{\partial\varphi}{\partial t}$ gives the following:

$$\begin{aligned} &\varepsilon(z-ct)(u_0 + \varepsilon^2 a^0 + \varepsilon^4 a^1 + \cdots) \\ &+\left(u_0\varepsilon^4\tfrac{\partial\varphi^0}{\partial\tau} + u_0\varepsilon^6\tfrac{\partial\varphi^1}{\partial\tau} - u_0 c\varepsilon^2\tfrac{\partial\varphi^0}{\partial\zeta} - u_0 c\varepsilon^4\tfrac{\partial\varphi^1}{\partial\zeta} - a^0 c\varepsilon^4\tfrac{\partial\varphi^0}{\partial\zeta} - ca^0\varepsilon^6\tfrac{\partial\varphi^1}{\partial\zeta} + \cdots\right) \\ &+\varepsilon^3\tfrac{\partial^2 a^0}{\partial\zeta^2} + \varepsilon^3\tfrac{\partial^2 a^0}{\partial\tau^2} + \varepsilon^5\tfrac{\partial^2 a^1}{\partial\zeta^2} + \varepsilon^5\tfrac{\partial^2 a^1}{\partial\tau^2} + (u_0 + \varepsilon^2 a^0 + \varepsilon^4 a^1 + \cdots)\left(\varepsilon^2\tfrac{\partial\varphi^0}{\partial\zeta} + \varepsilon^4\tfrac{\partial\varphi^1}{\partial\zeta}\right)^2 = 0. \end{aligned} \tag{2.13}$$

Denote part of Eq (2.13) as β^j, which yields the following:

$$cu_0\frac{\partial\varphi^{(j)}}{\partial\zeta} = \beta^j. \tag{2.14}$$

Removing the first-order derivative with respect to ζ from Eq (2.13) eliminates ε^2. Neglecting higher-order terms ε^4 and ε^6, and dividing by ε^2 transforms Eq (2.13) into the following:

$$\frac{a_0c^2}{2u_0}\frac{\partial a^0}{\partial\zeta} + \frac{\partial^3 a^0}{\partial\zeta^3} - 6\sqrt{2}u_0 a^0\frac{\partial a^0}{\partial\zeta} = 0. \tag{2.15}$$

From Eq (2.11), $c\frac{\partial a^0}{\partial \zeta} + 2u_0\frac{\partial^2 \varphi^0}{\partial \zeta^2} = 0$. Substituting this into $-\frac{\partial a^0}{\partial \tau} + 2\frac{\partial \varphi^0}{\partial \zeta}\frac{\partial a^0}{\partial \zeta} + 2a_0\frac{\partial^2 \varphi^0}{\partial \zeta^2} = 0$ yields the following:

$$\frac{\partial a^0}{\partial \zeta} = -\frac{u_0}{2a_0 c}\frac{\partial a^0}{\partial \tau}. \tag{2.16}$$

Substituting Eq (2.16) into Eq (2.15) gives an evolution equation with a^0 as the dependent variable:

$$\frac{a_0 c^2}{2u_0}\frac{\partial a^0}{\partial \zeta} + \frac{\partial^3 a^0}{\partial \zeta^3} + 3\sqrt{2}\frac{u_0^2}{c}\frac{\partial a^0}{\partial \tau} = 0. \tag{2.17}$$

Equation (2.17) manifestly assumes the form of the classical KdV equation, characterized by the coupling of nonlinearity and higher-order dispersion.

Transforming it to standard form proceeds as follows. Dividing both sides by $3\sqrt{2}\frac{u_0^2}{c}$ gives the following:

$$\frac{\partial a^0}{\partial \tau} + \frac{c^3}{6\sqrt{2}u_0^3}a^0\frac{\partial a^0}{\partial \zeta} + \frac{c}{3\sqrt{2}u_0^2}\frac{\partial^3 a^0}{\partial \zeta^3} = 0. \tag{2.18}$$

Apply the following scaling transformation:

$$\begin{cases} a^0 = AU, \\ \zeta = Bx, \\ \tau = Ct. \end{cases} \tag{2.19}$$

After substituting Eq (2.19) into Eq (2.18), multiplying by $\frac{C}{A}$ yields the following:

$$U_t + \frac{Ac^3 C}{6\sqrt{2}u_0^3 B}UU_x + \frac{cC}{3\sqrt{2}B^3 u_0^2}U_{xxx} = 0. \tag{2.20}$$

Set $\frac{c^3 C}{6\sqrt{2}u_0^3}\frac{A}{B} = 6$ and $\frac{cC}{3\sqrt{2}u_0^2}\frac{1}{B^3} = 1$. These relations determine A and B in terms of C. Taking $C = 1$ (so $\tau = t$) gives the following:

$$A = \frac{36\sqrt{2}u_0^3}{c^3}\cdot\left(\frac{c}{3\sqrt{2}u_0^2}\right)^{\frac{1}{3}}, \quad B = \left(\frac{C}{3\sqrt{2}u_0^2}\right)^{\frac{1}{3}}, \quad C = 1. \tag{2.21}$$

With these choices, $U(x,t)$ satisfies the standard KdV equation:

$$U_t + 6UU_x + U_{xxx} = 0. \tag{2.22}$$

At this point, through the multi-scale perturbation expansion, the original BEC system Eq (2.2) can be reduced to the standard KdV equation (2.22) in the long-wave limit. The advantage of this reduction is as follows: the original GP equation (2.2) is a (3+1)-dimensional nonlinear partial differential equation, whose numerical solution requires spatial discretization and time marching (e.g., the Fourier spectral method combined with fourth-order Runge-Kutta integration), and the computational results considerably depend on the domain size and resolution. Contrastingly, the reduced KdV equation is a (1+1)-dimensional equation. It admits multi-soliton exact solutions via the Hirota bilinear method (see Section 3), and can also be solved numerically with high accuracy using spectral methods within a relatively short computation time. Furthermore, the reduction procedure is able to reveal the dominant dynamics of the system under the long-wave condition, namely the balance between the nonlinear term and the third-order dispersion term, which underlies the existence of soliton structures. This information is often not easily extracted directly from a large amount of data in pure numerical simulations. In this regard, the present reduction method not only reduces the computational cost but also helps to deepen the understanding of the relevant physical mechanisms.

3. Construction and property analysis of soliton solutions and positon solutions

Soliton solutions are traveling wave solutions of nonlinear evolution equations that maintain their waveform and velocity during propagation, only suffering a phase shift after collisions. This stability arises from the balance between nonlinear and dispersive effects, and solitons appear in models such as the KdV and nonlinear Schrödinger equations. Gao and Tian [16] introduced positons as weakly localized solutions with positive spectral eigenvalues embedded in the continuous spectrum, thereby exhibiting superreflectionless behavior. Within the KdV framework, this chapter constructs soliton, positon, and mixed interaction solutions, and analyzes their waveform structures and dynamical properties.

