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Research article

Estimating the dimension of the attractor for the complex Ginzburg-Landau system

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Abstract: In this paper, the upper bound of the dimension of the global attractor associated with the four-dimensional cubic complex Ginzburg-Landau system is derived using the Lyapunov exponent method. First, by employing the Faedo-Galerkin method and energy estimation, the existence and uniqueness of solutions under the variational form of the system are established. Subsequently, a specific quantity is constructed and transformed into an inequality, from which the upper bound of the fractal dimension corresponding to the global attractor of the dynamical system is obtained.

Keywords: complex Ginzburg-Landau system; attractor; dimension estimation; Lyapunov exponent

1. Introduction

The complex Ginzburg-Landau (CGL) system plays a fundamental role in the study of infinite-dimensional dynamical systems and serves as a canonical model for describing a wide variety of nonlinear phenomena. Originally introduced by Ginzburg and Landau [1] in 1950 to model superconductivity in type-I materials from a macroscopic perspective, the equation was later complemented by the microscopic Bardeen-Cooper-Schrieffer (BCS) theory, which significantly advanced the field [2].

With the advancement of research, the CGL system has demonstrated remarkable universality and has been applied in diverse areas such as fluid mechanics, optics, and materials science. In fluid mechanics, it provides an effective description of convective instabilities and turbulence [3]. In optics, it captures key mechanisms such as laser generation, mode selection, and output characteristics [4]. In materials science, the CGL framework models microstructural evolution during phase transitions, thereby offering theoretical guidance for the design and optimization of new materials [5].

From the perspective of dynamical systems, the long-time behavior of solutions to the CGL system is naturally linked to the notion of attractors, which characterize the system's asymptotic dynamics. The dimension of an attractor is a crucial indicator of its geometric complexity.Among various approaches to dimension estimation, the Lyapunov exponent method is particularly significant. The Lyapunov exponents, introduced by Lyapunov [6], quantify the average exponential rate of divergence or convergence of nearby trajectories in phase space [7, 8]. Their relation to fractal dimension was first proposed in the Kaplan-Yorke conjecture. Although not rigorously established in full generality, this relationship has been validated in numerous physical systems [9].

Considerable progress has been made in the analysis of attractors for the CGL system and its variants. For example, Guo and Wang [10, 11] established the existence of global attractors and initiated the study of their geometric properties for generalized two-dimensional CGL equations, including Hausdorff and fractal dimensions. More recent works have extended this line of research to fractional spatial and temporal operators, as well as discrete lattice settings. Lu et al. [12] introduced spatial fractional order into the complex Ginzburg-Landau equation, formulating it as

$$u_t = \rho u - (1 + i\nu)(-\Delta)^{\alpha} u - (1 + i\mu)|u|^{2\sigma} u$$

where $x \in \mathbb{R}^2$, $\rho > 0$, $\sigma > 0$ and $\frac{1}{2} < \alpha < 1$.
The fractional Laplacian $(-\Delta)^{\alpha}$ introduces non-local spatial interactions, altering diffusion and wave-propagation, and thereby reshaping attractor dynamics. In parallel, Zhang et al. [13] explored time-fractional variants such as

$$\begin{cases} e^{-i\omega} {}_0^C D_t^{\alpha} v - \Delta v = e^{i\gamma} |v|^{p-1} v, & x \in \mathbb{R}^N,\ t > 0, \\ v(0, x) = v_0(x), & x \in \mathbb{R}^N. \end{cases}$$

By defining two operators and establishing suitable estimates, they obtained the local and global existence of the Cauchy problem. In addition to the fractionalization of space and time, discrete equations have also been investigated. Zhao and Zhou [14] studied the global attractor of the complex Ginzburg-Landau equation on a discrete infinite lattice:

$$\begin{cases} i\dot{u}_m - (\alpha - i\varepsilon)(2u_m - u_{m+1} - u_{m-1}) + iku_m + \beta |u_m|^{2\sigma} u_m = g_m, & m \in \mathbb{Z}, \\ u_m(0) = u_{0,m}, & m \in \mathbb{Z}. \end{cases}$$

and established the upper semicontinuity of the associated semigroup of the discrete global attractor in 2008. Seven years later, Du et al. [15] further investigated these discrete equations and proved the existence of exponential attractors for the discrete complex Ginzburg-Landau equations with given parameters. In discrete cases, global and exponential attractors have been obtained, though dimension estimates remain largely unresolved. In contrast, continuous high-dimensional systems pose additional challenges, stemming from the complexity of high-dimensional spatial regions and the spatio-temporal chaos caused by coupling with the time dimension [16]. Specifically:

