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***Research article***

# Multiple positive solutions for nonlinear singular Katugampola fractional differential equations with nonlocal coupled Riemann-Stieltjes integral boundary conditions

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**Abstract:** This study explores a system of singular nonlinear higher-order katugampola fractional differential equations (KFDEs) with nonlocal coupled Riemann-Stieltjes integral boundary conditions. First, the addressed KFDEs are reformulated as the equivalent integral equations through explicit construction of some appropriate Green's functions. Second, through synergistic application of Schauder's and Guo-Krasnoselskii's fixed-point theorems, some explicit parameter interval-dependent existence criteria are established for at least one or two positive solutions to the considered KFDEs with the help of the properties of Green's functions. Finally, some concrete examples are provided to validate the effectiveness of the main theoretical results.

**Keywords:** Katugampola fractional differential equations; multiple positive solutions; Green's functions; nonlinear alternative of Leray-Schauder type; Guo-Krasnoselskii's fixed-point theorem

## 1. Introduction

Consider the following nonlinear singular higher-order Katugampola fractional differential equations (KFDEs) with nonlocal coupled Riemann-Stieltjes integral boundary conditions (IBCs):

$$\begin{cases} \mathcal{D}_{a+}^{\alpha_1,\rho}\phi_1(t)+\lambda f_1(t,\phi_1(t),\phi_2(t))=0, \ \ \mathcal{D}_{a+}^{\alpha_2,\rho}\phi_2(t)+\lambda f_2(t,\phi_1(t),\phi_2(t))=0, \ \ t\in(a,b), \ \lambda>0, \\ \gamma^i\phi_1(a)=\gamma^j\phi_2(a)=0, \ \ \phi_1(b)=\nu_2\displaystyle\int_a^b u_2(s)\phi_2(s)\frac{dU_2(s)}{s^{1-\rho}}, \ \ \phi_2(b)=\nu_1\int_a^b u_1(s)\phi_1(s)\frac{dU_1(s)}{s^{1-\rho}}, \end{cases} \tag{1.1}$$

where $0\le i\le m_1-2$, $0\le j\le m_2-2$, $\mathcal{D}_{a+}^{\alpha_\varrho,\rho}$ denotes the Katugampola fractional derivatives of order $\alpha_\varrho>0$; $\gamma=t^{1-\rho}d/dt$, $\rho>0$; $\int_a^b u_\varrho(s)\phi_\varrho(s)\frac{dU_\varrho(s)}{s^{1-\rho}}$ stands for the Riemann-Stieltjes integral for a nondecreasing function $U_\varrho$ of bounded variation and a nonnegative continuous function $u_\varrho$; $\lambda,\nu_\varrho$ are three parameters satisfying the conditions presented subsequently; $\alpha_\varrho\in(m_\varrho-1,m_\varrho]$ for some positive integer $m_\varrho\ge 3$; and $f_\varrho$ is sign-changing continuous and may be singular at $t=a$ or/and $t=b$, $\varrho=1,2$.

Over the past few decades, considerable attention has been drawn to the investigation of FDEs owing to their extensive applications across diverse fields including physics, engineering, and economics. The evolution of fractional calculus has furnished robust mathematical frameworks for characterizing intricate systems exhibiting memory effects and hereditary traits. Particularly, notable contributions from multiple scholars have propelled progress in this research domain. A number of notable contributions from multiple scholars have focused on the boundary value problems (BVPs) that incorporate diverse fractional derivatives such as the Riemann-Liouville [1], Caputo [2–4], Hadamard [5–7], Caputo-Hadamard [8], generalized Caputo [9], generalized Riemann-Liouville [10], and tempered fractional operators [11]. The reader can also refer to [12–15] and the reference cited here. For example, leveraging a contribution of Meir-Keeler's fixed-point theorem (FPT) alongside the principle of measures of noncompactness, Berrighi et al. [16] considered the realm of mild solutions for conformable fractional order functional evolution equations. Szajek and Sumelka [17] applied the Riesz-Caputo fractional derivative to the discrete mass-spring structure identification in nonlocal continuum space. Utilizing methodologies from stochastic FPTs, Malki et al. [18] studied the existence and uniqueness of solutions for systems of random sequential FDEs with IBCs. By using the FPTs, Jiang et al. [19] and Yuan et al. [20] obtained positive solutions for some nonlinear Riemann-Liouville FDEs with four-points and integral boundary conditions, respectively.

On the other hand, Hadamard FDEs and other generalized FDEs have attracted the attention of scholars both at home and abroad. Based on Leray-Schauder's alternative and Banach's contraction principle, Ahmad and Ntouyas [21] and Ahmad et al. [22] obtained the existence and uniqueness of solutions for a system of Hadamard and Caputo-type FDEs with Hadamard type IBCs and coupled nonlocal IBCs, respectively. Yang [23, 24] considered the existence of positive solutions for singular nonlinear Hadamard-type FDEs subject to four-point coupled boundary conditions and coupled IBCs, respectively. By employing some FPTs, Tudorache and Luca [25, 26] investigated the existence and uniqueness of solutions and positive solutions for systems of Hilfer-Hadamard and Hadamard-type FDEs with nonlocal coupled boundary conditions, respectively. By using standard FPTs, Kiataramkul et al. [27] studied the existence and uniqueness of solution for mixed Riemann-Liouville and Hadamard-Caputo FDEs associated to nonlocal coupled fractional IBCs. Mehmood et al. [28] used the FPTs to show the existence and uniqueness of solutions for a coupled nonlinear AB-Caputo FDEs with three-point boundary conditions. By means of Mawhin's coincidence degree theory, Salim et al. [29] studied some existence and uniqueness results for a class of problems systems for nonlinear $k$-generalized $\psi$-Hilfer fractional differential equations with periodic conditions. By employing the standard FPTs, Yang [30] considered the existence and nonexistence of positive solutions for nonlinear $p$-Laplacian Hadamard FDEs with coupled nonlocal Riemann-Stieltjes IBCs. Zibar et al. [31] investigated the existence, uniqueness, and stability of solutions for nonlinear mixed generalized FDEs with mixed fractional derivative boundary conditions.

Katugampola fractional calculus offers a unified approach that encompasses both Riemann-Liouville and Hadamard fractional calculus frameworks. Conversely, scholarly attention toward KFDEs has surged in recent years. Illustrating this trend, Subramanian et al. [9] employed the Leray-Schauder alternative and Krasnoselskii's FPTs to establish the existence, uniqueness, and Hyers-Ulam stability of solutions for a coupled system of generalized Liouville-Caputo FDEs incorporating Katugampola IBCs. Subsequent studies further expanded the theoretical framework: leveraging properties of Green's functions, researchers derived a fractional Lyapunov inequality for higher-order

KFDEs [32]. Classical FPTs were also applied to investigate the existence and uniqueness of solutions for nonlinear Katugampola fractional BVPs [33]. Meanwhile, Łupińska and Schmeidel [34] developed criteria for the existence and non-existence of solutions to FDEs with Katugampola derivative-based boundary conditions by constructing a Lyapunov-type inequality. Complementing these approaches, Srivastava et al. [35] utilized coincidence degree theory to prove the existence of solutions for resonant KFDEs involving Riemann-Stieltjes IBCs. In the paper [36], Sadek et al. introduced the general Caputo-Katugampola fractional derivative, which can be seen as the generalization of Riemann-Liouville-Katugampola, the Caputo, Caputo-Katugampola and Caputo-Hadamard-type fractional derivatives.

To the best of our knowledge, the study of higher-order KFDEs incorporating nonlocal coupled Riemann-Stieltjes IBCs remains unexplored in the literature. To bridge this gap, this work employs the nonlinear alternative of Leray-Schauder type and Guo-Krasnoselskii's FPTs to establish novel existence criteria for at least one or two positive solutions to KFDE (1.1) under some specified hypotheses. The main contributions of this paper are listed as follows:

- Incorporate nonlocal coupled Riemann-Stieltjes integral boundary conditions into higher-order KFDEs with singular nonlinearities.
- Convert the coupled KFDEs into the equivalent integral equations and obtain the Green's functions along with its properties.
- Establish the explicit parameter intervals for the existence of single and multiple positive solutions using Leray-Schauder and Guo-Krasnoselskii FPTs.
- Employ concrete examples to show the effectiveness of the theoretical findings.

The structure of this paper is listed below. Section 2 presents foundational concepts from Katugampola fractional calculus theory. At the same time, the Green's functions for the addressed KFDE (1.1) and their essential properties are also given. In Section 3, some explicit parameter interval-dependent existence criteria for at least one or two positive solutions to the proposed KFDE (1.1) are developed. As applications, some concrete examples are verified to validate the feasibility of the main theoretical findings in Section 4. Finally, the conclusions of this paper are outlined in Section 5.

## 2. Preliminaries

To facilitate subsequent analysis, we begin by reviewing the fundamental principles from Katugampola fractional calculus theory. For more comprehensive overviews, the reader may refer to [32, 37, 38].

**Definition 1** ( [37, 38]). *Let $\alpha, \rho > 0$, $-\infty < a < b \leq \infty$, $p \geq 1$, and $\phi \in L^p(a, b)$. Define the following left-sided and right-sided Katugampola fractional integrals of order $\alpha$ for $t \in (a, b)$, respectively,*

$$\mathcal{I}_{a+}^{\alpha,\rho}\phi(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t (t^\rho - s^\rho)^{\alpha-1}\phi(s)\frac{ds}{s^{1-\rho}} \text{ and } \mathcal{I}_{b-}^{\alpha,\rho}\phi(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_t^b (s^\rho - t^\rho)^{\alpha-1}\phi(s)\frac{ds}{s^{1-\rho}}.$$

**Definition 2** ( [32, 38]). *Let $\alpha, \rho > 0$, $n = [\alpha] + 1$, $0 < a < b \leq \infty$, $p \geq 1$, and $\phi \in L^p(a, b)$. Define the following left-sided and right-sided Katugampola fractional derivatives of order $\alpha$, respectively,*

$$\mathcal{D}_{a+}^{\alpha,\rho}\phi(t) = \gamma^n \mathcal{I}_{a+}^{n-\alpha,\rho}\phi(t) = \frac{\gamma^n}{\Gamma(n-\alpha)}\int_a^t \Big(\frac{t^\rho - s^\rho}{\rho}\Big)^{n-\alpha-1}\phi(s)\frac{ds}{s^{1-\rho}}, \quad \gamma = t^{1-\rho}\frac{d}{dt},$$

$$\mathcal{D}^{\alpha,\rho}_{b-}\phi(t) =(-\gamma)^n \mathcal{I}^{n-\alpha,\rho}_{b-}\phi(t) = \frac{(-\gamma)^n}{\Gamma(n-\alpha)}\int_t^b \Big(\frac{s^\rho - t^\rho}{\rho}\Big)^{n-\alpha-1}\phi(s)\frac{ds}{s^{1-\rho}} \quad \textit{for } t\in(a,b).$$

Let $\mathbb{AC}[a,b]$ denote the collection of all absolutely continuous functions on the interval $[a,b]$, we introduce $\mathbb{AC}^n_\gamma[a,b] = \Big\{\phi : [a,b]\to\mathbb{R} \text{ and } \gamma^{n-1}\phi\in\mathbb{AC}[a,b], \gamma = t^{1-\rho}d/dt\Big\}$, $\mathbb{AC}^1_\gamma[a,b] = \mathbb{AC}[a,b]$.

**Lemma 1** ( [10]). *Let $\alpha>0$, $n=[\alpha]+1$, $\phi\in L(a,b)$, $\mathcal{I}^{\alpha,\rho}_{a+}\phi\in\mathbb{AC}^n_\gamma[a,b]$ and $\mathcal{I}^{\alpha,\rho}_{b-}\phi\in\mathbb{AC}^n_\gamma[a,b]$. Then the following equations holds*

$$\mathcal{I}^{\alpha,\rho}_{a+}\mathcal{D}^{\alpha,\rho}_{a+}\phi(t) = \phi(t) + \sum_{j=1}^n \hat{c}_j\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\alpha-j} \quad \textit{and} \quad \mathcal{I}^{\alpha,\rho}_{b-}\mathcal{D}^{\alpha,\rho}_{b-}\phi(t) = \phi(t) + \sum_{j=1}^n \check{c}_j\Big(\frac{b^\rho - t^\rho}{\rho}\Big)^{\alpha-j},$$

*where $\hat{c}_j = -\mathcal{D}^{\alpha-j,\rho}_{a+}\phi(a)/\Gamma(\alpha-j+1)$ and $\check{c}_j = (-1)^{j+1}\mathcal{D}^{\alpha-j,\rho}_{b-}\phi(b)/\Gamma(\alpha-j+1)$.*

**Lemma 2** ( [10,38]). *Let $a,\alpha,\rho>0$, and $\kappa>\alpha-1$. Then, for $i=1,2,\dots,[\alpha]+1$, we have*

$$\mathcal{D}^{\alpha,\rho}_{a+}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\kappa} = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\alpha+1)}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\kappa-\alpha} \quad \textit{and} \quad \mathcal{D}^{\alpha,\rho}_{b-}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\alpha-i} = 0.$$

**Lemma 3** ( [39]). *Let $\varphi$ be a continuous function on $[a,b]$, $0<a<b<\infty$, $\hbar>2$. Then the unique solution of KFDE $\mathcal{D}^{\hbar,\rho}_{a+}\phi(t)+\varphi(t) = 0$, $\gamma^i\phi(a) = \phi(b) = 0$, $0\le i\le[\hbar]-1$, is $\phi(t) = \int_a^b \mathcal{G}_\hbar(t,s)\varphi(s)\frac{ds}{s^{1-\rho}}$, where*

$$\mathcal{G}_\hbar(t,s) = \frac{\rho^{1-\hbar}}{\Gamma(\hbar)}\begin{cases}\Big(\dfrac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\hbar-1}(b^\rho - s^\rho)^{\hbar-1} - (t^\rho - s^\rho)^{\hbar-1}, & a\le s\le t\le b,\\ \Big(\dfrac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\hbar-1}(b^\rho - s^\rho)^{\hbar-1}, & a\le t\le s\le b.\end{cases} \tag{2.1}$$

**Lemma 4** ( [39]). *The Green's function $\mathcal{G}_\hbar(t,s)$ defined by (2.1) has the following properties:*

(I) *$\mathcal{G}_\hbar(t,s)$ is continuous function on $(t,s)\in[a,b]^2$ and $G_\hbar(t,s)>0$ for $t,s\in(a,b)$;*
(II) *$\varrho_\hbar(t)\sigma_\hbar(s)\le\frac{\rho^{\hbar-1}\Gamma(\hbar)}{(b^\rho-a^\rho)^{\hbar-1}}\mathcal{G}_\hbar(t,s)\le(\hbar-1)\sigma_\hbar(s)$ and $\varrho_\hbar(t)\sigma_\hbar(s)\le\frac{\rho^{\hbar-1}\Gamma(\hbar)}{(b^\rho-a^\rho)^{\hbar-1}}\mathcal{G}_\hbar(t,s)\le(\hbar-1)\varrho_\hbar(t)$ for $t,s\in[a,b]$, where $\varrho_\hbar(t) = (\frac{t^\rho-a^\rho}{b^\rho-a^\rho})^{\hbar-1}\frac{b^\rho-t^\rho}{b^\rho-a^\rho}$ and $\sigma_\hbar(t) = (\frac{b^\rho-t^\rho}{b^\rho-a^\rho})^{\hbar-1}\frac{t^\rho-a^\rho}{b^\rho-a^\rho}$ for $\hbar>2$ and $t\in[a,b]$.*

Now, we will investigate the corresponding Green's function for KFDE (1.1) and its fundamental properties.

