



Research article

Asymptotic behavior of solutions to the inflow problem for the bipolar compressible quantum Navier-Stokes-Poisson equations

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Abstract: This paper investigates the asymptotic behavior of solutions to the inflow problem for the one-dimensional bipolar compressible quantum Navier-Stokes-Poisson system, a model describing the motion of two-species charged particles, electrons and holes, in ultra-small sub-micron semiconductor devices where quantum effects are significant. First, with the aid of stable manifold theory and center manifold theory, we established the existence and spatial-decay properties of the boundary layer to the inflow problem for the transonic and subsonic cases. Next, under suitable assumptions on the boundary data and the space-asymptotic states, we proved the asymptotic stability of the boundary layer and the superposition of the boundary layer and the rarefaction wave in the case that the initial perturbation and the strength of the boundary layer are sufficiently small. The proof was completed by the L^2 -energy method with the help of the spatial-decay properties of the boundary layer and the time-decay properties of the smooth approximate rarefaction wave.

Keywords: bipolar compressible quantum Navier-Stokes-Poisson equations; inflow problem; asymptotic behavior; boundary layer; rarefaction wave; energy method

1. Introduction

The bipolar compressible quantum Navier-Stokes-Poisson system is commonly employed to model the motion of two-species charged particles, electrons and holes, in ultra-small sub-micron semiconductor devices where quantum effects are significant [1, 2]. In a one-dimensional setting, this system can be

formulated in Eulerian coordinates as

$$\begin{cases} \rho_{1t} + (\rho_1 u_1)_x = 0, \\ (\rho_1 u_1)_t + (\rho_1 u_1^2 + p_1)_x - 2\hbar_1^2 \rho_1 \left(\frac{(\sqrt{\rho_1})_{xx}}{\sqrt{\rho_1}} \right)_x = \rho_1 E + (\mu_1(\rho_1) u_{1x})_x, \\ \rho_{2t} + (\rho_2 u_2)_x = 0, \\ (\rho_2 u_2)_t + (\rho_2 u_2^2 + p_2)_x - 2\hbar_2^2 \rho_2 \left(\frac{(\sqrt{\rho_2})_{xx}}{\sqrt{\rho_2}} \right)_x = -\rho_2 E + (\mu_2(\rho_2) u_{2x})_x, \\ E_x = \rho_1 - \rho_2 - D(x), \end{cases} \quad (1.1)$$

where $(t, x) \in (0, +\infty) \times (0, +\infty)$. The unknown functions are the electron density $\rho_1 = \rho_1(t, x) > 0$, the hole density $\rho_2 = \rho_2(t, x) > 0$, the electron velocity $u_1 = u_1(t, x)$, the hole velocity $u_2 = u_2(t, x)$, and the electric field $E = E(t, x)$. The terms $p_1 = \hat{p}_1(\rho_1)$ and $p_2 = \hat{p}_2(\rho_2)$ denote the pressures. $\mu_1(\rho_1)$ and $\mu_2(\rho_2)$ represent the viscosity coefficients, and the positive constants \hbar_1 and \hbar_2 are the scaled Planck constants. The terms $2\hbar_1^2 \rho_1 \left(\frac{(\sqrt{\rho_1})_{xx}}{\sqrt{\rho_1}} \right)_x$ and $2\hbar_2^2 \rho_2 \left(\frac{(\sqrt{\rho_2})_{xx}}{\sqrt{\rho_2}} \right)_x$ in the momentum equations (1.1)₂ and (1.1)₄ are referred to as quantum correction terms (or dispersive terms), first derived by Wigner [3]. The given function $D(x)$ represents the doping profile, describing the density of impurities in semiconductor devices. When modeling the transport of a single charged particle in semiconductors, system (1.1) reduces to the unipolar compressible quantum Navier-Stokes-Poisson equations. Moreover, in the absence of quantum effects, namely $\hbar_i = 0$ ($i = 1, 2$), system (1.1) then simplifies to its classical counterpart, the bipolar compressible Navier-Stokes-Poisson equations. A detailed discussion of the compressible quantum Navier-Stokes-Poisson equations' applications can be found in, for example, [4, 5].

Over the past few years, there has been growing interest in the mathematical analysis of the compressible quantum Navier-Stokes-Poisson equations, leading to substantial progress. Now, let us review some of these results. For the unipolar compressible quantum Navier-Stokes-Poisson system, Yang and Ju [6] first investigated the global-in-time existence of weak solutions and combined a quasi-neutral and vanishing damping limit on a three-dimensional torus. Subsequently, Yang and Ju [7] and Kwon [8] considered the combined quasi-neutral, inviscid limit with general initial data on a three-dimensional torus and an unbounded domain, respectively. Tang et al. [9] discussed the global existence of weak solutions to the two-dimensional unipolar compressible quantum Navier-Stokes-Poisson equations with large initial data. Anotonelli et al. [10] showed the convergence of the weak solutions to the unipolar compressible quantum Navier-Stokes-Poisson system toward the weak solutions to the quantum drift-diffusion equations as relaxation time tends to zero. Li and Sun in [11] demonstrated the asymptotic stability of rarefaction waves to the initial value problem for the one-dimensional system. Furthermore, Tong and Xia [12] and Wu and Hou [13] considered the stability of stationary solutions to the initial value problem for the three-dimensional system. However, the research on the bipolar compressible quantum Navier-Stokes-Poisson equations remains limited due to the system's inherent complexity. Wu [14] first investigated the large-time behavior of solutions to the Cauchy problem for the three-dimensional system with a non-trivial doping profile. Later, Wu and Zhu [15] studied the existence and asymptotic stability of the boundary layer to the outflow problem for the one-dimensional bipolar compressible quantum Navier-Stokes-Poisson equations, and in [16], they proved the nonlinear stability of the superposition of the boundary layer and rarefaction wave.

To our knowledge, the asymptotic behavior of solutions to the inflow problem for the bipolar compressible quantum Navier-Stokes-Poisson equations remains unexplored, forming the primary focus

of this work. For simplicity, we assume throughout this paper that

$$\mu_1(\rho_1) = \mu_2(\rho_2) = \mu, \quad \hbar_1 = \hbar_2 = 1, \quad \hat{p}_1(s) = \hat{p}_2(s) = \hat{p}(s) = as^\gamma, \quad D(x) \equiv 0, \quad (1.2)$$

where $\mu > 0$, $a > 0$, and $\gamma > 1$ are constants. Under this assumption, system (1.1) reduces to

$$\begin{cases} \rho_{1t} + (\rho_1 u_1)_x = 0, \\ (\rho_1 u_1)_t + (\rho_1 u_1^2 + \hat{p}(\rho_1))_x - 2\rho_1 \left(\frac{(\sqrt{\rho_1})_{xx}}{\sqrt{\rho_1}} \right)_x = \rho_1 E + \mu u_{1xx}, \\ \rho_{2t} + (\rho_2 u_2)_x = 0, \\ (\rho_2 u_2)_t + (\rho_2 u_2^2 + \hat{p}(\rho_2))_x - 2\rho_2 \left(\frac{(\sqrt{\rho_2})_{xx}}{\sqrt{\rho_2}} \right)_x = -\rho_2 E + \mu u_{2xx}, \\ E_x = \rho_1 - \rho_2. \end{cases} \quad (1.3)$$

We investigate the inflow problem for system (1.3) with the initial data

$$(\rho_1, u_1, \rho_2, u_2)(0, x) = (\rho_{10}, u_{10}, \rho_{20}, u_{20})(x) \rightarrow (\rho_+, u_+, \rho_+, u_+) \text{ as } x \rightarrow +\infty, \quad (1.4)$$

and the boundary conditions

$$\rho_1(t, 0) = \rho_2(t, 0) = \rho_-, \quad u_1(t, 0) = u_2(t, 0) = u_-, \quad \rho_{1x}(t, 0) = \rho_{2x}(t, 0) = \rho_b, \quad (1.5)$$

$$E(t, 0) = 0. \quad (1.6)$$

Here, ρ_\pm , u_\pm , and ρ_b are given constants satisfying

$$u_- > 0, \quad \rho_\pm > 0, \quad \rho_+ \neq \rho_-,$$

and the initial data $\rho_{10}(x)$ and $\rho_{20}(x)$ satisfy the following conditions:

$$\inf_{x \in \mathbb{R}_+} \rho_{i0}(x) > 0, \quad i = 1, 2, \quad \int_0^{+\infty} (\rho_{10} - \rho_{20})(x) dx = 0.$$

Since $u_- > 0$ signifies that charged particles blow into the region through the boundary $\{x = 0\}$, the initial-boundary value problem (1.3)–(1.6) is termed an inflow problem [17]. Conversely, the case of $u_- < 0$ is termed an outflow problem. Unlike the outflow problem, the inflow boundary condition implies that the characteristics of the hyperbolic equations (1.3)₁ and (1.3)₃ are positive around the boundary $\{x = 0\}$. Consequently, the boundary conditions must be imposed on both velocities (u_1, u_2) and densities (ρ_1, ρ_2) for well-posedness. Additionally, the quantum correction terms necessitate boundary conditions on the density gradients ρ_{1x} and ρ_{2x} . Of course, there exists an extensive body of literature addressing the asymptotic analysis of solutions to the inflow problem for both the compressible Navier-Stokes equations and the compressible Navier-Stokes-Poisson equations. For example, [18–22] considered the isentropic compressible Navier-Stokes equations, [23–27] investigated the full compressible Navier-Stokes equations, and [28–32] studied the compressible Navier-Stokes-Poisson equations.

In this paper, inspired by [15, 16, 20], we will investigate the asymptotic behavior of solutions to the inflow problem (1.3)–(1.6). Here, we are interested particularly in the case that the time-asymptotic state of the solution to the inflow problem (1.3)–(1.6) is the boundary layer (also called the stationary wave) and the superposition of the boundary layer and the rarefaction wave. We note that the previous

studies [20–22, 26, 27] addressed the inflow problem for compressible Navier-Stokes equations in Lagrangian coordinates, however, this approach is inapplicable to the bipolar compressible quantum Navier-Stokes-Poisson system because of the complexity of the system itself. Inspired by [25, 28], we consider the inflow problem for the bipolar compressible quantum Navier-Stokes-Poisson equations in Eulerian coordinates.

The first purpose of this article is to establish the existence of the boundary layer $(\tilde{\rho}, \tilde{u})(x)$ to the inflow problem of the bipolar compressible quantum Navier-Stokes-Poisson equations (1.3)–(1.6), which is the solution to the following boundary value problem:

$$\begin{cases} (\tilde{\rho}\tilde{u})_x = 0, \\ (\tilde{\rho}\tilde{u}^2 + \hat{p}(\tilde{\rho}))_x - 2\tilde{\rho}\left(\frac{(\sqrt{\tilde{\rho}})_{xx}}{\sqrt{\tilde{\rho}}}\right)_x = \mu\tilde{u}_{xx}, \\ \tilde{\rho}(0) = \rho_-, \quad \tilde{u}(0) = u_-, \quad \tilde{\rho}_x(0) = \rho_b, \quad (\tilde{\rho}, \tilde{u})(+\infty) = (\rho_+, u_+). \end{cases} \quad (1.7)$$

The results on the existence and properties of the boundary layer $(\tilde{\rho}, \tilde{u})(x)$ are stated as follows.

Theorem 1.1. *Suppose that $u_+ > 0$, and that the boundary data (ρ_-, u_-, ρ_b) satisfies*

$$\rho_- u_- = \rho_+ u_+ \quad (1.8)$$

and

$$(\rho_-, \rho_b) \in \mathcal{M} := \left\{ (\rho_1, \rho_2) \in \mathbb{R}^2 \mid 0 < |\rho_1 - \rho_+| + |\rho_2| \leq \varepsilon_0 \right\}, \quad (1.9)$$

where $\varepsilon_0 > 0$ is a sufficiently small constant. Set $\delta := |\rho_- - \rho_+| + |\rho_b|$. Note that condition (1.9) is equivalent to $\delta \leq \varepsilon_0$.

- 1) For the supersonic case, i.e., (ρ_+, u_+) satisfies $u_+ > \sqrt{\hat{p}'(\rho_+)}$, there is no solution to problem (1.7).
- 2) For the transonic case, i.e., (ρ_+, u_+) satisfies $u_+ = \sqrt{\hat{p}'(\rho_+)}$, there exists a curve $\Gamma_1 \subset \mathcal{M}$ such that if

$$(\rho_-, \rho_b) \in \Gamma_1, \quad (1.10)$$

then problem (1.7) has a unique smooth solution $(\tilde{\rho}, \tilde{u})(x)$ which satisfies

$$|\partial_x^k (\tilde{\rho} - \rho_+, \tilde{u} - u_+)(x)| \leq \frac{C\delta^{k+1}}{(1 + \delta x)^{k+1}}, \quad k = 0, 1, 2, \dots, \quad (1.11)$$

and

$$\tilde{\rho}_x(x) < 0, \quad \tilde{u}_x(x) > 0. \quad (1.12)$$

- 3) For the subsonic case, i.e., (ρ_+, u_+) satisfies $0 < u_+ < \sqrt{\hat{p}'(\rho_+)}$, there exists a curve $\Gamma_2 \subset \mathcal{M}$ such that if

$$(\rho_-, \rho_b) \in \Gamma_2, \quad (1.13)$$

then problem (1.7) has a unique smooth solution $(\tilde{\rho}, \tilde{u})(x)$ which satisfies

$$|\partial_x^k (\tilde{\rho} - \rho_+, \tilde{u} - u_+)(x)| \leq C\delta e^{-Cx}, \quad k = 0, 1, 2, \dots$$

The second aim of this work is to establish the asymptotic stability of the boundary layer $(\tilde{\rho}, \tilde{u})(x)$ to the inflow problem (1.3)–(1.6). This result is formulated in the theorem below.

Theorem 1.2. *Suppose that (1.8) and one of the following conditions hold: (1) $u_+ = \sqrt{\hat{p}'(\rho_+)}$ and (1.10); (2) $0 < u_+ < \sqrt{\hat{p}'(\rho_+)}$ and (1.13). Assume that $(\rho_{10} - \tilde{\rho}, u_{10} - \tilde{u}, \rho_{20} - \tilde{\rho}, u_{20} - \tilde{u}, E_0) \in H^2(\mathbb{R}_+) \times H^1(\mathbb{R}_+) \times H^2(\mathbb{R}_+) \times H^1(\mathbb{R}_+) \times L^2(\mathbb{R}_+)$, and set*

$$M_1 := \|(\rho_{10} - \tilde{\rho}, \rho_{20} - \tilde{\rho})\|_2 + \|(u_{10} - \tilde{u}, u_{20} - \tilde{u})\|_1 + \|E_0\|.$$

Then there exists a sufficiently small positive constant ε_1 such that if $M_1 + \delta \leq \varepsilon_1$, then the inflow problem (1.3)–(1.6) admits a unique global solution $(\rho_1, u_1, \rho_2, u_2, E)(t, x)$, which satisfies

$$\begin{aligned} \rho_i - \tilde{\rho} &\in C([0, +\infty); H^2(\mathbb{R}_+)), & \partial_x(\rho_i - \tilde{\rho}) &\in L^2(0, +\infty; H^2(\mathbb{R}_+)), & i = 1, 2, \\ u_i - \tilde{u} &\in C([0, +\infty); H^1(\mathbb{R}_+)), & \partial_x(u_i - \tilde{u}) &\in L^2(0, +\infty; H^1(\mathbb{R}_+)), & i = 1, 2, \\ E &\in C([0, +\infty); H^1(\mathbb{R}_+)), & E_x &\in L^2(0, +\infty; L^2(\mathbb{R}_+)), \end{aligned}$$

and

$$\begin{aligned} &\sup_{t \geq 0} \left\{ \|(\rho_1 - \tilde{\rho}, \rho_2 - \tilde{\rho})(t)\|_2^2 + \|(u_1 - \tilde{u}, u_2 - \tilde{u}, E)(t)\|_1^2 \right\} \\ &+ \int_0^{+\infty} \left(\|\partial_x(\rho_1 - \tilde{\rho}, \rho_2 - \tilde{\rho})(t)\|_2^2 + \|\partial_x(u_1 - \tilde{u}, u_2 - \tilde{u})(t)\|_1^2 + \|E_x(t)\|^2 \right) dt \leq C(M_1^2 + \delta). \end{aligned}$$

Here, $E_0(x) = \int_0^x (\rho_{10} - \rho_{20})(y) dy$. Moreover, it holds that

$$\lim_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}_+} |(\rho_1, u_1, \rho_2, u_2, E)(t, x) - (\tilde{\rho}, \tilde{u}, \tilde{\rho}, \tilde{u}, 0)(x)| = 0.$$

The final purpose of this article is to establish the asymptotic stability of the superposition of the boundary layer and the rarefaction wave to the inflow problem (1.3)–(1.6). For this aim, we assume that

$$u_+ > \sqrt{\hat{p}'(\rho_+)}, \quad (1.14)$$

and then there exists a unique constant state (ρ_*, u_*) which satisfies

$$u_+ = u_* - \int_{\rho_*}^{\rho_+} \frac{\sqrt{\hat{p}'(\xi)}}{\xi} d\xi, \quad u_* = \sqrt{\hat{p}'(\rho_*)}, \quad \rho_+ < \rho_*, \quad u_+ > u_* > 0. \quad (1.15)$$

We need to emphasize that in what follows, (ρ_+, u_+) in (1.7) and Theorem 1.1 is replaced by (ρ_*, u_*) . Now we assume that the boundary data (ρ_-, u_-, ρ_b) satisfies the conditions (1.8) and (1.10), and then there exists a boundary layer $(\tilde{\rho}, \tilde{u})(x)$ to (1.7) which connects (ρ_-, u_-, ρ_b) and (ρ_*, u_*) and satisfies (1.11) and (1.12). Moreover, there exists a rarefaction wave $(\rho^r, u^r)(\frac{x}{t})$ connecting (ρ_*, u_*) and (ρ_+, u_+) , which is the unique entropy solution to the Riemann problem for the compressible Euler equations given by

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + \hat{p}(\rho))_x = 0, \end{cases} \quad (1.16)$$

with

$$(\rho, u)(0, x) = \begin{cases} (\rho_*, u_*), & \text{if } x < 0, \\ (\rho_+, u_+), & \text{if } x > 0. \end{cases} \quad (1.17)$$

Then we can define the nonlinear wave $(\rho^s, u^s)(t, x)$ as

$$(\rho^s, u^s)(t, x) = (\tilde{\rho}, \tilde{u})(x) + (\rho^r, u^r)\left(\frac{x}{t}\right) - (\rho_*, u_*), \quad (1.18)$$

which is the superposition of two different wave patterns: the boundary layer and the rarefaction wave. To show the asymptotic stability of this nonlinear wave, we have to introduce the following smooth approximate nonlinear wave:

$$(\hat{\rho}, \hat{u})(t, x) := (\tilde{\rho}, \tilde{u})(x) + (\bar{\rho}, \bar{u})(t, x) - (\rho_*, u_*), \quad (1.19)$$

where $(\bar{\rho}, \bar{u})(t, x)$ is the smooth approximate rarefaction wave, which is defined in (3.3) and (3.4). The stability result of the nonlinear wave $(\rho^s, u^s)(t, x)$ to the inflow problem (1.3)–(1.6) is stated in the following theorem.

