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Research article

Double thresholds for blowup and global existence of the solution to a system of parabolic equations

Xiaowei An[1,2] **and Xianfa Song**[3,4,*]

[1] School of Intelligence Policing, China People's Police University, Langfang 065000, China
[2] Hebei Key Laboratory of Information Support Technology for Smart Policing, China People's Police University, Langfang 065000, China
[3] Department of Mathematics, School of Mathematics, Tianjin University, Tianjin 300072, China
[4] Xinjiang Production and Construction Corps Key Laboratory of Green and Intelligent Development and Efficient Utilization of Strategic Mineral Resources, Xinjiang University of Technology, Hotan 84800, China

* **Correspondence:** Email: songxianfa@tju.edu.cn, songxianfa2004@163.com.

Abstract: We considered the following parabolic system:

$$\begin{cases} u_t = d_1\Delta u - a(x)\cdot\nabla u + f(u,v), & x\in\Omega,\ t>0,\\ v_t = d_2\Delta v - b(x)\cdot\nabla v + g(u,v), & x\in\Omega,\ t>0,\\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), & x\in\Omega, \end{cases}$$

subject to Dirichlet (or Neumann) boundary conditions. Here $\Omega\subset\mathbb{R}^N (N\ge 1)$ is a bounded smooth domain. In addition to some results on blowup and global existence of the solution, we found some more interesting results as follows: (1) There exists double thresholds for blowup and global existence of the solution. Under certain conditions, if $f(u,v)=f_1(v)g_1(u)$ and $g(u,v)=f_2(v)g_2(u)$, then the first watershed is

$$\int_{c_1}^{+\infty}\frac{du}{g_1(u)}=+\infty \quad \text{and} \quad \int_{c_2}^{+\infty}\frac{dv}{f_2(v)}=+\infty,$$

and the second watershed is

$$\int_{\tilde c_1}^{+\infty}\frac{dU}{\tilde f(\tilde F^{-1}(K\tilde G(U)))}=+\infty \quad \text{and} \quad \int_{\tilde c_2}^{+\infty}\frac{dV}{\tilde g(\tilde G^{-1}(\frac{1}{\epsilon}\tilde F(V)))}=+\infty.$$

Here $\tilde f, \tilde g, \tilde F$ and $\tilde G$ will be defined in Section 2.2. (2) If there exist nonnegative smooth functions $h(u)$, $l(v)$ and $H(s)$ such that

$$f(u,v)h'(u)l(v)+g(u,v)h(u)l'(v)=H[h(u)l(v)]\ge 0,$$

then the watershed for blowup in finite time and global existence of the solution is

$$\int_0^{+\infty} \frac{ds}{H(s)} = +\infty.$$

Keywords: system of parabolic equations; global existence; blowup; double thresholds

1. Introduction

In this paper, we consider the following problem:

$$\begin{cases} u_t = d_1\Delta u - a(x)\cdot\nabla u + f(u,v), & x\in\Omega,\ t>0, \\ v_t = d_2\Delta v - b(x)\cdot\nabla v + g(u,v), & x\in\Omega,\ t>0, \\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

subject to Neumann boundary condition $\frac{\partial u}{\partial\eta} = \frac{\partial v}{\partial\eta} = 0$ or Dirichlet boundary condition $u = v = 0$ on $\partial\Omega$, $t > 0$, where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary, η is the outer normal vector with respect to $\partial\Omega$, $d_1, d_2 \in \mathbb{R}^+$, $a(x)$ and $b(x)$ are continuous vector valued functions on Ω, $0 \leq u_0(x), v_0(x) \in C(\Omega)$, and f, g are continuous functions. It is well known that in heat transfer, u and v in (1.1) are the temperature of two bodies, $a(x)\cdot\nabla u$ and $b(x)\cdot\nabla v$ describe the heat convection. Under some assumptions on $a(x)$, $b(x)$, $f(u,v)$, $g(u,v)$, $u_0(x)$ and $v_0(x)$, the local well-posedness of problems (1.1) is true, see [1] and the references therein.

In this paper, we are mainly concerned with the phenomena of global existence and blowup in finite time. Global existence means that the solution equipped with L^∞-norm is always finite for all $0 < t < \infty$; blowup in finite time means that the L^∞-norm of solution is unbounded as $t \to T$ for some $T < \infty$.

Our results on (1.1) keep close contact with those of the following ODE problem

$$\begin{cases} u_t = f(u,v), \quad v_t = g(u,v), \quad t>0, \\ u(0)=u_0\geq 0, \quad v(0)=v_0\geq 0. \end{cases} \tag{1.2}$$

Deriving properties of solution to a system of parabolic equations from the associated ODE has been considered by many authors. Among them, the pioneering work of Smoller and his collaborators have played an important role, we refer to [2–4]. Inspired by their results, considering (1.2), we find a very interesting phenomenon: Roughly speaking, we show that, if the solution of (1.2) exists globally (or blows up in finite time) under certain assumptions on f and g, then we prove that the solution of (1.1) exists globally (or blows up in finite time) with suitable conditions on $a(x)$ and $b(x)$.

Let us recall some history in this direction. Let $u(t) \in C^1(0,\infty) \cap C[0,\infty)$ the solution of the following ODE problem:

$$u_t = f(u), \quad t>0, \quad u(0)=u_0\geq 0, \tag{1.3}$$

where $f \in C^1[0,\infty)$. If

$$\int^{+\infty} \frac{du}{f(u)} < +\infty, \tag{1.4}$$

then the solution will blow up in finite time provided that either $f(0) > 0$ or $u_0 > 0$. While if

$$\int^{+\infty} \frac{du}{f(u)} = +\infty, \tag{1.5}$$

then the solution exists globally. Here the notation $\int^{\infty}$ means that we take the integral near infinity (see [5]).

Let's recall some results on the following scalar semi-linear parabolic equation problems:

$$\begin{cases} u_t = d\Delta u + f(u), \quad x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \eta} = 0, \quad x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{1.6}$$

and

$$\begin{cases} u_t = d\Delta u + f(u), \quad x \in \Omega, \quad t > 0, \\ u = 0, \quad x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega. \end{cases} \tag{1.7}$$

In [5–9], the global existence and nonexistence results were obtained. By their results, if (1.4) holds, then the solution to (1.6) and (1.7) blows up in finite time. Similar results on semi-linear parabolic equations in the spatial inhomogeneous case were also obtained in [10, 11]. Parallel results on the following problem

$$\begin{cases} u_t = d\Delta u, \quad x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \eta} = f(u), \quad x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{1.8}$$

were established in [12, 13]. The blowup criterion of the solution to (1.8) is also (1.4). Some authors considered the continuity of blowup solutions to nonlinear heat equations, see [14].

Blowups in finite time and global existence phenomena have already been noticed in some special models of (1.1) when $a(x) = b(x) = 0$. In regards to the results on the global existence and nonexistence of the solutions to (1.1) when f, g enjoy the form of $f(u,v) = v^{p_1}u^{q_1}$, $g(u,v) = v^{p_2}u^{q_2}$ for some nonnegative constants p_1, p_2, q_1, q_2, we can see [15–20]. In [21], the blowup set was considered. In regards to the results on the global existence and nonexistence of the solution to (1.1) when $f(u,v) = g(u,v) = F(u)(1-v)$ or $f(u,v) = g(u,v) = F(u)\exp(\frac{v-1}{v})$ for some suitable function F, or $f(u,v) = -g(u,v)$, or Gierer–Meinhardt model (i.e., $f(u,v) = \frac{u^q}{v^r}$ and $g(u,v) = \frac{u^p}{v^s}$), we can refer to [22–25] respectively. In regards to the global and non-global existence results on (1.1) when the nonlinear terms are $h_1(t)v^{p_1}u^{q_1}$ and $h_2(t)v^{p_2}u^{q_2}$ in the governing equations, we can see [26, 27]. Some authors considered the blowup mechanism for a chemotaxis model, see [28], while some authors considered the Fujita critical exponent for the Cauchy problem of $u_t = \Delta u + v^{p_1}u^{q_1}$, $v_t = \Delta v + v^{p_2}u^{q_2}$, see [29, 30]. In regards to the results of the blow up phenomenon of the solution to parabolic systems which include the nonlinear boundary condition case, see [31, 32]. Some other results on global existence and blowup in finite time of solution to parabolic equations also can be seen in [33–37]. To see more information on the blowup results about parabolic equations, we can see the books [38, 39] and the references therein.