3.1. Soliton solutions of the KdV equation

To construct soliton solutions of Eq (2.22), we employ the standard Hirota bilinear method. Introduce the following transformation:

$$U = 2(\ln f)_{xx}, \tag{3.1}$$

where $U(x,t)$ is the wave amplitude function, and $f(x,t)$ is an auxiliary function to be determined. By substituting Eq (3.1) into Eq (2.22), the equation is converted into the following bilinear form:

$$(D_tD_x + D_x^4)f \cdot f = 0. \tag{3.2}$$

Expanding f as a power series in a small parameter ϵ leads to the following:

$$f = 1 + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \cdots, \tag{3.3}$$

where $f^{(m)}(x,t)$ $(m = 1, 2, \ldots)$ are the expansion coefficient functions. Substituting Eq (3.3) into Eq (3.2) and equating coefficients of like powers of ϵ yields a system of linear recurrence relations for $f^{(m)}$. Solving these relations order by order, the series truncates to a finite number of terms. Taking $\epsilon = 1$, one obtains the following compact n-soliton solution:

$$U = 2\left[\ln\left(\sum_{\mu_j=0,1}\exp\left(\sum_{j=1}^{n}\mu_j\xi_j + \sum_{1\le j<l\le n}\mu_j\mu_l A_{jl}\right)\right)\right]_{xx}, \tag{3.4}$$

where n is the number of solitons, $\mu_j \in \{0,1\}$ are summation indices, $\xi_j = k_jx - k_j^3t + \xi_j^{(0)}$ is the phase function of the j-th soliton, k_j is the wave number, $\xi_j^{(0)}$ is the initial phase, and $A_{jl} = \left(\frac{k_j-k_l}{k_j+k_l}\right)^2$ is the interaction coefficient between the j-th and l-th solitons.

For $n = 1$, Eq (3.4) reduces to the classical one-soliton solution

$$U = \frac{k_1^2}{2} \operatorname{sech}^2 \frac{\xi_1}{2}, \tag{3.5}$$

with $\xi_1 = k_1 x - k_1^3 t + \xi_1^{(0)}$. The profiles for $n = 2, 3$ are illustrated in Figures 1–3, which reflect the elastic, particle-like collision behaviors characteristic of integrable systems.

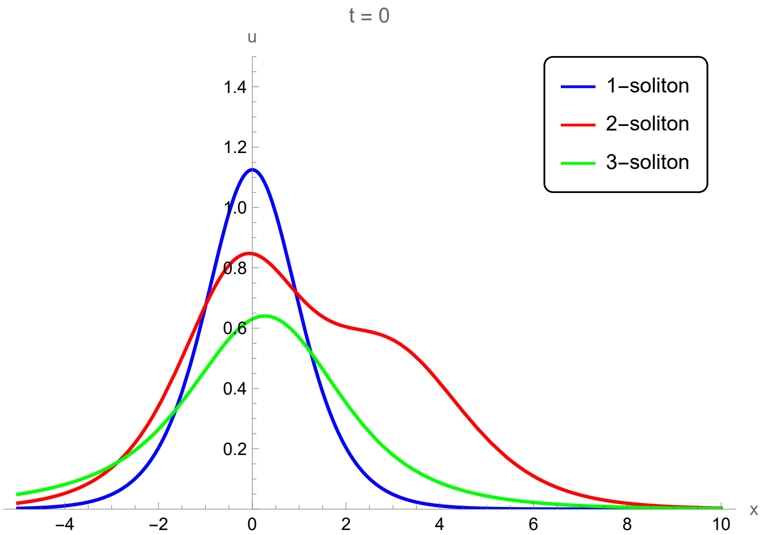

Figure 1. This figure shows the wave profiles at $t = 0$, where the horizontal axis x is the position and the vertical axis u is the wave height. The blue line (one-soliton) has a maximum of approximately 1.125 at x approximately 0, the red line (two-soliton) is a broad peak with a maximum of approximately 1.6 at x approximately 0.5, and the green line (three-soliton) has a main peak with a maximum of approximately 1.8 at x approximately 0.5, with a small shoulder peak on the left side.

The three figures show the comparison and spatiotemporal evolution of the one-soliton solution in Eq (3.5) and the n-soliton solutions in Eq (3.4) for $n = 2, 3$. Figures 1 and 2 are the waveforms at $t = 0$ and $t = 1$, respectively: the blue one-soliton always appears as a bell-shaped peak with a maximum value of approximately 1.125; the red two-soliton forms a broad peak with a maximum value of approximately 1.6 at $t = 0$ due to collision, and separates into two peaks at $t = 1$; and the green three-soliton has a main peak with a maximum value of approximately 1.8 at $t = 0$ accompanied by a small shoulder on the left, and separates into three peaks at $t = 1$. Figure 3 is the three-dimensional surface of the three-soliton solution, which shows the whole process of three solitons propagating rightward over time, colliding and overlapping near $t \approx 0$, and then separating again with phase shifts. The figures presented illustrate the nonlinear dynamical laws governing KdV soliton solutions from two complementary perspectives: parameter modulation and temporal evolution. In the parameter dimension, the soliton parameter k modulates the amplitude, characteristic width, and propagation velocity through clear scaling relations. In the evolution dimension, the soliton maintains waveform and energy integrity with uniform propagation, thereby exhibiting particle-like dynamic characteristics. These properties establish the KdV soliton as a paradigm of analytically solvable nonlinear waves in integrable systems.

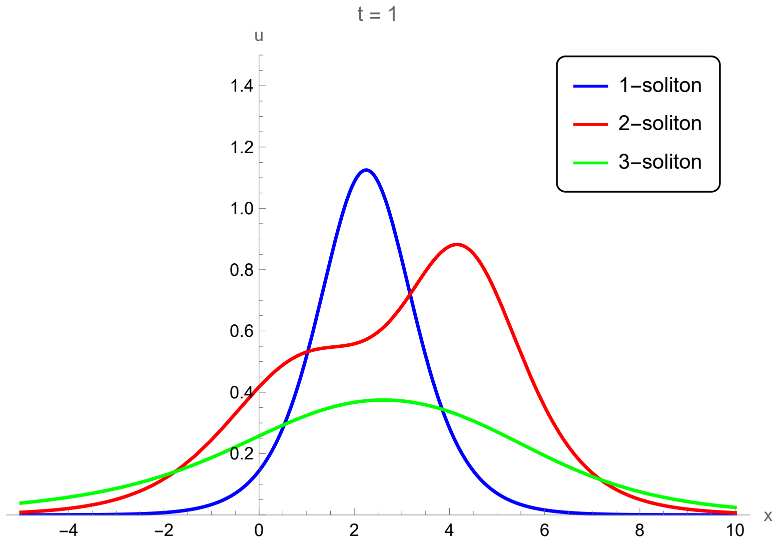

Figure 2. This figure shows three curves at $t = 1$: the blue one-soliton has a maximum value of approximately 1.125 and is located at x approximately 2.5; the red two-soliton has two peaks, with the right peak coinciding with the blue one, and the left peak approximately 0.4 at x approximately 0.3; and the green three-soliton has three peaks, with the rightmost peak coinciding with the blue one, the middle peak coinciding with the left peak of the red one, and the leftmost peak approximately 0.08 at x approximately −2.2.