Theorem 1.3. *Suppose that (1.14) and (1.15) hold, and that (ρ_-, u_-, ρ_b) satisfies the conditions (1.8) and (1.10). Assume that $(\rho_{10} - \hat{\rho}_0, u_{10} - \hat{u}_0, \rho_{20} - \hat{\rho}_0, u_{20} - \hat{u}_0, E_0) \in H^2(\mathbb{R}_+) \times H^1(\mathbb{R}_+) \times H^2(\mathbb{R}_+) \times H^1(\mathbb{R}_+) \times L^2(\mathbb{R}_+)$, and set*

$$M_2 := \|(\rho_{10} - \hat{\rho}_0, \rho_{20} - \hat{\rho}_0)\|_2 + \|(u_{10} - \hat{u}_0, u_{20} - \hat{u}_0)\|_1 + \|E_0\|. \quad (1.20)$$

Then there exists a sufficiently small positive constant ε_2 such that if $M_2 + \delta + \varepsilon^{\frac{1}{10}} \leq \varepsilon_2$, then the inflow problem (1.3)–(1.6) has a unique global solution $(\rho_1, u_1, \rho_2, u_2, E)(t, x)$, which satisfies

$$\begin{aligned} \rho_i - \hat{\rho} &\in C([0, +\infty); H^2(\mathbb{R}_+)), & \partial_x(\rho_i - \hat{\rho}) &\in L^2(0, +\infty; H^2(\mathbb{R}_+)), & i = 1, 2, \\ u_i - \hat{u} &\in C([0, +\infty); H^1(\mathbb{R}_+)), & \partial_x(u_i - \hat{u}) &\in L^2(0, +\infty; H^1(\mathbb{R}_+)), & i = 1, 2, \\ E &\in C([0, +\infty); H^1(\mathbb{R}_+)), & E_x &\in L^2(0, +\infty; L^2(\mathbb{R}_+)), \end{aligned}$$

and

$$\begin{aligned} &\sup_{t \geq 0} \left\{ \|(\rho_1 - \hat{\rho}, \rho_2 - \hat{\rho})(t)\|_2^2 + \|(u_1 - \hat{u}, u_2 - \hat{u}, E)(t)\|_1^2 \right\} \\ &+ \int_0^{+\infty} \left(\|\partial_x(\rho_1 - \hat{\rho}, \rho_2 - \hat{\rho})(t)\|_2^2 + \|\partial_x(u_1 - \hat{u}, u_2 - \hat{u})(t)\|_1^2 + \|E_x(t)\|^2 \right) dt \leq C(M_2^2 + \delta + \varepsilon^{\frac{1}{10}}). \end{aligned}$$

Here, $(\hat{\rho}_0, \hat{u}_0)(x) = (\hat{\rho}, \hat{u})(0, x)$, $E_0(x) = \int_0^x (\rho_{10} - \rho_{20})(y) dy$, and ε is given in (3.2). Moreover, it holds that

$$\lim_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}_+} |(\rho_1, u_1, \rho_2, u_2, E)(t, x) - (\rho^s, u^s, \rho^s, u^s, 0)(t, x)| = 0. \quad (1.21)$$

Remark 1.4. *Theorem 1.1 shows the existence and spatial-decay estimates of the boundary layer to the inflow problem of the one-dimensional bipolar compressible quantum Navier-Stokes-Poisson equations (1.3)–(1.6) for the transonic and subsonic cases. Theorems 1.2 and 1.3 show that the boundary layer and the superposition of the boundary layer and the rarefaction wave to the inflow problem (1.3)–(1.6) are time-asymptotically stable when the initial perturbation and the strength of the boundary layer are small enough. We should mention that the proof of Theorem 1.2 is analogous to but simpler than that of Theorem 1.3, thus, for brevity, we omit the proof of Theorem 1.2 and only present the proof of Theorems 1.1 and 1.3 in this article.*

Remark 1.5. *In this article, we only consider the case that the initial perturbation is small enough. In fact, it is more fascinating to consider the corresponding stability results for a large initial perturbation. Moreover, we should mention that if the doping profile $D(x)$ is a non-trivial function, the asymptotic behavior of solutions to the inflow problem will be more interesting but more challenging than the case considered in this article. Investigating similar problems for the bipolar full compressible quantum Navier-Stokes-Poisson equations and the bipolar compressible quantum Navier-Stokes-Maxwell equations also remains an important direction for future research. These cases will be addressed in our subsequent work.*

We now sketch the proofs for Theorems 1.1 and 1.3. First, inspired by [25], to establish the existence of the boundary layer, we rewrite the boundary value problem (1.7) into a 2×2 autonomous system of first-order ordinary differential equations. Then, with the aid of stable manifold theory and center manifold theory, we show the existence and spatial-decay properties of the boundary layer. Next, making use of the L^2 -energy method with the help of the spatial-decay estimates of the boundary layer and the time-decay rates of the smooth approximate rarefaction wave, we prove the asymptotic stability of the superposition of the boundary layer and the rarefaction wave in the case that the initial perturbation and the strength of the boundary layer are small enough. Compared to the previous study [20], the main difficulties of this article are stated as follows. The first difficulty is the occurrence of quantum correction terms, which yields highly nonlinear terms in establishing the a priori estimates. Our strategy to overcome this difficulty is to decompose the quantum correction terms into two parts: third-order linear terms and lower-order nonlinear terms. This allows us to employ the third-order linear terms, together with the linear part of the system, to control the nonlinear terms. Moreover, we need more regularities of the densities to obtain the energy estimates (see Proposition 3.3). The second difficulty is the intersection of the boundary layer and the rarefaction wave. To overcome this difficulty, we will apply the wave interaction estimates (see Lemma 3.5). The third difficulty comes from the electric field. More precisely, the term $\int_0^{+\infty} \frac{1}{2} \hat{u}_x E^2 dx$ cannot be estimated directly. To address this difficulty, inspired by [15, 16, 28], we shall employ the equations (1.3)₂ and (1.3)₄ to derive a new form of E (see (3.41)), and then we will use this form to estimate the term $\int_0^{+\infty} \frac{1}{2} \hat{u}_x E^2 dx$.

Finally, we explain the new points of the inflow problem, compared to the outflow problem [15, 16], in analyzing the asymptotic behavior of solutions to the one-dimensional bipolar compressible quantum Navier-Stokes-Poisson equations. The first one is the existence result of the boundary layer. More precisely, for the outflow problem, [15] shows that there exists a smooth boundary layer for the supersonic case. But for the inflow problem, as clearly shown in Theorem 1.1, there is no solution for the supersonic case. Moreover, for the transonic case, we can obtain the monotonicity of the boundary layer for the inflow problem, but we do not expect that for the outflow problem. The second one lies in the stability analysis for nonlinear waves. For the outflow problem, [16] indicates that $u_1(t, 0) = u_2(t, 0) = u_- < 0$ is essential in establishing the a priori estimates. For example, in (3.30) and (3.40), one can deduce using $u_- < 0$ that

$$\int_0^{+\infty} (\rho_i u_i e_i)_x dx = -\rho_i(t, 0) u_- e_i(t, 0) \geq C |\phi_i(t, 0)|^2 > 0$$

and

$$\frac{1}{2} \hat{u} E^2 \Big|_{x=0}^{+\infty} = -\frac{1}{2} u_- |E(t, 0)|^2 > 0.$$

However, for the inflow problem, the above two terms are non-positive and cannot be controlled due to $u_- > 0$. We overcome such bad terms using the boundary conditions $\phi_i(t, 0) = \rho_i(t, 0) - \hat{\rho}(t, 0) = 0$ and $E(t, 0) = 0$. Next, under the condition $u_- > 0$, we can derive the estimate for the boundary term $\phi_{ixx}(t, 0)$ directly (see Lemma 3.9). Moreover, to overcome the interaction terms between the electron field and the densities, we develop a new strategy to establish the energy estimates (see Eqs (3.50), (3.51), (3.66), and (3.67)), which is different from the previous techniques [15, 16, 28]. These are the main novel aspects of this study.

The remainder of this paper is organized as follows. In Section 2, we establish the existence and spatial-decay properties of the boundary layer, and complete the proof of Theorem 1.1. Then we present the proof of Theorem 1.3 in Section 3.

Notations. $C > 0$ denotes a universal constant, which is independent of time and may vary from line to line. $C_i > 0$ ($i = 1, 2, \dots$) denotes some specific constant. $L^p(\mathbb{R}_+)$ ($1 \leq p \leq \infty$) represents the usual Lebesgue space on \mathbb{R}_+ , with the norm $\|\cdot\|_{L^p}$. For the special case $p = 2$, we simply denote $\|\cdot\|_{L^2}$ by $\|\cdot\|$. Furthermore, for a nonnegative integer l , $H^l(\mathbb{R}_+)$ denotes the usual L^2 -type Sobolev space of order l . We write $\|\cdot\|_l$ for the standard norm of $H^l(\mathbb{R}_+)$. Finally, for $T > 0$, we use $C([0, T]; H^l(\mathbb{R}_+))$ (resp., $L^2(0, T; H^l(\mathbb{R}_+))$) to denote the space of continuous (resp., square integrable) functions on $[0, T]$ with values taken in $H^l(\mathbb{R}_+)$.

2. Existence of the boundary layer

In this section, we are going to establish the existence of the boundary layer, which is the solution to the boundary value problem (1.7), thereby proving Theorem 1.1. First, integrating (1.7)₁ over $(0, +\infty)$ yields

$$\rho_- u_- = \rho_+ u_+,$$

which, together with $u_- > 0$, implies that $u_+ > 0$ is necessary for the existence of the boundary layer. Next, we reformulate problem (1.7) into a 2×2 autonomous first-order ODE system. By integrating Eqs (1.7)₁ and (1.7)₂ over $(x, +\infty)$ respectively, we derive

$$\begin{cases} \tilde{u} = \frac{\rho_+ u_+}{\tilde{\rho}}, \\ \rho_+ u_+ (u_+ - \tilde{u}) + a \rho_+^\gamma - a \tilde{\rho}^\gamma + (\tilde{\rho}_{xx} - \frac{\tilde{\rho}_x^2}{\tilde{\rho}}) = -\mu \tilde{u}_x, \end{cases} \quad (2.1)$$

where we have used the fact that $2\tilde{\rho}\left(\frac{(\sqrt{\tilde{\rho}})_{xx}}{\sqrt{\tilde{\rho}}}\right)_x = \left(\tilde{\rho}_{xx} - \frac{\tilde{\rho}_x^2}{\tilde{\rho}}\right)_x$. Now we introduce the new variables

$$W_1 = \tilde{\rho} - \rho_+, \quad W_2 = \tilde{\rho}_x. \quad (2.2)$$

Then (2.1) can be rewritten into the following form:

$$\frac{d}{dx} \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = P \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} + \begin{pmatrix} 0 \\ G(W_1, W_2) \end{pmatrix}, \quad (2.3)$$

where

$$P := \begin{pmatrix} 0 & 1 \\ \hat{p}'(\rho_+) - u_+^2 & \frac{\mu u_+}{\rho_+} \end{pmatrix},$$

$$G(W_1, W_2) = a((W_1 + \rho_+)^{\gamma} - \rho_+^{\gamma} - \gamma \rho_+^{\gamma-1} W_1) + \frac{u_+^2 W_1^2}{\rho_+} - \frac{u_+^2 W_1^3}{\rho_+(W_1 + \rho_+)} - \frac{2\mu u_+ W_1 W_2}{(W_1 + \rho_+)^2} + \frac{W_2^2}{W_1 + \rho_+} - \frac{\mu u_+ W_1^2 W_2}{\rho_+(W_1 + \rho_+)^2}.$$

Moreover, it follows from (1.7)₃ and (2.2) that

$$(W_1, W_2)(0) = (W_{1-}, W_{2-}) := (\rho_- - \rho_+, \rho_b), \quad (W_1, W_2)(+\infty) = (0, 0). \quad (2.4)$$

Let λ_1 and λ_2 denote eigenvalues of the matrix P , which come from the characteristic equation

$$\det(\lambda I - P) = \lambda^2 - \frac{\mu u_+}{\rho_+} \lambda + (u_+^2 - \hat{p}'(\rho_+)) = 0, \quad (2.5)$$

and let r_1 and r_2 be the corresponding eigenvectors of P , associated with λ_1 and λ_2 , respectively. According to elementary linear algebra, there exists a real nonsingular matrix Q such that

$$Q^{-1} P Q = \bar{P},$$

where \bar{P} is a 2×2 real matrix with two eigenvalues λ_1 and λ_2 . Especially, if λ_1 and λ_2 are different real eigenvalues, then $Q = (r_1, r_2)$, $\bar{P} = \text{diag}(\lambda_1, \lambda_2)$. Now we set

$$\begin{pmatrix} \bar{W}_1(x) \\ \bar{W}_2(x) \end{pmatrix} := Q^{-1} \begin{pmatrix} W_1(x) \\ W_2(x) \end{pmatrix}, \quad (2.6)$$

and

$$\begin{pmatrix} \bar{W}_{1-} \\ \bar{W}_{2-} \end{pmatrix} := Q^{-1} \begin{pmatrix} W_{1-} \\ W_{2-} \end{pmatrix}, \quad \begin{pmatrix} \bar{G}_1(\bar{W}_1, \bar{W}_2) \\ \bar{G}_2(\bar{W}_1, \bar{W}_2) \end{pmatrix} := Q^{-1} \begin{pmatrix} 0 \\ G(W_1, W_2) \end{pmatrix}. \quad (2.7)$$

Then problem (2.3)–(2.4) can be reformulated into

$$\frac{d}{dx} \begin{pmatrix} \bar{W}_1 \\ \bar{W}_2 \end{pmatrix} = \bar{P} \begin{pmatrix} \bar{W}_1 \\ \bar{W}_2 \end{pmatrix} + \begin{pmatrix} \bar{G}_1(\bar{W}_1, \bar{W}_2) \\ \bar{G}_2(\bar{W}_1, \bar{W}_2) \end{pmatrix}, \quad (2.8)$$

with

$$(\bar{W}_1, \bar{W}_2)(0) = (\bar{W}_{1-}, \bar{W}_{2-}), \quad (\bar{W}_1, \bar{W}_2)(+\infty) = (0, 0). \quad (2.9)$$

Thus, proving the existence of the boundary layer $(\bar{\rho}, \bar{u})(x)$ reduces to establishing the existence of the solution $(\bar{W}_1, \bar{W}_2)(x)$ to problem (2.8)–(2.9).

First, we consider the supersonic case, namely, $u_+ > \sqrt{\hat{p}'(\rho_+)}$. It is easy to see from (2.5) that the matrix P has two positive real eigenvalues or two complex conjugate eigenvalues with positive real parts. Therefore, there is no solution to problem (2.8)–(2.9).