We establish some deeper results and find more interesting phenomena of the solution to (1.1) and (1.2), for general functions f and g. Under certain conditions, if $f(u,v) = f_1(v)g_1(u)$ and $g(u,v) =$

$f_2(v)g_2(u)$, then the first watershed is

$$\int_{c_1}^{+\infty} \frac{du}{g_1(u)} = +\infty, \quad \text{and} \quad \int_{c_2}^{+\infty} \frac{dv}{f_2(v)} = +\infty,$$

and the second watershed is

$$\int_{\tilde{c}_1}^{+\infty} \frac{dU}{\tilde{f}(\tilde{F}^{-1}(K\tilde{G}(U)))} = +\infty, \quad \text{and} \quad \int_{\tilde{c}_2}^{+\infty} \frac{dV}{\tilde{g}(\tilde{G}^{-1}(\frac{1}{\epsilon}\tilde{F}(V)))} = +\infty.$$

Here $\tilde{f}, \tilde{g}, \tilde{F}$, and $\tilde{G}$ will be defined in Section 2.2. Additionally, when the nonlinear Neumann boundary conditions are $\frac{\partial u}{\partial \eta} = r(u, v)$ and $\frac{\partial v}{\partial \eta} = s(u, v)$ under the assumptions on $r(u, v) > 0$ and $s(u, v) > 0$ in [40, 41], the properties for the solutions are very dependent on $r(u, v)$ and $s(u, v)$, while in this paper, (1.1) have homogeneous Neumann or Dirichlet boundary conditions, consequently, the properties for the solution to (1.1) are heavily dependent on nonlinearities $f(u, v)$ and $g(u, v)$ in the equations.

Deferring to the restricted condition $f + g \leq 0$ in [42] which dealt with the blowup phenomenon of the solution to (1.1) in dissipation mass case, we do not need to require $f + g \leq 0$ here, and we will establish the blowup results on the solution to (1.1) with more general $f(u, v)$ and $g(u, v)$ as follows. If there exist nonnegative smooth functions $h(u)$, $l(v)$ and $H(s)$ such that

$$f(u, v)h'(u)l(v) + g(u, v)h(u)l'(v) \leq H[h(u)l(v)], \quad \text{and} \quad \int_0^{+\infty} \frac{ds}{H(s)} = +\infty,$$

then the solution is global; if

$$f(u, v)h'(u)l(v) + g(u, v)h(u)l'(v) \geq H[h(u)l(v)], \quad \text{and} \quad \int_0^{+\infty} \frac{ds}{H(s)} < +\infty,$$

then the solution will blow up in finite time. Especially, if

$$f(u, v)h'(u)l(v) + g(u, v)h(u)l'(v) = H[h(u)l(v)] \geq 0,$$

then the watershed is $\int_0^{+\infty} \frac{ds}{H(s)} = +\infty$.

In conclusion, we will analyze parabolic problems as well as ODE in this paper. Similar ideas can be seen in [43, 44], which develop new methods of studying nonlocal partial differential equations based on the results about nonlocal ODE.

We will use Theorem A and Theorem B to state the results on ODE problem (1.2), while Theorem 1, 2 and 3 state the results on parabolic equation problem (1.1). In the next section, $(x_1, y_1) \geq (x_2, y_2)$ means that $x_1 \geq x_2$ and $y_1 \geq y_2$.

The rest of this paper is organized as follows: in Section 2, we will give the results on (1.1) and (1.2) when f and g are separable in the variables u and v. In Section 3, we will give the results on (1.1) and (1.2) when f and g are not separable in the variables u and v. In Section 4, we will consider the roles of the convection terms $a(x) \cdot \nabla u$ and $b(x) \cdot \nabla v$.

2. The results on the case of $f(u, v) = f_1(v)g_1(u)$ and $g(u, v) = f_2(v)g_2(u)$

In this section, we deal with problems (1.1) and (1.2), when f and g are in the forms of $f(u, v) = f_1(v)g_1(u)$ and $g(u, v) = f_2(v)g_2(u)$ with nonnegative $f_1, g_1, f_2, g_2 \in C^1(\mathbb{R}^+)$.

2.1. A special case of (1.1) and (1.2)

We consider a special case of (1.1) and (1.2) as follows:

$$\begin{cases} u_t = d_1\Delta u - a(x)\cdot\nabla u + f(v), & x\in\Omega,\ t>0,\\ v_t = d_2\Delta v - b(x)\cdot\nabla v + g(u), & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\eta} = \frac{\partial v}{\partial\eta} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\quad v(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{2.1}$$

$$\begin{cases} u_t = d_1\Delta u - a(x)\cdot\nabla u + f(v), & x\in\Omega,\ t>0,\\ v_t = d_2\Delta v - b(x)\cdot\nabla v + g(u), & x\in\Omega,\ t>0,\\ u = v = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\quad v(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{2.2}$$

and

$$\begin{cases} u_t = f(v),\quad v_t = g(u),\quad t>0,\\ u(0)=u_0\ge 0,\quad v(0)=v_0\ge 0. \end{cases} \tag{2.3}$$

Here $f(v), f'(v)\ge 0$ for $v\ge 0$, $g(u), g'(u)\ge 0$ for $u\ge 0$.

We first establish the results on the ODE problem (2.3), then we will apply them to the problems of (2.1) and (2.2).

About (2.3), we have

Lemma 2.1.1. *Assume that there exist* $0\le c_3\le u_0$, $0\le c_4\le v_0$ *such that* $f(v),\ f'(v)>0$ *for* $v\ge c_3$, $g(u),\ g'(u)>0$ *for* $u\ge c_4$. *Define*

$$F(v)=\int_{c_4}^{v} f(v)dv,\quad G(u)=\int_{c_3}^{u} g(u)du,\quad u,v\ge 0.$$

Suppose that there exist positive constants $0<\epsilon<1$, $K>1$ *and* $C>0$ *such that*

$$\epsilon G(u_0)\le F(v_0)\le KG(u_0), \tag{2.4}$$

$KG(\infty)\le CF(\infty)$, *and* $\frac{1}{\epsilon}F(\infty)\le CG(\infty)$. *Then we have:*

(i) If

$$\int_{u_0}^{+\infty}\frac{du}{f[F^{-1}(KG(u))]}=+\infty,\quad\text{and}\quad\int_{v_0}^{+\infty}\frac{dv}{g[G^{-1}(\frac{1}{\epsilon}F(v))]}=+\infty, \tag{2.5}$$

both hold, then the solution to (2.3) exists globally.

(ii) If either one of the following

$$\int_{u_0}^{+\infty}\frac{du}{f[F^{-1}(\epsilon G(u))]}<+\infty,\quad\text{or}\quad\int_{v_0}^{+\infty}\frac{dv}{g[G^{-1}(\frac{1}{K}F(v))]}<+\infty, \tag{2.6}$$

holds, then the solution to (2.3) blows up in finite time.