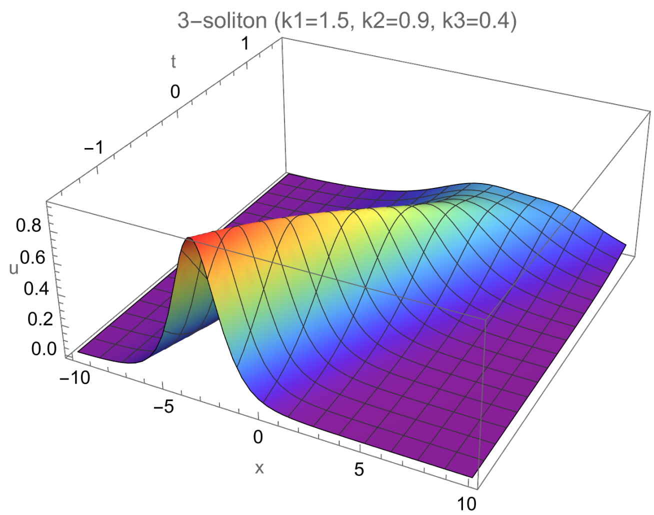

Figure 3. This three-dimensional figure shows the spatiotemporal evolution of the three-soliton solution Eq (3.4) with parameters $k_1 = 1.5$, $k_2 = 0.9$, and $k_3 = 0.4$. The horizontal axis x ranges from −10 to 10 and represents the spatial position, the vertical axis t ranges from −1.5 to 1.5 and represents time, and the height axis u represents the wave amplitude. The colors range from purple to red, corresponding to u values from small to large.

3.2. Positon solutions of the KdV equation

Gao and Tian [16] proposed a family of positonic, exact analytic solutions for the variable-coefficient KdV equation

$$u_t + a(t)uu_x + b(t)u_{xxx} = 0, \tag{3.6}$$

where $u(x,t)$ is the wave function, x and t are the spatial and temporal coordinates, respectively, and $a(t)$ and $b(t)$ are the coefficients of the nonlinear and dispersive terms, respectively. The positonic, exact analytic solutions read as follows:

$$\begin{aligned} u(x,t) = &-\frac{12\tau^2\Lambda b(t)}{a(t)\left[\chi_1\cos\mathcal{H}\sqrt{\Lambda^2-\omega(t)^2}+\omega(t)\sin\mathcal{H}+3\chi_2\tau^3\Lambda\int b(t)dt+\chi_2\tau x\Lambda\right]^2} \\ &\times\left\{2\Lambda+\chi_1\chi_2\tau x\cos\mathcal{H}\sqrt{\Lambda^2-\omega(t)^2}-2\chi_1\chi_2\sqrt{\Lambda^2-\omega(t)^2}\sin\mathcal{H}\right. \\ &\left.+\chi_2\omega(t)\left[2\cos\mathcal{H}+\tau x\sin\mathcal{H}+3\tau^3\sin\mathcal{H}\int b(t)dt\right]+3\chi_1\chi_2\tau^3\cos\mathcal{H}\sqrt{\Lambda^2-\omega(t)^2}\int b(t)dt\right\}, \end{aligned} \tag{3.7}$$

with

$$\mathcal{H}(x,t) = \tau x + \chi_1 \arcsin\left[\frac{\omega(t)}{\Lambda}\right] + \tau^3\int b(t)dt. \tag{3.8}$$

In the above expressions, τ is a constant, $\Lambda > 0$ is a constant, $\omega(t)$ is a function of time, and $\chi_1 = \pm1$, $\chi_2 = \pm1$.

Taking

$$a(t) = 6, \quad b(t) = 1, \quad \omega(t) = -\chi_2\Lambda, \quad \tau = 2\chi,$$

Eq (3.7) reduces to a solution of the constant-coefficient KdV equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{3.9}$$

namely the first-order positon solution

$$\begin{aligned} u(x,t) = &\frac{32\chi^2\sin[\chi(4t\chi^2+x_1+x)]}{\{\sin[2\chi(4t\chi^2+x_1+x)]-2\chi(12t\chi^2+y_1+x)\}^2} \\ &\times\left\{\chi(12t\chi^2+y_1+x)\cos[\chi(4t\chi^2+x_1+x)]-\sin[\chi(4t\chi^2+x_1+x)]\right\}, \end{aligned} \tag{3.10}$$

where $\chi > 0$, and x_1, y_1 are arbitrary constants.

Figures 4–6 illustrate the evolution of the function described in Eq (3.10) over different spatiotemporal ranges. Figure 4 shows the overall morphology of the function over a larger spatial range $x \in [-12, 12]$ and time range $t \in [0, 1]$. Figures 5 and 6 focus on a more localized region $x \in [-5, 5]$, $t \in [-2, 2]$, where Figure 5 exhibits a distinct valley at the center $(0, 0)$, with symmetric oscillations and rapid outward decay, while Figure 6 displays a sharp symmetric peak at the center $(0, 0)$, also rapidly decaying to the background level. Together, these three figures reflect the localized waveform characteristics of the positon solution from different perspectives.

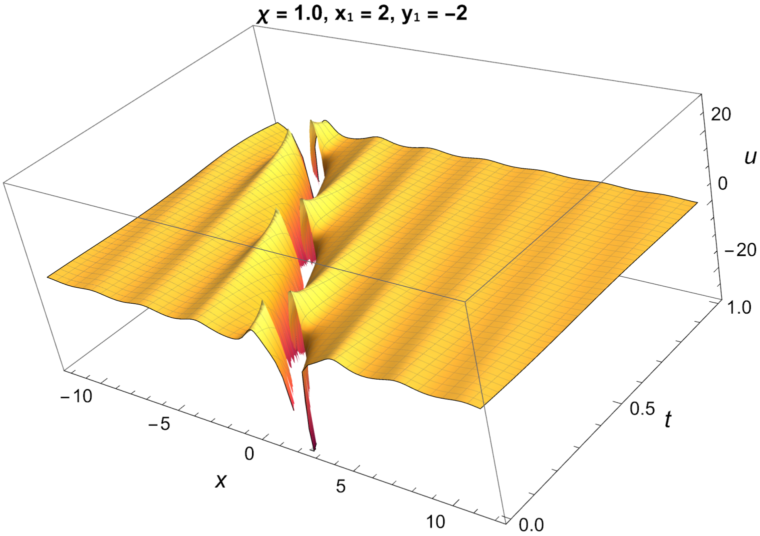

Figure 4. This surface plot shows the evolution of the positon solution Eq (3.10) with respect to the spatial coordinate x (horizontal axis, range -12 to 12) and time t (vertical axis, range 0 to 1).