Next, we investigate the transonic case that $u_+ = \sqrt{\hat{p}'(\rho_+)}$. In this case, the eigenvalues of P are

$$\lambda_1 = 0, \quad \lambda_2 = \frac{\mu u_+}{\rho_+} > 0,$$

and the corresponding eigenvectors are

$$r_1 = (1, 0)^t, \quad r_2 = \left(1, \frac{\mu u_+}{\rho_+}\right)^t,$$

respectively. Then

$$Q = \begin{pmatrix} 1 & 1 \\ 0 & \frac{\mu u_+}{\rho_+} \end{pmatrix}.$$

According to manifold theory (see, for instance, Theorem 3.2.1 on page 38 of [33]), there exists a local center manifold $W_{\text{loc}}^c(0, 0)$ and a local unstable manifold $W_{\text{loc}}^u(0, 0)$ corresponding to $\lambda_1 = 0$ and $\lambda_2 > 0$, respectively. More specifically,

$$W_{\text{loc}}^c(0, 0) = \{(\bar{W}_1, \bar{W}_2) \in \mathbb{R}^2 \mid \bar{W}_2 = h^c(\bar{W}_1), |\bar{W}_1| \text{ sufficiently small}\},$$

and

$$W_{\text{loc}}^u(0, 0) = \{(\bar{W}_1, \bar{W}_2) \in \mathbb{R}^2 \mid \bar{W}_1 = h^u(\bar{W}_2), |\bar{W}_2| \text{ sufficiently small}\},$$

where $h^c(x)$ and $h^u(x)$ are smooth functions with $h^c(0) = (h^c)'(0) = 0$ and $h^u(0) = (h^u)'(0) = 0$. Thus, to establish the existence of the solution $(\bar{W}_1, \bar{W}_2)(x)$ for problem (2.8)–(2.9), it is necessary to analyze the dynamics on the center manifold. To this end, let $z = z(x)$ denote the solution to the first equation of (2.8) restricted to the center manifold, that is, $z(x)$ satisfies the following equation:

$$z_x = \bar{G}_1(z, h^c(z)). \quad (2.10)$$

Then, based on center manifold theory (see Theorem 2 on page 4 of [34]), the solution to problem (2.8)–(2.9) can be expressed as

$$\bar{W}_1(x) = z(x), \quad (2.11)$$

$$\bar{W}_2(x) = h^c(z(x)). \quad (2.12)$$

Thus, to show the existence of the solution to (2.8)–(2.9), we only need to prove the existence of the solution $z(x)$ to (2.10) satisfying $z(x) \rightarrow 0$ as $x \rightarrow +\infty$. It follows from (2.6) and (2.7), and the fact

$$h^c(z) = O(z^2), \quad (2.13)$$

that

$$z_x = -\frac{a\gamma(\gamma+1)\rho_+^{\gamma-1}}{2\mu u_+} z^2 + O(z^3). \quad (2.14)$$

Since $u_+ > 0$, $z(x)$ decreases monotonically when $z(x)$ is sufficiently small. To ensure $z(x) \rightarrow 0$ as $x \rightarrow +\infty$, the boundary data $z(0)$ must be positive. Consequently, for the solution (\bar{W}_1, \bar{W}_2) to exist, the boundary data $(\bar{W}_{1-}, \bar{W}_{2-})$ should be located in the right region from the local unstable manifold, that is, $(\bar{W}_{1-}, \bar{W}_{2-})$ must satisfy

$$\bar{W}_{2-} = h^c(\bar{W}_{1-}), \quad \bar{W}_{1-} > h^u(\bar{W}_{2-}). \quad (2.15)$$

Next, Eq (2.14) gives

$$0 < \frac{C_1\delta}{1+\delta x} \leq z(x) \leq \frac{C_2\delta}{1+\delta x}, \quad |\partial_x^k z(x)| \leq \frac{C\delta^{k+1}}{(1+\delta x)^{k+1}}, \quad k = 1, 2, \dots \quad (2.16)$$

The combination of (2.11)–(2.13) with (2.16) yields

$$|\partial_x^k (\bar{W}_1, \bar{W}_2)(x)| \leq \frac{C\delta^{k+1}}{(1+\delta x)^{k+1}}, \quad k = 0, 1, 2, \dots \quad (2.17)$$

Finally, we study the subsonic case $u_+ < \sqrt{\hat{p}'(\rho_+)}$. It is obvious that the eigenvalues of the matrix P satisfy $\lambda_1 < 0$, $\lambda_2 > 0$. Then there exists a local stable manifold $W_{\text{loc}}^s(0, 0)$ and a local unstable manifold $W_{\text{loc}}^u(0, 0)$ corresponding to λ_1 and λ_2 , respectively. The local stable manifold is presented by

$$W_{\text{loc}}^s(0, 0) = \{(\bar{W}_1, \bar{W}_2) \in \mathbb{R}^2 \mid \bar{W}_2 = h^s(\bar{W}_1), |\bar{W}_1| \text{ sufficiently small}\},$$

where $h^s(x)$ is a smooth function with $h^s(0) = (h^s)'(0) = 0$. Therefore, if the boundary data $(\bar{W}_{1-}, \bar{W}_{2-})$ is located on the local stable manifold, namely,

$$\bar{W}_{2-} = h^s(\bar{W}_{1-}), \quad (2.18)$$

then problem (2.8)–(2.9) has a unique smooth solution $(\bar{W}_1, \bar{W}_2)(x)$ which satisfies

$$|\partial_x^k(\bar{W}_1, \bar{W}_2)(x)| \leq C\delta e^{-Cx}, \quad k = 0, 1, 2, \dots \quad (2.19)$$

We consolidate the preceding analysis into the following lemma.

Lemma 2.1. *Suppose that the boundary data $(\bar{W}_{1-}, \bar{W}_{2-})$ is sufficiently small.*

(i) *For the supersonic case $u_+ > \sqrt{\hat{p}'(\rho_+)}$, problem (2.8)–(2.9) has no solution.*

(ii) *For the transonic case $u_+ = \sqrt{\hat{p}'(\rho_+)}$, if $(\bar{W}_{1-}, \bar{W}_{2-})$ satisfies (2.15), then problem (2.8)–(2.9) has a unique smooth solution $(\bar{W}_1, \bar{W}_2)(x)$ which satisfies (2.17).*

(iii) *For the subsonic case $u_+ < \sqrt{\hat{p}'(\rho_+)}$, if $(\bar{W}_{1-}, \bar{W}_{2-})$ satisfies (2.18), then problem (2.8)–(2.9) has a unique smooth solution $(\bar{W}_1, \bar{W}_2)(x)$ which satisfies (2.19).*

Proof of Theorem 1.1. The existence and spatial-decay properties of the solution $(\tilde{\rho}, \tilde{u})(x)$ to problem (1.7) follow immediately from Lemma 2.1. We should mention that property (1.12) can be proved as follows. From (2.6), (2.11)–(2.13), we see that

$$\tilde{\rho} - \rho_+ = W_1 = \bar{W}_1 + \bar{W}_2 = z + O(z^2),$$

which, together with (2.14), implies

$$\begin{aligned} \tilde{\rho}_x &= -\frac{a\gamma(\gamma+1)\rho_+^{\gamma-1}}{2\mu u_+} z^2 + O(z^3) \leq -\frac{a\gamma(\gamma+1)\rho_+^{\gamma-1}}{4\mu u_+} z^2 < 0, \\ \tilde{u}_x &= -\frac{\rho_+ u_+ \tilde{\rho}_x}{\tilde{\rho}^2} > 0, \end{aligned}$$

where we have used the smallness of $|z|$.

Finally, we shall show that (2.15) and (2.18) yield (1.10) and (1.13), respectively. We define $\hat{W}_1 = \hat{W}_1(\tilde{\rho}, \tilde{\rho}_x)$ and $\hat{W}_2 = \hat{W}_2(\tilde{\rho}, \tilde{\rho}_x)$ by

$$(\hat{W}_1, \hat{W}_2)^t = Q^{-1}(\tilde{\rho} - \rho_+, \tilde{\rho}_x)^t,$$

and then, we may write that $\bar{W}_1(x) = \hat{W}_1(\tilde{\rho}, \tilde{\rho}_x)$, $\bar{W}_2(x) = \hat{W}_2(\tilde{\rho}, \tilde{\rho}_x)$. The curves Γ_1 and Γ_2 in Theorem 1.1 are defined by

$$\Gamma_1 := \{(\rho_1, \rho_2) \in \mathbb{R}^2 \mid \hat{W}_2(\rho_1, \rho_2) = h^c(\hat{W}_1(\rho_1, \rho_2)), \hat{W}_1(\rho_1, \rho_2) > h^u(\hat{W}_2(\rho_1, \rho_2))\},$$

$$\Gamma_2 := \{(\rho_1, \rho_2) \in \mathbb{R}^2 \mid \hat{W}_2(\rho_1, \rho_2) = h^s(\hat{W}_1(\rho_1, \rho_2))\}.$$

Thus, we have completed the proof of Theorem 1.1.

3. Proof of Theorem 1.3

This section is devoted to proving Theorem 1.3. For this aim, we first review the time-decay properties of the smooth approximate rarefaction wave and show some useful properties of the smooth approximate nonlinear wave in Subsection 3.1. Then we reformulate the original problem using perturbation variables in Subsection 3.2. Finally, we establish the a priori estimates and thereby complete the proof of Theorem 1.3 in Subsection 3.3.

3.1. The nonlinear wave

It is well known that the rarefaction wave $(\rho^r, u^r)(\frac{x}{t})$ is not smooth. Thus, to study the asymptotic behavior of solutions to the inflow problem (1.3)–(1.6), it is necessary to construct a smooth approximate rarefaction wave $(\bar{\rho}, \bar{u})(t, x)$ and a smooth approximate nonlinear wave $(\hat{\rho}, \hat{u})(t, x)$. In this subsection, we first review the properties of $(\bar{\rho}, \bar{u})(t, x)$, which have been established in [35], and then show some useful properties of $(\hat{\rho}, \hat{u})(t, x)$.

We begin with an analysis of the following Cauchy problem for the Burgers equation:

$$w_t + ww_x = 0, \quad (3.1)$$

with

$$w(0, x) = w_0(x) = \begin{cases} w_-, & x < 0, \\ w_- + C_q(w_+ - w_-) \int_0^{\varepsilon x} y^q e^{-y} dy, & x > 0, \end{cases} \quad (3.2)$$

where $w_- = u_* - \sqrt{\hat{p}'(\rho_*)} = 0$, $w_+ = u_+ - \sqrt{\hat{p}'(\rho_*)} > 0$, $C_q > 0$ is a constant satisfying $C_q \int_0^{+\infty} y^q e^{-y} dy = 1$ with $q \geq 10$ being a positive constant, and ε is a small positive constant to be determined later. As shown in [35], problem (3.1)–(3.2) has a unique global smooth solution $\bar{w}(t, x)$. Then the smooth approximate rarefaction wave $(\bar{\rho}, \bar{u})(t, x)$ can be constructed as follows:

$$\bar{w}(1+t, x) = \bar{u} - \sqrt{\hat{p}'(\bar{\rho})}, \quad (3.3)$$

$$\frac{d\bar{u}}{d\bar{\rho}} = -\frac{\sqrt{\hat{p}'(\bar{\rho})}}{\bar{\rho}}, \quad u_* = \bar{u}(\rho_*). \quad (3.4)$$

It is well known that $(\bar{\rho}, \bar{u})(t, x)$ has the following properties.

Lemma 3.1. (See [35]) Let $\tilde{\delta} = |\rho_+ - \rho_*| + |u_+ - u_*|$, and then $(\bar{\rho}, \bar{u})(t, x)$ satisfies

(i) $\bar{\rho}_x \leq 0$, $\bar{u}_x \geq 0$, $\rho_+ \leq \bar{\rho}(t, x) \leq \rho_*$, $u_* \leq \bar{u}(t, x) \leq u_+$, $x \in \mathbb{R}$, $t \geq 0$.

(ii) $(\bar{\rho}, \bar{u})|_{x \leq w_-(1+t)} = (\rho_*, u_*)$, $\frac{\partial^j}{\partial x^j}(\bar{\rho}, \bar{u})|_{x \leq w_-(1+t)} = (0, 0)$, $j = 1, 2, 3$.

(iii) For any p ($1 \leq p \leq +\infty$), there exists a constant $C_{p,q} > 0$ such that

$$\|(\bar{\rho}_x, \bar{u}_x)(t)\|_{L^p} \leq C_{p,q} \min\{\tilde{\delta}\varepsilon^{1-\frac{1}{p}}, \tilde{\delta}^{\frac{1}{p}}(1+t)^{-1+\frac{1}{p}}\}, \quad (3.5)$$

$$\|(\bar{\rho}_{xx}, \bar{u}_{xx})(t)\|_{L^p} \leq C_{p,q} \min\{\tilde{\delta}\varepsilon^{2-\frac{1}{p}}, \tilde{\delta}^{\frac{1}{q}}\varepsilon^{1-\frac{1}{p}+\frac{1}{q}}(1+t)^{-1+\frac{1}{q}}\}, \quad (3.6)$$

$$\|(\bar{\rho}_{xxx}, \bar{u}_{xxx})(t)\|_{L^p} \leq C_{p,q} \min\{\tilde{\delta}\varepsilon^{3-\frac{1}{p}}, \tilde{\delta}^{\frac{2}{q}}\varepsilon^{2-\frac{1}{p}+\frac{2}{q}}(1+t)^{-1+\frac{2}{q}}\}, \quad (3.7)$$

$$\|(\bar{\rho}_{xxxx}, \bar{u}_{xxxx})(t)\|_{L^p} \leq C_{p,q} \min\{\tilde{\delta}\varepsilon^{4-\frac{1}{p}}, \tilde{\delta}^{\frac{3}{q}}\varepsilon^{3-\frac{1}{p}+\frac{3}{q}}(1+t)^{-1+\frac{3}{q}}\}. \quad (3.8)$$

(iv) $\lim_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}} |(\bar{\rho}, \bar{u})(t, x) - (\rho^r, u^r)(\frac{x}{t})| = 0$.

We restrict $(\bar{\rho}, \bar{u})(t, x)$ on the half space $x \geq 0$, and it is easy to verify that $(\bar{\rho}, \bar{u})(t, x)$ satisfies

$$\begin{cases} \bar{\rho}_t + (\bar{\rho}\bar{u})_x = 0, \\ (\bar{\rho}\bar{u})_t + (\bar{\rho}\bar{u}^2 + \hat{p}(\bar{\rho}))_x = 0, \\ (\bar{\rho}, \bar{u})(t, 0) = (\rho_*, u_*), \quad (\bar{\rho}, \bar{u})(t, +\infty) = (\rho_+, u_+). \end{cases} \quad (3.9)$$

Next, from (1.7) and (3.9), we see that the smooth approximate nonlinear wave $(\hat{\rho}, \hat{u})(t, x)$, which is defined in (1.19), satisfies

$$\begin{cases} \hat{\rho}_t + (\hat{\rho}\hat{u})_x = \hat{f}, \\ \hat{\rho}(\hat{u}_t + \hat{u}\hat{u}_x) + \hat{p}'(\hat{\rho})\hat{\rho}_x = \mu\hat{u}_{xx} + \hat{\rho}_{xxx} + \hat{g}, \\ (\hat{\rho}, \hat{u})(0, x) = (\hat{\rho}_0, \hat{u}_0)(x), \\ (\hat{\rho}, \hat{u})(t, 0) = (\rho_-, u_-), \quad \hat{\rho}_x(t, 0) = \rho_b, \quad (\hat{\rho}, \hat{u})(t, +\infty) = (0, 0), \end{cases} \quad (3.10)$$

where

$$\begin{aligned} \hat{f} &:= \tilde{\rho}_x(\bar{u} - u_*) + \tilde{u}_x(\bar{\rho} - \rho_*) + \bar{\rho}_x(\tilde{u} - u_*) + \bar{u}_x(\tilde{\rho} - \rho_*), \\ \hat{g} &:= \tilde{u}\tilde{u}_x(\bar{\rho} - \rho_*) + \hat{\rho}\tilde{u}_x(\tilde{u} - u_*) + \hat{\rho}\tilde{u}_x(\bar{u} - u_*) + \bar{\rho}_x(\hat{p}'(\hat{\rho}) - \hat{p}'(\bar{\rho})) + \tilde{\rho}_x(\hat{p}'(\hat{\rho}) - \hat{p}'(\tilde{\rho})) \\ &\quad - \frac{2\tilde{\rho}_x\tilde{\rho}_{xx}}{\tilde{\rho}} + \frac{\tilde{\rho}_x^3}{\tilde{\rho}^2} - \frac{\hat{p}'(\bar{\rho})\bar{\rho}_x(\tilde{\rho} - \rho_*)}{\bar{\rho}} - \mu\bar{u}_{xx} - \bar{\rho}_{xxx}. \end{aligned}$$