Here F^{-1} and G^{-1} are the inverse functions of F and G defined on $[0, F(\infty))$ and $[0, G(\infty))$ respectively.

Proof: Noticing that $f(v) \geq 0$ and $g(u) \geq 0$, we know that $F(v)$, $G(u)$, F^{-1} and G^{-1} are all increasing functions.

Let

$$I(t) = F(v(t)) - \epsilon G(u(t)), \quad J(t) = F(v(t)) - KG(u(t)).$$

Then $I(0) \geq 0$, $J(0) \leq 0$ and by the definitions of $F(v)$ and $G(u)$,

$$I'(t) = (1-\epsilon) f(v(t)) g(u(t)) \geq 0, \quad J'(t) = (1-K) f(v(t)) g(u(t)) \leq 0.$$

Consequently,

$$\epsilon G(u(t)) \leq F(v(t)) \leq KG(u(t)), \quad \frac{1}{K} F(v(t)) \leq G(u(t)) \leq \frac{1}{\epsilon} F(v(t)),$$

and

$$F^{-1}(\epsilon G(u(t))) \leq v(t) \leq F^{-1}(KG(u(t))),$$
$$G^{-1}\left(\frac{1}{K} F(v(t))\right) \leq u(t) \leq G^{-1}\left(\frac{1}{\epsilon} F(v(t))\right).$$

$f'(v) \geq 0$ and $g'(u) \geq 0$(and $\not\equiv 0$ in any subinterval of $(0, +\infty)$) imply that $f(v)$ and $g(u)$ are increasing functions, so

$$f[F^{-1}(\epsilon G(u(t)))] \leq f(v(t)) \leq f[F^{-1}(KG(u(t)))],$$
$$g[G^{-1}\left(\frac{1}{K} F(v(t))\right)] \leq g(u(t)) \leq g[G^{-1}\left(\frac{1}{\epsilon} F(v(t))\right)].$$

That is,

$$f[F^{-1}(\epsilon G(u(t)))] \leq u_t \leq f[F^{-1}(KG(u(t)))], \tag{2.7}$$
$$g[G^{-1}\left(\frac{1}{K} F(v(t))\right)] \leq v_t \leq g[G^{-1}\left(\frac{1}{\epsilon} F(v(t))\right)]. \tag{2.8}$$

Using (2.7) and (2.8), by the sub-solution and super-solution theory of ODE, it is easy to verify that the solution of (1.2) is global if

$$\int_c^{+\infty} \frac{du}{f[F^{-1}(KG(u))]} = +\infty \quad \text{and} \quad \int_c^{+\infty} \frac{dv}{g[G^{-1}(\frac{1}{\epsilon} F(v))]} = +\infty,$$

and it will blow up in finite time for large initial data if

$$\int_c^{+\infty} \frac{du}{f[F^{-1}(\epsilon G(u))]} < +\infty \quad \text{or} \quad \int_c^{+\infty} \frac{dv}{g[G^{-1}(\frac{1}{K} F(v))]} < +\infty. \qquad \square$$

We can apply the results of Lemma 2.1.1 to (2.1) and (2.2).

Lemma 2.1.2. *(1) Under the condition (2.5), then the solutions of (2.1) and (2.2) are global for any $a(x)$, $b(x)$, and nonnegative initial data (u_0, v_0).*

(2) Suppose that

$$d_1\Delta u_0 - a(x)\cdot\nabla u_0 + f(v_0) \geq 0 \quad \text{for} \quad x \in \Omega, \tag{2.9}$$

$$d_2\Delta v_0 - b(x)\cdot\nabla v_0 + g(u_0) \geq 0 \quad \text{for} \quad x \in \Omega. \tag{2.10}$$

Under the condition (2.6), if the solution of (2.3) blows up in finite time for initial data $(c_1, c_2) \geq \mathbf{0}$*, then the solution of (2.1) will blow up in finite time for initial data* $(u_0, v_0) \geq (c_1, c_2)$*; parallelly, if the solution of (2.3) blows up in finite time for initial data* $(c_1, c_2) = \mathbf{0}$*, then the solution of (2.2) will blow up in finite time for nonnegative initial data* (u_0, v_0)*.*

Proof: (1) Under the conditions of (2.5), the solution of (2.3) with $c_1 = \max_{x\in\bar{\Omega}} u_0(x)$ and $c_2 = \max_{x\in\bar{\Omega}} v_0(x)$ can be taken as a super-solution of (2.1), which implies that the solution of (2.1) is global for any nonnegative initial data (u_0, v_0).

(2) Suppose that (2.9) and (2.10) hold. Then by the comparison principle, we have $u_t \geq 0$ and $v_t \geq 0$. Under the conditions of (2.6), if the solution of (2.3) will blow up in finite time for initial data (c_1, c_2), then the solution of (2.3) with $c_1 \leq c_1' = \min_{x\in\bar{\Omega}} u_0(x)$ and $c_2 \leq c_2' = \min_{x\in\bar{\Omega}} v_0(x)$ can be taken as a sub-solution of (2.1), which implies that the solution of (2.1) will blow up in finite time for initial data $u_0(x) \geq c_1$ and $v_0(x) \geq c_2$.

If the solution of (2.3) with initial data $(c_1, c_2) = \mathbf{0}$ will blow up in finite time, then we can take the sub-solution in the form of

$$(\underline{u}(x,t),\ \underline{v}(x,t)) = (\underline{u}(\alpha t\delta(x)),\ \underline{v}(\beta t\xi(x))).$$

Here the functions $\underline{u}(\cdot)$ and $\underline{v}(\cdot)$ satisfy

$$\begin{aligned} &\frac{d\underline{u}}{dt} = f(\underline{v}), \quad \frac{d\underline{v}}{dt} = g(\underline{u}), \quad t > 0, \\ &\underline{u}(0) = 0, \quad \underline{v}(0) = 0, \end{aligned}$$

i.e., $(\underline{u}(\cdot), \underline{v}(\cdot))$ is the blowup solution of (2.3) with initial data $(c_1, c_2) = \mathbf{0}$. And α, β are small positive constants to be determined later, the functions $\delta(x)$ and $\xi(x)$ are arbitrary independent nonnegative functions satisfy

$$d_1\Delta\delta - a(x)\cdot\nabla\delta + p(\delta) = 0 \quad \text{in } \Omega, \quad \delta = 0 \quad \text{on } \partial\Omega, \tag{2.11}$$

and

$$d_2\Delta\xi - b(x)\cdot\nabla\xi + q(\xi) = 0 \quad \text{in } \Omega, \quad \xi = 0 \quad \text{on } \partial\Omega, \tag{2.12}$$

where $p(\delta) > 0$ and $q(\xi) > 0$ are continuous functions such that (2.11) and (2.12) have nonnegative solutions. Obviously, $\underline{u}(x,t) = \underline{v}(x,t) = 0$ for $x \in \partial\Omega$, $t > 0$. After some computations, we have

$$\begin{aligned} &\underline{u}_t - d_1\Delta\underline{u} + a(x)\cdot\nabla\underline{u} \\ &= \alpha f(\underline{v}(t\beta\xi(x)))[\delta(x) + tp(\delta)] - t^2 d_1\alpha\beta g(\underline{u})f'(\underline{v})\nabla\delta\cdot\nabla\xi \\ &\leq f(\underline{v}(t\beta\xi(x))) = f(\underline{v}), \qquad (2.13) \\ &\underline{v}_t - d_2\Delta\underline{v} + b(x)\cdot\nabla\underline{v} \end{aligned}$$

$$= \beta g(\underline{u}(t\alpha\delta(x)))[\xi(x) + tq(\xi)] - t^2 d_2\alpha\beta f(\underline{v})g'(\underline{u})\nabla\delta \cdot \nabla\xi$$
$$\leq g(\underline{u}(t\alpha\delta(x))) = g(\underline{u}) \tag{2.14}$$

for $x \in \Omega$ and t small enough if $\alpha << 1, \beta << 1$. That is, $(\underline{u}, \underline{v})$ is a sub-solution of (2.2). Therefore, there exist some $x_0 \in \Omega$ and $T > 0$ such that

$$\lim_{t\to T^-}[\underline{u}(x_0, t) + \underline{v}(x_0, t)] = +\infty,$$

and

$$T \leq \frac{T^*}{\max(\alpha\delta_0, \beta\xi_0)},$$

where T^* is the blowup time of the solution to (2.3), $\delta_0 = \max_{x\in\Omega}\delta(x)$ and $\xi_0 = \max_{x\in\Omega}\xi(x)$. □

Some examples. We will present some examples to illustrate Lemmas 2.1.1 and 2.1.2.