**Lemma 5.** *Let $x_1,x_2\in L^p(a,b)$ be given functions. Then the following KFDE*

$$\begin{cases}\mathcal{D}^{\alpha_1,\rho}_{a+}\phi_1(t)+x_1(t)=0, \quad \mathcal{D}^{\alpha_2,\rho}_{a+}\phi_2(t)+x_2(t)=0, \quad t\in(a,b),\\ \gamma^i\phi_1(a)=\gamma^j\phi_2(a)=0, \;\; \phi_1(b)=v_2\displaystyle\int_a^b u_2(s)\phi_2(s)\frac{dU_2(s)}{s^{1-\rho}}, \;\; \phi_2(b)=v_1\int_a^b u_1(s)\phi_1(s)\frac{dU_1(s)}{s^{1-\rho}},\end{cases} \tag{2.2}$$

*where $0\le i\le m_1-2$, $0\le j\le m_2-2$, has an integral representation*

$$\begin{cases}\phi_1(t) = \displaystyle\int_a^b G_1(t,s)x_1(s)\frac{ds}{s^{1-\rho}} + \int_a^b H_2(t,s)x_2(s)\frac{ds}{s^{1-\rho}},\\ \phi_2(t) = \displaystyle\int_a^b G_2(t,s)x_2(s)\frac{ds}{s^{1-\rho}} + \int_a^b H_1(t,s)x_1(s)\frac{ds}{s^{1-\rho}},\end{cases} \tag{2.3}$$

*where* $c_{\alpha_1}^{\alpha_2} = 1 - v_1 v_2 c_{\alpha_1} c_{\alpha_2} > 0$,

$$G_1(t,s) = \mathcal{G}_{\alpha_1}(t,s) + \frac{v_1 v_2 c_{\alpha_2}}{c_{\alpha_1}^{\alpha_2}}\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_1-1}\int_a^b u_1(t)\mathcal{G}_{\alpha_1}(t,s)\frac{dU_1(t)}{t^{1-\rho}}, \tag{2.4a}$$

$$G_2(t,s) = \mathcal{G}_{\alpha_2}(t,s) + \frac{v_1 v_2 c_{\alpha_1}}{c_{\alpha_1}^{\alpha_2}}\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_2-1}\int_a^b u_2(t)\mathcal{G}_{\alpha_2}(t,s)\frac{dU_2(t)}{t^{1-\rho}}, \tag{2.4b}$$

$$H_1(t,s) = \frac{v_1}{c_{\alpha_1}^{\alpha_2}}\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_2-1}\int_a^b u_1(t)\mathcal{G}_{\alpha_1}(t,s)\frac{dU_1(t)}{t^{1-\rho}}, \quad c_{\alpha_1} = \int_a^b (\frac{t^\rho - a^\rho}{b^\rho - a^\rho})^{\alpha_1-1} u_1(t)\frac{dU_1(t)}{t^{1-\rho}}, \tag{2.4c}$$

$$H_2(t,s) = \frac{v_2}{c_{\alpha_1}^{\alpha_2}}\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_1-1}\int_a^b u_2(t)\mathcal{G}_{\alpha_2}(t,s)\frac{dU_2(t)}{t^{1-\rho}}, \quad c_{\alpha_2} = \int_a^b \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_2-1} u_2(t)\frac{dU_2(t)}{t^{1-\rho}}. \tag{2.4d}$$

*Proof.* It follows from Lemma 1 that the solution of KFDE (2.2) can be rewritten the equivalent integral systems as follows

$$\begin{cases} \phi_1(t) = c_{11}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\alpha_1-1} + \cdots + c_{1m_1}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\alpha_1-m_1} - \frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_a^t (t^\rho - s^\rho)^{\alpha_1-1}x_1(s)\frac{ds}{s^{1-\rho}}, \\ \phi_2(t) = c_{21}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\alpha_2-1} + \cdots + c_{2m_2}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\alpha_2-m_2} - \frac{\rho^{1-\alpha_2}}{\Gamma(\alpha_2)}\int_a^t (t^\rho - s^\rho)^{\alpha_2-1}x_2(s)\frac{ds}{s^{1-\rho}}. \end{cases} \tag{2.5}$$

From $\gamma^i\phi_1(a) = \gamma^j\phi_2(a) = 0$, $0 \le i \le m_1 - 2$, $0 \le j \le m_2 - 2$, we have $c_{\varrho m_\varrho} = c_{\varrho(m_\varrho-1)} = \cdots = c_{\varrho 2} = 0$ ($\varrho = 1, 2$). Thus, Eq (2.5) becomes the following form

$$\begin{cases} \phi_1(t) = c_{11}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\alpha_1-1} - \frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_a^t (t^\rho - s^\rho)^{\alpha_1-1}x_1(s)\frac{ds}{s^{1-\rho}}, \\ \phi_2(t) = c_{21}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\alpha_2-1} - \frac{\rho^{1-\alpha_2}}{\Gamma(\alpha_2)}\int_a^t (t^\rho - s^\rho)^{\alpha_2-1}x_2(s)\frac{ds}{s^{1-\rho}}. \end{cases} \tag{2.6}$$

It follows from $\phi_1(b) = v_2\int_a^b u_2(s)\phi_2(s)\frac{dU_2(s)}{s^{1-\rho}}$ and $\phi_2(b) = v_1\int_a^b u_1(s)\phi_1(s)\frac{dU_1(s)}{s^{1-\rho}}$ that we obtain by means of Eq (2.6)

$$\begin{cases} \Big(\frac{b^\rho - a^\rho}{\rho}\Big)^{\alpha_1-1}c_{11} = v_2\int_a^b u_2(s)\phi_2(s)\frac{dU_2(s)}{s^{1-\rho}} + \frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_a^b (b^\rho - s^\rho)^{\alpha_1-1}x_1(s)\frac{ds}{s^{1-\rho}}, \\ \Big(\frac{b^\rho - a^\rho}{\rho}\Big)^{\alpha_2-1}c_{21} = v_1\int_a^b u_1(s)\phi_1(s)\frac{dU_1(s)}{s^{1-\rho}} + \frac{\rho^{1-\alpha_2}}{\Gamma(\alpha_2)}\int_a^b (b^\rho - s^\rho)^{\alpha_2-1}x_2(s)\frac{ds}{s^{1-\rho}}. \end{cases} \tag{2.7}$$

Combining (2.5) and (2.7), we derive

$$\begin{aligned} \phi_1(t) =& \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_1-1}\left(v_2\int_a^b u_2(s)\phi_2(s)\frac{dU_2(s)}{s^{1-\rho}} + \frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_a^b (b^\rho - s^\rho)^{\alpha_1-1}x_1(s)\frac{ds}{s^{1-\rho}}\right) \\ &- \frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_a^t (t^\rho - s^\rho)^{\alpha_1-1}x_1(s)\frac{ds}{s^{1-\rho}} \\ =& v_2\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_1-1}\int_a^b u_2(s)\phi_2(s)\frac{dU_2(s)}{s^{1-\rho}} + \int_a^b \mathcal{G}_{\alpha_1}(t,s)x_1(s)\frac{ds}{s^{1-\rho}}, \end{aligned} \tag{2.8a}$$

$$\phi_2(t) = \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_2-1}\left(v_1\int_a^b u_1(s)\phi_1(s)\frac{dU_1(s)}{s^{1-\rho}} + \frac{\rho^{1-\alpha_2}}{\Gamma(\alpha_2)}\int_a^b (b^\rho - s^\rho)^{\alpha_2-1}x_2(s)\frac{ds}{s^{1-\rho}}\right)$$

$$-\frac{\rho^{1-\alpha_2}}{\Gamma(\alpha_2)}\int_a^t (t^\rho - s^\rho)^{\alpha_2-1}x_2(s)\frac{ds}{s^{1-\rho}}$$
$$=v_1\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_2-1}\int_a^b u_1(s)\phi_1(s)\frac{dU_1(s)}{s^{1-\rho}} + \int_a^b \mathcal{G}_{\alpha_2}(t,s)x_2(s)\frac{ds}{s^{1-\rho}}. \tag{2.8b}$$

Integrating the above Eq (2.8) with respect from $a$ to $b$, we obtain

$$\int_a^b u_1(t)\phi_1(t)\frac{dU_1(t)}{t^{1-\rho}} = \int_a^b u_1(t)\left(v_2\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_1-1}\int_a^b u_2(s)\phi_2(s)\frac{dU_2(s)}{s^{1-\rho}} + \int_a^b \mathcal{G}_{\alpha_1}(t,s)x_1(s)\frac{ds}{s^{1-\rho}}\right)\frac{dU_1(t)}{t^{1-\rho}}$$
$$=v_2c_{\alpha_1}\int_a^b u_2(s)\phi_2(s)\frac{dU_2(s)}{s^{1-\rho}} + \int_a^b\int_a^b u_1(t)\mathcal{G}_{\alpha_1}(t,s)\frac{dU_1(t)}{t^{1-\rho}}x_1(s)\frac{ds}{s^{1-\rho}}, \tag{2.9a}$$
$$\int_a^b u_2(t)\phi_2(t)\frac{dU_2(t)}{t^{1-\rho}} = \int_a^b u_2(t)\left(v_1\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_2-1}\int_a^b u_1(s)\phi_1(s)\frac{dU_1(s)}{s^{1-\rho}} + \int_a^b \mathcal{G}_{\alpha_2}(t,s)x_2(s)\frac{ds}{s^{1-\rho}}\right)\frac{dU_2(t)}{t^{1-\rho}}$$
$$=v_1c_{\alpha_2}\int_a^b u_1(s)\phi_1(s)\frac{dU_1(s)}{s^{1-\rho}} + \int_a^b\int_a^b u_2(t)\mathcal{G}_{\alpha_2}(t,s)\frac{dU_2(t)}{t^{1-\rho}}x_2(s)\frac{ds}{s^{1-\rho}}, \tag{2.9b}$$

Solving for $\int_a^b u_1(s)\phi_1(s)\frac{dU_1(s)}{s^{1-\rho}}$ and $\int_a^b u_2(s)\phi_2(s)\frac{dU_2(s)}{s^{1-\rho}}$ in Eq (2.9), we acquire

$$\begin{cases}\displaystyle\int_a^b u_1(s)\phi_1(s)\frac{dU_1(s)}{s^{1-\rho}} = \frac{v_1c_{\alpha_2}}{c_{\alpha_1}^{\alpha_2}}\int_a^b\int_a^b u_1(t)\mathcal{G}_{\alpha_1}(t,s)\frac{dU_1(t)}{t^{1-\rho}}x_1(s)\frac{ds}{s^{1-\rho}} \\ \displaystyle\qquad + \frac{1}{c_{\alpha_1}^{\alpha_2}}\int_a^b\int_a^b u_2(t)\mathcal{G}_{\alpha_2}(t,s)\frac{dU_2(t)}{t^{1-\rho}}x_2(s)\frac{ds}{s^{1-\rho}}, \\ \displaystyle\int_a^b u_2(s)\phi_2(s)\frac{dU_2(s)}{s^{1-\rho}} = \frac{v_2c_{\alpha_1}}{c_{\alpha_1}^{\alpha_2}}\int_a^b\int_a^b u_2(t)\mathcal{G}_{\alpha_2}(t,s)\frac{dU_2(t)}{t^{1-\rho}}x_2(s)\frac{ds}{s^{1-\rho}} \\ \displaystyle\qquad + \frac{1}{c_{\alpha_1}^{\alpha_2}}\int_a^b\int_a^b u_1(t)\mathcal{G}_{\alpha_1}(t,s)\frac{dU_1(t)}{t^{1-\rho}}x_1(s)\frac{ds}{s^{1-\rho}}.\end{cases} \tag{2.10}$$

Combining (2.6), (2.7) and (2.10), we get

$$\phi_1(t) = -\frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_a^t (t^\rho - s^\rho)^{\alpha_1-1}x_1(s)\frac{ds}{s^{1-\rho}} + \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_1-1}\Big(\frac{v_1v_2c_{\alpha_2}}{c_{\alpha_1}^{\alpha_2}}\int_a^b\int_a^b u_1(t)\mathcal{G}_{\alpha_1}(t,s)\frac{dU_1(t)}{t^{1-\rho}}x_1(s)\frac{ds}{s^{1-\rho}}$$
$$+ \frac{v_2}{c_{\alpha_1}^{\alpha_2}}\int_a^b\int_a^b u_2(t)\mathcal{G}_{\alpha_2}(t,s)\frac{dU_2(t)}{t^{1-\rho}}x_2(s)\frac{ds}{s^{1-\rho}} + \frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_a^b (b^\rho - s^\rho)^{\alpha_1-1}x_1(s)\frac{ds}{s^{1-\rho}}\Big)$$
$$= \int_a^b\Big(\mathcal{G}_{\alpha_1}(t,s) + \frac{v_1v_2c_{\alpha_2}}{c_{\alpha_1}^{\alpha_2}}\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_1-1}\int_a^b u_1(t)\mathcal{G}_{\alpha_1}(t,s)\frac{dU_1(t)}{t^{1-\rho}}\Big)x_1(s)\frac{ds}{s^{1-\rho}}$$
$$+ \int_a^b\Big(\frac{v_2}{c_{\alpha_1}^{\alpha_2}}\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_1-1}\int_a^b u_2(t)\mathcal{G}_{\alpha_2}(t,s)\frac{dU_2(t)}{t^{1-\rho}}\Big)x_2(s)\frac{ds}{s^{1-\rho}},$$
$$\phi_2(t) = -\frac{\rho^{1-\alpha_2}}{\Gamma(\alpha_2)}\int_a^t (t^\rho - s^\rho)^{\alpha_2-1}x_2(s)\frac{ds}{s^{1-\rho}} + \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_2-1}\Big(\frac{v_1v_2c_{\alpha_1}}{c_{\alpha_1}^{\alpha_2}}\int_a^b\int_a^b u_2(t)\mathcal{G}_{\alpha_2}(t,s)\frac{dU_2(t)}{t^{1-\rho}}x_2(s)\frac{ds}{s^{1-\rho}}$$
$$+ \frac{v_1}{c_{\alpha_1}^{\alpha_2}}\int_a^b\int_a^b u_1(t)\mathcal{G}_{\alpha_1}(t,s)\frac{dU_1(t)}{t^{1-\rho}}x_1(s)\frac{ds}{s^{1-\rho}} + \frac{\rho^{1-\alpha_2}}{\Gamma(\alpha_2)}\int_a^b (b^\rho - s^\rho)^{\alpha_2-1}x_2(s)\frac{ds}{s^{1-\rho}}\Big)$$

$$= \int_a^b \Big(\mathcal{G}_{\alpha_2}(t,s) + \frac{v_1 v_2 c_{\alpha_1}}{c_{\alpha_1}^{\alpha_2}}\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_2-1}\int_a^b u_2(t)\mathcal{G}_{\alpha_2}(t,s)\frac{dU_2(t)}{t^{1-\rho}}\Big)x_2(s)\frac{ds}{s^{1-\rho}}$$
$$+ \int_a^b \Big(\frac{v_1}{c_{\alpha_1}^{\alpha_2}}\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_2-1}\int_a^b u_1(t)\mathcal{G}_{\alpha_1}(t,s)\frac{dU_1(t)}{t^{1-\rho}}\Big)x_1(s)\frac{ds}{s^{1-\rho}}.$$