Now we derive some properties of the smooth approximate nonlinear wave $(\hat{\rho}, \hat{u})$. First, from $|\tilde{\rho} - \rho_*| + |\tilde{u} - u_*| \leq C\delta \ll 1$, $\rho_+ \leq \bar{\rho} \leq \rho_*$, and $u_* \leq \bar{u} \leq u_+$, we obtain that

$$0 < \frac{\rho_+}{2} \leq \hat{\rho} = \tilde{\rho} + \bar{\rho} - \rho_* \leq \frac{3}{2}\rho_*, \quad 0 < \frac{u_*}{2} \leq \hat{u} = \tilde{u} + \bar{u} - u_* \leq \frac{3}{2}u_+. \quad (3.11)$$

Next, employing (1.12) and the first item in Lemma 3.1, one gets

$$\hat{\rho}_x < 0, \quad \hat{u}_x > 0. \quad (3.12)$$

3.2. Reformulation of the original problem

In this subsection, we are going to reformulate the original problem (1.3)–(1.6) using perturbation variables. First, we define

$$\phi_i(t, x) = \rho_i(t, x) - \hat{\rho}(t, x), \quad \psi_i(t, x) = u_i(t, x) - \hat{u}(t, x), \quad i = 1, 2. \quad (3.13)$$

Then, from (1.3)–(1.6) and (1.19), we see that $(\phi_1, \psi_1, \phi_2, \psi_2, E)(t, x)$ satisfies

$$\begin{cases} \phi_{1t} + u_1\phi_{1x} + \rho_1\psi_{1x} = F_1, \\ \rho_1(\psi_{1t} + u_1\psi_{1x}) + \hat{p}'(\rho_1)\phi_{1x} = \rho_1E + \mu\psi_{1xx} + \phi_{1xxx} + G_1, \\ \phi_{2t} + u_2\phi_{2x} + \rho_2\psi_{2x} = F_2, \\ \rho_2(\psi_{2t} + u_2\psi_{2x}) + \hat{p}'(\rho_2)\phi_{2x} = -\rho_2E + \mu\psi_{2xx} + \phi_{2xxx} + G_2, \\ E_x = \phi_1 - \phi_2, \end{cases} \quad (3.14)$$

with

$$(\phi_1, \psi_1, \phi_2, \psi_2, E)(0, x) = (\phi_{10}, \psi_{10}, \phi_{20}, \psi_{20}, E_0)(x), \quad (3.15)$$

$$\phi_1(t, 0) = \phi_2(t, 0) = 0, \quad \psi_1(t, 0) = \psi_2(t, 0) = 0, \quad \phi_{1x}(t, 0) = \phi_{2x}(t, 0) = 0, \quad E(t, 0) = 0. \quad (3.16)$$

Here

$$\phi_{i0}(x) = \rho_{i0}(x) - \hat{\rho}_0(x), \quad \psi_{i0}(x) = u_{i0}(x) - \hat{u}_0(x), \quad i = 1, 2,$$

and

$$\begin{aligned} F_i &= -\hat{\rho}_x \psi_i - \hat{u}_x \phi_i - \hat{f}, \\ G_i &= -\rho_i \hat{u}_x \psi_i - (\hat{p}'(\rho_i) - \hat{p}'(\hat{\rho})) \hat{\rho}_x - \frac{1}{\hat{\rho}} (\mu \hat{u}_{xx} + \hat{\rho}_{xxx} - \hat{p}'(\hat{\rho}) \hat{\rho}_x + \hat{g}) \phi_i - (\bar{\rho} - \rho_*) \tilde{u} \tilde{u}_x \\ &\quad - \hat{\rho} (\tilde{u} - u_*) \tilde{u}_x - \hat{\rho} (\bar{u} - u_*) \tilde{u}_x - (\hat{p}'(\hat{\rho}) - \hat{p}'(\bar{\rho})) \bar{\rho}_x - (\hat{p}'(\hat{\rho}) - \hat{p}'(\bar{\rho})) \tilde{\rho}_x \\ &\quad + \frac{\hat{p}'(\bar{\rho})}{\bar{\rho}} (\bar{\rho} - \rho_*) \bar{\rho}_x + \mu \bar{u}_{xx} + \bar{\rho}_{xxx} - \frac{2}{\rho_i} (\phi_{ix} \phi_{ixx} + \hat{\rho}_{xx} \phi_{ix} + \hat{\rho}_x \phi_{ixx}) \\ &\quad + \frac{1}{\rho_i^2} (\phi_{ix}^3 + 3\hat{\rho}_x \phi_{ix}^2 + 3\hat{\rho}_x^2 \phi_{ix}) - \frac{2}{\rho_i} (\bar{\rho}_x \bar{\rho}_{xx} + \bar{\rho}_x \tilde{\rho}_{xx} + \tilde{\rho}_x \bar{\rho}_{xx}) + \frac{2}{\tilde{\rho} \rho_i} \tilde{\rho}_x \tilde{\rho}_{xx} (\phi_i + \bar{\rho} - \rho_*) \\ &\quad + \frac{1}{\rho_i^2} (\bar{\rho}_x^3 + 3\tilde{\rho}_x^2 \tilde{\rho}_x + 3\tilde{\rho}_x \tilde{\rho}_x^2) - \frac{1}{\tilde{\rho}^2 \rho_i^2} \tilde{\rho}_x^3 (\phi_i^2 + 2\hat{\rho} \phi_i + 2\tilde{\rho} (\bar{\rho} - \rho_*) + (\bar{\rho} - \rho_*)^2) \end{aligned}$$

for $i = 1, 2$.

To prove that the solution $(\rho_1, u_1, \rho_2, u_2, E)(t, x)$ to the inflow problem (1.3)–(1.6) exists globally, it suffices to show the global existence of the solution $(\phi_1, \psi_1, \phi_2, \psi_2, E)(t, x)$ to problem (3.14)–(3.16). To this end, we apply the standard continuation argument, which hinges on two key components: a local existence theorem and a priori estimates. Using standard techniques, namely the dual argument and the iteration method, as employed in [36, 37], we establish the local existence of the solution in the following proposition and omit the detailed proof.

Proposition 3.2. (Local existence) *If the conditions of Theorem 1.3 are satisfied, then there exists a positive constant T_0 such that problem (3.14)–(3.16) has a unique solution $(\phi_1, \psi_1, \phi_2, \psi_2, E)(t, x)$ on the time interval $[0, T_0]$, which satisfies*

$$\begin{aligned} \phi_i &\in C([0, T_0]; H^2(\mathbb{R}_+)), & \phi_{ix} &\in L^2(0, T_0; H^2(\mathbb{R}_+)), & i = 1, 2, \\ \psi_i &\in C([0, T_0]; H^1(\mathbb{R}_+)), & \psi_{ix} &\in L^2(0, T_0; H^1(\mathbb{R}_+)), & i = 1, 2, \\ E &\in C([0, T_0]; H^1(\mathbb{R}_+)), & E_x &\in L^2(0, T_0; L^2(\mathbb{R}_+)). \end{aligned}$$

The main effort in the rest of this article is to prove the following a priori estimates.

Proposition 3.3. (A priori estimates) *Suppose that the assumptions of Theorem 1.3 hold, and let $(\phi_1, \psi_1, \phi_2, \psi_2, E)(t, x)$ be the solution to problem (3.14)–(3.16) on the time interval $[0, T]$ for some constant $T > 0$. Set*

$$M(T) := \sup_{0 \leq t \leq T} \left\{ \|(\phi_1, \phi_2)(t)\|_2 + \|(\psi_1, \psi_2)(t)\|_1 + \|E(t)\| \right\}. \quad (3.17)$$

Then there exists a sufficiently small positive constant ε_3 such that if $M(T) + \delta + \varepsilon^{\frac{1}{10}} \leq \varepsilon_3$, then the following estimate holds:

$$\begin{aligned} & \|(\phi_1, \phi_2)(t)\|_2^2 + \|(\psi_1, \psi_2, E)(t)\|_1^2 + \int_0^t \left(\|(\phi_{1x}, \phi_{2x})(\tau)\|_2^2 + \|(\psi_{1x}, \psi_{2x})(\tau)\|_1^2 + \|E_x(\tau)\|_1^2 \right. \\ & \left. + \|\sqrt{\hat{u}_x}(\phi_1, \phi_2, \psi_1, \psi_2)(\tau)\|^2 + |(\phi_{1xx}, \phi_{2xx})(\tau, 0)|^2 \right) d\tau \leq C(M_2^2 + \delta + \varepsilon^{\frac{1}{10}}) \end{aligned} \quad (3.18)$$

for any $t \in [0, T]$, where M_2 is given in (1.20).

3.3. A priori estimates

This subsection is devoted to proving Proposition 3.3. For brevity, we take $q = 10$ in Lemma 3.1. First of all, we derive from (3.17), (3.14)₅, and the Sobolev inequality $\|f\|_{L^\infty} \leq C\|f\|_2^{\frac{1}{2}}\|f_x\|_2^{\frac{1}{2}}$ that

$$\|(\phi_1, \phi_2, \phi_{1x}, \phi_{2x}, \psi_1, \psi_2, E)(t)\|_{L^\infty} \leq CM(T) \ll 1, \quad t \in [0, T], \quad (3.19)$$

which, together with (3.11), implies

$$0 < \frac{\rho_+}{4} \leq \rho_i(t, x) \leq 2\rho_*, \quad 0 < \frac{u_*}{4} \leq u_i(t, x) \leq 2u_+, \quad i = 1, 2, \quad (t, x) \in [0, T] \times \mathbb{R}_+. \quad (3.20)$$

The following Poincaré-type inequality, whose proof is given in [38], will be employed in deriving the a priori estimates.

Lemma 3.4. Assume that $f \in \{\tilde{\rho}, \tilde{u}\}$, and then for any $h \in H^1(\mathbb{R}_+)$, it holds that

$$\int_{\mathbb{R}_+} |\partial_x^j f|^k |h|^2 dx \leq C\delta(|h(0)|^2 + \|h_x\|^2), \quad (3.21)$$

for $k, j = 1, 2, \dots$, except $k = j = 1$.

Moreover, to control the interaction terms originating from the boundary layer and the rarefaction wave, we require the following inequalities, whose proofs can be found in [39].

Lemma 3.5. (Wave interaction estimates) For any $f, g \in \{\rho, u\}$, it holds that

$$\|\tilde{f}_x(\bar{g} - g_*)(t)\|_{L^1} + \|\tilde{f}_x(\bar{g} - g_*)(t)\|_{L^1} \leq C(\delta + \varepsilon^{\frac{1}{10}})(1+t)^{-\frac{4}{5}}, \quad (3.22)$$

$$\|\tilde{f}_x(\bar{g} - g_*)(t)\| + \|\tilde{f}_x(\bar{g} - g_*)(t)\| \leq C(\delta + \varepsilon^{\frac{1}{10}})(1+t)^{-\frac{9}{10}}, \quad (3.23)$$

$$\|\tilde{f}_{xx}(\bar{g} - g_*)(t)\|_{L^1} + \|\tilde{f}_{xx}(\bar{g} - g_*)(t)\| + \|\tilde{f}_{xxx}(\bar{g} - g_*)(t)\| \leq C\delta(1+t)^{-1}, \quad (3.24)$$

$$\|\tilde{f}_x \bar{g}_x(t)\|_{L^1} + \|\tilde{f}_x \bar{g}_x(t)\| + \|\tilde{f}_{xx} \bar{g}_x(t)\| \leq C\delta(1+t)^{-1}. \quad (3.25)$$

In what follows, we shall derive some a priori estimates for $(\phi_1, \psi_1, \phi_2, \psi_2, E)(t, x)$, which will be used to prove Proposition 3.3. The first step is to establish the basic energy estimate for the solution $(\phi_1, \psi_1, \phi_2, \psi_2, E)(t, x)$ to problem (3.14)–(3.16).

Lemma 3.6. Suppose that the conditions listed in Proposition 3.3 are met, and then it holds that

$$\begin{aligned} & \|(\phi_1, \phi_2, \phi_{1x}, \phi_{2x}, \psi_1, \psi_2, E)(t)\|^2 + \int_0^t \left(\|\sqrt{\hat{u}_x}(\phi_1, \phi_2, \psi_1, \psi_2)(\tau)\|^2 + \|(\psi_{1x}, \psi_{2x})(\tau)\|_1^2 \right) d\tau \\ & \leq C(M(T) + \delta + \varepsilon^{\frac{1}{10}}) \int_0^t \|(\phi_{1x}, \phi_{2x}, \phi_{1xx}, \phi_{2xx}, E_x(\tau))\|^2 d\tau + C(M_2 + \delta + \varepsilon^{\frac{1}{10}}) \end{aligned} \quad (3.26)$$

for any $t \in [0, T]$.

Proof. As in [16], we define

$$e_i = \frac{1}{2}\psi_i^2 + \Phi(\rho_i, \hat{\rho}), \quad (3.27)$$

where

$$\Phi(\rho_i, \hat{\rho}) = \int_{\hat{\rho}}^{\rho_i} \frac{\hat{\rho}(\xi) - \hat{\rho}(\hat{\rho})}{\xi^2} d\xi.$$

Hereafter, the subscript i takes values 1, 2 and suppose i is not summed. Combining (3.27) with (3.20) gives

$$C_3(\phi_i^2 + \psi_i^2) \leq e_i \leq C_4(\phi_i^2 + \psi_i^2), \quad (3.28)$$

for some positive constants C_3 and C_4 . Next, a tedious but direct computation yields

$$\begin{aligned} & \sum_{i=1}^2 \left((\rho_i e_i)_t + (\rho_i u_i e_i)_x + ((\hat{\rho}(\rho_i) - \hat{\rho}(\hat{\rho}))\psi_i - \mu\psi_i\psi_{ix})_x + \mu\psi_{ix}^2 \right) \\ &= \sum_{i=1}^2 (\phi_{ixxx}\psi_i + H_i - \hat{u}_x \rho_i \psi_i^2) + (\rho_1\psi_1 - \rho_2\psi_2)E, \end{aligned} \quad (3.29)$$

where

$$\begin{aligned} H_i &= -(\hat{\rho}(\rho_i) - \hat{\rho}(\hat{\rho}) - \hat{\rho}'(\hat{\rho})\phi_i)\hat{u}_x - \frac{\hat{\rho}'(\hat{\rho})}{\hat{\rho}}\hat{f}\phi_i + (G_i + \rho_i\hat{u}_x\psi_i + (\hat{\rho}'(\rho_i) - \hat{\rho}'(\hat{\rho}))\hat{\rho}_x) \\ &\quad - \frac{\hat{\rho}'(\hat{\rho})}{\hat{\rho}}\hat{\rho}_x\phi_i)\psi_i. \end{aligned}$$

Integrating (3.29) over $(0, +\infty)$ with respect to x and applying the boundary conditions (3.16), we arrive at

$$\begin{aligned} & \sum_{i=1}^2 \left(\frac{d}{dt} \int_0^{+\infty} \rho_i e_i dx + \int_0^{+\infty} \mu\psi_{ix}^2 dx \right) = \sum_{i=1}^2 \left(\int_0^{+\infty} \phi_{ixxx}\psi_i dx + \int_0^{+\infty} H_i dx \right. \\ & \left. - \int_0^{+\infty} \hat{u}_x \rho_i \psi_i^2 dx \right) + \int_0^{+\infty} (\rho_1\psi_1 - \rho_2\psi_2)E dx. \end{aligned} \quad (3.30)$$

Taking integration by parts, and employing (3.14)₁ and (3.14)₃, one gets

$$\begin{aligned} \int_0^{+\infty} \phi_{ixxx}\psi_i dx &= - \int_0^{+\infty} \phi_{ixx}\psi_{ix} dx \\ &= \int_0^{+\infty} \frac{1}{\rho_i} \phi_{ixx}\phi_{it} dx + \int_0^{+\infty} \frac{1}{\rho_i} \phi_{ixx}u_i\phi_{ix} dx - \int_0^{+\infty} \frac{1}{\rho_i} \phi_{ixx}F_i dx \\ &=: I_{i1} + I_{i2} + I_{i3}. \end{aligned} \quad (3.31)$$