Example 2.1. $f(v) = v^p$, $g(u) = u^q$. Then $F(v) = \frac{v^{p+1}}{p+1}$, $G(u) = \frac{u^{q+1}}{q+1}$,

$$f[F^{-1}(KG(u))] = C_1(p, q, K)u^{\frac{p(q+1)}{p+1}}, \quad f[F^{-1}(\epsilon G(u))] = C_1'(p, q, \epsilon)u^{\frac{p(q+1)}{p+1}},$$

$$g[G^{-1}(\frac{1}{\epsilon}F(v))] = C'(p, q, \epsilon)v^{\frac{q(p+1)}{q+1}}, \quad g[G^{-1}(\frac{1}{K}F(v))] = C_2'(p, q, K)v^{\frac{q(p+1)}{q+1}}.$$

The solutions of ODE (2.3) is global if and only if $pq \leq 1$. Consequently, the solution of (2.1) is global for any initial data if $pq \leq 1$, while the solution of (2.1) will blow up in finite time if $pq > 1$ for the initial data which satisfy Lemma 2.1.2 with $u_0(x) \geq c_1 > 0$ and $v_0(x) \geq c_2 > 0$.

Example 2.2. $f(v) = e^v$, $g(u) = e^u$. Then $F(v) = e^v - 1 \geq \frac{e^v}{2}$ for $v > \ln 2$, $G(u) = e^u - 1 \geq \frac{e^u}{2}$ for $u > \ln 2$,

$$f[F^{-1}(KG(u))] \geq C_1(K)e^u, \quad f[F^{-1}(\epsilon G(u))] \geq C_1'(\epsilon)e^u,$$

$$g[G^{-1}(\frac{1}{\epsilon}F(v))] \geq C_2(\epsilon)e^v, \quad g[G^{-1}(\frac{1}{K}F(v))] \geq C_2'(K)e^v,$$

for w, z large enough. The solution of ODE (2.3) will blow up in finite time for nonnegative initial data. Consequently, the solutions of (2.1) and (2.2) will blow up in finite time for the nonnegative initial data which satisfy Lemma 2.1.2.

Example 2.3. (i) $f(v) = e^v$, $g(u) = \ln u$, $c_3 = 0$, $c_4 = 1$. Then $F(v) \leq e^v$, $G(u) = u\ln u + 1 - u$,

$$f[F^{-1}(KG(u))] \leq C_1(K)[u\ln u + 1 - u], \quad f[F^{-1}(\epsilon G(u))] \leq C_1'(\epsilon)[u\ln u + 1 - u].$$

Obviously,

$$\int_1^{+\infty} \frac{du}{f[F^{-1}(KG(u))]} \geq \int_1^{+\infty} \frac{du}{C_1(K)[u\ln u + 1 - u]} = +\infty.$$

Although we cannot give the explicit expression of G^{-1}, since $G(u) \leq u\ln u$ for $u > 1$, we can obtain

$$g[G^{-1}(\frac{1}{\epsilon}F(v))] \leq C_1'(p, q, \epsilon)v(1 + \ln v) \quad \text{for} \quad v >> 1.$$

Therefore,

$$\int_C^{+\infty} \frac{dv}{g[G^{-1}(\frac{1}{\epsilon}F(v))]} = +\infty.$$

So the solution of ODE (2.3) is global for initial data $c_1 > 1$ and $c_2 \geq 0$. In fact, we even can construct the following super-solution of ODE (2.3)

$$\bar{u}(t) = e^{Me^{Kt}}, \quad \bar{v}(t) = Me^{Kt}, \quad K > 1, \quad M = \max(c_1, c_2),$$

and verify that the solution of ODE (2.3) is global. Consequently, the solution of (2.1) is global for initial data $u_0 > 1$ and $v_0 \geq 0$.

(ii) $f(v) = e^{e^v}$, $g(u) = \ln u$. Although we cannot write out the explicit expression of F^{-1} and G^{-1}, we can construct a sub-solution of ODE (2.3) with the form of

$$\underline{u}(t) = e^{\frac{a}{(1-ct)^K}}, \quad \underline{v}(t) = \ln\frac{b}{(1-ct)^L}, \quad L > K > 1, \quad c \leq \min(\frac{a}{L}, \frac{b}{a}),$$

and prove that the solution of ODE (2.3) blows up in finite time. Consequently, the solution of (2.1) blows up in finite time for large initial data.

(iii) $f(z) = e^{e^v}$, $g(w) = \ln(\ln u)$. Although we cannot write out the explicit expression of F^{-1} and G^{-1}, we can construct a super-solution of ODE (2.3) with the form of

$$\bar{u}(t) = e^{e^{Me^{Kt}}}, \quad \bar{v}(t) = Me^{Kt}, \quad K > 1, \quad M = \max(c_1, c_2),$$

and verify that the solution of ODE (2.3) is global. Consequently, the solution of (2.1) is global for initial data $u_0 > 1$ and $v_0 \geq 0$.

(iv) Generally, consider the ODE problem

$$\begin{cases} u_t = \exp(\exp(...(\exp(v))...)), \\ v_t = \ln(\ln(...(\ln(M+u))...)), \quad t > 0, \\ u(0) = c_1, \quad v(0) = c_2. \end{cases} \tag{2.15}$$

Here $\exp(\exp(...(\exp(v))...))$ is m-multiple contained function, $\ln(\ln(...(\ln(M+u))...))$ is n-multiple contained function, M is large enough such that

$$\ln(\ln(...(\ln(M))...)) \geq 0.$$

If $m > n$, and c_1, c_2 are large enough, we can construct the blowup sub-solution having the form of

$$\underline{u}(t) = \exp(\exp(...(\exp(\frac{a}{(1-ct)^K}))...)), \ \underline{v}(t) = \ln(\frac{b}{(1-ct)^L}).$$

Here $\exp(\exp(...(\exp(\cdot))...))$ is $(m-1)$-multiple contained function. The solution of ODE (2.15) will blow up in finite time. Consider

$$\begin{cases} u_t = \Delta u + a(x)\cdot\nabla u + \exp(\exp(...(\exp(v))...)), \quad x \in \Omega, \ t > 0, \\ v_t = \Delta v + b(x)\cdot\nabla v + \ln(\ln(...(\ln(M+u))...)), \quad x \in \Omega, \ t > 0, \\ \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, \quad x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega. \end{cases} \tag{2.16}$$

By Lemma 2.1.2, the solution of (2.16) will blow up in finite time if $a(x)$, $b(x)$ and the initial data satisfy the assumptions of Lemma 2.1.2.