Hence, we have

$$\begin{cases}\phi_1(t) = \int_a^b G_1(t,s)x_1(s)\frac{ds}{s^{1-\rho}} + \int_a^b H_2(t,s)x_2(s)\frac{ds}{s^{1-\rho}},\\ \phi_2(t) = \int_a^b G_2(t,s)x_2(s)\frac{ds}{s^{1-\rho}} + \int_a^b H_1(t,s)x_1(s)\frac{ds}{s^{1-\rho}}.\end{cases}$$

The proof of Lemma 5 is completed. □

Lemma 5 implies that KFDE (1.1) admits an integral representation as follows.

$$\begin{cases}\phi_1(t) = \lambda\Big(\int_a^b G_1(t,s)f_1(s,\phi_1(s),\phi_2(s))\frac{ds}{s^{1-\rho}} + \int_a^b H_2(t,s)f_2(s,\phi_1(s),\phi_2(s))\frac{ds}{s^{1-\rho}}\Big),\\ \phi_2(t) = \lambda\Big(\int_a^b G_2(t,s)f_2(s,\phi_1(s),\phi_2(s))\frac{ds}{s^{1-\rho}} + \int_a^b H_1(t,s)f_1(s,\phi_1(s),\phi_2(s))\frac{ds}{s^{1-\rho}}\Big).\end{cases} \tag{2.11}$$

**Lemma 6.** *For $t, s \in [a,b]$, the functions $G_\varrho(t,s)$ and $H_\varrho(t,s)$ ($\varrho = 1,2$) defined by* (2.4) *satisfy*

$$\frac{v_1 v_2 c_{\alpha_2}(b^\rho - a^\rho)^{\alpha_1-1}}{\rho^{\alpha_1-1}c_{\alpha_1}^{\alpha_2}\Gamma(\alpha_1)}\int_a^b u_1(t)\varrho_{\alpha_1}(t)\frac{dU_1(t)}{t^{1-\rho}}\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_1-1}\sigma_{\alpha_1}(s) \le G_1(t,s)$$
$$\le \frac{(b^\rho - a^\rho)^{\alpha_1-1}}{\rho^{\alpha_1-1}\Gamma(\alpha_1-1)}\Big(1 + \frac{v_1 v_2 c_{\alpha_2}}{c_{\alpha_1}^{\alpha_2}}\int_a^b u_1(t)\frac{dU_1(t)}{t^{1-\rho}}\Big)\sigma_{\alpha_1}(s), \tag{2.12a}$$

$$\frac{v_1 v_2 c_{\alpha_1}(b^\rho - a^\rho)^{\alpha_2-1}}{\rho^{\alpha_2-1}c_{\alpha_1}^{\alpha_2}\Gamma(\alpha_2)}\int_a^b u_2(t)\varrho_{\alpha_2}(t)\frac{dU_2(t)}{t^{1-\rho}}\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_2-1}\sigma_{\alpha_2}(s) \le G_2(t,s)$$
$$\le \frac{(b^\rho - a^\rho)^{\alpha_2-1}}{\rho^{\alpha_2-1}\Gamma(\alpha_2-1)}\Big(1 + \frac{v_1 v_2 c_{\alpha_1}}{c_{\alpha_1}^{\alpha_2}}\int_a^b u_2(t)\frac{dU_2(t)}{t^{1-\rho}}\Big)\sigma_{\alpha_2}(s), \tag{2.12b}$$

$$\frac{v_1(b^\rho - a^\rho)^{\alpha_1-1}}{\rho^{\alpha_1-1}c_{\alpha_1}^{\alpha_2}\Gamma(\alpha_1)}\int_a^b u_1(t)\varrho_{\alpha_1}(t)\frac{dU_1(t)}{t^{1-\rho}}\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_2-1}\sigma_{\alpha_1}(s) \le H_1(t,s)$$
$$\le \frac{v_1(b^\rho - a^\rho)^{\alpha_1-1}}{\rho^{\alpha_1-1}c_{\alpha_1}^{\alpha_2}\Gamma(\alpha_1-1)}\int_a^b u_1(t)\frac{dU_1(t)}{t^{1-\rho}}\sigma_{\alpha_1}(s), \tag{2.13a}$$

$$\frac{v_2(b^\rho - a^\rho)^{\alpha_2-1}}{\rho^{\alpha_2-1}c_{\alpha_1}^{\alpha_2}\Gamma(\alpha_2)}\int_a^b u_2(t)\varrho_{\alpha_2}(t)\frac{dU_2(t)}{t^{1-\rho}}\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_1-1}\sigma_{\alpha_2}(s) \le H_2(t,s)$$
$$\le \frac{v_2(b^\rho - a^\rho)^{\alpha_2-1}}{\rho^{\alpha_2-1}c_{\alpha_1}^{\alpha_2}\Gamma(\alpha_2-1)}\int_a^b u_2(t)\frac{dU_2(t)}{t^{1-\rho}}\sigma_{\alpha_2}(s), \tag{2.13b}$$

$$G_1(t,s) \le \frac{(b^\rho - a^\rho)^{\alpha_1-1}}{\rho^{\alpha_1-1}\Gamma(\alpha_1-1)}\Big(1 + \frac{v_1 v_2 c_{\alpha_2}}{c_{\alpha_1}^{\alpha_2}}\int_a^b u_1(t)\frac{dU_1(t)}{t^{1-\rho}}\Big)\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_1-1}, \tag{2.14a}$$

$$G_2(t,s) \le \frac{(b^\rho - a^\rho)^{\alpha_2-1}}{\rho^{\alpha_2-1}\Gamma(\alpha_2-1)}\Big(1+\frac{v_1v_2c_{\alpha_1}}{c_{\alpha_1}^{\alpha_2}}\int_a^b u_2(t)\frac{dU_2(t)}{t^{1-\rho}}\Big)\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_2-1}, \tag{2.14b}$$

$$H_1(t,s) \le \frac{v_1(b^\rho - a^\rho)^{\alpha_1-1}}{\rho^{\alpha_1-1}c_{\alpha_1}^{\alpha_1}\Gamma(\alpha_1)}\int_a^b u_1(t)\frac{dU_1(t)}{t^{1-\rho}}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_2-1}, \tag{2.15a}$$

$$H_2(t,s) \le \frac{v_2(b^\rho - a^\rho)^{\alpha_2-1}}{\rho^{\alpha_2-1}c_{\alpha_1}^{\alpha_2}\Gamma(\alpha_2)}\int_a^b u_2(t)\frac{dU_2(t)}{t^{1-\rho}}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}. \tag{2.15b}$$

*Proof.* First, we will demonstrate the validity of Eq (2.12a).

$$\begin{aligned}
G_1(t,s) =& \mathcal{G}_{\alpha_1}(t,s) + \frac{v_1v_2c_{\alpha_2}}{c_{\alpha_1}^{\alpha_2}}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\int_a^b u_1(t)\mathcal{G}_{\alpha_1}(t,s)\frac{dU_1(t)}{t^{1-\rho}}\\
\le& \frac{(b^\rho-a^\rho)^{\alpha_1-1}}{\rho^{\alpha_1-1}\Gamma(\alpha_1)}(\alpha_1-1)\sigma_{\alpha_1}(s) + \frac{v_1v_2c_{\alpha_2}}{c_{\alpha_1}^{\alpha_2}}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\int_a^b u_1(t)\frac{(b^\rho-a^\rho)^{\alpha_1-1}}{\rho^{\alpha_1-1}\Gamma(\alpha_1)}(\alpha_1-1)\sigma_{\alpha_1}(s)\frac{dU_1(t)}{t^{1-\rho}}\\
\le& \frac{(b^\rho-a^\rho)^{\alpha_1-1}}{\rho^{\alpha_1-1}\Gamma(\alpha_1-1)}\sigma_{\alpha_1}(s) + \frac{v_1v_2c_{\alpha_2}}{c_{\alpha_1}^{\alpha_2}}\frac{(b^\rho-a^\rho)^{\alpha_1-1}}{\rho^{\alpha_1-1}\Gamma(\alpha_1-1)}\int_a^b u_1(t)\frac{dU_1(t)}{t^{1-\rho}}\sigma_{\alpha_1}(s)\\
=& \frac{(b^\rho-a^\rho)^{\alpha_1-1}}{\rho^{\alpha_1-1}\Gamma(\alpha_1-1)}\Big(1+\frac{v_1v_2c_{\alpha_2}}{c_{\alpha_1}^{\alpha_2}}\int_a^b u_1(t)\frac{dU_1(t)}{t^{1-\rho}}\Big)\sigma_{\alpha_1}(s) \ \text{ for } \forall t,s\in[a,b],\\
G_1(t,s) =& \mathcal{G}_{\alpha_1}(t,s) + \frac{v_1v_2c_{\alpha_2}}{c_{\alpha_1}^{\alpha_2}}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\int_a^b u_1(t)\mathcal{G}_{\alpha_1}(t,s)\frac{dU_1(t)}{t^{1-\rho}}\\
\ge& \frac{v_1v_2c_{\alpha_2}}{c_{\alpha_1}^{\alpha_2}}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\int_a^b u_1(t)\frac{(b^\rho-a^\rho)^{\alpha_1-1}}{\rho^{\alpha_1-1}\Gamma(\alpha_1)}\varrho_{\alpha_1}(t)\sigma_{\alpha_1}(s)\frac{dU_1(t)}{t^{1-\rho}}\\
=& \frac{v_1v_2c_{\alpha_2}(b^\rho-a^\rho)^{\alpha_1-1}}{\rho^{\alpha_1-1}c_{\alpha_1}^{\alpha_2}\Gamma(\alpha_1)}\int_a^b u_1(t)\varrho_{\alpha_1}(t)\frac{dU_1(t)}{t^{1-\rho}}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\sigma_{\alpha_1}(s) \ \text{ for } \forall t,s\in[a,b].
\end{aligned}$$

Second, we will demonstrate the validity of Eq (2.13b).

$$\begin{aligned}
H_2(t,s) =& \frac{v_2}{c_{\alpha_1}^{\alpha_2}}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\int_a^b u_2(t)\mathcal{G}_{\alpha_2}(t,s)\frac{dU_2(t)}{t^{1-\rho}}\\
\le& \frac{v_2}{c_{\alpha_1}^{\alpha_2}}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\int_a^b u_2(t)\frac{(b^\rho-a^\rho)^{\alpha_2-1}}{\rho^{\alpha_2-1}\Gamma(\alpha_2)}(\alpha_2-1)\sigma_{\alpha_2}(s)\frac{dU_2(t)}{t^{1-\rho}}\\
\le& \frac{v_2(b^\rho-a^\rho)^{\alpha_2-1}}{\rho^{\alpha_2-1}c_{\alpha_1}^{\alpha_2}\Gamma(\alpha_2-1)}\int_a^b u_2(t)\frac{dU_2(t)}{t^{1-\rho}}\sigma_{\alpha_2}(s) \ \text{ for } \forall t,s\in[a,b],\\
H_2(t,s) =& \frac{v_2}{c_{\alpha_1}^{\alpha_2}}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\int_a^b u_2(t)\mathcal{G}_{\alpha_2}(t,s)\frac{dU_2(t)}{t^{1-\rho}}\\
\ge& \frac{v_2}{c_{\alpha_1}^{\alpha_2}}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\int_a^b u_2(t)\frac{(b^\rho-a^\rho)^{\alpha_2-1}}{\rho^{\alpha_2-1}\Gamma(\alpha_2)}\varrho_{\alpha_2}(t)\sigma_{\alpha_2}(s)\frac{dU_2(t)}{t^{1-\rho}}\\
=& \frac{v_2(b^\rho-a^\rho)^{\alpha_2-1}}{\rho^{\alpha_2-1}c_{\alpha_1}^{\alpha_2}\Gamma(\alpha_2)}\int_a^b u_2(t)\varrho_{\alpha_2}(t)\frac{dU_2(t)}{t^{1-\rho}}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\sigma_{\alpha_2}(s) \ \text{ for } \forall t,s\in[a,b].
\end{aligned}$$

Third, we will demonstrate the validity of inequalities (2.14a) and (2.15b).

$$G_1(t,s) = \mathcal{G}_{\alpha_1}(t,s) + \frac{v_1v_2c_{\alpha_2}}{c_{\alpha_1}^{\alpha_2}}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\int_a^b u_1(t)\mathcal{G}_{\alpha_1}(t,s)\frac{dU_1(t)}{t^{1-\rho}}$$

$$
\begin{aligned}
&\le \frac{(b^\rho-a^\rho)^{\alpha_1-1}}{\rho^{\alpha_1-1}\Gamma(\alpha_1)}(\alpha_1-1)\varrho_{\alpha_1}(t)+\frac{v_1v_2c_{\alpha_2}}{c_{\alpha_1}^{\alpha_2}}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\int_a^b u_1(t)\frac{(b^\rho-a^\rho)^{\alpha_1-1}}{\rho^{\alpha_1-1}\Gamma(\alpha_1)}(\alpha_1-1)\varrho_{\alpha_1}(t)\frac{dU_1(t)}{t^{1-\rho}}\\
&\le \frac{(b^\rho-a^\rho)^{\alpha_1-1}}{\rho^{\alpha_1-1}\Gamma(\alpha_1-1)}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}+\frac{v_1v_2c_{\alpha_2}(b^\rho-a^\rho)^{\alpha_1-1}}{\rho^{\alpha_1-1}c_{\alpha_1}^{\alpha_2}\Gamma(\alpha_1-1)}\int_a^b u_1(t)\frac{dU_1(t)}{t^{1-\rho}}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\\
&= \frac{(b^\rho-a^\rho)^{\alpha_1-1}}{\rho^{\alpha_1-1}\Gamma(\alpha_1-1)}\Big(1+\frac{v_1v_2c_{\alpha_2}}{c_{\alpha_1}^{\alpha_2}}\int_a^b u_1(t)\frac{dU_1(t)}{t^{1-\rho}}\Big)\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\quad \text{for } \forall t,s\in[a,b],\\
H_2(t,s) &= \frac{v_2}{c_{\alpha_1}^{\alpha_2}}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\int_a^b u_2(t)\mathcal{G}_{\alpha_2}(t,s)\frac{dU_2(t)}{t^{1-\rho}}\le \frac{v_2}{c_{\alpha_1}^{\alpha_2}}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\int_a^b u_2(t)\frac{(b^\rho-a^\rho)^{\alpha_2-1}}{\rho^{\alpha_2-1}\Gamma(\alpha_2)}\varrho_{\alpha_2}(t)\frac{dU_2(t)}{t^{1-\rho}}\\
&\le \frac{v_2(b^\rho-a^\rho)^{\alpha_2-1}}{\rho^{\alpha_2-1}c_{\alpha_1}^{\alpha_2}\Gamma(\alpha_2)}\int_a^b u_2(t)\frac{dU_2(t)}{t^{1-\rho}}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\quad \text{for } \forall t,s\in[a,b].
\end{aligned}
$$