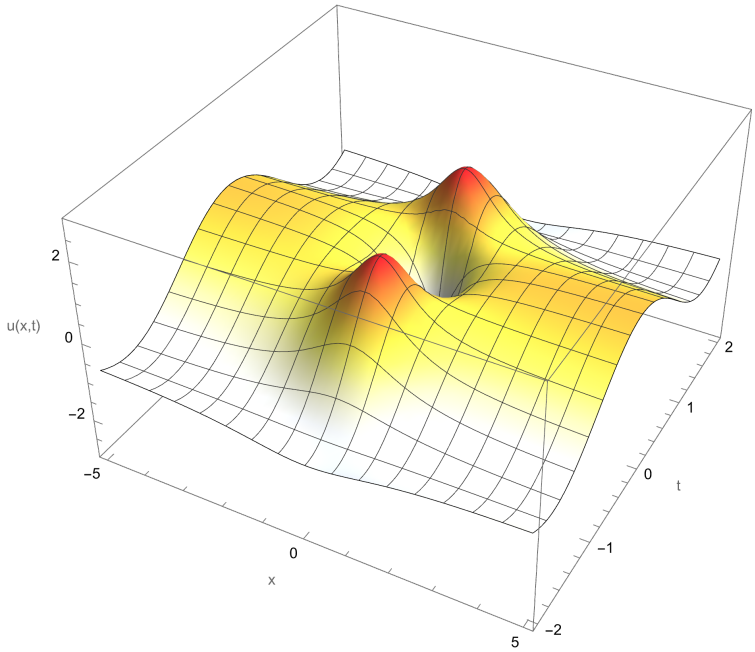

Figure 5. This figure depicts the evolution of $u(x,t)$ in the range $x \in [-5,5]$, $t \in [-2,2]$. The waveform exhibits a valley at the center $(0,0)$, with symmetric oscillations and rapid decay outward.

The positon solution further reveals an underlying nonlocal symmetry inherent to the KdV equation. The local pulse response described by such solutions inherently relates to the spectral structure of the equation and the evolution of scattering data.

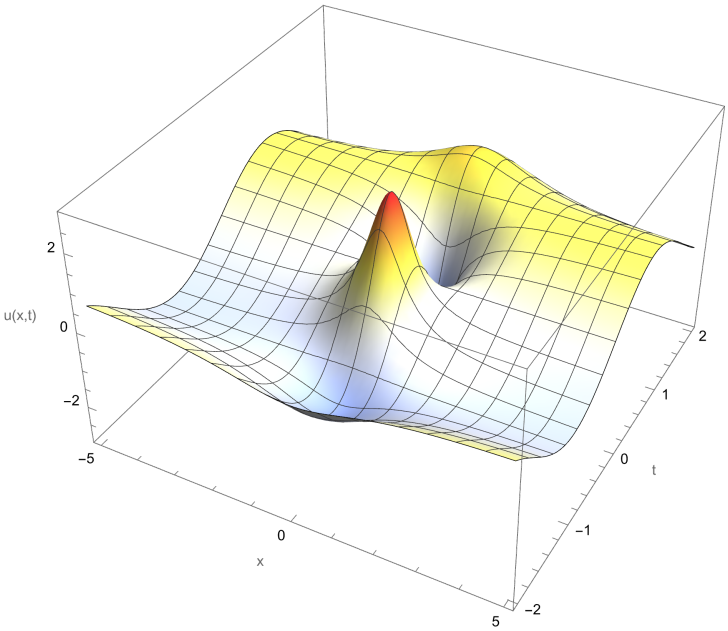

Figure 6. This figure shows the evolution of $u(x,t)$ with respect to space x and time t. The function displays a sharp symmetric peak at the center $(0,0)$, rapidly decaying to the background level outward.

3.3. Construction and derivation of the first-order positon-soliton mixed interaction solution

By combining Eq (3.5), Eq (3.10), and the Hirota transformation $u = 2(\ln f)_{xx}$, the tau function corresponding to a single soliton is obtained as

$$f_1 = 1 + e^{\xi_1}, \quad \xi_1 = k_1 x - 4k_1^3 t + \delta_1, \tag{3.11}$$

and the tau function for the first-order positon solution is constructed as

$$f_p = \tau \cos\theta - \sin\theta, \tag{3.12}$$

with

$$\tau = \chi(12\chi^2 t + y_1 + x), \qquad \theta = \chi(4\chi^2 t + x_1 + x),$$

where χ denotes the resonant eigenvalue associated with the positon solution, x_1 and y_1 are two independent translation constants, τ represents the linear combination of spatiotemporal variables, and θ characterises the oscillatory phase of the positon solution.

To construct the mixed interaction solution between the soliton and the positon wave, we assume that the soliton eigenvalue k_1 and the positon resonant eigenvalue χ are distinct, so as to avoid singular degeneracy caused by resonant coupling. By employing the Wronskian determinant to couple the two tau functions, we define the mixed tau function as follows:

$$f_{\text{mix}} = \begin{vmatrix} f_1 & f_p \\ \partial_x f_1 & \partial_x f_p \end{vmatrix} = f_1 \partial_x f_p - f_p \partial_x f_1. \tag{3.13}$$

By substituting Eqs (3.11) and (3.12) into Eq (3.13), we obtain the explicit form of the mixed tau function:

$$f_{\text{mix}} = -\chi\tau(1+e^{\xi_1})\sin\theta - k_1 e^{\xi_1}(\tau\cos\theta - \sin\theta). \tag{3.14}$$

This function simultaneously contains exponential soliton terms and trigonometric positon-wave terms, thus serving as the fundamental kernel function to describe the interaction between the two types of waves.