If $m \le n$, we can construct the global super-solution with the form of

$$\bar{u}(t) = \exp(\exp(...(M\exp(Kt)...)),\ \bar{v}(t) = Me^{Lt}.$$

Here exp(exp(...(exp(·))...)) is $(m+1)$-multiple contained function. The solution of ODE (2.15) is global. Consequently, by Lemma 2.1.2, the solution of (2.16) is global for any $a(x)$, $b(x)$ and initial data.

2.2. The results on (1.1) and (1.2) when the nonlinearities are separable in the variables u and v

In this case, the corresponding ODE problem (1.2) becomes

$$\begin{cases} u_t = f_1(v)g_1(u), \quad v_t = f_2(v)g_2(u), \quad t > 0, \\ u(0) = u_0 \ge 0, \quad v(0) = v_0 \ge 0. \end{cases} \tag{2.17}$$

Here

$$f_i(v) > 0, \quad g_i(u) > 0 (i = 1, 2), \quad f_1'(v) \ge 0 \text{ and } g_2'(u) \ge 0, \tag{2.18}$$

for $v > 0$, $u > 0$.

If $\int_{v_0}^{+\infty} \frac{dv}{f_2(v)} < +\infty$ or $\int_{u_0}^{+\infty} \frac{du}{g_1(u)} < +\infty$, then the solution to (2.17) will blow up in finite time if $u_0 > 0$ and $v_0 > 0$. In fact, (2.17) implies that $u_t \ge cg_1(u)$ or $v_t \ge cf_2(v)$ for some $c > 0$ under the assumptions of (2.18), while solution to the initial value problem $u_t = cg_1(u)$, $u(0) = u_0$ will blow up in finite time if $\int_{u_0}^{+\infty} \frac{du}{g_1(u)} < +\infty$, so does the solution to $v_t = cf_2(v)$, $v(0) = v_0$ if $\int_{v_0}^{+\infty} \frac{dv}{f_2(v)} < +\infty$.

Consequently, we consider the case of

$$\int_{u_0}^{+\infty} \frac{du}{g_1(u)} = +\infty, \quad \int_{v_0}^{+\infty} \frac{dv}{f_2(v)} = +\infty,$$

in the remaining part of this section below.

Define

$$G_1(u) = \int_{u_0}^{u} \frac{du}{g_1(u)}, \quad F_2(v) = \int_{v_0}^{v} \frac{dv}{f_2(v)}. \tag{2.19}$$

Thus, (2.17) can be written as

$$\begin{cases} (G_1(u))_t = f_1(v), \quad (F_2(v))_t = g_2(u), \quad t > 0, \\ u(0) = u_0 \ge 0, \quad v(0) = v_0 \ge 0. \end{cases} \tag{2.20}$$

Denote

$$U(t) = G_1(u(t)), \quad V(t) = F_2(v(t)), \tag{2.21}$$

$$\tilde{f}(V) = f_1(v) = f_1[F_2^{-1}(V)], \quad \tilde{g}(U) = g_2(u) = g_2[G_1^{-1}(U)]. \tag{2.22}$$

Then (2.17) becomes

$$\begin{cases} U_t = \tilde{f}(V), \quad V_t = \tilde{g}(U), \quad t > 0, \\ U(0) = 0, \quad V(0) = 0. \end{cases} \tag{2.23}$$

Since (2.23) has the form of (2.3), by the results and the proof of Lemma 2.1.1, we can obtain

Theorem A. *Assume that* $f(u, v) = f_1(v)g_1(u)$ *and* $g(u, v) = f_2(v)g_2(u)$*,* $f_i(v), f_i'(v) \ge 0$ *for* $v \ge 0$ *and* $f_i(v), f_i'(v) \not\equiv 0$ *in any subinterval of* $(0, +\infty)$*,* $g_i(u), g_i'(u) \ge 0$ *for* $u \ge 0$ *and* $g_i(u), g_i'(u) \not\equiv 0$ *in any subinterval of* $(0, +\infty)$*,* $i = 1, 2$*.*

(1) If $\int_{u_0}^{+\infty} \frac{du}{g_1(u)} < +\infty$ *or* $\int_{v_0}^{+\infty} \frac{dv}{f_2(v)} < +\infty$*. Then the solution of (2.17) will blow up in finite time for positive initial data.*

(2) Assume that $\int_{u_0}^{+\infty} \frac{du}{g_1(u)} = +\infty$ *and* $\int_{v_0}^{+\infty} \frac{dv}{f_2(v)} = +\infty$*. And there exist* $0 \le \tilde{c}_3 \le \tilde{c}_1$, $0 \le \tilde{c}_4 \le \tilde{c}_2$, $\tilde{f}(V), \tilde{f}'(V) \ge 0$ *for all* $V \ge \tilde{c}_3$, $\tilde{g}(U), \tilde{g}'(U) \ge 0$ *for all* $U \ge \tilde{c}_4$*. Let* $\tilde{F}(V(t)) = \int_{\tilde{c}_4}^{V(t)} \tilde{f}(V)dV$ *and* $\tilde{G}(U(t)) = \int_{\tilde{c}_3}^{U(t)} \tilde{g}(U)dU$ *for* $t \ge 0$*. Suppose that there exist positive constants* ϵ *and* K *such that*

$$\epsilon\tilde{G}(\tilde{c}_1) \le \tilde{F}(\tilde{c}_2) \le K\tilde{G}(\tilde{c}_1). \tag{2.24}$$

Then the solution of (2.23) exists globally provided that

$$\int_{\tilde{c}_1}^{+\infty} \frac{dU}{\tilde{f}[\tilde{F}^{-1}(K\tilde{G}(U))]} = +\infty \quad \text{and} \quad \int_{\tilde{c}_2}^{+\infty} \frac{dV}{\tilde{g}[\tilde{G}^{-1}(\frac{1}{\epsilon}\tilde{F}(V))]} = +\infty, \tag{2.25}$$

and it will blow up in finite time for large initial data if

$$\int_{\tilde{c}_1}^{+\infty} \frac{dU}{\tilde{f}[\tilde{F}^{-1}(\epsilon\tilde{G}(U))]} < +\infty \quad \text{or} \quad \int_{\tilde{c}_2}^{+\infty} \frac{dV}{\tilde{g}[\tilde{G}^{-1}(\frac{1}{K}\tilde{F}(V))]} < +\infty. \tag{2.26}$$

Here $\tilde{F}^{-1}$ *and* $\tilde{G}^{-1}$ *are the inverse functions of* $\tilde{F}$ *and* $\tilde{G}$ *respectively. Therefore, the solution of (2.17) exists globally or blows up in finite time correspondingly.*

We can use the above results on ODE (2.17) to (1.1) when the nonlinearities are separable in u and v as follows.

Theorem 1. *Assume that* $f(u,v) = f_1(v)g_1(u)$ *and* $g(u,v) = f_2(v)g_2(u)$, $f_i(v), f_i'(v) \ge 0$ *for* $v \ge 0$ *and* $f_i(v), f_i'(v) \not\equiv 0$ *in any subinterval of* $(0,+\infty)$, $g_i(u), g_i'(u) \ge 0$ *for* $u \ge 0$ *and* $g_i(u), g_i'(u) \not\equiv 0$ *in any subinterval of* $(0,+\infty)$, $i = 1,2$.

(1) If the assumptions of Theorem A(2) and (2.25) hold, then the solutions of (1.1) are global for any nonnegative initial data (u_0, v_0).

(2) Suppose that

$$d_1\Delta u_0 - a(x)\cdot\nabla u_0 + f(u_0,v_0) \ge 0 \quad \text{for} \quad x \in \Omega, \tag{2.27}$$

$$d_2\Delta v_0 - b(x)\cdot\nabla v_0 + g(u_0,v_0) \ge 0 \quad \text{for} \quad x \in \Omega. \tag{2.28}$$

If the assumptions of Theorem A(1) hold or the assumptions of Theorem A(2) hold, and (2.26) are true, then the solution of (1.1) will blow up in finite time for initial data $(u_0,v_0) \ge (w_0,z_0)$*. If the solution of (2.17) blows up in finite time for initial data* $(w_0,z_0) = \mathbf{0}$*, then the solutions of (1.1) will blow up in finite time for any nonnegative initial data* (u_0,v_0).