First, we obtain, by applying the Hölder inequality, Lemma 3.1, (3.19)–(3.21), and (3.23), that

$$\begin{aligned} I_{i1} &= - \int_0^{+\infty} \frac{\phi_{ix}\phi_{ixt}}{\rho_i} dx + \int_0^{+\infty} \frac{\phi_{ix}\phi_{it}(\phi_{ix} + \hat{\rho}_x)}{\rho_i^2} dx \\ &= - \frac{d}{dt} \int_0^{+\infty} \frac{\phi_{ix}^2}{2\rho_i} dx + \int_0^{+\infty} \frac{\phi_{ix}^2(\phi_{it} - \hat{\rho}_t)}{2\rho_i^2} dx + \int_0^{+\infty} \frac{\phi_{ix}\phi_{it}\hat{\rho}_x}{\rho_i^2} dx \end{aligned}$$

$$\begin{aligned}
&= -\frac{d}{dt} \int_0^{+\infty} \frac{\phi_{ix}^2}{2\rho_i} dx + \int_0^{+\infty} \frac{1}{2\rho_i^2} \phi_{ix}^2 (F_i - u_i \phi_{ix} - \rho_i \psi_{ix} - \hat{f} + \hat{\rho}_x \hat{u} + \hat{\rho} \hat{u}_x) dx \\
&\quad + \int_0^{+\infty} \frac{1}{\rho_i^2} \phi_{ix} \hat{\rho}_x (F_i - u_i \phi_{ix} - \rho_i \psi_{ix}) dx \\
&\leq -\frac{d}{dt} \int_0^{+\infty} \frac{\phi_{ix}^2}{2\rho_i} dx + C(\|F_i(t)\|_{L^\infty} + \|\phi_{ix}(t)\|_{L^\infty} + \|\hat{f}(t)\|_{L^\infty} + \|\hat{\rho}_x(t)\|_{L^\infty} \\
&\quad + \|\hat{u}_x(t)\|_{L^\infty}) \|\phi_{ix}(t)\|^2 + C(\|\phi_{ix}(t)\|_{L^\infty} + \|\hat{\rho}_x(t)\|_{L^\infty}) \|\phi_{ix}(t)\| \|\psi_{ix}(t)\| \\
&\quad + C \int_0^{+\infty} (\bar{\rho}_x^2 + \bar{\rho}_x^2 + \bar{u}_x^2 + \bar{u}_x^2) (\phi_{ix}^2 + \phi_i^2 + \psi_i^2) dx + C \|\hat{\rho}_x(t)\|_{L^\infty} \|\phi_{ix}(t)\| \|\hat{f}(t)\| \\
&\leq -\frac{d}{dt} \int_0^{+\infty} \frac{\phi_{ix}^2}{2\rho_i} dx + C(M(T) + \delta + \varepsilon) \|(\phi_{ix}, \psi_{ix})(t)\|^2 \\
&\quad + C \int_0^{+\infty} (\bar{\rho}_x^2 + \bar{u}_x^2) (\phi_i^2 + \psi_i^2) dx + C \|\phi_{ix}(t)\| (\|\bar{\rho}_x(\bar{u} - \bar{u}_*)(t)\| + \|\bar{u}_x(\bar{\rho} - \rho_*)(t)\| \\
&\quad + \|\bar{\rho}_x(\bar{u} - u_*)(t)\| + \|\bar{u}_x(\bar{\rho} - \rho_*)(t)\|) \\
&\leq -\frac{d}{dt} \int_0^{+\infty} \frac{\phi_{ix}^2}{2\rho_i} dx + C(M(T) + \delta + \varepsilon) \|(\phi_{ix}, \psi_{ix})(t)\|^2 \\
&\quad + C(\delta + \varepsilon^{\frac{1}{10}}) (1+t)^{-\frac{9}{10}} \|\phi_{ix}(t)\| + C(\|\phi_i(t)\|_{L^\infty} + \|\psi_i(t)\|_{L^\infty}) (\|\bar{\rho}_x(t)\|^2 + \|\bar{u}_x(t)\|^2) \\
&\leq -\frac{d}{dt} \int_0^{+\infty} \frac{\phi_{ix}^2}{2\rho_i} dx + C(M(T) + \delta + \varepsilon^{\frac{1}{10}}) \|(\phi_{ix}, \psi_{ix})(t)\|^2 + C(\delta + \varepsilon^{\frac{1}{10}}) (1+t)^{-\frac{9}{5}} \\
&\quad + C(\|\phi_i(t)\| \|\phi_{ix}(t)\| + \|\psi_i(t)\| \|\psi_{ix}(t)\|) \varepsilon^{\frac{1}{10}} (1+t)^{-\frac{9}{10}} \\
&\leq -\frac{d}{dt} \int_0^{+\infty} \frac{\phi_{ix}^2}{2\rho_i} dx + C(M(T) + \delta + \varepsilon^{\frac{1}{10}}) \|(\phi_{ix}, \psi_{ix})(t)\|^2 + C(\delta + \varepsilon^{\frac{1}{10}}) (1+t)^{-\frac{9}{5}} \\
&\quad + CM(T) (\|\phi_{ix}(t)\| + \|\psi_{ix}(t)\|) \varepsilon^{\frac{1}{10}} (1+t)^{-\frac{9}{10}} \\
&\leq -\frac{d}{dt} \int_0^{+\infty} \frac{\phi_{ix}^2}{2\rho_i} dx + C(M(T) + \delta + \varepsilon^{\frac{1}{10}}) \|(\phi_{ix}, \psi_{ix})(t)\|^2 + C(\delta + \varepsilon^{\frac{1}{10}}) (1+t)^{-\frac{9}{5}}. \quad (3.32)
\end{aligned}$$

Second, one obtains after using (3.19) that

$$\begin{aligned}
I_{i2} &= \int_0^{+\infty} \left(\frac{u_i \phi_{ix}^2}{2\rho_i} \right)_x dx - \int_0^{+\infty} \frac{u_{ix} \phi_{ix}^2}{2\rho_i} dx + \int_0^{+\infty} \frac{u_i \phi_{ix}^2 \rho_{ix}}{2\rho_i^2} dx \\
&= -\int_0^{+\infty} \frac{\psi_{ix} \phi_{ix}^2}{2\rho_i} dx - \int_0^{+\infty} \frac{\hat{u}_x \phi_{ix}^2}{2\rho_i} dx + \int_0^{+\infty} \frac{u_i \phi_{ix}^3}{2\rho_i^2} dx + \int_0^{+\infty} \frac{u_i \hat{\rho}_x \phi_{ix}^2}{2\rho_i^2} dx \\
&\leq C \|\phi_{ix}(t)\|_{L^\infty} (\|\phi_{ix}(t)\| \|\psi_{ix}(t)\| + \|\phi_{ix}(t)\|^2) + C(\delta + \varepsilon) \|\phi_{ix}(t)\|^2 \\
&\leq C(M(T) + \delta + \varepsilon) \|(\phi_{ix}, \psi_{ix})(t)\|^2. \quad (3.33)
\end{aligned}$$

Third, taking integration by parts, and using the Hölder inequality, the Poincaré-type inequality (3.21), and (3.23) and (3.24), one arrives at

$$\begin{aligned}
I_{i3} &= \int_0^{+\infty} \frac{F_{ix} \phi_{ix}}{\rho_i} dx - \int_0^{+\infty} \frac{\phi_{ix} \rho_{ix} F_i}{\rho_i^2} dx \\
&\leq C(\delta + \varepsilon) \|(\phi_{ix}, \psi_{ix})(t)\|^2 + C \int_0^{+\infty} |\bar{\rho}_{xx}| \psi_i^2 dx + C \int_0^{+\infty} |\bar{u}_{xx}| \phi_i^2 dx
\end{aligned}$$

$$\begin{aligned}
& + C\|\phi_{ix}(t)\|(\|\hat{f}_x(t)\| + \|\hat{f}(t)\|) + C\|F_i(t)\|_{L^\infty}\|\phi_{ix}(t)\|^2 + C\int_0^{+\infty}(\bar{\rho}_x^2 + \bar{u}_x^2)(\phi_i^2 + \psi_i^2)dx \\
\leq & C(\delta + \varepsilon^{\frac{1}{10}})\|(\phi_{ix}, \psi_{ix})(t)\|^2 + C(\delta + \varepsilon^{\frac{1}{10}})(1+t)^{-\frac{9}{5}} \\
& + C(\|\phi_i(t)\|_{L^\infty} + \|\psi_i(t)\|_{L^\infty})(\|\bar{\rho}_{xx}(t)\|_{L^1} + \|\bar{u}_{xx}(t)\|_{L^1} + \|\bar{\rho}_x(t)\|^2 + \|\bar{u}_x(t)\|^2) \\
\leq & C(\delta + \varepsilon^{\frac{1}{10}})\|(\phi_{ix}, \psi_{ix})(t)\|^2 + C(\delta + \varepsilon^{\frac{1}{10}})(1+t)^{-\frac{9}{5}} \\
& + C(\|\phi_i(t)\|\|\phi_{ix}(t)\| + \|\psi_i(t)\|\|\psi_{ix}(t)\|)\varepsilon^{\frac{1}{10}}(1+t)^{-\frac{9}{10}} \\
\leq & C(\delta + \varepsilon^{\frac{1}{10}})\|(\phi_{ix}, \psi_{ix})(t)\|^2 + C(\delta + \varepsilon^{\frac{1}{10}})(1+t)^{-\frac{9}{5}}. \tag{3.34}
\end{aligned}$$

Putting (3.32)–(3.34) into (3.31) leads to

$$\begin{aligned}
\sum_{i=1}^2 \int_0^{+\infty} \phi_{ixxx} \psi_i dx & \leq \sum_{i=1}^2 \left(-\frac{d}{dt} \int_0^{+\infty} \frac{\phi_{ix}^2}{2\rho_i} dx + C(M(T) + \delta + \varepsilon^{\frac{1}{10}})\|(\phi_{ix}, \psi_{ix})(t)\|^2 \right) \\
+ C(\delta + \varepsilon^{\frac{1}{10}})(1+t)^{-\frac{9}{5}}. & \tag{3.35}
\end{aligned}$$

Next, we shall estimate the term $\int_0^{+\infty} H_i dx$. First, it is easy to verify from Taylor's formula that

$$\hat{p}(\rho_i) - \hat{p}(\hat{\rho}) - \hat{p}'(\hat{\rho})\phi_i \geq C\phi_i^2,$$

which, together with $\hat{u}_x > 0$, implies

$$-\int_0^{+\infty} (\hat{p}(\rho_i) - \hat{p}(\hat{\rho}) - \hat{p}'(\hat{\rho})\phi_i)\hat{u}_x dx \leq -C\int_0^{+\infty} \hat{u}_x \phi_i^2 dx. \tag{3.36}$$

Next, using (3.17) and (3.22), we arrive at

$$\begin{aligned}
-\int_0^{+\infty} \frac{\hat{p}'(\hat{\rho})}{\hat{\rho}} \hat{f} \phi_i dx & \leq C\|\phi_i(t)\|_{L^\infty}\|\hat{f}(t)\|_{L^1} \\
& \leq C\|\phi_i(t)\|^{\frac{1}{2}}\|\phi_{ix}(t)\|^{\frac{1}{2}}(\|\bar{\rho}_x(\bar{u} - \bar{u}_*)(t)\|_{L^1} + \|\bar{u}_x(\bar{\rho} - \rho_*)(t)\|_{L^1} \\
& \quad + \|\bar{\rho}_x(\bar{u} - u_*)(t)\|_{L^1} + \|\bar{u}_x(\bar{\rho} - \rho_*)(t)\|_{L^1}) \\
& \leq CM(T)^{\frac{1}{2}}(\delta + \varepsilon^{\frac{1}{10}})(1+t)^{-\frac{4}{5}}\|\phi_{ix}(t)\|^{\frac{1}{2}} \\
& \leq C(\delta + \varepsilon^{\frac{1}{10}})\|\phi_{ix}(t)\|^2 + C(\delta + \varepsilon^{\frac{1}{10}})(1+t)^{-\frac{16}{15}}. \tag{3.37}
\end{aligned}$$

Finally, employing (3.22), (3.24), and (3.25), and using an analysis similar to the proof of (3.32)–(3.34), we can show

$$\begin{aligned}
& \int_0^{+\infty} \left(G_i + \rho_i \hat{u}_x \psi_i + (\hat{p}'(\rho_i) - \hat{p}'(\hat{\rho}))\hat{\rho}_x - \frac{\hat{p}'(\hat{\rho})}{\hat{\rho}} \hat{\rho}_x \phi_i \right) dx \\
\leq & C(M(T) + \delta + \varepsilon^{\frac{1}{10}})\|(\phi_{ix}, \psi_{ix})(t)\|^2 + C(\delta + \varepsilon^{\frac{1}{10}})(1+t)^{-\frac{16}{15}}. \tag{3.38}
\end{aligned}$$

Combining (3.38) with (3.36) and (3.37) yields

$$\sum_{i=1}^2 \int_0^{+\infty} H_i dx \leq \sum_{i=1}^2 \left(-C\int_0^{+\infty} \hat{u}_x \phi_i^2 dx + C(M(T) + \delta + \varepsilon^{\frac{1}{10}})\|(\phi_{ix}, \psi_{ix})(t)\|^2 \right)$$

$$+ C(\delta + \varepsilon^{\frac{1}{10}})(1+t)^{-\frac{16}{15}}. \quad (3.39)$$

Next, differentiating (1.3)₅ in t , one obtains using (1.3)₁ and (1.3)₃

$$E_{xt} = (\rho_2\psi_2 - \rho_1\psi_1)_x - (\hat{u}E_x)_x.$$

Integrating it with respect to x over $(x, +\infty)$ yields

$$\rho_1\psi_1 - \rho_2\psi_2 = -E_t - \hat{u}E_x.$$

Then we have

$$\begin{aligned} \int_0^{+\infty} (\rho_1\psi_1 - \rho_2\psi_2)E dx &= - \int_0^{+\infty} EE_t dx - \int_0^{+\infty} \hat{u}EE_x dx \\ &= - \frac{d}{dt} \int_0^{+\infty} \frac{1}{2}E^2 dx - \frac{1}{2}\hat{u}E^2 \Big|_{x=0}^{+\infty} + \int_0^{+\infty} \frac{1}{2}\hat{u}_x E^2 dx \\ &= - \frac{d}{dt} \int_0^{+\infty} \frac{1}{2}E^2 dx + \int_0^{+\infty} \frac{1}{2}\hat{u}_x E^2 dx. \end{aligned} \quad (3.40)$$

Now we focus on the estimate of the term $\int_0^{+\infty} \frac{1}{2}\hat{u}_x E^2 dx$. Performing $\frac{(1.3)_1}{\rho_1} - \frac{(1.3)_2}{\rho_2}$ gives

$$\begin{aligned} E &= \frac{1}{2}(\psi_1 - \psi_2)_t + \frac{1}{4}(u_1^2 - u_2^2)_x + \frac{a\gamma}{2(\gamma-1)}(\rho_1^{\gamma-1} - \rho_2^{\gamma-1})_x - \frac{\mu}{2}\left(\frac{u_{1xx}}{\rho_1} - \frac{u_{2xx}}{\rho_2}\right) \\ &- \left(\frac{(\sqrt{\rho_1})_{xx}}{\sqrt{\rho_1}} - \frac{(\sqrt{\rho_2})_{xx}}{\sqrt{\rho_2}}\right)_x. \end{aligned} \quad (3.41)$$

Using (3.41), we can rewrite $\int_0^{+\infty} \frac{1}{2}\hat{u}_x E^2 dx$ as follows:

$$\begin{aligned} \int_0^{+\infty} \frac{1}{2}\hat{u}_x E^2 dx &= \frac{d}{dt} \int_0^{+\infty} \frac{\hat{u}_x E}{4}(\psi_1 - \psi_2) dx + \int_0^{+\infty} \frac{\hat{u}_x}{8}(\rho_1 + \rho_2)(\psi_1 - \psi_2)^2 dx \\ &- \int_0^{+\infty} \frac{\hat{u}\hat{u}_{xx}E}{4}(\psi_1 - \psi_2) dx + \int_0^{+\infty} \frac{\hat{u}_x}{8}(E(\psi_1^2 - \psi_2^2))_x dx - \int_0^{+\infty} \frac{\hat{u}_{xt}E}{4}(\psi_1 - \psi_2) dx \\ &+ \int_0^{+\infty} \frac{a\gamma\hat{u}_x E}{4(\gamma-1)}(\rho_1^{\gamma-1} - \rho_2^{\gamma-1})_x dx - \int_0^{+\infty} \frac{\mu\hat{u}_x E}{4}\left(\frac{u_{1xx}}{\rho_1} - \frac{u_{2xx}}{\rho_2}\right) dx \\ &- \int_0^{+\infty} \frac{\hat{u}_x E}{2}\left(\frac{(\sqrt{\rho_1})_{xx}}{\sqrt{\rho_1}} - \frac{(\sqrt{\rho_2})_{xx}}{\sqrt{\rho_2}}\right)_x dx. \end{aligned} \quad (3.42)$$