By employing a comparable methodology, we derive (2.12b), (2.13a), (2.14b), and (2.15a). The arguments above suffice to prove the lemma. □

**Remark 1.** *It follows from Lemma 6 that for any $t,s\in[a,b]$, we obtain*

$$
\begin{aligned}
&\ell_1\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\sigma_{\alpha_1}(s)\le G_1(t,s)\le \ell_2\sigma_{\alpha_1}(s), && \ell_1\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_2-1}\sigma_{\alpha_2}(s)\le G_2(t,s)\le \ell_2\sigma_{\alpha_2}(s),\\
&\ell_1\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_2-1}\sigma_{\alpha_1}(s)\le H_1(t,s)\le \ell_2\sigma_{\alpha_1}(s), && \ell_1\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\sigma_{\alpha_2}(s)\le H_2(t,s)\le \ell_2\sigma_{\alpha_2}(s),\\
&G_1(t,s),H_2(t,s)\le \ell_2\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}, && G_2(t,s),H_1(t,s)\le \ell_2\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_2-1},
\end{aligned}
$$

*where*

$$
\ell_1=\min\left\{\begin{array}{l}\frac{v_1v_2c_{\alpha_2}(b^\rho-a^\rho)^{\alpha_1-1}}{\rho^{\alpha_1-1}c_{\alpha_1}^{\alpha_2}\Gamma(\alpha_1)}\int_a^b u_1(t)\varrho_{\alpha_1}(t)\frac{dU_1(t)}{t^{1-\rho}},\ \frac{v_1(b^\rho-a^\rho)^{\alpha_1-1}}{\rho^{\alpha_1-1}c_{\alpha_1}^{\alpha_2}\Gamma(\alpha_1)}\int_a^b u_1(t)\varrho_{\alpha_1}(t)\frac{dU_1(t)}{t^{1-\rho}},\\ \frac{v_1v_2c_{\alpha_1}(b^\rho-a^\rho)^{\alpha_2-1}}{\rho^{\alpha_2-1}c_{\alpha_1}^{\alpha_2}\Gamma(\alpha_2)}\int_a^b u_2(t)\varrho_{\alpha_2}(t)\frac{dU_2(t)}{t^{1-\rho}},\ \frac{v_2(b^\rho-a^\rho)^{\alpha_2-1}}{\rho^{\alpha_2-1}c_{\alpha_1}^{\alpha_2}\Gamma(\alpha_2)}\int_a^b u_2(t)\varrho_{\alpha_2}(t)\frac{dU_2(t)}{t^{1-\rho}}\end{array}\right\},
$$

$$
\ell_2=\max\left\{\begin{array}{l}\frac{(b^\rho-a^\rho)^{\alpha_1-1}}{\rho^{\alpha_1-1}\Gamma(\alpha_1-1)}\Big(1+\frac{v_1v_2c_{\alpha_2}}{c_{\alpha_1}^{\alpha_2}}\int_a^b u_1(t)\frac{dU_1(t)}{t^{1-\rho}}\Big),\ \frac{v_1(b^\rho-a^\rho)^{\alpha_1-1}}{\rho^{\alpha_1-1}c_{\alpha_1}^{\alpha_2}\Gamma(\alpha_1-1)}\int_a^b u_1(t)\frac{dU_1(t)}{t^{1-\rho}},\\ \frac{(b^\rho-a^\rho)^{\alpha_2-1}}{\rho^{\alpha_2-1}\Gamma(\alpha_2-1)}\Big(1+\frac{v_1v_2c_{\alpha_1}}{c_{\alpha_1}^{\alpha_2}}\int_a^b u_2(t)\frac{dU_2(t)}{t^{1-\rho}}\Big),\ \frac{v_2(b^\rho-a^\rho)^{\alpha_2-1}}{\rho^{\alpha_2-1}c_{\alpha_1}^{\alpha_2}\Gamma(\alpha_2-1)}\int_a^b u_2(t)\frac{dU_2(t)}{t^{1-\rho}}\end{array}\right\}.
$$

Throughout the remainder of this paper, we assume always the following hypotheses are satisfied:

($C_1$) Let $f_\varrho(t,\phi_1,\phi_2)\in C([a,b]\times[0,+\infty)^2,(-\infty,+\infty))$, furthermore, there exists a function $g_\varrho(t)\in L^1([a,b],(0,+\infty))$ such that $f_\varrho(t,\phi_1,\phi_2)\ge -g_\varrho(t)$ for any $t\in[a,b]$, $\phi_\varrho\in[0,+\infty)$, $\varrho=1,2$.

($C_1^*$) Let $f_\varrho(t,\phi_1,\phi_2)\in C((a,b)\times[0,+\infty)^2,(-\infty,+\infty))$, $f_\varrho$ may be singular at $t=a,b$, furthermore, there exists two functions $g_\varrho(t)\in L^1([a,b],(0,+\infty))$ such that $f_\varrho(t,\phi_1,\phi_2)\ge -g_\varrho(t)$ for any $t\in[a,b]$, $\phi_\varrho\in[0,+\infty)$, $\varrho=1,2$.

($C_2$) Let $f_\varrho(t,0,0)>0$ for $t\in[a,n]$, $\varrho=1,2$.

($C_3$) There exists $[\epsilon_1,\epsilon_2]\subset(a,b)$ satisfying $\liminf\limits_{\phi_1\uparrow+\infty}\min\limits_{t\in[\epsilon_1,\epsilon_2]}\frac{f_1(t,\phi_1,\phi_2)}{\phi_1}=+\infty$ and $\liminf\limits_{\phi_2\uparrow+\infty}\min\limits_{t\in[\epsilon_1,\epsilon_2]}\frac{f_2(t,\phi_1,\phi_2)}{\phi_2}=+\infty$.

($C_3^*$) There exists $[\epsilon_1,\epsilon_2]\subset(a,b)$ satisfying $\liminf\limits_{\phi_2\uparrow+\infty}\min\limits_{t\in[\epsilon_1,\epsilon_2]}\frac{f_1(t,\phi_1,\phi_2)}{\phi_2}=+\infty$ and $\liminf\limits_{\phi_1\uparrow+\infty}\min\limits_{t\in[\epsilon_1,\epsilon_2]}\frac{f_2(t,\phi_1,\phi_2)}{\phi_1}=+\infty$.

($C_4$) $\int_a^b\sigma_{\alpha_\varrho}(s)g_\varrho(s)\frac{ds}{s^{1-\rho}}<+\infty$ and $\int_a^b\sigma_{\alpha_\varrho}(s)f_\varrho(s,\phi_1,\phi_2)\frac{ds}{s^{1-\rho}}<+\infty$ for any $\phi_\varrho\in[0,m]$, where $m>0$ is any constant, $\varrho=1,2$.

**Lemma 7.** *Assume the condition* ($C_1$) *or* ($C_1^*$) *holds, then the following KFDE*

$$\begin{cases} -\mathcal{D}_{a+}^{\alpha_1,\rho}\varpi_1(t)=\lambda g_1(t), \quad -\mathcal{D}_{a+}^{\alpha_2,\rho}\varpi_2(t)=\lambda g_2(t), \quad t\in(a,b),\\ \gamma^i\varpi_1(a)=\gamma^j\varpi_2(a)=0, \ \ \varpi_1(b)=\nu_2\int_a^b u_2(s)\varpi_2(s)\frac{dU_2(s)}{s^{1-\rho}}, \ \ \varpi_2(b)=\nu_1\int_a^b u_1(s)\varpi_1(s)\frac{dU_1(s)}{s^{1-\rho}}, \end{cases}$$

*where* $0\le i\le m_1-2$*,* $0\le j\le m_2-2$*, has an unique solution*

$$\begin{cases} \varpi_1(t)=\lambda\left(\int_a^b G_1(t,s)g_1(s)\frac{ds}{s^{1-\rho}}+\int_a^b H_2(t,s)g_2(s)\frac{ds}{s^{1-\rho}}\right),\\ \varpi_2(t)=\lambda\left(\int_a^b G_2(t,s)g_2(s)\frac{ds}{s^{1-\rho}}+\int_a^b H_1(t,s)g_1(s)\frac{ds}{s^{1-\rho}}\right). \end{cases} \tag{2.16}$$

*which satisfy*

$$\begin{cases} \varpi_1(t)\le\lambda\ell_2\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\int_a^b(g_1(s)+g_2(s))\frac{ds}{s^{1-\rho}}, \quad t\in[a,b],\\ \varpi_2(t)\le\lambda\ell_2\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_2-1}\int_a^b(g_1(s)+g_2(s))\frac{ds}{s^{1-\rho}}, \quad t\in[a,b]. \end{cases} \tag{2.17}$$

*Proof.* By invoking Lemma 5, Remark 1, and the condition ($C_1$) or ($C_1^*$), we verify that (2.16) and (2.17) are satisfied. □

Let $\mathbb{E}=[a,b]^2$, then $\mathbb{E}$ is a Banach space equipped with the norm $\|(\phi_1,\phi_2)\|_1=\|\phi_1\|+\|\phi_2\|$, $\|\phi_\varrho\|=\max_{t\in[a,b]}|\phi_\varrho(t)|$ for any $(\phi_1,\phi_2)\in\mathbb{E}$, $\varrho=1,2$. Let $\mathbb{P}=\{(\phi_1,\phi_2)\in\mathbb{E}:\phi_\varrho(t)\ge\ell(\frac{t^\rho-a^\rho}{b^\rho-a^\rho})^{\alpha_\varrho-1}\|\phi_\varrho\|$ for $t\in[a,b],\varrho=1,2\}$, where $0<\ell=\ell_1/\ell_2<1$. Then $\mathbb{P}$ is a cone of $\mathbb{E}$.

Next we only consider the following singular KFDE

$$\begin{cases} \mathcal{D}_{a+}^{\alpha_1,\rho}\psi_1(t)+\lambda(f_1(t,[\psi_1(t)-\varpi_1(t)]^*,[\psi_2(t)-\varpi_2(t)]^*)+g_1(t))=0, \ \ \lambda>0,\\ \mathcal{D}_{a+}^{\alpha_2,\rho}\psi_1(t)+\lambda(f_2(t,[\psi_1(t)-\varpi_1(t)]^*,[\psi_2(t)-\varpi_2(t)]^*)+g_2(t))=0, \ \ t\in(a,b),\\ \gamma^i\psi_1(a)=\gamma^j\psi_2(a)=0, \ \ \psi_1(b)=\nu_2\int_a^b u_2(s)\psi_2(s)\frac{dU_2(s)}{s^{1-\rho}}, \ \ \psi_2(b)=\nu_1\int_a^b u_1(s)\psi_1(s)\frac{dU_1(s)}{s^{1-\rho}}, \end{cases} \tag{2.18}$$

where $0\le i\le m_1-2$, $0\le j\le m_2-2$, an adjusted function $[\aleph(t)]^*$ is defined by $[\aleph(t)]^*=\aleph(t)$, if $\aleph(t)\ge0$, and $[\aleph(t)]^*=0$, if $\aleph(t)<0$ for any $\aleph\in C[a,b]$.

**Lemma 8.** *If* $(\psi_1,\psi_2)\in C[a,b]^2$ *with* $\psi_i(t)>\varpi_i(t)$ $(i=1,2)$ *for any* $t\in(a,b)$ *is a positive solution of KFDE* (2.18)*, then* $(\psi_1-\varpi_1,\psi_2-\varpi_2)$ *is a positive solution of KFDE* (1.1)*.*

*Proof.* In fact, if $(\psi_1,\psi_2)\in C[a,b]^2$ is a positive solution of KFDE (2.18) such that $\psi_\varrho(t)>\varpi_\varrho(t)$ for any $t\in(a,b]$ and $\varrho=1,2$, then, from (2.18) and the definition of $[\cdot]^*$, we derive

$$\begin{cases} \mathcal{D}_{a+}^{\alpha_1,\rho}\psi_1(t)+\lambda(f_1(t,\psi_1(t)-\varpi_1(t),\psi_2(t)-\varpi_2(t))+g_1(t))=0, \ \ \lambda>0,\\ \mathcal{D}_{a+}^{\alpha_2,\rho}\psi_1(t)+\lambda(f_2(t,\psi_1(t)-\varpi_1(t),\psi_2(t)-\varpi_2(t))+g_2(t))=0, \ \ t\in(a,b),\\ \gamma^i\psi_1(a)=\gamma^j\psi_2(a)=0, \ \ \psi_1(b)=\nu_2\int_a^b u_2(s)\psi_2(s)\frac{dU_2(s)}{s^{1-\rho}}, \ \ \psi_2(b)=\nu_1\int_a^b u_1(s)\psi_1(s)\frac{dU_1(s)}{s^{1-\rho}}, \end{cases} \tag{2.19}$$

where $0 \le i \le m_1 - 2$, $0 \le j \le m_2 - 2$. Let $\phi_\varrho = \psi_\varrho - \varpi_\varrho$, from Lemma 7, then we have $\mathcal{D}_{a+}^{\alpha_\varrho,\rho}\phi_\varrho(t) = \mathcal{D}_{a+}^{\alpha_\varrho,\rho}\psi_\varrho(t) - \mathcal{D}_{a+}^{\alpha_\varrho,\rho}\varpi_\varrho(t) = \mathcal{D}_{a+}^{\alpha_\varrho,\rho}\psi_\varrho(t) + \lambda g_\varrho(t)$ for $t \in (a,b)$ and $\varrho = 1,2$. So KFDE (2.19) can be rewritten as

$$\begin{cases} \mathcal{D}_{a+}^{\alpha_1,\rho}\phi_1(t) + \lambda f_1(t,\phi_1(t),\phi_2(t)) = 0, \quad \mathcal{D}_{a+}^{\alpha_2,\rho}\phi_2(t) + \lambda f_2(t,\phi_1(t),\phi_2(t)) = 0, \quad t \in (a,b), \ \lambda > 0, \\ \gamma^i\phi_1(a) = \gamma^j\phi_2(a) = 0, \quad \phi_1(b) = v_2\int_a^b u_2(s)\phi_2(s)\frac{dU_2(s)}{s^{1-\rho}}, \quad \phi_2(b) = v_1\int_a^b u_1(s)\phi_1(s)\frac{dU_1(s)}{s^{1-\rho}}, \end{cases}$$

where $0 \le i \le m_1 - 2$, $0 \le j \le m_2 - 2$, i.e., $(\psi_1 - \varpi_1, \psi_2 - \varpi_2)$ is a positive solution of KFDE (1.1). This proves Lemma 8. □

It follows from Lemma 3 that KFDE (2.18) can be reformulated as

$$\begin{aligned} \psi_1(t) =& \lambda\Big(\int_a^b G_1(t,s)(f_1(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*) + g_1(s))\frac{ds}{s^{1-\rho}} \\ &+ \int_a^b H_2(t,s)(f_2(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*) + g_2(s))\frac{ds}{s^{1-\rho}}\Big), \end{aligned} \tag{2.20a}$$

$$\begin{aligned} \psi_2(t) =& \lambda\Big(\int_a^b G_2(t,s)(f_2(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*) + g_2(s))\frac{ds}{s^{1-\rho}} \\ &+ \int_a^b H_1(t,s)(f_1(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*) + g_1(s))\frac{ds}{s^{1-\rho}}\Big). \end{aligned} \tag{2.20b}$$

A solution to KFDE (2.18) is defined as a solution to the corresponding system of integral equations (2.20). We introduce an operator $\mathrm{T} : \mathbb{P} \to \mathbb{P}$ defined by

$$\mathrm{T}(\psi_1,\psi_2) = (\mathrm{T}_1(\psi_1,\psi_2), \mathrm{T}_2(\psi_1,\psi_2)),$$

where the operators $\mathrm{T}_\varrho : \mathbb{P} \to C[a,b]$ ($\varrho = 1,2$) are defined by

$$\begin{aligned} \mathrm{T}_1(\psi_1,\psi_2)(t) =& \lambda\Big(\int_a^b G_1(t,s)(f_1(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*) + g_1(s))\frac{ds}{s^{1-\rho}} \\ &+ \int_a^b H_2(t,s)(f_2(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*) + g_2(s))\frac{ds}{s^{1-\rho}}\Big), \end{aligned} \tag{2.21a}$$

$$\begin{aligned} \mathrm{T}_2(\psi_1,\psi_2)(t) =& \lambda\Big(\int_a^b G_2(t,s)(f_2(s,[\psi_1(s)-\varpi_2(s)]^*,[\psi_2(s)-\varpi_2(s)]^*) + g_2(s))\frac{ds}{s^{1-\rho}} \\ &+ \int_a^b H_1(t,s)(f_1(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*) + g_1(s))\frac{ds}{s^{1-\rho}}\Big). \end{aligned} \tag{2.21b}$$

Evidently, if the pair $(\psi_1,\psi_2) \in \mathbb{P}$ is a fixed point of the operator T, then it necessarily solves KFDE (2.21).