The proof of Theorem 1: (1) Let $(\bar{u}(t),\bar{v}(t))$ be the global solution of (2.17) with initial data

$$c_1 = \max_{x\in\bar{\Omega}} u_0(x), \quad c_2 = \max_{x\in\bar{\Omega}} v_0(x).$$

It is to verify that $(\bar{u}(t),\bar{v}(t))$ is a super-solution of (1.1) and

$$u(x,t) \le \bar{u}(t), \quad v(x,t) \le \bar{v}(t) \quad \text{for} \quad x \in \Omega,\ t > 0,$$

which implies that the solutions of (1.1) are global.

(2) Let $(\underline{u}(t), \underline{v}(t))$ be the blowup solution of (2.17) with initial data

$$c_1 \le \min_{x\in\bar{\Omega}} u_0(x), \quad c_2 \le \min_{x\in\bar{\Omega}} v_0(x).$$

Since $\frac{\partial f}{\partial v} \ge 0$ and $\frac{\partial g}{\partial u} \ge 0$, we can apply the comparison principle to (1.1). It is easy to verify that $(\underline{u}(t), \underline{v}(t))$ is a sub-solution of (1.1). And

$$u(x,t) \ge \underline{u}(t), \quad v(x,t) \ge \underline{v}(t) \quad \text{for} \quad x \in \Omega,\ t > 0,$$

which implies that the solutions of (1.1) will blow up in finite time. □

Some examples. We would like to give some examples to illustrate the results of Theorem A and Theorem 1.

Example 2.4. $f_1(v)g_1(u) = v^p u^q$, $f_2(v)g_2(u) = u^r v^s$, $pr = (1-q)(1-s)$ is the watershed for the blowup in finite time and global existence of the solutions to (1.1) and (1.2) when the nonlinearities are separable in variables u and v.

Example 2.5. $f_1(v)g_1(u) = \frac{v^p}{u^q}$, $f_2(v)g_2(u) = \frac{u^r}{v^s}$, $pr = (q+1)(s+1)$ is the watershed for the blowup in finite time and global existence of the solutions to (1.1) and (1.2) when the nonlinearities are separable in variables u and v.

3. The results on (1.1) when f and g are not separable in variables u and v

In this section, we will consider (1.1) when f and g are not separable in variables u and v. Due to the complexity of nonlinearities, in this case, whether the solution is global or blowup cannot be judged directly by the structures of f and g, therefore, we need some auxiliary functions. First, we state the results on ODE (1.2) as follows.

Theorem B *(1) Assume that $f(u,v)$ and $g(u,v)$ are locally Lipschitz continuous functions for $(u,v) \ge$ $\mathbf{0}$, and there exist nonnegative smooth functions $h(u)$ and $l(v)$ such that*

$$f(u,v)h'(u)l(v) + g(u,v)h(u)l'(v) \le H[h(u)l(v)], \tag{3.1}$$

where $H(s)$ is a continuous function for $s \ge 0$. If $H(s) > 0$ for $s \ge 0$ and

$$\int_0^{+\infty} \frac{ds}{H(s)} = +\infty, \tag{3.2}$$

or $-\infty < H(s) \le 0$ for $s \ge 0$, then $h(u)l(v)$ is global.

Especially, if (3.2) holds, $f(u,v) \ge 0$, $g(u,v) \ge 0$, $h'(u) \ge 0 (\not\equiv 0)$ and $l'(v) \ge 0 (\not\equiv 0)$, then the solution of (1.2) is global for any nonnegative initial data.

(2) Assume that $f(u,v)$ and $g(u,v)$ are locally Lipchitz continuous functions for all $(u,v) \ge \mathbf{0}$, and there exist nonnegative smooth functions $h(u)$ and $l(v)$ such that

$$f(u,v)h'(u)l(v) + g(u,v)h(u)l'(v) \ge H[h(u)l(v)] \ge 0, \tag{3.3}$$

where $H(s)$ is a continuous function for $s \ge 0$. If

$$\int_c^{+\infty} \frac{ds}{H(s)} < +\infty \quad \text{for some given} \quad c > 0, \tag{3.4}$$

then $h(u)l(v)$ and the solution of (1.2) will blow up in finite time for large initial data.

Proof: (1) Multiplying the equation $u_t = f(u,v)$ by $h'(u)l(v)$, the equation $v_t = g(u,v)$ by $h(u)l'(v)$, then adding the results, we have

$$h'(u)l(v)u_t + h(u)l'(v)v_t = f(u,v)h'(u)l(v) + g(u,v)h(u)l'(v) \le H(h(u)l(v)). \tag{3.5}$$

By the classic results of ODE, if $H(s) \ge 0$ with the condition of (3.2) or $-\infty < H(s) \le 0$, then the solution of the following ODE problem

$$s_t = H(s(t)), \quad s(0) = c > 0, \tag{3.6}$$

is global. However, the solution of (3.6) with $s(0) = h(c_1)l(c_2)$ is a super-solution of (3.5). Consequently, $h(u(t))l(v(t))$ is global.

Especially, if (3.2) holds, $f(u,v)$, $g(u,v)$, $h'(u)$ and $l'(v)$ are nonnegative, then we have $u_t \ge 0$, $v_t \ge 0$, $h(u) \ge h(u(t_0)) > 0$ and $l(v) \ge l(v(t_0)) > 0$ for some $t_0 > 0$. Therefore,

$$h(u(t_0))l(v(t)) \le h(u(t))l(v(t)), \quad h(u(t))l(v(t_0)) \le h(u(t))l(v(t)),$$

which implies that $h(u(t))$ and $l(v(t))$ are global. By the continuity and monotonicity of $h(u)$ and $l(v)$, we know that h^{-1} and l^{-1} exist, if $h(u(t)) \le M(t)$ and $l(v(t)) \le M(t)$ for any $0 < t < +\infty$, then $u(t) \le h^{-1}(M(t))$ and $v(t) \le l^{-1}(M(t))$ for any given $0 < t < +\infty$. That is, $(u(t), v(t))$ is global for any nonnegative initial data.

(2) Multiplying the equation $u_t = f(u,v)$ by $h'(u)l(v)$, the equation $v_t = g(u,v)$ by $h(u)l'(v)$, then adding the results, we have

$$h'(u)l(v)u_t + h(u)l'(v)v_t = f(u,v)h'(u)l(v) + g(u,v)h(u)l'(v) \ge H(h(u)l(v)). \tag{3.7}$$

By the classic theory of ODE, under the condition of (3.4), the solution of the following ODE problem

$$s_t = H(s(t)), \quad s(0) = c > 0 \tag{3.8}$$

will blow up in finite time for large initial data. The solution of (3.8) with $s(0) = h(c_1)l(c_2)$ is a sub-solution of (3.7). Consequently, $h(u(t))l(v(t))$ will blow up in finite time for large initial data. By the continuities of $h(u)$ and $l(v)$, we know that $(u(t), v(t))$ will blow up in finite time for large initial data.□

Remark 3.1: By the result of (1) and (2), if there exist nonnegative smooth functions $h(u)$ and $l(v)$ such that $f(u,v)h'(u)l(v) + g(u,v)h(u)l'(v) = H[h(u)l(v)] \ge 0$, then the watershed for blowup in finite time and global existence of the solution is $\int_0^{+\infty} \frac{ds}{H(s)} = +\infty$.