First, taking integration by parts, and using Young's inequality and the Poincaré-type inequality (3.21), we see

$$\begin{aligned} &- \int_0^{+\infty} \frac{\hat{u}\hat{u}_{xx}E}{4}(\psi_1 - \psi_2) dx + \int_0^{+\infty} \frac{\hat{u}_x}{8}(E(\psi_1^2 - \psi_2^2))_x dx \\ &= - \int_0^{+\infty} \frac{\hat{u}\hat{u}_{xx}E}{4}(\psi_1 - \psi_2) dx - \int_0^{+\infty} \frac{\hat{u}_{xx}E}{8}(\psi_1^2 - \psi_2^2) dx \\ &\leq C \int_0^{+\infty} (|\hat{u}_{xx}| + |\bar{u}_{xx}|)(E^2 + \psi_1^2 + \psi_2^2) dx + C\|E(t)\|_{L^\infty} \int_0^{+\infty} (|\bar{u}_{xx}| + |\hat{u}_{xx}|)(\psi_1^2 + \psi_2^2) dx \end{aligned}$$

$$\begin{aligned}
&\leq C\delta\|(\psi_{1x}, \psi_{2x}, E_x)(t)\|^2 + C(\|E(t)\|_{L^\infty} + \|\psi_1(t)\|_{L^\infty} + \|\psi_2(t)\|_{L^\infty})\|\bar{u}_{xx}(t)\|_{L^1} \\
&\leq C\delta\|(\psi_{1x}, \psi_{2x}, E_x)(t)\|^2 + CM(T)(\|E_x(t)\| + \|\psi_{1x}(t)\| + \|\psi_{2x}(t)\|)\varepsilon^{\frac{1}{10}}(1+t)^{-\frac{9}{10}} \\
&\leq C(\delta + \varepsilon^{\frac{1}{10}})\|(\psi_{1x}, \psi_{2x}, E_x)(t)\|^2 + C\varepsilon^{\frac{1}{10}}(1+t)^{-\frac{9}{5}}.
\end{aligned} \tag{3.43}$$

Next, it follows in a similar way as for the proof of (3.43) that

$$\begin{aligned}
&-\int_0^{+\infty} \frac{\hat{u}_x E}{4}(\psi_1 - \psi_2)dx + \int_0^{+\infty} \frac{a\gamma\hat{u}_x E}{4(\gamma-1)}(\rho_1^{\gamma-1} - \rho_2^{\gamma-1})_x dx \\
&-\int_0^{+\infty} \frac{\mu\hat{u}_x E}{4}\left(\frac{u_{1xx}}{\rho_1} - \frac{u_{2xx}}{\rho_2}\right)dx - \int_0^{+\infty} \frac{\hat{u}_x E}{2}\left(\frac{(\sqrt{\rho_1})_{xx}}{\sqrt{\rho_1}} - \frac{(\sqrt{\rho_2})_{xx}}{\sqrt{\rho_2}}\right)_x dx \\
&\leq C(\delta + \varepsilon^{\frac{1}{10}})\|(\phi_{1x}, \phi_{2x}, \phi_{1xx}, \phi_{2xx}, \psi_{1x}, \psi_{2x}, E_x)(t)\|^2 + C\varepsilon^{\frac{1}{10}}(1+t)^{-\frac{9}{5}}.
\end{aligned}$$

This combined with (3.42) and (3.43) gives

$$\begin{aligned}
&\int_0^{+\infty} \frac{1}{2}\hat{u}_x E^2 dx \leq \frac{d}{dt} \int_0^{+\infty} \frac{\hat{u}_x E}{4}(\psi_1 - \psi_2)dx + \int_0^{+\infty} \frac{\hat{u}_x}{8}(\rho_1 + \rho_2)(\psi_1 - \psi_2)^2 dx \\
&+ C\varepsilon^{\frac{1}{10}}(1+t)^{-\frac{9}{5}} + C(\delta + \varepsilon^{\frac{1}{10}})\|(\phi_{1x}, \phi_{2x}, \phi_{1xx}, \phi_{2xx}, \psi_{1x}, \psi_{2x}, E_x)(t)\|^2.
\end{aligned} \tag{3.44}$$

Then, (3.44) together with (3.40) implies

$$\begin{aligned}
&\int_0^{+\infty} (\rho_1\psi_1 - \rho_2\psi_2)Edx \leq -\frac{d}{dt} \int_0^{+\infty} \frac{1}{2}E^2 dx + \frac{d}{dt} \int_0^{+\infty} \frac{\hat{u}_x E}{4}(\psi_1 - \psi_2)dx \\
&+ \int_0^{+\infty} \frac{\hat{u}_x}{8}(\rho_1 + \rho_2)(\psi_1 - \psi_2)^2 dx + C(\delta + \varepsilon^{\frac{1}{10}})\|(\phi_{1x}, \phi_{2x}, \phi_{1xx}, \phi_{2xx}, \psi_{1x}, \psi_{2x}, E_x)(t)\|^2 \\
&+ C\varepsilon^{\frac{1}{10}}(1+t)^{-\frac{9}{5}}.
\end{aligned} \tag{3.45}$$

Moreover, applying the Young inequality, one has

$$\begin{aligned}
&-\int_0^{+\infty} \hat{u}_x(\rho_1\psi_1^2 + \rho_2\psi_2^2)dx + \int_0^{+\infty} \frac{\hat{u}_x}{8}(\rho_1 + \rho_2)(\psi_1 - \psi_2)^2 dx \\
&= -\int_0^{+\infty} \hat{u}_x\hat{\rho}(\psi_1^2 + \psi_2^2)dx - \int_0^{+\infty} \hat{u}_x(\phi_1\psi_1^2 + \phi_2\psi_2^2)dx + \int_0^{+\infty} \frac{\hat{u}_x}{8}(\phi_1 + \phi_2 + 2\hat{\rho})(\psi_1 - \psi_2)^2 dx \\
&\leq -\int_0^{+\infty} \hat{u}_x\hat{\rho}\left(\frac{3}{4}\psi_1^2 + \frac{3}{4}\psi_2^2 + \frac{1}{2}\psi_1\psi_2\right)dx + C(\|\phi_1(t)\|_{L^\infty} + \|\phi_2(t)\|_{L^\infty}) \int_0^{+\infty} \hat{u}_x(\psi_1^2 + \psi_2^2)dx \\
&\leq -\frac{1}{2} \int_0^{+\infty} \hat{u}_x\hat{\rho}(\psi_1^2 + \psi_2^2)dx + CM(T) \int_0^{+\infty} \hat{u}_x(\psi_1^2 + \psi_2^2)dx \\
&\leq -C \int_0^{+\infty} \hat{u}_x(\psi_1^2 + \psi_2^2)dx.
\end{aligned} \tag{3.46}$$

The combination of (3.46) with (3.45) shows

$$\begin{aligned}
&-\int_0^{+\infty} \hat{u}_x(\rho_1\psi_1^2 + \rho_2\psi_2^2)dx + \int_0^{+\infty} (\rho_1\psi_1 - \rho_2\psi_2)Edx \leq -\frac{d}{dt} \int_0^{+\infty} \frac{1}{2}E^2 dx \\
&+ \frac{d}{dt} \int_0^{+\infty} \frac{\hat{u}_x E}{4}(\psi_1 - \psi_2)dx - C \int_0^{+\infty} \hat{u}_x(\psi_1^2 + \psi_2^2)dx + C\varepsilon^{\frac{1}{10}}(1+t)^{-\frac{9}{5}}
\end{aligned}$$

$$+ C(\delta + \varepsilon^{\frac{1}{10}}) \|(\phi_{1x}, \phi_{2x}, \phi_{1xx}, \phi_{2xx}, \psi_{1x}, \psi_{2x}, E_x)(t)\|^2. \quad (3.47)$$

Inserting (3.35), (3.39), and (3.47) into (3.30), we have

$$\begin{aligned} & \frac{d}{dt} \int_0^{+\infty} \left(\rho_1 e_1 + \rho_2 e_2 + \frac{\phi_{1x}^2}{2\rho_1} + \frac{\phi_{2x}^2}{2\rho_2} + \frac{1}{2} E^2 - \frac{\hat{u}_x E}{4} (\psi_1 - \psi_2) \right) dx \\ & + \int_0^{+\infty} \mu (\psi_{1x}^2 + \psi_{2x}^2) dx + C \int_0^{+\infty} \hat{u}_x (\phi_1^2 + \phi_2^2 + \psi_1^2 + \psi_2^2) dx \leq C(\delta + \varepsilon^{\frac{1}{10}}) (1+t)^{-\frac{16}{15}} \\ & + C(M(T) + \delta + \varepsilon^{\frac{1}{10}}) \|(\phi_{1x}, \phi_{2x}, \phi_{1xx}, \phi_{2xx}, \psi_{1x}, \psi_{2x}, E_x)(t)\|^2. \end{aligned} \quad (3.48)$$

Integrating it over $(0, t)$, and employing (3.20), (3.28), and the smallness of $M(T) + \delta + \varepsilon^{\frac{1}{10}}$, we then obtain the desired estimate (3.26). This completes the proof.

Next, we will derive the estimate of $\int_0^t \|(\phi_{1x}, \phi_{2x}, \phi_{1xx}, \phi_{2xx}, E_x(\tau))\|^2 d\tau$.

Lemma 3.7. *Under the hypotheses of Proposition 3.3, the following estimate holds:*

$$\begin{aligned} & \|(\phi_{1x}, \phi_{2x})(t)\|^2 + \int_0^t \|(\phi_{1x}, \phi_{2x}, \phi_{1xx}, \phi_{2xx}, E_x(\tau))\|^2 d\tau \\ & \leq C \|(\psi_1, \psi_2)(t)\|^2 + C \int_0^t \|(\psi_{1x}, \psi_{2x})(\tau)\|^2 d\tau + C(M_2 + \delta + \varepsilon^{\frac{1}{10}}) \end{aligned} \quad (3.49)$$

for any $t \in [0, T]$.

Proof. Multiplying (3.14)₂ by $\frac{\phi_{1x}}{\rho_1}$, (3.14)₄ by $\frac{\phi_{2x}}{\rho_2}$, adding them together, and then integrating the resultant expression with respect to x over $(0, +\infty)$, we obtain

$$\begin{aligned} & \sum_{i=1}^2 \left(\frac{d}{dt} \int_0^{+\infty} \phi_{ix} \psi_i dx + \int_0^{+\infty} \frac{\hat{p}'(\rho_i)}{\rho_i} \phi_{ix}^2 dx - \int_0^{+\infty} \frac{\mu \phi_{ix} \psi_{ixx}}{\rho_i} dx \right) = \int_0^{+\infty} (\phi_{1x} - \phi_{2x}) E dx \\ & + \sum_{i=1}^2 \left(\int_0^{+\infty} (\phi_{ixt} \psi_i - u_i \phi_{ix} \psi_{ix} + \frac{\phi_{ixxx} \phi_{ix}}{\rho_i}) dx + \int_0^{+\infty} \frac{\phi_{ix} G_i}{\rho_i} dx \right). \end{aligned} \quad (3.50)$$

Let us estimate the terms on the right-hand side of (3.50). First, taking integration by parts and using (3.14)₅, we have

$$\begin{aligned} \int_0^{+\infty} (\phi_{1x} - \phi_{2x}) E dx & = - \int_0^{+\infty} (\phi_1 - \phi_2) E_x dx \\ & = - \int_0^{+\infty} E_x^2 dx. \end{aligned} \quad (3.51)$$

Next, taking integration by parts, and then applying the Young inequality, (3.19), (3.21), and (3.23), one has

$$\begin{aligned} & \int_0^{+\infty} \left(\phi_{ixt} \psi_i - u_i \phi_{ix} \psi_{ix} + \frac{\phi_{ixxx} \phi_{ix}}{\rho_i} \right) dx \\ & = - \int_0^{+\infty} \left(\psi_{ix} \phi_{it} + u_i \phi_{ix} \psi_{ix} + \phi_{ixx} \left(\frac{\phi_{ix}}{\rho_i} \right)_x \right) dx \end{aligned}$$

$$\begin{aligned}
&= \int_0^{+\infty} \psi_{ix}(\rho_i \psi_{ix} - F_i) dx - \int_0^{+\infty} \frac{\phi_{ixx}^2}{\rho_i} dx + \int_0^{+\infty} \frac{\phi_{ixx} \phi_{ix} (\phi_{ix} + \hat{\rho}_x)}{\rho_i^2} dx \\
&\leq - \int_0^{+\infty} \frac{\phi_{ixx}^2}{\rho_i} dx + C \|\psi_{ix}(t)\|^2 + C \|F_i(t)\|^2 + C (\|\phi_{ix}(t)\|_{L^\infty} + \|\hat{\rho}_x(t)\|_{L^\infty}) \|\phi_{ix}(t)\| \|\phi_{ixx}(t)\| \\
&\leq - \int_0^{+\infty} \frac{\phi_{ixx}^2}{\rho_i} dx + C \|\psi_{ix}(t)\|^2 + C \int_0^{+\infty} (\hat{\rho}_x^2 \psi_i^2 + \hat{u}_x^2 \phi_i^2 + \hat{f}^2) dx \\
&\quad + C(M(T) + \delta + \varepsilon) \|(\phi_{ix}, \phi_{ixx})(t)\|^2 \\
&\leq - \int_0^{+\infty} \frac{\phi_{ixx}^2}{\rho_i} dx + C \|\psi_{ix}(t)\|^2 + C(M(T) + \delta + \varepsilon) \|(\phi_{ix}, \phi_{ixx})(t)\|^2 \\
&\quad + C(\delta + \varepsilon^{\frac{1}{10}})(1+t)^{-\frac{9}{5}} + C \int_0^{+\infty} (\bar{\rho}_x^2 \psi_i^2 + \bar{u}_x^2 \phi_i^2) dx \\
&\leq - \int_0^{+\infty} \frac{\phi_{ixx}^2}{\rho_i} dx + C \|\psi_{ix}(t)\|^2 + C(M(T) + \delta + \varepsilon) \|(\phi_{ix}, \phi_{ixx})(t)\|^2 \\
&\quad + C(\delta + \varepsilon^{\frac{1}{10}})(1+t)^{-\frac{9}{5}} + CM(T) (\|\psi_{ix}(t)\| \|\bar{\rho}_x(t)\|^2 + \|\phi_{ix}(t)\| \|\bar{u}_x(t)\|^2) \\
&\leq - \int_0^{+\infty} \frac{\phi_{ixx}^2}{\rho_i} dx + C \|\psi_{ix}(t)\|^2 + C(M(T) + \delta + \varepsilon^{\frac{1}{10}}) \|(\phi_{ix}, \phi_{ixx})(t)\|^2 \\
&\quad + C(\delta + \varepsilon^{\frac{1}{10}})(1+t)^{-\frac{9}{5}}. \tag{3.52}
\end{aligned}$$

Finally, similar to the proofs of (3.35) and (3.39), we can show

$$\begin{aligned}
\int_0^{+\infty} \frac{\phi_{ix} G_i}{\rho_i} dx &\leq \int_0^{+\infty} \frac{\hat{p}'(\rho_i)}{2\rho_i} \phi_{ix}^2 dx + C(M(T) + \delta + \varepsilon^{\frac{1}{10}}) \|(\phi_{ix}, \phi_{ixx})(t)\|^2 \\
&\quad + C \|\psi_{ix}(t)\|^2 + C(\delta + \varepsilon^{\frac{1}{10}})(1+t)^{-\frac{8}{5}}. \tag{3.53}
\end{aligned}$$

Putting (3.51)–(3.53) into (3.50) gives

$$\begin{aligned}
&\sum_{i=1}^2 \left(\frac{d}{dt} \int_0^{+\infty} \phi_{ix} \psi_i dx + \int_0^{+\infty} \frac{\hat{p}'(\rho_i)}{2\rho_i} \phi_{ix}^2 dx + \int_0^{+\infty} \frac{\phi_{ixx}^2}{\rho_i} dx - \int_0^{+\infty} \frac{\mu \phi_{ix} \psi_{ixx}}{\rho_i} dx \right) \\
&\quad + \int_0^{+\infty} E_x^2 dx \leq \sum_{i=1}^2 \left(C \|\psi_{ix}(t)\|^2 + C(M(T) + \delta + \varepsilon^{\frac{1}{10}}) \|(\phi_{ix}, \phi_{ixx})(t)\|^2 \right) \\
&\quad + C(\delta + \varepsilon^{\frac{1}{10}})(1+t)^{-\frac{8}{5}}. \tag{3.54}
\end{aligned}$$

Next, differentiating (3.14)₁ and (3.14)₃ in x , respectively, we have

$$\phi_{1xt} + u_{1x} \phi_{1x} + u_1 \phi_{1xx} + \rho_{1x} \psi_{1x} + \rho_1 \psi_{1xx} = F_{1x}, \tag{3.55}$$

$$\phi_{2xt} + u_{2x} \phi_{2x} + u_2 \phi_{2xx} + \rho_{2x} \psi_{2x} + \rho_2 \psi_{2xx} = F_{2x}. \tag{3.56}$$

Taking $\int_0^{+\infty} ((3.55) \times \frac{\mu \phi_{1x}}{\rho_1^2} + (3.56) \times \frac{\mu \phi_{2x}}{\rho_2^2}) dx$, we obtain after using an analysis similar to the proof of (3.54) that

$$\sum_{i=1}^2 \left(\frac{d}{dt} \int_0^{+\infty} \frac{\mu \phi_{ix}^2}{2\rho_i^2} dx + \int_0^{+\infty} \frac{\mu \phi_{ix} \psi_{ixx}}{\rho_i} dx \right) \leq C(\delta + \varepsilon^{\frac{1}{10}})(1+t)^{-\frac{9}{5}} + \sum_{i=1}^2 \left(C \|\psi_{ix}(t)\|^2 \right)$$

$$+ C(M(T) + \delta + \varepsilon^{\frac{1}{10}})\|\phi_{ix}(t)\|^2. \quad (3.57)$$

Then adding (3.57) to (3.54) yields

$$\begin{aligned} & \frac{d}{dt} \int_0^{+\infty} \left(\frac{\mu\phi_{1x}^2}{2\rho_1^2} + \frac{\mu\phi_{2x}^2}{2\rho_2^2} + \phi_{1x}\psi_1 + \phi_{2x}\psi_2 \right) dx + \int_0^{+\infty} \left(\frac{\hat{p}'(\rho_1)}{2\rho_1} \phi_{1x}^2 + \frac{\hat{p}'(\rho_2)}{2\rho_2} \phi_{2x}^2 \right. \\ & + \left. \frac{\phi_{1xx}^2}{\rho_1} + \frac{\phi_{2xx}^2}{\rho_2} + E_x^2 \right) dx \leq C(M(T) + \delta + \varepsilon^{\frac{1}{10}})\|(\phi_{1x}, \phi_{2x}, \phi_{1xx}, \phi_{2xx})(t)\|^2 \\ & + C\|(\psi_{1x}, \psi_{2x})(t)\|^2 + C(\delta + \varepsilon^{\frac{1}{10}})(1+t)^{-\frac{8}{5}}. \end{aligned}$$

Integrating it over $(0, t)$ and using the smallness of $M(T) + \delta + \varepsilon^{\frac{1}{10}}$, one obtains (3.49). This finishes the proof of Lemma 3.7.