- *Enhanced Coupling of Nonlocal Terms:* The phase field interaction term $|u|^2 u$ leads to higher-order nonlinear coupling.
- *Criticality of Sobolev Embedding:* The embedding $\mathbb{H}^1(\Omega) \hookrightarrow L^4(\Omega)$ loses compactness in $\mathbb{R}^4$.
- *Estimation of Trace Operators:* The estimation of the volume deformation rate $\mathrm{Tr}(F' \circ Q_m)$ requires anisotropic embedding techniques.

Despite these difficulties, the study of high-dimensional CGL systems remains of substantial interest, both for its theoretical implications and for its relevance to spatio temporal chaotic behavior. For example, Cheng et al. [17] studied the trichotomy solution of the energy-critical complex Ginzburg Landau equation in three and four dimensions. Cheng and Zheng [18] proved the global well-posedness of the energy-critical complex Ginzburg-Landau equations in the exterior domain in three and four dimensions. Yang et al. [19] established the existence of the global attractor for the complex Ginzburg-Landau system by using Galerkin's approximation method and the theory of compact embedding in Sobolev spaces, but did not investigate the geometrical properties of the attractor, including its dimension.

The present paper focuses on the four-dimensional cubic complex Ginzburg-Landau system. By employing the Lyapunov exponent method, we derive an explicit upper bound for the fractal dimension of its global attractor under standard boundary conditions. This work extends existing results from lower-dimensional or fractional-order settings to the critical four-dimensional case. Related studies can be found in [20–23].

In this paper, we consider the following system of equations.

$$\begin{cases} \partial_t \boldsymbol{v} - (\lambda + i\alpha)\Delta \boldsymbol{v} + (\kappa + i\beta)|\boldsymbol{v}|^2 \boldsymbol{v} - \gamma \boldsymbol{v} = \boldsymbol{0}, \\ \boldsymbol{v}(0) = \boldsymbol{v_0} \end{cases} \tag{1.1}$$

in which $\boldsymbol{v} = (v_1, v_2)^T$, $v_1 := v_1(x,t)$, $v_2 := v_2(x,t)$, $(x,t) \in \Omega \times \mathbb{R}_+$, Ω is a bounded opening set of $\mathbb{R}^4$; $\lambda > 0$, $\kappa > 0$, $\gamma < 0$, α, β can be any real number.

The m-dimensional volume element is constructed using the first variational equation of (1.1), and the main theorem of this paper is established by combining the semi-group technique with inequality estimation.

Theorem 1.1. *Consider the dynamical system generated by (1.1) under one of the following boundary conditions: Dirichlet, Neumann, or space-periodicity. Let O denote the corresponding global attractor and let m be a constant. Then the following theorem holds:*

(i) Within the phase space $H \times H$, the m-dimensional volume element exhibits exponential decay as $t \to \infty$;

(ii) The Hausdorff dimension of O is bounded above by m, and its fractal dimension is less than or equal to $2m$.

The proof of this theorem primarily employs the Lyapunov exponent method, which is central to analyzing the system's dynamics. The core idea is to examine the behavior of the volume element in phase space under semigroup dynamics. We proceed by establishing estimates for the deformation rate of the volume element, leveraging the semi-group properties together with inequality techniques.

Step 1: Lyapunov exponent estimate

Consider the time evolution of the volume element corresponding to the phase space $H \times H$. For each initial condition $\boldsymbol{v_0} \in H \times H$, the semigroup $S(t)$ evolves the system according to the equation:

$$\frac{d}{dt} S(t)\boldsymbol{v_0} = F(S(t)\boldsymbol{v_0}),$$

here, F denotes the non-linear operator defined by the system. By analyzing the eigenvalues of the linearized operator at each point in phase space, we obtain the Lyapunov exponents, which describe the average exponential rate of divergence or convergence of neighboring trajectories.

Step 2: Volume deformation

Using the Lyapunov exponents, the deformation of the m-dimensional volume element can be estimated. Specifically, we consider the quantity $\omega_m(t)$, which represents the volume expansion rate in the phase space:

$$\omega_m(t) = \sup_{\|\boldsymbol{v_0}\|\leq 1} \left| \frac{d}{dt} \log |\boldsymbol{V}_1(t) \wedge \boldsymbol{V}_2(t) \wedge \cdots \wedge \boldsymbol{V}_m(t)| \right|.$$

We use this quantity to establish exponential decay, which ultimately shows that the volume element decays exponentially as $t \to \infty$. This exponential decay implies that the global attractor $\mathcal{O}$ is finite-dimensional, with its dimension bounded above by m.