Some examples. We will give some examples to illustrate the results.

Example 3.1. $f(u,v) = Au^2v - Bu^p$, $g(u,v) = Cuv^2 - Dv^q (A, B, C, D > 0)$. If $p \ne 2$ and $q \ne 2$, we cannot write $f(u,v)$ and $g(u,v)$ in the form of $f(u,v) = f_1(v)g_1(u)$ and $g(u,v) = f_2(v)g_2(u)$.

(1) However, if $0 < p < 2$ and $0 < q < 2$, and $(u(0), v(0)) > (M, N)$ with (M, N) is large enough such that $f(u(0), v(0)) \ge 0$ and $g(u(0), v(0)) \ge 0$, taking $h(u) = u$ and $l(v) = v$, then we can get

$$(uv)_t = (A + C)u^2v^2 - Bu^pv - Duv^q, \tag{3.9}$$

and prove that uv and the solution of ODE (1.2) will blow up in finite time for large initial data. In fact, by continuities, we have $u_t \ge 0$ and $v_t \ge 0$ for all $t \ge 0$, and we can write (3.9) as

$$(uv)_t = u^2v^2[(A + C) - \frac{B}{u^{2-p}v} - \frac{D}{uv^{2-q}}] \ge \frac{(A + C)}{2}(uv)^2 := L(uv). \tag{3.10}$$

Obviously, (uv) will blow up in finite time for large initial data. Moreover, we even can construct the sub-solution of (1.2) has the form of

$$(\underline{u}(t),\ \underline{v}(t)) = (\frac{M}{(1-ct)},\ \frac{N}{(1-ct)}),$$

with $u(0) = M$, $v(0) = N$ and

$$AM^2N > BM^p, \quad CMN^2 > DN^q,$$

if $0 < p < 2$ and $0 < q < 2$. Here

$$c = \min(AMN - BM^{p-1},\ CMN - DN^{q-1}).$$

(2) On the other hand, if $0 < p < 2$, $0 < q < 2$, the initial data $u(0) = \epsilon_1$ and $v(0) = \epsilon_2$ are small enough such that $\epsilon_1^{2-p}\epsilon_2 \le \frac{B}{A}$ and $\epsilon_1\epsilon_2^{2-q} \le \frac{D}{C}$, then (3.9) can be written as

$$(uv)_t = u^p v(Au^{2-p}v - B) + uv^q(Cuv^{2-q} - D) \le 0 \equiv H(uv), \tag{3.11}$$

the solution of ODE (1.2) is global. In fact, we even can let (ϵ_1, ϵ_2) be a super-solution of (1.2), and obtain $u(t) \le \epsilon_1$, $v(t) \le \epsilon_2$.

Example 3.2. (1) The case of $f(u, v) \le 0$ and $g(u, v) \ge 0$ is contained in the cases of Theorem B. For example: Consider the following problem

$$\begin{cases} u_t = -Au^k(t)v^l(t), \quad v_t = Bu^m(t)v^n(t), \quad t > 0, \\ u(0) = c_1 \ge 0, \quad v(0) = c_2 \ge 0. \end{cases} \tag{3.12}$$

If there exist $\alpha > 0$ and $\beta > 0$ such that

$$\frac{k+\alpha-1}{\alpha} = \frac{l+\beta}{\beta} := p, \quad \frac{m+\alpha}{\alpha} = \frac{n+\beta-1}{\beta} := q. \tag{3.13}$$

Then

$$(u^\alpha v^\beta)_t = -A\alpha(u^\alpha v^\beta)^p + B\beta(u^\alpha v^\beta)^q. \tag{3.14}$$

If $1 < p \le q$ and $A\alpha < B\beta$, $(u^\alpha v^\beta)$ will blow up in finite time for positive initial data .

(2) In Theorem B and Theorem 2, the conditions $f(u, v) \ge 0$, $g(u, v) \ge 0$, $h'(u) \ge 0$ and $l'(v) \ge 0$ are needed to keep the solution (u, v) of (1.2) be global. We consider (3.12) again, but we will show that: There exist A, B, k, l, m and l such that $\lim_{t\to T^-} u(t) = 0$, $\lim_{t\to T^-} v(t) = +\infty$ but uv keeps bounded in $[0, T)$.

Let

$$u(t) = (T-t)^a, \quad v(t) = \frac{1}{(T-t)^b}.$$

Suppose that a, b, k, l, m and n satisfy

$$a(1-k) + bl = 1, \quad am + b(1-n) = -1,$$

and

$$a = \frac{l+1-n}{(1-k)(1-n)-lm} > 0, \quad b = \frac{k-1-m}{(1-k)(1-n)-lm} > 0.$$

(For example, $k = 1, l = 2, m = 1, n = 4, a = b = \frac{1}{2}$).

If $A = a$, $B = b$ and initial data

$$c_1 = T^a, \quad c_2 = \frac{1}{T^b},$$

then (3.12) has the exact solution $(u(t), v(t)) = ((T-t)^a, \frac{1}{(T-t)^b})$. Moreover, $\lim_{t\to T^-} u(t) = 0$, $\lim_{t\to T^-} v(t) = +\infty$ but wz keeps bounded in $[0, T)$ if $a \geq b > 0$.

(3) Applying these results to Example 3.2, we see that: Some kind of nonlinearities $f(u, v)$ and $g(u, v)$ make (u, v) blow up but $(u^\alpha v^\beta)$ exist globally, while some kind of nonlinearities $f(u, v)$ and $g(u, v)$ can make both (u, v) and $(u^\alpha v^\beta)$ blow up.

We can apply the conclusions of Theorem B to (1.1) as follows.

Theorem 2. *Assume that $f(u, v)$ and $g(u, v)$ are locally Lipchitz continuous functions for $(u, v) \geq \mathbf{0}$, $\frac{\partial f(u,v)}{\partial v} \geq 0$ and $\frac{\partial g(u,v)}{\partial u} \geq 0$. Suppose that $h(u) \geq 0$ and $h'(u) \geq 0$ for $u \geq 0$, $l(v) \geq 0$ and $l'(v) \geq 0$ for $v \geq 0$.*

(1) If (3.1) and (3.2) hold, then the solutions of (1.1) are global for any nonnegative initial data;

(2) Suppose that (2.27) and (2.28) hold. If (3.3) and (3.4) hold and the solution of (1.2) blows up in finite time for initial data (c_1, c_2), then the solution of (1.1) will blow up in finite time for initial data $(u_0(x), v_0(x)) \geq (c_1, c_2)$. If the solution of (1.2) blows up in finite time for initial data $(c_1, c_2) = \mathbf{0}$, then the solutions of (1.1) will blow up in finite time for initial data $(u_0(x), v_0(x)) \geq \mathbf{0}$.

Proof: The proof is entirely similar to that of Theorem 1. We can take the solution of (1.2) as the sub-solution or super-solution of (1.1), then we can obtain the corresponding conclusions. □

4. Some kinds of convection terms make the solution exist globally

In this section, we will show that some kinds of convection terms $a(x) \cdot \nabla u$ and $b(x) \cdot \nabla v$ make the solution exist globally.

Theorem 3. *Assume that $f(u, v)$ and $g(u, v)$ are smooth functions for $(u, v) \geq \mathbf{0}$, $\frac{\partial f}{\partial v} \geq 0$ and $\frac{\partial g}{\partial u} \geq 0$.*

(1) Suppose that the initial data $(u_0, v_0) \in C^2(\Omega) \times C^2(\Omega)$, $(u_0(x), v_0(x)) \geq \mathbf{0}$, $\nabla u_0(x) \neq \mathbf{0}$ and $\nabla v_0(x) \neq \mathbf{0}$ for all $x \in \Omega$. Moreover, for (1.1) suppose that $\frac{\partial u_0}{\partial \eta} \geq 0$ and $\frac{\partial v_0}{\partial \eta} \geq 0$ on $\partial\Omega$. Then there exist $a(x)$ and $b(x)$ such that the solution of (1.1) is global.