Now we can establish the following basic energy estimate.

Corollary 3.8. *If the conditions of Proposition 3.3 are satisfied, then it holds that*

$$\begin{aligned} & \|(\phi_1, \phi_2, \phi_{1x}, \phi_{2x}, \psi_1, \psi_2, E)(t)\|^2 + \int_0^t \|(\phi_{1x}, \phi_{2x}, \phi_{1xx}, \phi_{2xx}, \psi_{1x}, \psi_{2x}, E_x)(\tau)\|^2 \\ & + \|\sqrt{\hat{u}_x}(\phi_1, \phi_2, \psi_1, \psi_2)(\tau)\|^2 d\tau \leq C(M_2 + \delta + \varepsilon^{\frac{1}{10}}) \end{aligned} \quad (3.58)$$

for any $t \in [0, T]$.

Proof. By taking $\lambda \times (3.26) + (3.49)$ for some suitably large positive constant λ , and using the smallness of $M(T) + \delta + \varepsilon^{\frac{1}{10}}$, we obtain (3.58). The proof of Corollary 3.8 is finished.

Next, we derive the estimates for $\|(\phi_{1xx}, \phi_{2xx}, \psi_{1x}, \psi_{2x})(t)\|$ and $\int_0^t \|(\psi_{1xx}, \psi_{2xx})(\tau)\|^2 d\tau$.

Lemma 3.9. *Suppose that the conditions of Proposition 3.3 are satisfied, then it holds that*

$$\begin{aligned} & \|(\phi_{1xx}, \phi_{2xx}, \psi_{1x}, \psi_{2x})(t)\|^2 + \int_0^t (\|(\psi_{1xx}, \psi_{2xx})(\tau)\|^2 + |(\phi_{1xx}, \phi_{2xx})(\tau, 0)|^2) d\tau \\ & \leq C(M(T) + \delta + \varepsilon^{\frac{1}{10}}) \int_0^t \|(\phi_{1xxx}, \phi_{2xxx})(\tau)\|^2 d\tau + C(M_2 + \delta + \varepsilon^{\frac{1}{10}}) \end{aligned} \quad (3.59)$$

for any $t \in [0, T]$.

Proof. Taking $\int_0^{+\infty} (-\psi_{1xx} \times (3.14)_2 - \psi_{2xx} \times (3.14)_4) dx$ leads to the following equation:

$$\begin{aligned} & \sum_{i=1}^2 \left(\frac{d}{dt} \int_0^{+\infty} \frac{1}{2} \rho_i \psi_{ix}^2 dx + \int_0^{+\infty} \mu \psi_{ixx}^2 dx \right) = - \int_0^{+\infty} (\rho_1 \psi_{1xx} - \rho_2 \psi_{2xx}) E dx \\ & - \int_0^{+\infty} (\rho_{1x} \psi_{1x} - \rho_{2x} \psi_{2x}) E dx + \sum_{i=1}^2 \left[\int_0^{+\infty} \left(\frac{1}{2} \rho_{ii} \psi_{ix}^2 + \rho_i u_i \psi_{ix} \psi_{ixx} + \hat{p}'(\rho_i) \phi_{ix} \psi_{ixx} \right) dx \right. \\ & \left. - \int_0^{+\infty} \left(\frac{1}{\rho_i} \rho_{ix} \psi_{ix} \phi_{ixxx} + \phi_{ixxx} \psi_{ixx} \right) dx - \int_0^{+\infty} \rho_{ix} \psi_{ix} \left(\frac{\mu \psi_{ixx}}{\rho_i} + \frac{G_i}{\rho_i} - \frac{\hat{p}'(\rho_i)}{\rho_i} \phi_{ix} \right) dx \right] \end{aligned}$$

$$- u_i \psi_{ix}) dx - \int_0^{+\infty} G_i \psi_{ixx} dx]. \quad (3.60)$$

First, integration by parts implies that

$$\begin{aligned} & - \int_0^{+\infty} (\rho_1 \psi_{1xx} - \rho_2 \psi_{2xx}) E dx - \int_0^{+\infty} (\rho_{1x} \psi_{1x} - \rho_{2x} \psi_{2x}) E dx \\ &= - \int_0^{+\infty} (\rho_1 \psi_{1x} - \rho_2 \psi_{2x})_x E dx \\ &= \int_0^{+\infty} (\rho_1 \psi_{1x} - \rho_2 \psi_{2x}) E_x dx \\ &\leq C \|(\psi_{1x}, \psi_{2x}, E_x)(t)\|^2. \end{aligned} \quad (3.61)$$

Next, making use of the Young inequality and (3.17), we arrive at

$$\begin{aligned} & \int_0^{+\infty} \left(\frac{1}{2} \rho_{ii} \psi_{ix}^2 + \rho_i u_i \psi_{ix} \psi_{ixx} + \hat{p}'(\rho_i) \phi_{ix} \psi_{ixx} \right) dx \\ &\leq C \int_0^{+\infty} (|\phi_{ix}| + |\psi_{ix}| + |\hat{\rho}_x| + |\hat{u}_x|) \psi_{ix}^2 dx + \frac{\mu}{4} \|\psi_{ixx}(t)\|^2 + C \|(\phi_{ix}, \psi_{ix})(t)\|^2 \\ &\leq C \|\psi_{ix}(t)\|_{L^\infty} \|\psi_{ix}(t)\|^2 + \frac{\mu}{4} \|\psi_{ixx}(t)\|^2 + C \|(\phi_{ix}, \psi_{ix})(t)\|^2 \\ &\leq C \|\psi_{ix}(t)\|^{\frac{5}{2}} \|\psi_{ixx}(t)\|^{\frac{1}{2}} + \frac{\mu}{4} \|\psi_{ixx}(t)\|^2 + C \|(\phi_{ix}, \psi_{ix})(t)\|^2 \\ &\leq CM(T) \|\psi_{ix}(t)\|^{\frac{3}{2}} \|\psi_{ixx}(t)\|^{\frac{1}{2}} + \frac{\mu}{4} \|\psi_{ixx}(t)\|^2 + C \|(\phi_{ix}, \psi_{ix})(t)\|^2 \\ &\leq \frac{\mu}{4} \|\psi_{ixx}(t)\|^2 + CM(T) \|\psi_{ixx}(t)\|^2 + C \|(\phi_{ix}, \psi_{ix})(t)\|^2. \end{aligned} \quad (3.62)$$

Moreover, taking integration by parts and applying (3.17), (3.21), (3.24), (3.25), (3.55), and (3.56), one has

$$\begin{aligned} & - \int_0^{+\infty} \left(\frac{1}{\rho_i} \rho_{ix} \psi_{ix} \phi_{ixxx} + \phi_{ixxx} \psi_{ixx} \right) dx \\ &= \int_0^{+\infty} \frac{1}{\rho_i} \phi_{ixxx} (\phi_{ixt} + u_{ix} \phi_{ix} + u_i \phi_{ixx} - F_{ix}) dx \\ &= - \frac{d}{dt} \int_0^{+\infty} \frac{\phi_{ixx}^2}{2\rho_i} dx + \int_0^{+\infty} \frac{1}{2\rho_i^2} (\phi_{ix} u_i + \hat{\rho}_x u_i + \rho_i \psi_{ix} + \rho_i \hat{u}_x) \phi_{ixx}^2 dx \\ &\quad - \int_0^{+\infty} \frac{1}{\rho_i^2} \phi_{ixx} (\phi_{ix} + \hat{\rho}_x) (u_i \phi_{ixx} + 2\psi_{ix} \phi_{ix} + 2\hat{u}_x \phi_{ix} + 2\hat{\rho}_x \psi_{ix} + \rho_i \psi_{ixx} + \hat{\rho}_{xx} \psi_i + \hat{u}_{xx} \phi_i \\ &\quad + \hat{f}_x) dx + \int_0^{+\infty} \frac{1}{\rho_i} \phi_{ixxx} \phi_{ix} (\psi_{ix} + \hat{u}_x) dx - \frac{u_-}{2\rho_-} |\phi_{ixx}(t, 0)|^2 - \int_0^{+\infty} \frac{1}{2} \phi_{ixx}^2 \left(\frac{\psi_{ix} + \hat{u}_x}{\rho_i} \right. \\ &\quad \left. - \frac{\phi_{ix} + \hat{\rho}_x}{\rho_i^2} \right) dx + \int_0^{+\infty} \frac{1}{\rho_i} \phi_{ixxx} (\hat{\rho}_x \psi_{ix} + \hat{\rho}_{xx} \psi_i + \hat{u}_x \phi_{ix} + \hat{u}_{xx} \phi_i + \hat{f}_x) dx \\ &\leq - \frac{d}{dt} \int_0^{+\infty} \frac{\phi_{ixx}^2}{2\rho_i} dx + C \|(\phi_{ix}, \phi_{ixx}, \psi_{ix})(t)\|^2 + C(M(T) + \delta + \varepsilon) \|(\phi_{ixxx}, \psi_{ixx})(t)\|^2 \end{aligned}$$

$$\begin{aligned}
& + C\|\psi_{ix}(t)\|_{L^\infty}\|\phi_{ixx}(t)\|^2 + C\|\psi_i(t)\|_{L^\infty}^2\|\bar{\rho}_{xx}(t)\|_{L^1} + C\|\phi_i(t)\|_{L^\infty}^2\|\bar{u}_{xx}(t)\|_{L^1} \\
& + C(\|\phi_{ixx}(t)\| + \|\phi_{ixxx}(t)\|)\|\hat{f}_x(t)\| - \frac{u_-}{2\rho_-}|\phi_{ixx}(t, 0)|^2 \\
\leq & -\frac{d}{dt}\int_0^{+\infty}\frac{\phi_{ixx}^2}{2\rho_i}dx - \frac{u_-}{2\rho_-}|\phi_{ixx}(t, 0)|^2 + C\|(\phi_{ix}, \phi_{ixx}, \psi_{ix})(t)\|^2 \\
& + C(M(T) + \delta + \varepsilon^{\frac{1}{10}})\|(\phi_{ixxx}, \psi_{ixx})(t)\|^2 + CM(T)\|\psi_{ix}(t)\|^{\frac{1}{2}}\|\psi_{ixx}(t)\|^{\frac{1}{2}}\|\phi_{ixx}(t)\| \\
& + C(\delta + \varepsilon^{\frac{1}{10}})(1+t)^{-\frac{9}{5}} + CM(T)(\|\phi_{ix}(t)\| + \|\psi_{ix}(t)\|)\varepsilon^{\frac{1}{10}}(1+t)^{-\frac{9}{10}} \\
\leq & -\frac{d}{dt}\int_0^{+\infty}\frac{\phi_{ixx}^2}{2\rho_i}dx - \frac{u_-}{2\rho_-}|\phi_{ixx}(t, 0)|^2 + C\|(\phi_{ix}, \phi_{ixx}, \psi_{ix})(t)\|^2 \\
& + C(M(T) + \delta + \varepsilon^{\frac{1}{10}})\|(\phi_{ixxx}, \psi_{ixx})(t)\|^2 + C(\delta + \varepsilon^{\frac{1}{10}})(1+t)^{-\frac{9}{5}}. \tag{3.63}
\end{aligned}$$

Finally, using an analysis similar to the proofs of (3.35) and (3.39), we can prove that

$$\begin{aligned}
& -\int_0^{+\infty}\rho_{ix}\psi_{ix}\left(\frac{\mu\psi_{ixx}}{\rho_i} + \frac{G_i}{\rho_i} - \frac{\hat{p}'(\rho_i)}{\rho_i}\psi_{ix} - u_i\psi_{ix}\right)dx - \int_0^{+\infty}G_i\psi_{ixx}dx \\
\leq & \frac{\mu}{4}\|\psi_{ixx}(t)\|^2 + C(M(T) + \delta + \varepsilon^{\frac{1}{10}})\|\psi_{ixx}(t)\|^2 + C\|(\phi_{ix}, \phi_{ixx}, \psi_{ix})(t)\|^2 \\
& + C(\delta + \varepsilon^{\frac{1}{10}})(1+t)^{-\frac{8}{5}}. \tag{3.64}
\end{aligned}$$

Inserting (3.61)–(3.64) into (3.60) yields

$$\begin{aligned}
& \frac{d}{dt}\int_0^{+\infty}\left(\frac{1}{2}\rho_1\psi_{1x}^2 + \frac{1}{2}\rho_2\psi_{2x}^2 + \frac{\phi_{1xx}^2}{2\rho_1} + \frac{\phi_{2xx}^2}{2\rho_2}\right)dx + \int_0^{+\infty}\frac{\mu}{2}(\psi_{1xx}^2 + \psi_{2xx}^2)dx \\
& + \frac{u_-}{2\rho_-}|\phi_{1xx}, \phi_{2xx}(t, 0)|^2 \leq C(M(T) + \delta + \varepsilon^{\frac{1}{10}})\|(\phi_{1xxx}, \phi_{2xxx}, \psi_{1xx}, \psi_{2xx})(t)\|^2 \\
& + C\|(\phi_{1x}, \phi_{2x}, \phi_{1xx}, \phi_{2xx}, \psi_{1x}, \psi_{2x}, E_x)(t)\|^2 + C(\delta + \varepsilon^{\frac{1}{10}})(1+t)^{-\frac{8}{5}}.
\end{aligned}$$

Integrating it over $(0, t)$, and making use of the estimate (3.58) and the smallness of $M(T) + \delta + \varepsilon^{\frac{1}{10}}$, we arrive at (3.59), thereby completing the proof of Lemma 3.9.

Finally, we derive the estimate of $\int_0^t\|(\phi_{1xxx}, \phi_{2xxx})(\tau)\|^2d\tau$.

Lemma 3.10. *Under the assumptions of Proposition 3.3, it holds that*

$$\int_0^t\|(\phi_{1xxx}, \phi_{2xxx})(\tau)\|^2d\tau \leq C(M_2 + \delta + \varepsilon^{\frac{1}{10}}) \tag{3.65}$$

for any $t \in [0, T]$.