Taking the first-order spatial derivative of Eq (3.14) and applying the product rule term by term yields the following:

$$\partial_x f_{\text{mix}} = f_1 \partial_{xx} f_p - f_p \partial_{xx} f_1, \tag{3.15}$$

where the cross terms $\partial_x f_1 \partial_x f_p$ cancel each other, giving the simplified first-order derivative expression. Substituting Eqs (3.11) and (3.12) into Eq (3.15) gives the following:

$$\partial_x f_{\text{mix}} = -\chi^2(1+e^{\xi_1})(\tau\cos\theta + \sin\theta) - k_1^2 e^{\xi_1}(\tau\cos\theta - \sin\theta). \tag{3.16}$$

Furthermore, taking the second-order spatial derivative, we obtain the following:

$$\partial_{xx} f_{\text{mix}} = -\chi^3(2\cos\theta - \tau\sin\theta) + e^{\xi_1}\left[-k_1\chi^2(\tau\cos\theta + \sin\theta) + k_1^2\chi\tau\sin\theta - k_1^3(\tau\cos\theta - \sin\theta)\right]. \tag{3.17}$$

Using the logarithmic second-order differential identity

$$(\ln f)_{xx} = \frac{f f_{xx} - (f_x)^2}{f^2},$$

and substituting Eq (3.14), Eq (3.16), and Eq (3.17) into the Hirota transformation, we finally obtain the mixed interaction solution of a single soliton and a positon wave:

$$u_{\text{mix}} = 2 \cdot \frac{f_{\text{mix}} \cdot \partial_{xx} f_{\text{mix}} - (\partial_x f_{\text{mix}})^2}{f_{\text{mix}}^2}. \tag{3.18}$$

The numerator of this solution can be decomposed into three parts: purely exponential soliton terms, purely trigonometric positon-wave terms, and nonlinear cross terms coupling exponentials and trigonometric functions. Among these, the cross-coupling terms represent the distinctive structure arising from the nonlinear collision between the soliton and the positon wave, and embody the dynamical characteristics of their mutual interaction.

This section presents the soliton solutions and positon solutions of the KdV equation, constructs their mixed interaction solutions, and analyzes the corresponding properties. The soliton solutions are derived via the Hirota bilinear method. Through the logarithmic transformation, the KdV equation is converted into a bilinear form, and the general expression for multi-soliton solutions is obtained by combining perturbation expansion with series truncation. The wave profiles and spatiotemporal evolution diagrams illustrate the elastic collision behavior during multi-soliton propagation. The positon solutions are taken from the family of solutions to the variable-coefficient KdV equation proposed by Gao and Tian [16], and their explicit expressions are given under the reduction to constant coefficients. The wave profiles reveal their weakly localized oscillatory structures and superreflectionless properties. The mixed solutions are constructed by coupling the one-soliton tau function and the first-order positon tau function via the Wronskian determinant. The explicit expression contains purely exponential terms, purely trigonometric terms, and nonlinear cross terms between the two. The cross terms reflect the energy exchange and phase modulation during the interaction between the two types of heterogeneous waves.

4. Nonlocal symmetry

A symmetry analysis is an important tool to study integrable structures and construct exact solutions of nonlinear evolution equations. Unlike classical Lie point symmetries, the generators of nonlocal symmetries also depend on nonlocal information such as eigenfunctions of spectral problems. This chapter takes the KdV equation as a model to examine the internal relations among its nonlocal symmetries, Darboux transformations, Bäcklund transformations, and infinite conservation laws, and proves the superposition invariance of conserved quantities for soliton-positon composite waves based on the Wronskian coupling framework.

4.1. Nonlocal symmetries of the KdV equation

In the study of the KdV equation, a symmetry analysis serves as an important tool to reveal its integrable structure and construct exact solutions. Consider the standard form of the KdV equation:

$$U_t + 6UU_x + U_{xxx} = 0, \tag{4.1}$$

where $U = U(x,t)$ is the wave field variable. This equation can be used to describe the propagation behavior of weakly nonlinear long waves. The corresponding Lax pair can be written as follows:

$$\psi_{xx} + (U - \lambda)\psi = 0, \tag{4.2}$$

$$\psi_t + \psi_{xxx} + 3(U + \lambda)\psi = 0, \tag{4.3}$$

where ψ is an auxiliary function, and λ, μ are parameters. Considering the infinitesimal transformation $U \to U + \epsilon\sigma_1$, the symmetry σ_1 of the KdV equation must satisfy the following linearized equation:

$$\sigma_{1,t} + 6(U\sigma_1)_x + \sigma_{1,xxx} = 0. \tag{4.4}$$

Ma and Fei [19] investigated the nonlocal symmetry of the KdV equation within this framework. They assumed that σ_1 depends on U and its derivatives, as well as on the auxiliary variables ψ and ψ_x:

$$\sigma_1 = \eta(x,t,U,\psi,\psi_x)U_x + \theta(x,t,U,\psi,\psi_x)U_t - \Upsilon(x,t,U,\psi,\xi). \tag{4.5}$$

In Eq (4.5), η, θ, and Υ are the three generating functions that characterize the infinitesimal transformations of the Lie group. The term ηU_x accounts for the contribution of spatial translations, θU_t accounts for the contribution of temporal translations, and Υ appears as a gauge correction term independent of the derivatives of U, which is typically employed to accommodate more intricate symmetry operations such as scaling or gauge transformations.Substituting Eq (4.5) into Eq (4.4) yields a vector symmetry expression containing several arbitrary constants:

$$\sigma_1 = \left(\frac{c_1 x}{2} - 6c_3 t - c_5\right)U_x + \left(\frac{3c_1 t}{2} - c_4\right)U_t + c_1 U + c_2\psi\psi_x e^{-2\mu t} + c_3. \tag{4.6}$$

Here, $c_i (i = 1, \ldots, 5)$ are arbitrary constants. The terms involving c_1, c_3, c_4, c_5 correspond to classical Lie point symmetries, while the term containing c_2 represents a type of nonlocal symmetry.

By taking $c_1 = c_3 = c_4 = c_5 = 0$ and $c_2 = 1$, Ma and Fei [19] obtained a simplified nonlocal symmetry expression:

$$\sigma_{\text{nonlocal}} = \psi\psi_x e^{-2\mu t}. \tag{4.7}$$

Thus, the construction of this nonlocal symmetry requires the auxiliary field ψ and its spatial derivative ψ_x, thus reflecting the intrinsic structure of the Lax pair framework.

Based on the two-fold Darboux transformation, the connection between the KdV equation and its Bäcklund transformation follows. The Darboux transformation can be regarded as a special form of the Bäcklund transformation, thereby generating new solutions through translation of spectral parameters. The relationship between the solution obtained by the Darboux transformation and the original solution yields the explicit expression of the corresponding Bäcklund transformation.

4.2. Darboux transformation

In the study of nonlinear integrable systems, the Darboux transformation is one of the core tools to construct exact solutions [20]. As its basic form, the first-order Darboux transformation can generate non-trivial soliton solutions from the trivial solutions of the system through the appropriate selection of spectral parameters and seed solutions, while keeping the spectral structure of the Lax pair unchanged.Specifically, given a particular solution of the linear spectral problem and its corresponding spectral parameter, the gauge transformation of the solution can be realized by constructing the Darboux matrix, thereby deriving the expression of the new potential function [21]. This method not only provides a rigorous analytical construction approach for soliton dynamics, but also lays a foundation for further research on higher-order rogue waves and multi-soliton interactions, and has important application value in complex nonlinear systems such as the Aizhan-Gudekli-Nurshuak-Zhanbota model [21].