**Lemma 9.** *If the condition* $(\mathrm{C}_1)$ *or* $(\mathrm{C}_1^*)$ *holds, then* $\mathrm{T} : \mathbb{P} \to \mathbb{P}$ *is a completely continuous operator.*

*Proof.* For any fixed $(\psi_1,\psi_2) \in \mathbb{P}$, there exists a positive constant $\mathcal{L}$ such that $\|(\psi_1,\psi_2)\|_1 \le \mathcal{L}$. And then we obtain $[\psi_\varrho(s) - \varpi_\varrho(s)]^* \le \psi_\varrho(s) \le \|\psi_\varrho\| \le \|(\psi_1,\psi_2)\|_1 \le \mathcal{L}$ for $s \in [a,b]$ and $\varrho = 1,2$. For any $t \in [a,b]$, it follows from (2.21) and Remark 1 that

$$\mathrm{T}_1(\psi_1,\psi_2)(t) = \lambda\Big(\int_a^b G_1(t,s)(f_1(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*) + g_1(s))\frac{ds}{s^{1-\rho}}$$

$$
\begin{aligned}
&+\int_a^b H_2(t,s)(f_2(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*)+g_2(s))\frac{ds}{s^{1-\rho}}\Big)\\
\leq&\lambda\Big(\ell_2\int_a^b \sigma_{\alpha_1}(s)(f_1(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*)+g_1(s))\frac{ds}{s^{1-\rho}}\\
&+\ell_2\int_a^b \sigma_{\alpha_2}(s)(f_2(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*)+g_2(s))\frac{ds}{s^{1-\rho}}\Big)\\
\leq&\lambda\mathfrak{M}\ell_2\int_a^b(\sigma_{\alpha_1}(s)+\sigma_{\alpha_2}(s))\frac{ds}{s^{1-\rho}}+\lambda\ell_2\int_a^b(\sigma_{\alpha_1}(s)g_1(s)+\sigma_{\alpha_2}(s)g_2(s))\frac{ds}{s^{1-\rho}}\\
\leq&2\lambda\mathfrak{M}\ell_2\frac{b^\rho-a^\rho}{\rho}+\lambda\ell_2\int_a^b(g_1(s)+g_2(s))\frac{ds}{s^{1-\rho}}<+\infty,
\end{aligned}
$$

where $\mathfrak{M}=\max\{\max_{t\in[a,b],\phi_1,\phi_2\in[0,\mathcal{L}]} f_1(t,\phi_1,\phi_2),\max_{t\in[a,b],\phi_1,\phi_2\in[0,\mathcal{L}]} f_2(t,\phi_1,\phi_2)\}+1$. Similarly, we can observe that $|\mathrm{T}_2(\psi_1,\psi_2)(t)|\leq 2\lambda\mathfrak{M}\ell_2\frac{b^\rho-a^\rho}{\rho}+\lambda\ell_2\int_a^b(g_1(s)+g_2(s))\frac{ds}{s^{1-\rho}}<+\infty$. Thus $\mathrm{T}:\mathbb{P}\to\mathbb{E}$ is well defined.

Next, we prove that $\mathrm{T}:\mathbb{P}\to\mathbb{P}$. For any fixed $(\psi_1,\psi_2)\in\mathbb{P}$ and $t\in[a,b]$, using (2.21) and Remark 1, we acquire

$$
\begin{aligned}
\mathrm{T}_1(\psi_1,\psi_2)(t)\leq&\lambda\Big(\ell_2\int_a^b \sigma_{\alpha_1}(s)(f_1(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*)+g_1(s))\frac{ds}{s^{1-\rho}}\\
&+\ell_2\int_a^b \sigma_{\alpha_2}(s)(f_2(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*)+g_2(s))\frac{ds}{s^{1-\rho}}\Big),\\
\mathrm{T}_2(\psi_1,\psi_2)(t)\leq&\lambda\Big(\ell_2\int_a^b \sigma_{\alpha_2}(s)(f_2(s,[\psi_1(s)-\varpi_2(s)]^*,[\psi_2(s)-\varpi_2(s)]^*)+g_2(s))\frac{ds}{s^{1-\rho}}\\
&+\ell_2\int_a^b \sigma_{\alpha_1}(s)(f_1(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*)+g_1(s))\frac{ds}{s^{1-\rho}}\Big),
\end{aligned}
$$

which reveals that

$$
\begin{aligned}
\|\mathrm{T}_1(\psi_1,\psi_2)\|\leq&\lambda\ell_2\Big(\int_a^b \sigma_{\alpha_1}(s)(f_1(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*)+g_1(s))\frac{ds}{s^{1-\rho}}\\
&+\int_a^b \sigma_{\alpha_2}(s)(f_2(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*)+g_2(s))\frac{ds}{s^{1-\rho}}\Big),\\
\|\mathrm{T}_2(\psi_1,\psi_2)\|\leq&\lambda\ell_2\Big(\int_a^b \sigma_{\alpha_2}(s)(f_2(s,[\psi_1(s)-\varpi_2(s)]^*,[\psi_2(s)-\varpi_2(s)]^*)+g_2(s))\frac{ds}{s^{1-\rho}}\\
&+\int_a^b \sigma_{\alpha_1}(s)(f_1(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*)+g_1(s))\frac{ds}{s^{1-\rho}}\Big).
\end{aligned}
$$

Alternatively, combining (2.21) with Remark 1, we further deduce that

$$
\begin{aligned}
\mathrm{T}_1(\psi_1,\psi_2)(t)\geq&\lambda\ell_1\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\Big(\int_a^b \sigma_{\alpha_1}(s)(f_1(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*)+g_1(s))\frac{ds}{s^{1-\rho}}\\
&+\int_a^b \sigma_{\alpha_2}(s)(f_2(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*)+g_2(s))\frac{ds}{s^{1-\rho}}\Big),
\end{aligned}
$$

$$\mathrm{T}_2(\psi_1,\psi_2)(t) \geq \lambda\ell_1\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_2-1}\Big(\int_a^b \sigma_{\alpha_2}(s)(f_2(s,[\psi_1(s)-\varpi_2(s)]^*,[\psi_2(s)-\varpi_2(s)]^*)+g_2(s))\frac{ds}{s^{1-\rho}}$$
$$+\int_a^b \sigma_{\alpha_1}(s)(f_1(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*)+g_1(s))\frac{ds}{s^{1-\rho}}\Big).$$

The combination of the preceding two equations yields

$$\mathrm{T}_\varrho(\psi_1,\psi_2)(t) \geq \ell\Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_\varrho-1}\|\mathrm{T}_\varrho(\psi_1,\psi_2)\| \text{ for } t\in[a,b] \text{ and } \varrho = 1,2,$$

which implies that $\mathrm{T}(\mathbb{P}) \subset \mathbb{P}$. By invoking the Ascoli-Arzela theorem, we immediately establish the complete continuity of the operator $\mathrm{T}:\mathbb{P}\to\mathbb{P}$. This concludes the verification of the lemma. □

The following Leray-Schauder nonlinear alternative and Krasnoselskii's FPT will serve as foundational tools in our subsequent analysis.

**Theorem A** (Nonlinear alternative of Leray-Schauder type, see [40])**.** *Let $\mathbb{X}$ be a Banach space with closed convex $\Omega \in \mathbb{X}$, $\mathbb{U} \subset \Omega$ a relatively open neighborhood of 0, and $\mathrm{S} : \overline{\mathbb{U}} \to \Omega$ a compact continuous operator. Then either*

(a) *S has a fixed point in $\overline{\mathbb{U}}$, or*
(b) *there exists $u \in \partial\mathbb{U}$ and $v \in (0,1)$, with $u = v\mathrm{S}u$.*

**Theorem B** (Krasnoselskii's FPT, see [41])**.** *Let $\mathbb{X}$ be a Banach space equipped with a cone $\mathbb{P}$. Suppose $\Omega_1$ and $\Omega_2$ are open subsets of $\mathbb{X}$ satisfying the following conditions $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and $\mathrm{S} : \mathbb{P} \to \mathbb{P}$ is a completely continuous operator such that, either*

(a) $\|\mathrm{S}w\| \leq \|w\|$, $w \in \mathbb{P}\cap\partial\Omega_1$, $\|\mathrm{S}w\| \geq \|w\|$, $w \in \mathbb{P}\cap\partial\Omega_2$, *or*
(b) $\|\mathrm{S}w\| \geq \|w\|$, $w \in \mathbb{P}\cap\partial\Omega_1$, $\|\mathrm{S}w\| \leq \|w\|$, $w \in \mathbb{P}\cap\partial\Omega_2$.

*Then S has a fixed point $w \in \mathbb{P}\cap(\overline{\Omega}_2\backslash\Omega_1)$.*

## 3. Main results

In this section, we present our main existence results. The first theorem guarantees at least one solution for small values of the parameter $\lambda$, while the subsequent theorems, under stronger conditions, establish the existence of two distinct positive solutions.

**Theorem 1.** *Assume that hypotheses* $(C_1)$ *and* $(C_2)$ *are satisfied. Then there exists a positive constant $\lambda_*$ such that KFDE* (1.1) *admits at least one positive solution for any $0 < \lambda \leq \lambda_*$.*

*Proof.* Let $\varepsilon \in (0,1)$ and fixed $\delta \in (0,1)$, it follows from $(C_2)$ that

$$f_\varrho(t,\phi_1,\phi_2) \geq \delta f_\varrho(t,0,0) \quad \text{for } a\leq t\leq b,\ 0\leq\phi_\varrho\leq\varepsilon \text{ and } \varrho = 1,2. \tag{3.1}$$

Let $\overline{f_\varrho}(\varepsilon) = \max_{a\leq t\leq b, 0\leq\phi_\varrho\leq\varepsilon}\{f_\varrho(t,\phi_1,\phi_2)+g_\varrho(t)\}$ and $\vartheta_\varrho = \ell_2\int_a^b \sigma_{\alpha_\varrho}(s)\frac{ds}{s^{1-\rho}}$, we acquire $\lim_{\varepsilon\downarrow 0}(\overline{f_\varrho}(\varepsilon)/\varepsilon) = +\infty$ for $\varrho = 1,2$. Let $\lambda$ satisfy $0 < \lambda < \lambda_* := \varepsilon/(8\vartheta\overline{f}(\varepsilon))$, where $\vartheta = \max\{\vartheta_1,\vartheta_2\}$ and $\overline{f}(\varepsilon) = \max\{\overline{f_1}(\varepsilon), \overline{f_2}(\varepsilon)\}$, then we have $\lim_{\varepsilon\downarrow 0}(\overline{f}(\varepsilon)/\varepsilon) = +\infty$ and $\overline{f}(\varepsilon)/\varepsilon < 1/(8\vartheta\lambda)$. Furthermore, then exists a $R_0 \in (0,\varepsilon)$ such that $\overline{f}(R_0)/R_0 = 1/(8\vartheta\lambda)$.

For $\mathbb{U} = \{(\psi_1, \psi_2) \in \mathbb{P} : \|(\psi_1, \psi_2)\|_1 < R_0\}$, let $(\psi_1, \psi_2) \in \partial\mathbb{U}$ and $\theta \in (0, 1)$ be such that $(\psi_1, \psi_2) = \theta\mathrm{T}(\psi_1, \psi_2)$, i.e., $\psi_1 = \theta\mathrm{T}_1(\psi_1, \psi_2)$ and $\psi_2 = \theta\mathrm{T}_2(\psi_1, \psi_2)$. We will apply the Leray-Schauder nonlinear alternative (Theorem A) to prove T a fixed point in $\overline{\mathbb{U}}$. To do this, we show that no solution lies on its boundary $\partial\mathbb{U}$. We announce that $\|(\psi_1, \psi_2)\|_1 \neq R_0$. In fact, for $(\psi_1, \psi_2) \in \partial\mathbb{U}$ and $\|(\psi_1, \psi_2)\|_1 = R_0$, we obtian

$$\begin{aligned}\psi_1(t) = \theta\mathrm{T}_1(\psi_1, \psi_2)(t) \leq &\lambda\Big(\int_a^b G_1(t, s)(f_1(s, [\psi_1(s) - \varpi_1(s)]^*, [\psi_2(s) - \varpi_2(s)]^*) + g_1(s))\frac{ds}{s^{1-\rho}} \\ &+ \int_a^b H_2(t, s)(f_2(s, [\psi_1(s) - \varpi_1(s)]^*, [\psi_2(s) - \varpi_2(s)]^*) + g_2(s))\frac{ds}{s^{1-\rho}}\Big) \\ \leq &\lambda\Big(\int_a^b G_1(t, s)\overline{f_1}(R_0)\frac{ds}{s^{1-\rho}} + \int_a^b H_2(t, s)\overline{f_2}(R_0)\frac{ds}{s^{1-\rho}}\Big) \\ \leq &\lambda\ell_2\Big(\int_a^b \sigma_{\alpha_1}(s)\frac{ds}{s^{1-\rho}}\overline{f_1}(R_0) + \int_a^b \sigma_{\alpha_2}(s)\frac{ds}{s^{1-\rho}}\overline{f_2}(R_0)\Big) \leq 2\vartheta\lambda\overline{f}(R_0). \end{aligned} \tag{3.2}$$

Following an analogous approach, we obtain

$$\psi_2(t) = \theta\mathrm{T}_2(\psi_1, \psi_2)(t) \leq 2\vartheta\lambda\overline{f}(R_0). \tag{3.3}$$