Proof. Multiplying (3.14)₂ by $-\frac{\phi_{1xxx}}{\rho_1}$, (3.14)₄ by $-\frac{\phi_{2xxx}}{\rho_2}$, adding them together, and then integrating the resultant equation with respect to x over $(0, +\infty)$, we obtain

$$\begin{aligned}
& \sum_{i=1}^2\left(\frac{d}{dt}\int_0^{+\infty}\phi_{ixx}\psi_{ix}dx + \int_0^{+\infty}\frac{\phi_{ixxx}^2}{\rho_i}dx\right) = -\int_0^{+\infty}(\phi_{1xxx} - \phi_{2xxx})E dx \\
& + \sum_{i=1}^2\left[\int_0^{+\infty}\psi_{ix}\phi_{ixxx}dx + \int_0^{+\infty}\left(u_i\phi_{ixxx}\psi_i + \frac{\hat{p}'(\rho_i)}{\rho_i}\phi_{ix}\phi_{ixxx} - \frac{\mu\phi_{ixxx}\psi_{ix}}{\rho_i} - \frac{\phi_{ixxx}G_i}{\rho_i}\right)dx\right].
\end{aligned}$$

(3.66)

First, applying integration by parts, one gets

$$\begin{aligned} - \int_0^{+\infty} (\phi_{1xxx} - \phi_{2xxx}) E dx &= \int_0^{+\infty} (\phi_{1xx} - \phi_{2xx}) E_x dx \\ &\leq C \|(\phi_{1xx}, \phi_{2xx}, E_x)(t)\|^2. \end{aligned} \quad (3.67)$$

Next, taking integration by parts and utilizing (3.55) and (3.56), we have

$$\begin{aligned} \int_0^{+\infty} \psi_{ix} \phi_{ixxt} dx &= - \int_0^{+\infty} \psi_{ixx} \phi_{ixt} dx \\ &= \int_0^{+\infty} \psi_{ixx} (u_{ix} \phi_{ix} + u_i \phi_{ixx} + \rho_{ix} \psi_{ix} + \rho_i \psi_{ixx} - F_{ix}) dx \\ &\leq C \|(\phi_{ix}, \phi_{ixx}, \psi_{ix}, \psi_{ixx})(t)\|^2 + C(\delta + \varepsilon^{\frac{1}{10}})(1+t)^{-\frac{9}{5}}. \end{aligned} \quad (3.68)$$

Finally, employing Young's inequality and (3.23)–(3.25), one shows

$$\begin{aligned} &\sum_{i=1}^2 \int_0^{+\infty} \left(u_i \phi_{ixxx} \psi_i + \frac{\hat{p}'(\rho_i)}{\rho_i} \phi_{ix} \phi_{ixxx} - \frac{\mu \phi_{ixxx} \psi_{ix}}{\rho_i} - \frac{\phi_{ixxx} G_i}{\rho_i} \right) dx \\ &\leq \sum_{i=1}^2 \left(\int_0^{+\infty} \frac{\phi_{ixxx}^2}{2\rho_i} dx + C \|(\phi_{ix}, \phi_{ixx}, \psi_{ix}, \psi_{ixx})(t)\|^2 \right) + C(\delta + \varepsilon^{\frac{1}{10}})(1+t)^{-\frac{8}{5}}. \end{aligned} \quad (3.69)$$

Putting (3.67)–(3.69) into (3.66), one arrives at

$$\begin{aligned} &\frac{d}{dt} \int_0^{+\infty} (\phi_{1xx} \psi_{1x} + \phi_{2xx} \psi_{2x}) dx + \int_0^{+\infty} \left(\frac{\phi_{1xxx}^2}{2\rho_1} + \frac{\phi_{2xxx}^2}{2\rho_2} \right) dx \\ &\leq C \|(\phi_{1x}, \phi_{2x}, \phi_{1xx}, \phi_{2xx}, \psi_{1x}, \psi_{2x}, \psi_{1xx}, \psi_{2xx}, E_x)(t)\|^2 + C(\delta + \varepsilon^{\frac{1}{10}})(1+t)^{-\frac{8}{5}}. \end{aligned} \quad (3.70)$$

We obtain the desired estimate, thereby completing the proof, by integrating (3.70) over $(0, t)$ and making use of (3.58) and (3.59).

Proof of Proposition 3.3. Combining the estimates (3.58), (3.59), and (3.65), and using the fact that $\|E_x(t)\|^2 \leq C \|(\phi_1, \phi_2)(t)\|^2$, we directly establish the estimate (3.18), thereby completing the proof of Proposition 3.3.

Proof of Theorem 1.3. The global-in-time existence of the solution $(\rho_1, u_1, \rho_2, u_2, E)(t, x)$ to the inflow problem (1.3)–(1.6) follows from the standard continuation argument based on the local existence theorem (Proposition 3.2) and the a priori estimates (Proposition 3.3). Hence, to complete the proof of Theorem 1.3, we only need to prove (1.21). To this end, it suffices to show

$$\|(\phi_{1x}, \psi_{1x}, \phi_{2x}, \psi_{2x}, E_x)(t)\| \rightarrow 0 \quad \text{as } t \rightarrow +\infty. \quad (3.71)$$

In fact, if (3.71) is proven, then from the Sobolev inequality $\|f\|_{L^\infty} \leq C \|f\|^{\frac{1}{2}} \|f_x\|^{\frac{1}{2}}$ and the estimate

$$\|(\phi_1, \psi_1, \phi_2, \psi_2, E)(t)\| \leq C,$$

we infer that

$$\|(\phi_1, \psi_1, \phi_2, \psi_2, E)(t)\|_{L^\infty} \rightarrow 0 \quad \text{as } t \rightarrow +\infty,$$

namely,

$$\|(\rho_1 - \hat{\rho}, u_1 - \hat{u}, \rho_2 - \hat{\rho}, u_2 - \hat{u}, E)(t)\|_{L^\infty} \rightarrow 0 \quad \text{as } t \rightarrow +\infty,$$

which, together with the fact $\lim_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}} |(\bar{\rho}, \bar{u})(t, x) - (n^r, u^r)(\frac{x}{t})| = 0$, implies (1.21).

In what follows, we focus on the proof of (3.71). First, taking $\int_0^{+\infty} (\phi_{1x} \times (3.55) + \phi_{2x} \times (3.56)) dx$ and $\int_0^{+\infty} (-\frac{\psi_{1xx}}{\rho_1} \times (3.14)_2 - \frac{\psi_{2xx}}{\rho_2} \times (3.14)_4) dx$, we can show

$$\int_0^{+\infty} \left| \frac{d}{dt} \|(\phi_{1x}, \psi_{1x}, \phi_{2x}, \psi_{2x})(t)\|^2 \right| dt \leq C \quad (3.72)$$

with the help of the estimate (3.18). Next, using (3.14)₅ and (3.18), we can prove that

$$\begin{aligned} & \int_0^{+\infty} \left| \frac{d}{dt} \|E_x(t)\|^2 \right| dt \\ & \leq C \int_0^{+\infty} \int_{\mathbb{R}_+} E_x^2 dx dt + C \int_0^{+\infty} \int_{\mathbb{R}_+} E_{xt}^2 dx dt \\ & \leq C \int_0^{+\infty} \int_{\mathbb{R}_+} E_x^2 dx dt + C \int_0^{+\infty} \int_{\mathbb{R}_+} (\phi_{1t}^2 + \phi_{2t}^2) dx dt \\ & \leq C \int_0^{+\infty} \int_{\mathbb{R}_+} E_x^2 dx dt + C \int_0^{+\infty} \int_{\mathbb{R}_+} (\phi_{1x}^2 + \phi_{2x}^2 + \psi_{1x}^2 + \psi_{2x}^2 + \hat{\rho}_x^2 (\psi_1^2 + \psi_2^2) \\ & \quad + \hat{u}_x^2 (\phi_1^2 + \phi_2^2) + \hat{f}^2) dx dt \\ & \leq C \int_0^{+\infty} \|(\phi_{1x}, \phi_{2x}, \psi_{1x}, \psi_{2x}, E_x)(t)\|^2 dt + C \int_0^{+\infty} (1+t)^{-\frac{9}{5}} dt \\ & \leq C. \end{aligned} \quad (3.73)$$

The combination of (3.72) and (3.73) with (3.18) gives (3.71). This completes the proof of Theorem 1.3.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

References

1. D. K. Ferry, J. R. Zhou, Form of the quantum potential for use in hydrodynamic equations for semiconductor device modeling, *Phys. Rev. B*, **48** (1993), 7944–7950. <https://doi.org/10.1103/PhysRevB.48.7944>
2. N. C. Klusdahl, A. M. Krizan, D. K. Ferry, C. Ringhofer, Self-consistent study of the resonant-tunneling diode, *Phys. Rev. B*, **39** (1989), 7720–7735. <https://doi.org/10.1103/PhysRevB.39.7720>
3. E. Wigner, On the quantum correction for thermodynamic equilibrium, *Phys. Rev.*, **40** (1932), 749–759. <https://doi.org/10.1103/PhysRev.40.749>
4. S. Brull, F. Méhats, Derivation of viscous correction terms for the isothermal quantum Euler model, *Z. Angew. Math. Mech.*, **90** (2010), 219–230. <https://doi.org/10.1002/zamm.200900297>
5. A. Jüngel, J. P. Milišić, Full compressible Navier-Stokes equations for quantum fluids: derivation and numerical solution, *Kinet. Relat. Models*, **4** (2011), 785–807. <https://doi.org/10.3934/krm.2011.4.785>
6. J. W. Yang, Q. C. Ju, Existence of global weak solutions for Navier-Stokes-Poisson equations with quantum effect and convergence to incompressible Navier-Stokes equations, *Math. Meth. Appl. Sci.*, **38** (2015), 3629–3641. <https://doi.org/10.1002/mma.3304>
7. J. W. Yang, Q. C. Ju, Convergence of the quantum Navier-Stokes-Poisson equations to the incompressible Euler equations for general initial data, *Nonlinear Anal. Real World Appl.*, **23** (2015), 148–159. <https://doi.org/10.1016/j.nonrwa.2014.12.003>
8. Y. S. Kwon, From the degenerate quantum compressible Navier-Stokes-Poisson system to incompressible Euler equations, *J. Math. Phys.*, **59** (2018), 123101. <https://doi.org/10.1063/1.4996942>
9. T. Tang, H. J. Gao, Q. K. Xiao, On global existence of weak solutions to a viscous capillary model of plasma, *Nonlinear Anal.*, **185** (2019), 1–14. <https://doi.org/10.1016/j.na.2019.02.029>
10. P. Anotonelli, G. C. Carnevale, C. Lattanzio, Relaxation limit from the quantum Navier-Stokes equations to the quantum drift-diffusion equation, *J. Nonlinear Sci.*, **31** (2021), 71. <https://doi.org/10.1007/s00332-021-09728-y>
11. Y. P. Li, W. L. Sun, Asymptotic stability of the rarefaction wave for the compressible quantum Navier-Stokes-Poisson equations, *J. Math. Anal. Appl.*, **453** (2017), 174–194. <https://doi.org/10.1016/j.jmaa.2017.03.042>
12. L. L. Tong, Y. Xia, Global existence and the algebraic decay rate of the solution for the quantum Navier-Stokes-Poisson equations in \mathbb{R}^3 , *J. Math. Phys.*, **63** (2022), 091511. <https://doi.org/10.1063/5.0082375>
13. Q. W. Wu, X. F. Hou, Asymptotic stability of the stationary wave for the quantum Navier-Stokes-Poisson system, *Nonlinear Anal. Real World Appl.*, **69** (2023), 103713. <https://doi.org/10.1016/j.nonrwa.2022.103713>

14. Q. W. Wu, Large-time behavior of solutions to the bipolar quantum Navier-Stokes-Poisson equations, *Z. Angew. Math. Phys.*, **73** (2022), 214. <https://doi.org/10.1007/s00033-022-01851-4>
15. Q. W. Wu, P. C. Zhu, Stability of the stationary solution to an outflow problem for the bipolar quantum Navier-Stokes-Poisson equations, *Commun. Pure Appl. Anal.*, **22** (2023), 2400–2429. <https://doi.org/10.3934/cpaa.2023070>
16. Q. W. Wu, P. C. Zhu, Stability of a nonlinear wave for an outflow problem of the bipolar quantum Navier-Stokes-Poisson system, *Discrete Contin. Dyn. Syst.-Ser. B*, **29** (2024), 3346–3377. <https://doi.org/10.3934/dcdsb.2024007>
17. A. Matsumura, Inflow and outflow problems in the half space for a one-dimensional isentropic model system of compressible viscous gas, *Methods Appl. Anal.*, **8** (2001), 645–666. <https://dx.doi.org/10.4310/MAA.2001.v8.n4.a14>
18. D. F. Bian, L. L. Fan, L. He, H. J. Zhao, Viscous shock wave to an inflow problem for compressible viscous gas with large density oscillations, *Acta Math. Appl. Sin. Engl. Ser.*, **35** (2019), 129–157. <https://doi.org/10.1007/s10255-019-0801-2>
19. L. L. Fan, H. X. Liu, T. Wang, H.J. Zhao, Inflow problem for the one-dimensional compressible Navier-Stokes equations under large initial perturbation, *J. Differ. Equations*, **257** (2014), 3521–3553. <https://doi.org/10.1016/j.jde.2014.07.001>
20. A. Matsumura, K. Nishihara, Large-time behavior of solutions to an inflow problem in the half space for a one-dimensional system of compressible viscous gas, *Commun. Math. Phys.*, **222** (2001), 449–474. <https://doi.org/10.1007/s002200100517>
21. F. M. Huang, A. Matsumura, X. D. Shi, Viscous shock wave and boundary layer solution to an inflow problem for compressible viscous gas, *Commun. Math. Phys.*, **239** (2003), 261–285. <https://doi.org/10.1007/s00220-003-0874-9>
22. F. M. Huang, A. Matsumura, X. D. Shi, A gas-solid free boundary problem for a compressible viscous gas, *SIAM J. Math. Anal.*, **34** (2003), 1331–1355. <https://doi.org/10.1137/S0036141002403730>
23. H. Hong, T. Wang, Large-time behavior of solutions to the inflow problem of full compressible Navier-Stokes equations with large perturbation, *Nonlinearity*, **30** (2017), 3010–3039. <https://doi.org/10.1088/1361-6544/aa7739>
24. H. Hong, T. Wang, Stability of stationary solutions to the inflow problem for full compressible Navier-Stokes equations with a large initial perturbation, *SIAM J. Math. Anal.*, **49** (2017), 2138–2166. <https://doi.org/10.1137/16M108536X>
25. T. Nakamura, S. Nishibata, Stationary wave associated with an inflow problem in the half line for viscous heat-conductive gas, *J. Hyperbolic Differ. Equations*, **8** (2011), 657–670. <https://doi.org/10.1142/S0219891611002524>
26. X. H. Qin, Y. Wang, Stability of wave patterns to the inflow problem of full compressible Navier-Stokes equations, *SIAM J. Math. Anal.*, **41** (2009), 2057–2087. <https://doi.org/10.1137/09075425X>
27. X. H. Qin, Y. Wang, Large-time behavior of solutions to the inflow problem of full compressible Navier-Stokes equations, *SIAM J. Math. Anal.*, **43** (2011), 341–346. <https://doi.org/10.1137/100793463>

28. H. Hong, X. D. Shi, T. Wang, Stability of stationary solutions to the inflow problem for the two-fluid non-isentropic Navier-Stokes-Poisson system, *J. Differ. Equations*, **265** (2018), 1129–1155. <https://doi.org/10.1016/j.jde.2018.03.016>
29. H. Hong, The existence and stability of stationary solutions of the inflow problem for full compressible Navier-Stokes-Poisson system, *Acta Math. Sci.*, **41** (2021), 319–336. <https://doi.org/10.1007/s10473-021-0119-z>
30. H. Hong, J. Kim, Stability of stationary solutions for the unipolar isentropic compressible Navier-Stokes-Poisson system, *Math. Meth. Appl. Sci.*, **44** (2021), 11102–11123. <https://doi.org/10.1002/mma.7474>
31. Q. W. Wu, J. J. Zhang, Asymptotic stability of the rarefaction wave for the inflow problem of the bipolar compressible Navier-Stokes-Poisson system with large initial perturbation, *J. Math. Phys.*, **66** (2025), 051504. <https://doi.org/10.1063/5.0233303>
32. Y. P. Li, Q. W. Wu, Stationary solutions to an inflow problem for a compressible model of the viscous ions motion, *Stud. Appl. Math.*, **155** (2025), e70121. <https://doi.org/10.1111/sapm.70121>
33. S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, 2nd edition, Springer-Verlag, New York, 2003. <https://doi.org/10.1007/b97481>
34. J. Carr, *Applications of Centre Manifold Theory*, Springer-Verlag, New York, 1981. <https://doi.org/10.1007/978-1-4612-5929-9>
35. Y. P. Li, Z. Z. Chen, Asymptotic behavior of solutions to an impermeable wall problem of the compressible fluid models of Korteweg type with density-dependent viscosity and capillarity, *SIAM J. Math. Anal.*, **53** (2021), 1434–1473. <https://doi.org/10.1137/20M1340319>
36. H. Hattori, D. Li, Solutions for two dimensional system for materials of Korteweg type, *SIAM J. Math. Anal.*, **25** (1994), 85–98. <https://doi.org/10.1137/S003614109223413X>
37. H. Hattori, D. Li, Global solutions of a high dimensional system for Korteweg materials, *J. Math. Anal. Appl.*, **198** (1996), 84–97. <https://doi.org/10.1006/jmaa.1996.0069>
38. S. Kawashima, S. Nishibata, P. C. Zhu, Asymptotic stability of the stationary solution to the compressible Navier-Stokes equations in the half space, *Commun. Math. Phys.*, **240** (2003), 483–500. <https://doi.org/10.1007/s00220-003-0909-2>
39. Q. W. Wu, X. F. Hou, P. C. Zhu, Asymptotic stability of a nonlinear wave for an outflow problem of the bipolar Navier-Stokes-Poisson system under large initial perturbation, *Z. Angew. Math. Phys.*, **74** (2023), 146. <https://doi.org/10.1007/s00033-023-02029-2>



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