If ψ is a solution of Eq (4.7) at $\lambda = \lambda_1$, then the transformation

$$\tilde{\psi} = \psi_x - \frac{\psi_{1,x}}{\psi_1}\psi,$$

such that $\tilde{\psi}$ satisfies $\tilde{\psi}_{xx} + (\lambda - \tilde{u})\tilde{\psi} = 0$, where $\tilde{u} = u - 2\frac{d}{dx}\left(\frac{\psi_{1,x}}{\psi_1}\right) = u - 2\partial_x^2(\ln\psi_1)$. Proof: Let $v = \frac{\psi_{1,x}}{\psi_1}$, then $v_x + v^2 = u - \lambda_1$. Substitute into $\tilde{\psi}_{xx} + (\lambda - \tilde{u})\tilde{\psi}$, and using $\tilde{u} = u - 2v_x$ gives the following:

$$\begin{aligned}\tilde{\psi}_{xx} + (\lambda - \tilde{u})\tilde{\psi} = &[u_x - v_{xx} - v(u-\lambda)]\,\psi + (u - \lambda - 2v_x)\psi_x \\ &+ (\lambda - u + 2v_x)(\psi_x - v\psi).\end{aligned} \tag{4.8}$$

From $v_x + v^2 = u - \lambda_1$, we have $(u-\lambda_1)_x = u_x$. Therefore, $\tilde{\psi}_{xx} + (\lambda - \tilde{u})\tilde{\psi} = 0$, which completes the proof.

4.3. Crum's Theorem

Crum's theorem generalizes the Darboux transformation to n iterations [22]. Unlike the simple superposition of first-order transformations, the n-fold transformed potential can be written in a closed form involving the Wronskian determinant of the n eigenfunctions.

Let $\psi_1, \psi_2, \ldots, \psi_n$ be solutions of the spectral problem

$$\psi_{xx} + (U - \lambda_i)\psi = 0,$$

corresponding to n distinct spectral parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$. After applying n successive first-order Darboux transformations, the new potential is given by the following:

$$U^{(n)} = U - 2\partial_x^2 \ln W(\psi_1, \psi_2, \ldots, \psi_n),$$

where $W(\psi_1, \psi_2, \ldots, \psi_n)$ denotes the Wronskian determinant

$$W(\psi_1, \psi_2, \ldots, \psi_n) = \det \begin{pmatrix} \psi_1 & \psi_2 & \cdots & \psi_n \\ \partial_x \psi_1 & \partial_x \psi_2 & \cdots & \partial_x \psi_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial_x^{n-1} \psi_1 & \partial_x^{n-1} \psi_2 & \cdots & \partial_x^{n-1} \psi_n \end{pmatrix}.$$

The fact that the n-fold transformed potential can be expressed so compactly as the logarithmic derivative of a single Wronskian depends on the compatibility of the Darboux transformations at different spectral parameters and the properties of Wronskians. This result is an important foundation to construct multi-soliton solutions and study the algebraic structure of integrable systems.

4.4. Second-order Darboux transformation

For the case $n = 2$, the Wronskian simplifies to the following:

$$W(\psi_1, \psi_2) = \psi_1 \psi_{2,x} - \psi_2 \psi_{1,x}.$$

Using the spectral equations $\psi_{i,xx} = (U - \lambda_i)\psi_i$ for $i = 1, 2$, the explicit expression for the second-order transformed potential can be written. A direct calculation yields the following [21, 22]:

$$\tilde{U} = U - 2\left[\frac{(\lambda_1 - \lambda_2)\sigma_x}{W} - \frac{(\lambda_1 - \lambda_2)^2 \sigma^2}{W^2}\right], \qquad \text{where } \sigma = \psi_1 \psi_2. \tag{4.9}$$

This expression shows that the second-order transformation is not a simple superposition of two first-order transformations; the cross term proportional to σ^2/W^2 arises from the interaction between the two eigenfunctions.

Based on the two-fold Darboux transformation, the connection between the KdV equation and its Bäcklund transformation follows. The Darboux transformation can be regarded as a special form of the Bäcklund transformation, thereby generating new solutions through translation of spectral parameters. The relationship between the solution obtained by the Darboux transformation and the original solution yields the explicit expression of the corresponding Bäcklund transformation.

4.5. Bäcklund transformation

The Bäcklund transformation provides a powerful algebraic framework to generate new solutions to nonlinear evolution equations from known ones [23]. Based on the twofold Darboux transformation, the Bäcklund transformation for the KdV equation can be established. Starting from the wave functions of the Lax pair, define the auxiliary function $v = \psi_{1,x}/\psi_1$, where ψ_1 is the eigenfunction corresponding to the spectral parameter λ_1. Then, v satisfies the Riccati equation

$$v_x + v^2 = u - \lambda_1, \tag{4.10}$$

where $u(x,t)$ is a known solution of the KdV equation. The Darboux transformation maps u to a new solution $\tilde{u}$:

$$\tilde{u} = u - 2v_x. \tag{4.11}$$

Combining Eq (4.10) with its counterpart for $\tilde{u}$ and eliminating common terms yields the spatial part of the Bäcklund transformation:

$$u + \tilde{u} = 2v^2 + 2\lambda_1, \qquad u - \tilde{u} = 2v_x. \tag{4.12}$$

From the time evolution equation of the Lax pair, the temporal part is obtained as follows:

$$v_t = \tilde{u}_x + 2v_x(\tilde{u} + 2\lambda_1). \tag{4.13}$$

Combining Eqs (4.12) and (4.13), the complete Bäcklund transformation for the KdV equation is as follows:

$$\begin{cases} u + \tilde{u} = 2v^2 + 2\lambda_1, \\ u - \tilde{u} = 2v_x, \\ v_t = \tilde{u}_x + 2v_x(\tilde{u} + 2\lambda_1). \end{cases} \tag{4.14}$$

The compatibility of this transformation with the Lax pair guarantees the existence of an infinite hierarchy of conserved currents.

The compatibility of the Bäcklund transformation and the Lax pair implies a generation mechanism for infinitely many conserved currents. Based on this, conserved densities and currents can be recursively constructed.