It follows from (3.2) and (3.3) that $R_0 = \|(\psi_1, \psi_2)\|_1 = \|\psi_1\| + \|\psi_2\| \leq 4\vartheta\lambda\overline{f}(R_0)$, i.e., $\overline{f}(R_0)/R_0 \geq 1/(4\vartheta\lambda) > 1/(8\vartheta\lambda) = \overline{f}(R_0)/R_0$, which yields that $\|(\psi_1, \psi_2)\|_1 \neq R_0$. It follows from Theorem A that T has a fixed point $(\psi_1, \psi_2) \in \overline{\mathbb{U}}$. Moreover, combining (3.1) and the fact that $R_0 < \varepsilon$, we derive

$$\begin{aligned}\psi_1(t) = &\lambda\Big(\int_a^b G_1(t, s)(f_1(s, [\psi_1(s) - \varpi_1(s)]^*, [\psi_2(s) - \varpi_2(s)]^*) + g_1(s))\frac{ds}{s^{1-\rho}} \\ &+ \int_a^b H_2(t, s)(f_2(s, [\psi_1(s) - \varpi_1(s)]^*, [\psi_2(s) - \varpi_2(s)]^*) + g_2(s))\frac{ds}{s^{1-\rho}}\Big) \\ \geq &\lambda\Big(\int_a^b G_1(t, s)(\delta f_1(s, 0, 0) + g_1(s))\frac{ds}{s^{1-\rho}} + \int_a^b H_2(t, s)(\delta f_2(s, 0, 0) + g_2(s))\frac{ds}{s^{1-\rho}}\Big) \\ \geq &\lambda\Big(\int_a^b G_1(t, s)g_1(s)\frac{ds}{s^{1-\rho}} + \int_a^b H_2(t, s)g_2(s)\frac{ds}{s^{1-\rho}}\Big) = \varpi_1(t) \text{ for } t \in (a, b).\end{aligned}$$

In the same manner, we derive $\psi_2(t) \geq \varpi_2(t)$ for $t \in (a, b)$. Then T has a positive fixed point $(\psi_1, \psi_2)$ satisfying $\|(\psi_1, \psi_2)\|_1 \leq R_0 < 1$. Namely, $(\psi_1, \psi_2)$ is positive solution of KFDE (2.18) with $\psi_\varrho(t) \geq \varpi_\varrho(t)$ for $t \in (a, b)$ and $\varrho = 1, 2$. Let $\phi_\varrho(t) = \psi_\varrho(t) - \varpi_\varrho(t) \geq 0$, $\varrho = 1, 2$. Then $(\phi_1, \phi_1)$ is a nonnegative solution (positive on $(a, b)$) of KFDE (1.1). The arguments above suffice to prove Theorem 1. □

**Theorem 2.** *Assume that hypotheses* $(C_1^*)$ *and* $(C_3)$–$(C_4)$ *are satisfied. Then there exists a positive constant* $\lambda^*$ *such that KFDE* (1.1) *admits at least one positive solution for any* $0 < \lambda \leq \lambda^*$.

*Proof.* Let $\Omega_1 = \{(\psi_1, \psi_2) \in \mathbb{E} : \|\psi_\varrho\| < R_1, \varrho = 1, 2\}$, where $R_1 = \max\{1, r\}$, $r = \frac{\ell_2}{\ell}\int_a^b (g_1(s) + g_2(s))\frac{ds}{s^{1-\rho}}$. Set $\lambda^* = \min\{1, R_1/(2(R+1)), R_1/(2r)\}$, where $R = \ell_2\big(\int_a^b \sigma_{\alpha_1}(s)(\max_{0\leq\phi_1,\phi_2\leq R_1} f_1(s, \phi_1, \phi_2) + g_1(s))\frac{ds}{s^{1-\rho}} + \int_a^b \sigma_{\alpha_2}(s)(\max_{0\leq\phi_1,\phi_2\leq R_1} f_2(s, \phi_1, \phi_2) + g_2(s))\frac{ds}{s^{1-\rho}}\big) \geq 0$. Then, for any $(\psi_1, \psi_2) \in \mathbb{P} \cap \partial\Omega_1$, we derive $\|\psi_1\| = R_1$ or $\|\psi_2\| = R_1$. Furthermore, we obtain $\psi_\varrho(t) - \varpi_\varrho(t) \leq \psi_\varrho(t) \leq \|\psi_\varrho\| \leq R_1$, $\varrho = 1, 2$, and

$$\|\mathrm{T}_1(\psi_1, \psi_2)\| \leq \lambda\ell_2\Big(\int_a^b \sigma_{\alpha_1}(s)(f_1(s, [\psi_1(s) - \varpi_1(s)]^*, [\psi_2(s) - \varpi_2(s)]^*) + g_1(s))\frac{ds}{s^{1-\rho}}$$

$$
\begin{aligned}
&+\int_a^b \sigma_{\alpha_2}(s)(f_2(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*)+g_2(s))\frac{ds}{s^{1-\rho}}\Big)\\
\le&\lambda\ell_2\Big(\int_a^b \sigma_{\alpha_1}(s)\Big(\max_{0\le\phi_1,\phi_2\le R_1} f_1(s,\phi_1,\phi_2)+g_1(s)\Big)\frac{ds}{s^{1-\rho}}\\
&+\int_a^b \sigma_{\alpha_2}(s)\Big(\max_{0\le\phi_1,\phi_2\le R_1} f_2(s,\phi_1,\phi_2)+g_2(s)\Big)\frac{ds}{s^{1-\rho}}\Big)=\lambda R\le\frac{R_1}{2}.
\end{aligned}
$$

Similarly, we also get $\|\mathrm{T}_2(\psi_1,\psi_2)\|\le R_1/2$. Combining the previous results yields

$$
\|\mathrm{T}(\psi_1,\psi_2)\|_1=\|\mathrm{T}_1(\psi_1,\psi_2)\|+\|\mathrm{T}_2(\psi_1,\psi_2)\|\le R_1\le\|(\psi_1,\psi_2)\|_1,\ (\psi_1,\psi_2)\in\mathbb{P}\cap\partial\Omega_1. \tag{3.4}
$$

Alternatively, select two constants $N_1,N_2>1$ such that

$$
\begin{aligned}
\lambda\ell\ell_1\frac{N_1}{2}\Big(\frac{\epsilon_1^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\int_{\epsilon_1}^{\epsilon_2}\sigma_{\alpha_1}(s)\Big(\frac{s^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\frac{ds}{s^{1-\rho}}\ge&1,\\
\lambda\ell\ell_1\frac{N_2}{2}\Big(\frac{\epsilon_1^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_2-1}\int_{\epsilon_1}^{\epsilon_2}\sigma_{\alpha_2}(s)\Big(\frac{s^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_2-1}\frac{ds}{s^{1-\rho}}\ge&1.
\end{aligned}
$$

It follows from hypotheses ($C_3$) and ($C_4$) that there exists a constant $\mathcal{B}>R_1$ satisfying

$$
\frac{f_1(t,\phi_1,\phi_2)}{\phi_1}>N_1,\ \text{ namely }\ f_1(t,\phi_1,\phi_2)>N_1\phi_1\ \text{ for }\ t\in[\epsilon_1,\epsilon_2],\ \ \phi_1>\mathcal{B},\ \ \phi_2>0, \tag{3.5}
$$

$$
\frac{f_2(t,\phi_1,\phi_2)}{\phi_2}>N_2,\ \text{ namely }\ f_2(t,\phi_1,\phi_2)>N_2\phi_2\ \text{ for }\ t\in[\epsilon_1,\epsilon_2],\ \ \phi_1>0,\ \ \phi_2>\mathcal{B}. \tag{3.6}
$$

Let $R_2=\max\{R_1+1,2\lambda r,\frac{2}{\ell}\max\{(\frac{b^\rho-a^\rho}{\epsilon_1^\rho-a^\rho})^{\alpha_1-1},(\frac{b^\rho-a^\rho}{\epsilon_1^\rho-a^\rho})^{\alpha_2-1}\}(\mathcal{B}+1)\}$ and $\Omega_2=\{(\psi_1,\psi_2)\in\mathbb{E}:\|\psi_\varrho\|<R_2,\varrho=1,2\}$. Then for any $(\psi_1,\psi_2)\in\mathbb{P}\cap\partial\Omega_2$, we have $\|\psi_1\|=R_2$ or $\|\psi_2\|=R_2$. If $\|\psi_1\|=R_2$, we can state that

$$
\begin{aligned}
\psi_1(t)-\varpi_1(t)=&\psi_1(t)-\lambda\left(\int_a^b G_1(t,s)g_1(s)\frac{ds}{s^{1-\rho}}+\int_a^b H_2(t,s)g_2(s)\frac{ds}{s^{1-\rho}}\right)\\
\ge&\psi_1(t)-\lambda\ell_2\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\int_a^b(g_1(s)+g_2(s))\frac{ds}{s^{1-\rho}}\\
=&\psi_1(t)-\lambda\ell\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\frac{\ell_2}{\ell}\int_a^b(g_1(s)+g_2(s))\frac{ds}{s^{1-\rho}}\\
=&\psi_1(t)-\lambda\ell\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}r\ge\left(1-\frac{\lambda r}{R_2}\right)\psi_1(t)\ge\frac{1}{2}\psi_1(t)\ge0,\ \ t\in[a,b],
\end{aligned}
$$

and then

$$
\begin{aligned}
\min_{t\in[\epsilon_1,\epsilon_2]}\{[\psi_1(t)-\varpi_1(t)]^*\}&=\min_{t\in[\epsilon_1,\epsilon_2]}\{\psi_1(t)-\varpi_1(t)\}\ge\min_{t\in[\epsilon_1,\epsilon_2]}\Big\{\frac{1}{2}\psi_1(t)\Big\}\\
&\ge\min_{t\in[\epsilon_1,\epsilon_2]}\Big\{\frac{\ell}{2}(\frac{t^\rho-a^\rho}{b^\rho-a^\rho})^{\alpha_1-1}\|\psi_1\|\Big\}=\frac{\ell}{2}\Big(\frac{\epsilon_1^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}R_2\ge\mathcal{B}+1>\mathcal{B}.
\end{aligned}
$$

Since $\mathcal{B}>R_1\ge r$, from (3.5) and the above inequalities, we acquire

$$
f_1(t,[\psi_1(t)-\varpi_1(t)]^*,[\psi_2(t)-\varpi_2(t)]^*)\ge N_1[\psi_1(t)-\varpi_1(t)]^*\ge\frac{N_1}{2}\psi_1(t),\ \ t\in[\epsilon_1,\epsilon_2]. \tag{3.7}
$$

It follows from the previous inequality (3.7) that

$$\begin{aligned}\mathrm{T}_1(\psi_1,\psi_2)(t) =&\lambda\Big(\int_a^b G_1(t,s)(f_1(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*)+g_1(s))\frac{ds}{s^{1-\rho}}\\&+\int_a^b H_2(t,s)(f_2(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*)+g_2(s))\frac{ds}{s^{1-\rho}}\Big)\\\geq&\lambda\int_a^b G_1(t,s)(f_1(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*)+g_1(s))\frac{ds}{s^{1-\rho}}\\\geq&\lambda\int_a^b \ell_1\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\sigma_{\alpha_1}(s)f_1(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*)\frac{ds}{s^{1-\rho}}\\\geq&\lambda\ell_1\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\int_{\epsilon_1}^{\epsilon_2}\sigma_{\alpha_1}(s)\frac{N_1}{2}\psi_1(s)\frac{ds}{s^{1-\rho}}\\\geq&\lambda\ell_1\frac{N_1}{2}\Big(\frac{\epsilon_1^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\int_{\epsilon_1}^{\epsilon_2}\sigma_{\alpha_1}(s)\ell\Big(\frac{s^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\|\psi_1\|\frac{ds}{s^{1-\rho}}\\=&\lambda\ell\ell_1\frac{N_1}{2}\Big(\frac{\epsilon_1^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\int_{\epsilon_1}^{\epsilon_2}\sigma_{\alpha_1}(s)\Big(\frac{s^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_1-1}\frac{ds}{s^{1-\rho}}R_2\geq R_2,\ t\in[\epsilon_1,\epsilon_2].\end{aligned}$$

If $\|\psi_2\|=R_2$, we obtain

$$\psi_2(t)-\varpi_2(t)=\psi_2(t)-\lambda\left(\int_a^b\mathfrak{G}_2(t,s)g_2(s)\frac{ds}{s^{1-\rho}}+\int_a^b\mathfrak{H}_1(t,s)g_1(s)\frac{ds}{s^{1-\rho}}\right)\geq\frac{1}{2}\psi_2(t)\geq0,\quad t\in[a,b],$$

and then

$$\begin{aligned}\min_{t\in[\epsilon_1,\epsilon_2]}\{[\psi_2(t)-\varpi_2(t)]^*\}&=\min_{t\in[\epsilon_1,\epsilon_2]}\{\psi_2(t)-\varpi_2(t)\}\geq\min_{t\in[\epsilon_1,\epsilon_2]}\Big\{\frac{1}{2}\psi_2(t)\Big\}\\&\geq\min_{t\in[\epsilon_1,\epsilon_2]}\Big\{\frac{\ell}{2}(\frac{t^\rho-a^\rho}{b^\rho-a^\rho})^{\alpha_2-1}\|\psi_2\|\Big\}=\frac{\ell}{2}\Big(\frac{\epsilon_1^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_2-1}R_2\geq\mathcal{B}+1>\mathcal{B}.\end{aligned}$$

Since $\mathcal{B}>R_1\geq r$, from (3.6) and the above inequalities, we acquire

$$f_2(t,[\psi_1(t)-\varpi_1(t)]^*,[\psi_2(t)-\varpi_2(t)]^*)\geq N_2[\psi_2(t)-\varpi_2(t)]^*\geq\frac{N_2}{2}\psi_2(t),\ \ t\in[\epsilon_1,\epsilon_2].\tag{3.8}$$

It follows from (3.8) and the previous inequality that

$$\begin{aligned}\mathrm{T}_1(\psi_1,\psi_2)(t) =&\lambda\Big(\int_a^b G_1(t,s)(f_1(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*)+g_1(s))\frac{ds}{s^{1-\rho}}\\&+\int_a^b H_2(t,s)(f_2(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*)+g_2(s))\frac{ds}{s^{1-\rho}}\Big)\\\geq&\lambda\int_a^b H_2(t,s)(f_2(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*)+g_2(s))\frac{ds}{s^{1-\rho}}\\\geq&\lambda\int_a^b \ell_1\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\alpha_2-1}\sigma_{\alpha_2}(s)f_2(s,[\psi_1(s)-\varpi_1(s)]^*,[\psi_2(s)-\varpi_2(s)]^*)\frac{ds}{s^{1-\rho}}\end{aligned}$$

$$
\begin{aligned}
&\geq \lambda \ell_1 \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_2 - 1} \int_{\epsilon_1}^{\epsilon_2} \sigma_{\alpha_2}(s) \frac{N_2}{2} \psi_2(s) \frac{ds}{s^{1-\rho}} \\
&\geq \lambda \ell_1 \frac{N_2}{2} \Big(\frac{\epsilon_1^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_2 - 1} \int_{\epsilon_1}^{\epsilon_2} \sigma_{\alpha_2}(s) \ell \Big(\frac{s^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_2 - 1} \|\psi_2\| \frac{ds}{s^{1-\rho}} \\
&= \lambda \ell \ell_1 \frac{N_2}{2} \Big(\frac{\epsilon_1^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_2 - 1} \int_{\epsilon_1}^{\epsilon_2} \sigma_{\alpha_2}(s) \Big(\frac{s^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_2 - 1} \frac{ds}{s^{1-\rho}} R_2 \geq R_2, \ t \in [\epsilon_1, \epsilon_2].
\end{aligned}
$$