(2) Assume that there exist positive constants $K > 1$, $L > 1$, c_3, c_4, p_1, p_2, q_1 and q_2 such that $0 < p_1 < 1$, $0 < q_2 < 1$, $p_2 q_1 \geq (1-p_1)(1-q_2)$, $c_3^{p_1} c_4^{q_1} \leq \lambda_1 d_1 c_3 K$, $c_3^{p_2} c_4^{q_2} \leq \lambda_1 d_2 c_4 L$,

$$f(c_3\varphi^K, c_4\varphi^L) \leq c_3^{p_1} c_4^{q_1} \varphi^{Kp_1+Lq_1}, \quad g(c_3\varphi^K, c_4\varphi^L) \leq c_3^{p_2} c_4^{q_2} \varphi^{Kp_2+Lq_2}, \tag{4.1}$$

and

$$\varphi(x)a(x) \cdot \nabla\varphi(x) \geq (K-1)|\nabla\varphi(x)|^2, \tag{4.2}$$

$$\varphi(x)b(x) \cdot \nabla\varphi(x) \geq (L-1)|\nabla\varphi(x)|^2. \tag{4.3}$$

Here φ is the first eigenfunction of

$$-\Delta\varphi = \lambda_1\varphi \quad \text{in } \Omega, \quad \varphi = 0 \quad \text{on} \quad \partial\Omega. \tag{4.4}$$

normalised by $\max_{x\in\bar{\Omega}} \varphi(x) = 1$. Then the solution of (1.1) is global if the initial data $(u_0(x), v_0(x))$ satisfies $u_0(x) \leq c_3\varphi^K(x)$ and $v_0(x) \leq c_4\varphi^L(x)$.

Proof: (1) Note that $\frac{\partial f}{\partial v} \geq 0$ and $\frac{\partial g}{\partial u} \geq 0$, the comparison principle of a system of parabolic equations can be applied to (1.1). Since $\nabla u_0 \neq 0$ and $\nabla v_0 \neq 0$ for all $x \in \Omega$, there exist $a(x)$ and $b(x)$ such that

$$a(x) \cdot \nabla u_0 \geq d_1 \Delta u_0 + f(u_0, v_0), \quad b(x) \cdot \nabla v_0 \geq d_2 \Delta v_0 + g(u_0, v_0). \tag{4.5}$$

We write it as

$$0 \geq d_1 \Delta u_0 - a(x) \cdot \nabla u_0 + f(u_0, v_0), \quad 0 \geq d_2 \Delta v_0 - b(x) \cdot \nabla v_0 + g(u_0, v_0). \tag{4.6}$$

Combining this and the assumptions of $\frac{\partial u_0}{\partial \eta} \geq 0$ and $\frac{\partial v_0}{\partial \eta} \geq 0$ on $\partial\Omega$, we know that (u_0, v_0) is a super-solution of (1.1) with $a(x)$ and $b(x)$ satisfying (4.5) subject to Neumann boundary condition, hence the solutions of (1.1) is global.

Since $a(x)$ and $b(x)$ satisfy (4.6), and noticing that $u_0 \geq 0$ and $v_0 \geq 0$ on $\partial\Omega$, we can take (u_0, v_0) as a super-solution of (1.1) subject to Dirichlet boundary condition. Consequently, the solution of (1.1) subject to Dirichlet boundary condition is also global (globally bounded).

(2) Since $\frac{\partial f}{\partial v} \geq 0$ and $\frac{\partial g}{\partial u} \geq 0$, we can apply the comparison principle to (1.1).

Under the assumptions of $f(u, v)$, $g(u, v)$, $a(x)$ and $b(x)$, taking

$$(\bar{u}(x, t), \bar{v}(x, t)) = (c_3 \varphi^K(x), c_4 \varphi^L(x)),$$

we have

$$\begin{aligned}
& d_1 \Delta \bar{u} - a(x) \cdot \nabla \bar{u} + f(\bar{u}, \bar{v}) \\
= & -\lambda_1 d_1 c_3 K \varphi^K + c_3 K(K-1) \varphi^{K-2} |\nabla \varphi|^2 - c_3 K \varphi^{K-1} a(x) \cdot \nabla \varphi + f(c_3 \varphi^K, c_4 \varphi^L) \\
\leq & -\lambda_1 d_1 c_3 K \varphi^K + c_3 K \varphi^{K-2} [(K-1)|\nabla \varphi|^2 - \varphi a(x) \cdot \nabla \varphi] + c_3^{p_1} c_4^{q_1} \varphi^{K p_1 + L q_1} \\
\leq & [c_3^{p_1} c_4^{q_1} - \lambda_1 d_1 c_3 K] \varphi^K + c_3 K \varphi^{K-2} [(K-1)|\nabla \varphi|^2 - \varphi a(x) \cdot \nabla \varphi] \\
\leq & 0 \quad \text{for} \quad x \in \Omega, \ t > 0,
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
& d_2 \Delta \bar{v} - b(x) \cdot \nabla \bar{v} + g(\bar{u}, \bar{v}) \\
= & -\lambda_1 d_2 c_4 L \varphi^L + c_4 L(L-1) \varphi^{L-2} |\nabla \varphi|^2 - c_4 L \varphi^{L-1} b(x) \cdot \nabla \varphi + g(c_3 \varphi^K, c_4 \varphi^L) \\
\leq & -\lambda_1 d_2 c_4 L \varphi^L + c_4 L \varphi^{L-2} [(L-1)|\nabla \varphi|^2 - \varphi b(x) \cdot \nabla \varphi] + c_3^{p_2} c_4^{q_2} \varphi^{K p_2 + L q_2} \\
\leq & [c_3^{p_2} c_4^{q_2} - \lambda_1 d_2 c_4 L] \varphi^L + c_4 L \varphi^{L-2} [(L-1)|\nabla \varphi|^2 - \varphi b(x) \cdot \nabla \varphi] \\
\leq & 0 \quad \text{for} \quad x \in \Omega, \ t > 0.
\end{aligned} \tag{4.8}$$

Obviously,

$$(\bar{u}(x, t), \bar{v}(x, t)) = (c_3 \varphi^K(x), c_4 \varphi^L(x)) = (0, 0), \tag{4.9}$$

$$\frac{\partial \bar{u}}{\partial \eta} = c_3 K \varphi^{K-1} \frac{\partial \varphi}{\partial \eta} = 0, \quad \frac{\partial \bar{v}}{\partial \eta} = c_4 L \varphi^{L-1} \frac{\partial \varphi}{\partial \eta} = 0 \tag{4.10}$$

for $x \in \partial\Omega$ and $t > 0$.

Equations (4.7)–(4.9) (or Eq (4.10)) show that $(\bar{u}, \bar{v})$ is a super-solution of (1.1) for initial data $u_0(x) \leq c_3 \varphi^K(x)$ and $v_0(x) \leq c_4 \varphi^L(x)$, which means that the solution of (1.1)) is globally bounded. □

Remark 4.1. Theorem 3 shows that, the convection terms $a(x) \cdot \nabla u$ and $b(x) \cdot \nabla v$ can affect the properties for the solutions under certain conditions.

Remark 4.2. The typical $f(u, v)$ and $g(u, v)$ satisfying Theorem 3 are $f(u, v) = u^{p_1} v^{q_1}$ and $g(u, v) = u^{p_2} v^{q_2}$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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