4.6. Infinitely many conservation laws

The Lax pair formulation of the KdV equation provides a framework to derive its infinite conservation laws. From the spectral problem

$$(-\partial_x^2 + u)\psi = \lambda\psi, \tag{4.15}$$

where $\psi(x,t)$ is the eigenfunction of the Lax pair and λ is the spectral parameter. Setting $\lambda = k^2$ and $\psi = e^{-ikx}\chi$, with $\chi(x,k)$ the Jost function, leads to the Riccati-type equation

$$w_x + w^2 - 2ikw - u = 0, \qquad w = \chi_x/\chi. \tag{4.16}$$

Expanding w asymptotically as $k \to \infty$ in the form

$$w = \sum_{n=1}^{\infty} \frac{w_n}{(2ik)^n}, \tag{4.17}$$

and equating coefficients of like powers of k yields a recursive hierarchy for w_n. The first few coefficients are as follows:

$$w_1 = -u, \quad w_2 = -u_x, \quad w_3 = u^2 - u_{xx}, \quad w_4 = 4uu_x - u_{xxx}, \quad w_5 = -2u^3 + 6uu_{xx} + 5u_x^2 - u_{xxxx}. \tag{4.18}$$

Since the transmission coefficient $a(k) = \exp\left(\int_{-\infty}^{\infty} w\,dx\right)$ is independent of time, each $I_n = \int_{-\infty}^{\infty} w_n\,dx$ constitutes a conserved quantity. Thus, the conservation densities are as follows:

$$\rho_1 = -u, \quad \rho_2 = -u_x, \quad \rho_3 = u^2 - u_{xx}, \quad \rho_4 = 4uu_x - u_{xxx}, \quad \rho_5 = -2u^3 + 6uu_{xx} + 5u_x^2 - u_{xxxx}. \tag{4.19}$$

Under decaying boundary conditions, the physically relevant conserved integrals are

$$I_1 = -\int u\,dx, \qquad I_3 = \int u^2\,dx, \qquad I_5 = \int (-2u^3 - u_x^2)\,dx, \tag{4.20}$$

while I_2 and I_4 are total derivatives and thus vanish. This infinite hierarchy of conservation laws provides further support for the integrability of the reduced KdV equation and also lays a foundation for subsequent perturbation analyses.

4.7. Conservation law superposition invariance of soliton-positon composite waves

In the preceding sections, the exact solutions which describe the mixed interaction between first-order solitons and positon solutions were constructed via the Hirota transformation combined with the Wronskian determinant coupling method. Furthermore, by incorporating the prolongation structure theory, a recursive mapping between nonlocal symmetries and infinite conservation laws was established, thereby analyzing the integrable properties of the KdV equation from an algebraic perspective. To further investigate the global dynamical behavior of this nonlinear coupled composite wave and to verify the physical self-consistency of the multi-excitation exact solutions, the present section proposes and proves a superposition invariance law for the conserved quantities of the composite wave. The analysis demonstrates that the Wronskian bilinear coupling primarily governs the spatial distribution of localized waveforms, while only exerting a minor influence on the global conservation properties of the system.

Proposition 4.1. ***Superposition invariance of conserved quantities for soliton-positon composite waves.*** *Let u_s, u_p, and u_c be field solutions derived from the tau functions f_1, f_p, and f_{mix}, respectively, via the Hirota transformation $u = 2(\ln f)_{xx}$, where*

$$f_{\mathrm{mix}} = W(f_1, f_p) = f_1\partial_x f_p - f_p\partial_x f_1 \tag{4.21}$$

denotes the Wronskian of f_1 and f_p. Here, u_s corresponds to the pure soliton solution, u_p corresponds to the pure positon solution, and u_c corresponds to the coupled composite solution of the two. Then, for an arbitrary-order conserved density ρ_n of the KdV equation ($n \in \mathbb{N}^+$), the corresponding global conserved quantities satisfy the exact superposition relation

$$I_n[u_c] = I_n[u_s] + I_n[u_p], \tag{4.22}$$

where the conserved quantity functional is defined as

$$I_n[u] = \int_{-\infty}^{+\infty} \rho_n(u, u_x, u_{xx}, \ldots)\,dx, \tag{4.23}$$

with the integration covering the entire one-dimensional space.

Proof. The infinite conservation laws of the KdV equation can be uniformly expressed in the following divergence form:

$$\partial_t \rho_n + \partial_x J_n = 0, \tag{4.24}$$

where ρ_n is the n-th order conserved density, and J_n is the corresponding conserved flux density. The system satisfies the vanishing boundary conditions at infinity: as $|x| \to \infty$, the field u and all its spatial derivatives tend to zero.

We aim to compute the integral deviation

$$\Delta I_n := I_n[u_c] - I_n[u_s] - I_n[u_p] = \int_{-\infty}^{+\infty} [\rho_n(u_c) - \rho_n(u_s) - \rho_n(u_p)]\, dx. \tag{4.25}$$

To determine whether ΔI_n vanishes, we first examine the structure of the conserved density difference in the integrand. For the KdV equation, any conserved density ρ_n can be expressed as a polynomial in the field u and its spatial derivatives. Substituting $u = 2(\ln f)_{xx}$, the quantity $\rho_n(u)$ can be further expanded in terms of logarithmic derivatives of the tau function f. By utilizing the algebraic structure of the bilinear form, the conserved density differences which correspond to the three tau functions f_1, f_p, and f_{mix} can be treated in a unified manner.

A crucial algebraic fact is that for the arbitrary order n, the expression $\rho_n(u_c) - \rho_n(u_s) - \rho_n(u_p)$ can be cast into a total derivative with respect to x involving f_1, f_p, and their derivatives, that is, there exists a function Φ_n, only depending on f_1, f_p, and their derivatives up to finite order (not exceeding $2n$), such that

$$\rho_n(u_c) - \rho_n(u_s) - \rho_n(u_p) = \partial_x \Phi_n(f_1, f_p, \partial_x f_1, \partial_x f_p, \ldots, \partial_x^{2n} f_1, \partial_x^{2n} f_p). \tag{4.26}$$

The validity of this property relies on the bilinear identity satisfied by the Wronskian coupling:

$$(D_x D_t + D_x^4) f_1 \cdot f_p = 0, \tag{4.27}$$

where D_x and D_t are the Hirota bilinear operators. From this identity, it follows that for any expression constructed from f_1, f_p and their derivatives via polynomial and differential operations, the conserved density difference generated by the Wronskian coupling $f_{\text{mix}} = W(f_1, f_p)$ is always a total derivative. For low orders ($n = 1, 2, 3$), this can be verified by direct expansion; for the general n, the conclusion follows from Eq (4.27) combined with the recursive structure of the conserved densities, the key point being the commutativity of the recursion operator with the difference operation.