Based on the previous proof, for any $(\psi_1, \psi_2) \in \mathbb{P} \cap \partial\Omega_2$, we conclude that $\mathrm{T}_1(\psi_1, \psi_2)(t) \geq R_2$ for $t \in [\epsilon_1, \epsilon_2]$. Similarly, for any $(\psi_1, \psi_2) \in \mathbb{P} \cap \partial\Omega_2$, the following result also holds $\mathrm{T}_2(\psi_1, \psi_2)(t) \geq R_2$ for $t \in [\epsilon_1, \epsilon_2]$. Combining the two results yields

$$
\|\mathrm{T}(\psi_1, \psi_2)\|_1 = \|\mathrm{T}_1(\psi_1, \psi_2)\| + \|\mathrm{T}_2(\psi_1, \psi_2)\| \geq R_1 \geq \|(\psi_1, \psi_2)\|_1, \ (\psi_1, \psi_2) \in \mathbb{P} \cap \partial\Omega_2. \tag{3.9}
$$

It follows from (3.4) and (3.9) that condition (b) of Theorem B is satisfied. Consequently, T has a fixed point $(\psi_1, \psi_2)$ with $r \leq R_1 < \|\psi_\varrho\| < R_2, \varrho = 1, 2$. Because $r \leq R_1 < \|\psi_\varrho\| < R_2, \varrho = 1, 2$, we obtain

$$
\begin{aligned}
\psi_1(t) - \varpi_1(t) =& \psi_1(t) - \lambda \Big( \int_a^b G_1(t,s) g_1(s) \frac{ds}{s^{1-\rho}} + \int_a^b H_2(t,s) g_2(s) \frac{ds}{s^{1-\rho}} \Big) \\
\geq& \ell \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_1 - 1} \|\psi_1\| - \lambda \ell_2 \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_1 - 1} \int_a^b (g_1(s) + g_2(s)) \frac{ds}{s^{1-\rho}} \\
=& \ell \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_1 - 1} \|\psi_1\| - \lambda \ell \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_1 - 1} \frac{\ell_2}{\ell} \int_a^b (g_1(s) + g_2(s)) \frac{ds}{s^{1-\rho}} \\
\geq& \ell \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_1 - 1} r - \lambda \ell \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_1 - 1} r = (1 - \lambda) \ell \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_1 - 1} r \geq 0, \quad t \in (a, b), \\
\psi_2(t) - \varpi_2(t) =& \psi_2(t) - \lambda \Big( \int_a^b G_2(t,s) g_2(s) \frac{ds}{s^{1-\rho}} + \int_a H_1(t,s) g_1(s) \frac{ds}{s^{1-\rho}} \Big) \\
\geq& \ell \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_2 - 1} \|\psi_2\| - \lambda \ell_2 \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_2 - 1} \int_a^b (g_1(s) + g_2(s)) \frac{ds}{s^{1-\rho}} \\
=& \ell \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_2 - 1} \|\psi_2\| - \lambda \ell \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_2 - 1} \frac{\ell_2}{\ell} \int_a^b (g_1(s) + g_2(s)) \frac{ds}{s^{1-\rho}} \\
\geq& \ell \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_2 - 1} r - \lambda \ell \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_2 - 1} r = (1 - \lambda) \ell \Big(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha_2 - 1} r \geq 0, \quad t \in (a, b).
\end{aligned}
$$

Thus, $(\psi_1, \psi_2)$ is positive solution of KFDE (2.18) with $\psi_\varrho(t) > \varpi_\varrho(t)$ for $t \in (a, b)$, $\varrho = 1, 2$. Let $\phi_\varrho(t) = \psi_\varrho(t) - \varpi_\varrho(t) \geq 0$, $\varrho = 1, 2$. Then $(\phi_1, \phi_2)$ is a nonnegative solution (positive on $(a, b)$) of KFDE (1.1). This concludes the proof. □

Along with the proof of Theorem 2, it is apparent that $(\mathrm{C}_3^*)$ may substitute for $(\mathrm{C}_3)$. Consequently, we derive the theorem below.

**Theorem 3.** *Assume that hypotheses* $(\mathrm{C}_1^*)$, $(\mathrm{C}_3^*)$ *and* $(\mathrm{C}_4)$ *are satisfied. Then there exists a constant* $\lambda_* > 0$ *such that KFDE* (1.1) *has at least one positive solution for any* $0 < \lambda \leq \lambda_*$.

Given that hypothesis $(\mathrm{C}_1)$ entails both $(\mathrm{C}_1^*)$ and $(\mathrm{C}_4)$, the proofs of Theorems 1 and 2 suffice to derive the following result.

**Theorem 4.** *Assume that hypotheses* ($C_1$)–($C_3$) *are satisfied. Then KFDE* (1.1) *has at least two positive solutions for* $\lambda > 0$ *sufficiently small.*

In fact, it follows from Theorems 1 and 2 that KFDE (1.1) has at least two positive solutions for $0 < \lambda < \min\{\lambda_*, \lambda^*\}$. Correspondingly, it can be inferred that

**Theorem 5.** *Assume that hypotheses* ($C_1$)–($C_2$) *and* ($C_3^*$) *are satisfied. Then KFDE* (1.1) *has at least two positive solutions for* $\lambda > 0$ *sufficiently small.*

## 4. Some examples

**Example 1.** *Consider the following KFDE with coupled IBCs*

$$\begin{cases} \mathcal{D}_{4+}^{\alpha_1,\frac{1}{2}}\phi_1(t) + \lambda\Big(\phi_1^{\beta_1} + \dfrac{1}{\sqrt{(\sqrt{t}-2)(3-\sqrt{t})}}\cos(2\pi\phi_2)\Big) = 0, \quad \lambda > 0, \\ \mathcal{D}_{4+}^{\alpha_2,\frac{1}{2}}\phi_2(t) + \lambda\Big(\phi_2^{\beta_2} + \dfrac{1}{\sqrt{(\sqrt{t}-2)(3-\sqrt{t})}}\sin(2\pi\phi_1)\Big) = 0, \quad t \in (4,9), \\ \gamma^i\phi_1(4) = \gamma^j\phi_2(4) = 0, \ \ \phi_1(9) = \nu_2\displaystyle\int_4^9 u_2(s)\phi_2(s)\frac{dU_2(s)}{\sqrt{s}}, \ \ \phi_2(9) = \nu_1\int_4^9 u_1(s)\phi_1(s)\frac{dU_1(s)}{\sqrt{s}}, \end{cases} \tag{4.1}$$

*where* $0 \le i \le m_1 - 2$, $0 \le j \le m_2 - 2$, *the constant* $\beta_\varrho$ *satisfies* $\beta_\varrho > 1$; *the functions* $u_\varrho, U_\varrho$ *and the constant* $\nu_\varrho$ *are selected appropriately such that* $c_{\alpha_1}^{\alpha_2} > 0$ *is satisfied. Then KFDE* (4.1) *admits a strictly positive solution* $(\phi_1, \phi_2)$ *satisfying* $\phi_\varrho > 0$ *for sufficiently small* $\lambda > 0$ *and* $t \in (4,9)$, $\varrho = 1,2$.

*Proof.* Based on Eq (4.1), we derive

$$f_1(t,\phi_1,\phi_2) = \phi_1^{\beta_1} + \frac{1}{\sqrt{(\sqrt{t}-2)(3-\sqrt{t})}}\cos(2\pi\phi_2), \qquad g_1(t) = \frac{2}{\sqrt{(\sqrt{t}-2)(3-\sqrt{t})}},$$
$$f_2(t,\phi_1,\phi_2) = \phi_2^{\beta_2} + \frac{1}{\sqrt{(\sqrt{t}-2)(3-\sqrt{t})}}\sin(2\pi\phi_1), \qquad g_2(t) = \frac{2}{\sqrt{(\sqrt{t}-2)(3-\sqrt{t})}}.$$

Evidently, for $\phi_1, \phi_2 \ge 0$ and $t \in [\epsilon_1, \epsilon_2] \subset (4,9)$, we acquire

$$f_\varrho(t,\phi_1,\phi_2) + g_1(t) \ge \phi_\varrho^{\beta_\varrho} + \frac{1}{\sqrt{(\sqrt{t}-2)(3-\sqrt{t})}} > 0, \ \ \liminf_{\phi_\varrho\uparrow+\infty}\frac{f_\varrho(t,\phi_1,\phi_2)}{\phi_\varrho} = +\infty, \ \ \varrho = 1,2.$$

Thus hypotheses ($C_1^*$) and ($C_3$)–($C_4$) are satisfied.

Let $r = \frac{\ell_2}{\ell}\int_4^9 \frac{2}{\sqrt{(\sqrt{s}-2)(3-\sqrt{s})}}\frac{ds}{\sqrt{s}} = \frac{4\ell_2\pi}{\ell}$ and $R_1 = 1 + r$. Then we have

$$\begin{aligned} R =& \ell_2\Big(\int_4^9 \sigma_{\alpha_1}(s)\Big(\max_{0\le\phi_1,\phi_2\le R_1} f_1(s,\phi_1,\phi_2) + g_1(s)\Big)\frac{ds}{\sqrt{s}} + \int_4^9 \sigma_{\alpha_2}(s)\Big(\max_{0\le\phi_1,\phi_2\le R_1} f_2(s,\phi_1,\phi_2) + g_2(s)\Big)\frac{ds}{\sqrt{s}}\Big) \\ \le& \ell_2\Big(\int_4^9\Big(R_1^{\beta_1} + \frac{3}{\sqrt{(\sqrt{s}-2)(3-\sqrt{s})}}\Big)\frac{ds}{\sqrt{s}} + \int_4^9\Big(R_1^{\beta_2} + \frac{3}{\sqrt{(\sqrt{s}-2)(3-\sqrt{s})}}\Big)\frac{ds}{\sqrt{s}}\Big) = \ell_2(R_1^{\beta_1} + R_1^{\beta_2} + 12\pi). \end{aligned}$$

Let $\lambda^* = \min\{1, R_1/(2(R+1)), R_1/(2r)\}$. Then, for $0 < \lambda < \lambda^*$, Theorem 2 ensures that KFDE (4.1) admits a positive solution $(\phi_1, \phi_2)$ satisfying $\|\phi_1\| \geq 1$ and $\|\phi_2\| \geq 1$. □

Example 1 demonstrates that even when the nonlinearity $f_i$ $(i = 1, 2)$ is singular at the endpoints and changes sign, our theory ensures the existence of a strictly positive solution for a sufficiently small perturbation parameter $\lambda$.

**Example 2.** *Consider the following KFDE with coupled IBCs*

$$\begin{cases} \mathcal{D}_{4+}^{\alpha_1, \frac{1}{2}}\phi_1(t) + \lambda\Big(\dfrac{2}{\sqrt{t}-1}(\phi_1 - \theta_1)(\phi_1 - \theta_2) + \cos\Big(\dfrac{\pi}{2\theta_1}\phi_2\Big)\Big) = 0, \quad \lambda > 0, \\ \mathcal{D}_{4+}^{\alpha_2, \frac{1}{2}}\phi_2(t) + \lambda\Big(\dfrac{2}{\sqrt{t}-1}(\phi_2 - \theta_3)(\phi_2 - \theta_4) + \sin\Big(\dfrac{\pi}{2\theta_3}\phi_1\Big)\Big) = 0, \quad t \in (4, 9), \\ \gamma^i\phi_1(4) = \gamma^j\phi_2(4) = 0, \;\; \phi_1(9) = v_2\displaystyle\int_4^9 u_2(s)\phi_2(s)\frac{dU_2(s)}{\sqrt{s}}, \;\; \phi_2(9) = v_1\displaystyle\int_4^9 u_1(s)\phi_1(s)\frac{dU_1(s)}{\sqrt{s}}, \end{cases} \tag{4.2}$$

*where* $0 \leq i \leq m_1 - 2$, $0 \leq j \leq m_2 - 2$, *the constants satisfy* $\theta_2 > \theta_1 > 0$, $\theta_4 > \theta_3 > 0$; *the functions* $u_\varrho, U_\varrho$ *and the constant* $v_\varrho$ *are selected appropriately such that* $c_{\alpha_1}^{\alpha_2} > 0$ *is satisfied,* $\varrho = 1, 2$. *Then KFDE* (4.2) *admits two strictly positive solutions* $(\phi_{11}, \phi_{21})$ *and* $(\phi_{12}, \phi_{22})$ *satisfying* $\phi_{1\varrho} > 0$ *and* $\phi_{2\varrho} > 0$ $(\varrho = 1, 2)$ *for sufficiently small* $\lambda > 0$ *and* $t \in (4, 9)$.

*Proof.* It follows from (4.2) that we have

$$f_1(t, \phi_1, \phi_2) = \frac{2}{\sqrt{t}-1}(\phi_1 - \theta_1)(\phi_1 - \theta_2) + \cos\Big(\frac{\pi}{2\theta_1}\phi_2\Big),$$
$$f_2(t, \phi_1, \phi_2) = \frac{2}{\sqrt{t}-1}(\phi_2 - \theta_3)(\phi_2 - \theta_4) + \sin\Big(\frac{\pi}{2\theta_3}\phi_1\Big).$$

Obviously, we can select $g_1(t) = g_2(t) = m_0 > 0$ such that $f_1(t, \phi_1, \phi_2) + m_0 > 0$ and $f_2(t, \phi_1, \phi_2) + m_0 > 0$ for $\forall t \in (4, 9)$. Let $\delta = \min\{\theta_1\theta_2, \theta_3\theta_4\}/[16(\theta_1\theta_2 + \theta_3\theta_4 + 1)]$, $\varepsilon = \min\{1, \theta_1, \theta_3\}/4$, and $\vartheta = \max\{\ell_2\int_4^9 \sigma_{\alpha_1}(s)\frac{ds}{\sqrt{s}}, \ell_2\int_4^9 \sigma_{\alpha_2}(s)\frac{ds}{\sqrt{s}}\}$, we obtain $f_1(t, \phi_1, \phi_2) \geq \delta f_1(t, 0, 0) \geq \delta(\theta_1\theta_2 + 1)$ and $f_2(t, \phi_1, \phi_2) \geq \delta f_2(t, 0, 0) \geq \delta\theta_3\theta_4$ for $t \in (4, 9)$ and $0 \leq \phi_1, \phi_2 \leq \varepsilon$. Thus hypotheses ($C_1$)–($C_2$) hold. Since

$$\overline{f_\varrho}(\varepsilon) = \max_{4\leq t\leq 9, 0\leq\phi_1,\phi_2\leq\varepsilon}\{f_\varrho(t, \phi_1, \phi_2) + g_\varrho(t)\} \leq 2(\theta_1\theta_2 + \theta_3\theta_4) + m_0 + 1, \;\; \varrho = 1, 2,$$
$$\overline{f}(\varepsilon) = \max\{\overline{f_1}(\varepsilon), \overline{f_2}(\varepsilon)\} \leq 2(\theta_1\theta_2 + \theta_3\theta_4) + m_0 + 1.$$

We can choose $\lambda_* = \varepsilon/[8\vartheta(2(\theta_1\theta_2 + \theta_3\theta_4) + m_0 + 1)]$. Then, for $0 < \lambda < \lambda^*$, Theorem 1 guarantees that KFDE (4.2) has a positive solution $(\phi_{11}, \phi_{21})$ satisfying $\|\phi_{11}\| \leq 1/4$ and $\|\phi_{21}\| \leq 1/4$.