Now, we address the integral of the total derivative term. Substituting Eq (4.26) into Eq (4.25) yields the following:

$$\Delta I_n = \int_{-\infty}^{+\infty} \partial_x \Phi_n \, dx = \Phi_n \Big|_{-\infty}^{+\infty}. \tag{4.28}$$

Thus, the value of ΔI_n is determined by the boundary behavior of Φ_n at infinity. Now, we examine the asymptotic properties of Φ_n as $|x| \to \infty$. In this work, we restrict to the single-soliton case, for which the soliton tau function takes the form

$$f_1 = 1 + e^{\xi_1}, \quad \xi_1 = k_1 x - 4k_1^3 t + \delta_1, \tag{4.29}$$

where $k_1 > 0$ is the soliton wavenumber, and δ_1 is the initial phase. The first-order logarithmic derivative is

$$(\ln f_1)_x = \frac{k_1 e^{\xi_1}}{1 + e^{\xi_1}}, \tag{4.30}$$

which tends to 0 as $x \to -\infty$ and to k_1 as $x \to +\infty$, thus approaching the respective constant values exponentially at both ends. The second- and higher-order logarithmic derivatives

$$(\ln f_1)_{xx}, \quad (\ln f_1)_{xxx}, \quad \ldots$$

all decay to zero exponentially as $|x| \to \infty$. Since the field $u = 2(\ln f)_{xx}$ and its spatial derivatives $u_x, u_{xx}, \ldots$ only involve the second and higher derivatives of $(\ln f)$, and do not contain linear terms in $(\ln f)_x$, the function Φ_n constructed from ρ_n contains no nonzero constant asymptotic terms.

For the positon tau function f_p, the corresponding solution u_p exhibits algebraic decay of the form $u_p = O(1/|x|)$. From this, it follows that f_p behaves asymptotically as a polynomial growth (or tends to a constant) in x, and its logarithmic derivatives of all orders satisfy the following:

$$(\ln f_p)^{(m)} = O(1/|x|^m), \quad m \geq 1. \tag{4.31}$$

Since Φ_n is composed of logarithmic combinations of f_1, f_p, and their derivatives through polynomial and differential operations, it inherits the decay properties of both. For the soliton part, each term in Φ_n either contains an exponential decay factor or solely consists of second and higher derivatives of $(\ln f_1)$; for the positon part, each term contains at least one logarithmic derivative factor of the positon solution, whose decay order is no lower than $1/|x|$. Hence, in both limits $x \to \pm\infty$, we have the following:

$$\lim_{x\to\pm\infty} \Phi_n(x) = 0. \tag{4.32}$$

From Eqs (4.28) and (4.32), we obtain the following:

$$\Delta I_n = 0 - 0 = 0. \tag{4.33}$$

Combining the above analysis, the integral deviation vanishes, which directly yields the following:

$$I_n[u_c] = I_n[u_s] + I_n[u_p]. \tag{4.34}$$

Since n is arbitrary, the superposition relation holds for all orders of conserved quantities. This completes the proof. □

Corollary 4.1. *From Proposition 4.1, it follows that within the Wronskian bilinear integrable framework, the global conserved quantities of all orders remain stable throughout the nonlinear coupling, collision, and long-time evolution of soliton and positon solutions, thus satisfying the superposition invariance property. The coupling between the two types of nonlinear excitations primarily manifests as a reconstruction of the localized waveform and generally does not induce significant global energy dissipation or particle number variation. This provides a theoretical basis for the stable coexistence of multiple types of nonlinear excitations in two-component BEC systems.*

The proof relies on the total derivative form guaranteed by Eq (4.28). This property depends on the integrable structure of KdV and the Wronskian construction, and does not hold for general nonlinear couplings. For $n = 1, 2, 3$, Φ_n can be explicitly given; higher orders follow from the bilinear identity and recursive conserved density structure. Together with the preceding discussions on tau-function coupling, bilinear integrable structure, and nonlocal symmetries, this analysis establishes a self-consistent analytical framework of the exact solution structure, integrable symmetry, and global conservation dynamics for soliton-positon composite waves.

This section focuses on the integrable structure and conservation properties of the KdV equation, presenting the explicit expression of nonlocal symmetries based on the Lax pair, the Darboux transformation, the Bäcklund transformation, and the infinite conservation laws obtained via the Riccati expansion method. On this basis, the superposition invariance of conserved quantities for soliton-positon composite waves is proposed and proven. Together, these results establish a self-consistent analytical framework of the exact solution structure, integrable symmetry, and global conservation dynamics.

5. Conclusions

In this paper, starting from a nonlinear Schrödinger-type model for BECs, we rewrite the GP equation in quantum hydrodynamic form via a density-phase decomposition, and then reduce it asymptotically to the classical KdV equation in the long-wave limit through a multi-scale perturbation expansion. Based on this reduction, multi-soliton solutions are constructed by means of the Hirota bilinear method. The positon solutions of the KdV equation are also presented, which originate from the solution family of the variable-coefficient KdV equation proposed by Gao and Tian [16], and are reduced here to the constant-coefficient case. Furthermore, by coupling the tau function of a single soliton and that of a first-order positon via the Wronskian determinant, we construct the mixed interaction solution between the soliton and the positon. In the analysis of the integrable structure, with the aid of the Lax pair and prolongation structure theory, we give the explicit expression of the nonlocal symmetry, establish the Darboux transformation and the Bäcklund transformation, and derive infinitely many conservation laws through the Riccati expansion. In addition, we prove that within the Wronskian bilinear integrable framework, all orders of global conserved quantities for the soliton-positon composite wave satisfy an exact superposition relation.

Compared with direct numerical simulations of the full GP equation, the asymptotic reduction proposed here reduces the original (3+1)-dimensional problem to a (1+1)-dimensional KdV equation in the long-wave limit. This reduction not only lowers the computational cost, but also explicitly reveals the balance between nonlinearity and third-order dispersion, which underlies the formation of coherent structures such as solitons. The multi-scale perturbation framework adopted here is not limited to the case of a constant scattering length; for other quantum fluid systems, as long as their long-wave excitations satisfy similar scale-separation conditions, the framework can be accordingly generalized.

The Darboux transformation is a fundamental tool to elucidate the solution structure and symmetry properties of the KdV equation, as it preserves both the Lax pair formulation and the algebraic structure of the infinite conservation law hierarchy during the transformation process. From the Hamiltonian perspective, the KdV equation admits a well-defined Hamiltonian structure, with each conservation law corresponding to a Hamiltonian vector field. By preserving all conservation laws, the Darboux transformation constitutes a symmetry of the system. The nonlocal symmetries can be interpreted as Hamiltonian vector fields derived from specific conservation laws, and these symmetries collectively form a Virasoro algebra, thereby establishing a correspondence among Darboux transformations, conservation laws, and symmetry algebras.

It should be noted that all conclusions in this paper are obtained under the condition of a fixed scattering length, namely, the coupling constant is fixed at $g = -2$ in the nondimensionalization process. This restriction allows us to focus entirely on the asymptotic reduction from the BEC system to the KdV equation and the analysis of its integrable structure. For the case of a varying scattering length, one only needs to retain g as an adjustable parameter in the nondimensionalization and track its influence on the coefficients of the reduced equation, thereby directly extending the framework established in this paper. This generalization will be taken up in our future work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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