On the other hand, for $\phi_1, \phi_2 \geq 0$ and $t \in [\epsilon_1, \epsilon_2] \subset (4, 9)$, we have

$$\liminf_{\phi_2\uparrow+\infty}\frac{f_1(t, \phi_1, \phi_2)}{\phi_2} = +\infty \;\; \text{and} \;\; \liminf_{\phi_1\uparrow+\infty}\frac{f_2(t, \phi_1, \phi_2)}{\phi_1} = +\infty.$$

Thus hypotheses ($C_1^*$) and ($C_3$)–($C_4$) also hold. Let $r = 4m_0\ell_2/\ell$ and $R_1 = 1 + r$. We have

$$R = \ell_2\Big(\int_4^9 \sigma_{\alpha_1}(s)\Big(\max_{0\leq\phi_1,\phi_2\leq R_1} f_1(s, \phi_1, \phi_2) + m_0\Big)\frac{ds}{\sqrt{s}} + \int_4^9 \sigma_{\alpha_2}(s)\Big(\max_{0\leq\phi_1,\phi_2\leq R_1} f_2(s, \phi_1, \phi_2) + m_0\Big)\frac{ds}{\sqrt{s}}\Big).$$

Let $\lambda^* = \min\{1, R_1/(2(R+1)), R_1/(2r)\}$. Then, for $0 < \lambda < \lambda^*$, Theorem 2 ensures that KFDE (4.2) admits a positive solution $(\phi_{12}, \phi_{22})$ satisfying $\|\phi_{12}\| \geq 1$ and $\|\phi_{22}\| \geq 1$.

Combining the previous two results, all the hypotheses of Theorem 4 hold true. Then, if $0 < \lambda < \min\{\lambda_*, \lambda^*\}$, Theorem 4 guarantees that KFDE (4.2) has two positive solutions $(\phi_{11}, \phi_{21})$ and $(\phi_{12}, \phi_{22})$ satisfying $\phi_{1\varrho} > 0$ and $\phi_{2\varrho} > 0$ ($\varrho = 1, 2$) for $t \in (4, 9)$. □

**Example 3.** *Consider the following KFDE with coupled IBCs*

$$\begin{cases} \mathcal{D}_{8+}^{\alpha_1,\frac{2}{3}}\phi_1(t) + \lambda(\phi_2^{\beta_1} + \cos(2\pi\phi_1)) = 0, \ \ \mathcal{D}_{8+}^{\alpha_2,\frac{2}{3}}\phi_2(t) + \lambda(\phi_1^{\beta_2} + \cos(2\pi\phi_2)) = 0, \ \ t \in (8, 27), \ \ \lambda > 0, \\ \gamma^i\phi_1(8) = \gamma^j\phi_2(8) = 0, \ \ \phi_1(27) = \nu_2\int_8^{27} u_2(s)\phi_2(s)\dfrac{dU_2(s)}{\sqrt[3]{s}}, \ \ \phi_2(27) = \nu_1\int_8^{27} u_1(s)\phi_1(s)\dfrac{dU_1(s)}{\sqrt[3]{s}}, \end{cases} \tag{4.3}$$

*where $0 \leq i \leq m_1 - 2$, $0 \leq j \leq m_2 - 2$, the constant $\beta_\varrho$ satisfies $\beta_\varrho > 1$; the functions $u_\varrho, U_\varrho$ and the constant $\nu_\varrho$ are selected appropriately such that $c_{\alpha_1}^{\alpha_2} > 0$ is satisfied, $\varrho = 1, 2$. KFDE (4.3) admits two strictly positive solutions $(\phi_{11}, \phi_{21})$ and $(\phi_{12}, \phi_{22})$ satisfying $\phi_{1\varrho} > 0$ and $\phi_{2\varrho} > 0$ ($\varrho = 1, 2$) for sufficiently small $\lambda > 0$ and $t \in (8, 27)$.*

*Proof.* It follows from (4.3) that we obtain

$$f_1(t, \phi_1, \phi_2) = \phi_2^{\beta_1} + \cos(2\pi\phi_1), \quad f_2(t, \phi_1, \phi_2) = \phi_1^{\beta_2} + \cos(2\pi\phi_2), \quad g_1(t) = g_2(t) = 2.$$

Obviously, for $\phi_1, \phi_2 \geq 0$, we can acquire

$$f_1(t, \phi_1, \phi_2) + g_1(t) \geq \phi_2^{\beta_1} + 1 > 0, \qquad f_2(t, \phi_1, \phi_2) + g_2(t) \geq \phi_1^{\beta_2} + 1 > 0 \ \text{ for } \ t \in (8, 27),$$
$$\liminf_{\phi_2 \uparrow +\infty} \frac{f_1(t, \phi_1, \phi_2)}{\phi_2} = +\infty, \qquad \liminf_{\phi_1 \uparrow +\infty} \frac{f_2(t, \phi_1, \phi_2)}{\phi_1} = +\infty \ \text{ for } \ t \in [\epsilon_1, \epsilon_2] \subset (8, 27).$$

And $f_1(t, 0, 0) = f_2(t, 0, 0) = 1 > 0$, for $t \in [8, 27]$. Thus hypotheses ($C_1$)–($C_2$) are satisfied.

Let $\delta = 1/2$ and $\varepsilon = 1/8$, $\vartheta = \max\{\ell_2\int_8^{27}\sigma_{\alpha_1}(s)\frac{ds}{\sqrt[3]{s}}, \ell_2\int_8^{27}\sigma_{\alpha_2}(s)\frac{ds}{\sqrt[3]{s}}\}$, we obtain $\overline{f}(\varepsilon) = \max\{\overline{f_1}(\varepsilon), \overline{f_2}(\varepsilon)\}$, where $\overline{f_\varrho}(\varepsilon) = \max_{8 \leq t \leq 27, 0 \leq \phi_1, \phi_2 \leq \varepsilon}\{f_\varrho(t, u, v) + g_\varrho(t)\} \leq 8^{-\beta_\varrho} + 3$, $\varrho = 1, 2$. Then $\varepsilon/(8\vartheta\overline{h}(\varepsilon)) \geq 1/(64\vartheta(1+3)) = 1/(256\vartheta)$. Let $\lambda_* = 1/(256\vartheta)$. Now, if $0 < \lambda < \lambda_*$, Theorem 1 guarantees that KFDE (4.3) has a positive solution $(\phi_{11}, \phi_{21})$ with $\|\phi_{11}\| \leq 1/8$ and $\|\phi_{21}\| \leq 1/8$.

On the other hand, let $r = 30\ell_2/\ell$ and $R_1 = 1 + r$. Then we obtain

$$R = \ell_2\Big(\int_8^{27}\sigma_{\alpha_1}(s)\Big(\max_{0 \leq \phi_1, \phi_2 \leq R_1} f_1(s, \phi_1, \phi_2) + 2\Big)\frac{ds}{\sqrt[3]{s}} + \int_8^{27}\sigma_{\alpha_2}(s)\Big(\max_{0 \leq \phi_1, \phi_2 \leq R_1} f_2(s, \phi_1, \phi_2) + 2\Big)\frac{ds}{\sqrt[3]{s}}\Big).$$

Let $\lambda^* = \min\{1, R_1/(2(R+1)), R_1/(2r)\}$. Then, for $0 < \lambda < \lambda^*$, Theorem 3 ensures that KFDE (4.3) admits a positive solution $(\phi_{12}, \phi_{22})$ satisfying $\|\phi_{12}\| \geq 1$ and $\|\phi_{22}\| \geq 1$.

Combining the previous two results, all the hypotheses of Theorem 5 hold true. Then, if $0 < \lambda < \min\{\lambda_*, \lambda^*\}$, Theorem 5 guarantees that KFDE (4.3) has two positive solutions $(\phi_{11}, \phi_{21})$ and $(\phi_{12}, \phi_{22})$ satisfying $\phi_{1\varrho} > 0$ and $\phi_{2\varrho} > 0$ ($\varrho = 1, 2$) for $t \in (8, 27)$. □

**Example 4.** *Consider the following KFDE with coupled IBCs*

$$\begin{cases} \mathcal{D}_{8+}^{\alpha_1,\frac{2}{3}}\phi_1(t) + \lambda\left(e^{\phi_1} + \phi_2^2 + 7\cos(2\pi(t-1)\phi_1)\right) = 0, \quad t \in (8, 27), \ \ \lambda > 0, \\ \mathcal{D}_{8+}^{\alpha_2,\frac{2}{3}}\phi_2(t) + \lambda\left(e^{\phi_2} + \phi_1^2 + 7\cos(2\pi(t-1)\phi_2)\right) = 0, \quad t \in (8, 27), \ \ \lambda > 0, \\ \gamma^i\phi_1(8) = \gamma^j\phi_2(8) = 0, \ \ \phi_1(27) = \nu_2\int_8^{27} u_2(s)\phi_2(s)\dfrac{dU_2(s)}{\sqrt[3]{s}}, \ \ \phi_2(27) = \nu_1\int_8^{27} u_1(s)\phi_1(s)\dfrac{dU_1(s)}{\sqrt[3]{s}}, \end{cases} \tag{4.4}$$

*where* $0 \leq i \leq m_1-2$, $0 \leq j \leq m_2-2$, *the functions* $u_\varrho, U_\varrho$ *and the constant* $v_\varrho$ *are selected appropriately such that* $c_{\alpha_1}^{\alpha_2} > 0$ *is satisfied,* $\varrho = 1,2$. *KFDE* (4.4) *admits two strictly positive solutions* $(\phi_{11}, \phi_{21})$ *and* $(\phi_{12}, \phi_{22})$ *satisfying* $\phi_{1\varrho} > 0$ *and* $\phi_{2\varrho} > 0$ $(\varrho = 1,2)$ *for sufficiently small* $\lambda > 0$ *and* $t \in (8, 27)$.

*Proof.* It follows from (4.4) that we can see

$$f_1(t,\phi_1,\phi_2) = e^{\phi_1} + \phi_2^2 + 7\cos(2\pi(t-1)\phi_1), \quad f_2(t,\phi_1,\phi_2) = e^{\phi_2} + \phi_1^2 + 7\cos(2\pi(t-1)\phi_2).$$

Obviously, we can choose the functions $g_\varrho(t) = 8 > 0$ such that $f_\varrho(t,0,0) = 8$ and $f_\varrho(t,\phi_1,\phi_2) + 8 \geq 1 > 0$ for $\forall t \in (8,27)$, $\varrho = 1,2$. Let $\delta = 1/100$ and $\varepsilon = 1/8$, we acquire $f_\varrho(t,u,v) \geq \delta f_\varrho(t,0,0)$ for $t \in (8,27)$, $0 \leq \phi_\varrho \leq \varepsilon$, and $\varrho = 1,2$. Thus hypotheses ($C_1$) and ($C_2$) are satisfied. Additionally, we get

$$\overline{f}(\varepsilon) = \max\Big\{\max_{8\leq t\leq 27, 0\leq\phi_1,\phi_2\leq\varepsilon}\{f_1(t,\phi_1,\phi_2)+g_1(t)\}, \max_{8\leq t\leq 27, 0\leq\phi_1,\phi_2\leq\varepsilon}\{f_2(t,\phi_1,\phi_2)+g_2(t)\}\Big\} \leq e + 16.$$

Let $\vartheta = \max\{\ell_2\int_8^{27}\sigma_{\alpha_1}(s)\frac{ds}{\sqrt[3]{s}}, \ell_2\int_8^{27}\sigma_{\alpha_2}(s)\frac{ds}{\sqrt[3]{s}}\}$. Moreover, we set $\lambda_* = \varepsilon/(8\vartheta(e+16))$. Then, if $0 < \lambda < \lambda_*$, Theorem 1 ensures that KFDE (4.4) has a positive solution $(\phi_{11},\phi_{21})$ with $\|\phi_{11}\| \leq 1/8$ and $\|\phi_{21}\| \leq 1/8$.

On the other hand, for $\phi_1, \phi_2 \geq 0$ and $t \in [\epsilon_1, \epsilon_2] \subset (8,27)$, we can observe

$$\liminf_{\phi_1\uparrow+\infty}\frac{f_1(t,\phi_1,\phi_2)}{\phi_1} = +\infty \text{ and } \liminf_{\phi_2\uparrow+\infty}\frac{f_2(t,\phi_1,\phi_2)}{\phi_2} = +\infty.$$

Thus hypotheses ($C_1^*$) and ($C_3$)–($C_4$) also hold. Let $r = 60\ell_2/\ell$ and $R_1 = 1 + r$. We have

$$\begin{aligned} R =&\ell_2\Big(\int_8^{27}\sigma_{\alpha_1}(s)\Big(\max_{0\leq\phi_1,\phi_2\leq R_1}f_1(s,\phi_1,\phi_2)+8\Big)\frac{ds}{\sqrt[3]{s}} + \int_8^{27}\sigma_{\alpha_2}(s)\Big(\max_{0\leq\phi_1,\phi_2\leq R_1}f_2(s,\phi_1,\phi_2)+8\Big)\frac{ds}{\sqrt[3]{s}}\Big) \\ \leq&\ell_2\Big(\int_8^{27}(e^{R_1}+R_1^2+7+8)\frac{ds}{\sqrt[3]{s}} + \int_8^{27}(e^{R_1}+R_1^2+7+8)\frac{ds}{\sqrt[3]{s}}\Big) = 15\ell_2(e^{R_1}+R_1^2+15). \end{aligned}$$

Let $\lambda^* = \min\{1, R_1/(2(R+1)), R_1/(2r)\}$. Then, for $0 < \lambda < \lambda^*$, Theorem 2 ensures that KFDE (4.4) admits a positive solution $(\phi_{12},\phi_{22})$ satisfying $\|\phi_{12}\| \geq 1$ and $\|\phi_{22}\| \geq 1$.

Combining the previous two results, all the hypotheses of Theorem 4 hold true. Then, if $0 < \lambda < \min\{\lambda_*, \lambda^*\}$, Theorem 4 guarantees that KFDE (4.4) has two positive solutions $(\phi_{11},\phi_{21})$ and $(\phi_{12},\phi_{22})$ satisfying $\phi_{1\varrho} > 0$ and $\phi_{2\varrho} > 0$ $(\varrho = 1,2)$ for $t \in (8,27)$. □

It follows from that the preceding examples demonstrate the validity of the main results. Under different conditions, the existence of at least one or two positive solutions to some KFDE with coupled IBCs are shown for a sufficiently small parameter $\lambda$.

## 5. Conclusions

This study has investigated a system of singular nonlinear higher-order KFDEs governed by nonlocal coupled Riemann-Stieltjes IBCs. The following key contributions have been made: First, the targeted KFDEs have been successfully transformed into equivalent integral equations through systematic construction of specifically designed Green's functions. Second, by synergistically employing the Schauder's and Guo-Krasnoselskii's FPTs, comprehensive existence criteria dependent on explicit parameter intervals have been established. These criteria have been proven to guarantee at least one or two positive solutions, with their validity rigorously demonstrated through detailed analysis of Green's function properties. Finally, concrete numerical examples have been provided to substantiate the practical applicability and theoretical consistency of the developed criteria. Following the research presented in this paper, we will continue to investigate the existence of positive solutions for related KFDEs in future studies.

## Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The author declares there are no conflicts of interest.

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