Step 3: Dimension bounds

The final step involves estimating the Hausdorff dimension of the global attractor. Using the properties of Lyapunov exponents and volume distortion, we derive an upper bound for the Hausdorff dimension of the attractor $\mathcal{O}$ as m, and an upper bound for its fractal dimension as $2m$.

Hence, we conclude that the Hausdorff dimension of $\mathcal{O}$ is at most m, and its fractal dimension is at most $2m$, as stated in the theorem.

2. Background knowledge

In this paper, the function space represented by $\mathbb{X}$ is the complex space of the real function space X. In order to obtain the main results of this paper, the following assumptions and related theorems and lemmas are given, and the proof of them can be found [24].

Firstly, we introduce the three basic types of boundary conditions that are used in this paper:

- Dirichlet boundary condition:
$$v = 0 \quad \text{on } \Gamma \times \mathbb{R}_+; \tag{2.1}$$
- Neumann boundary condition:
$$\frac{\partial v}{\partial \boldsymbol{\nu}} = 0 \quad \text{on } \Gamma \times \mathbb{R}_+, \tag{2.2}$$
Where $\boldsymbol{\nu}$ is the unit outward normal on Γ;
- Space - periodicity, in which case,
$$v \text{ is } \Omega - \text{periodic}, \Omega = [0, L_1] \times [0, L_2] \times [0, L_3] \times [0, L_4]. \tag{2.3}$$

Under the above three boundary conditions, let $H = \mathbb{L}^2(\Omega)$, and

$$\begin{aligned}
&K = \mathbb{H}^1(\Omega),\ D(A) = \mathbb{H}_0^1(\Omega) \cap \mathbb{H}^2(\Omega) \times \mathbb{H}_0^1(\Omega) \cap \mathbb{H}^2(\Omega) \quad \text{in case (2.1)},\\
&K = \mathbb{H}^1(\Omega),\ D(A) = \left\{ v \in \mathbb{H}^2(\Omega) \times \mathbb{H}^2(\Omega), \frac{\partial v}{\partial \boldsymbol{\nu}} = 0 \right\}\\
&\qquad \text{when } \boldsymbol{\nu} \text{ is on the boundary in case (2.2)},\\
&K = \mathbb{H}_{\text{per}}^1(\Omega),\ D(A) = \mathbb{H}_{\text{per}}^2(\Omega) \times \mathbb{H}_{\text{per}}^2(\Omega) \quad \text{in case (2.3)}.
\end{aligned}$$

To construct a theoretical basis for the subsequent proofs, we shall consider an abstract framework. Let $\boldsymbol{f} \in L^2(0, T; H) \times L^2(0, T; H)$ and $\boldsymbol{v_0} \in H \times H$. Consider a Gelfand triple of Hilbert spaces

$$K \times K \subset H \times H \subset K' \times K',$$

where each embedding is dense and continuous. We seek a function $\boldsymbol{v} : [0,T] \to K \times K$ satisfying the abstract evolution equation

$$\frac{d\boldsymbol{v}(t)}{dt} + A(t)\boldsymbol{v}(t) = \boldsymbol{f}(t), \quad t \in (0,T). \tag{2.4}$$

with initial condition

$$\boldsymbol{v}(0) = \boldsymbol{v_0}. \tag{2.5}$$

For the above abstract initial value problem, we give the first theorem needed for the subsequent proof.

Theorem 2.1. *Assume that $\boldsymbol{v_0} \in H \times H$, $\boldsymbol{f} \in L^2(0,T;K') \times L^2(0,T;K')$ (or $L^2(0,T;H) \times L^2(0,T;H)$), and that the bilinear form $b(t,*,*)$satisfies the following conditions:*

(i) for every $\boldsymbol{u}, \boldsymbol{v} \in K \times K$, the mapping $t \mapsto b(t,\boldsymbol{u},\boldsymbol{v})$ is measurable;
(ii) there exists $M_T < \infty$, such that

$$|b(t,\boldsymbol{u},\boldsymbol{v})|_{H\times H} \le M_T \|\boldsymbol{u}\|_{K\times K}\|\boldsymbol{v}\|_{K\times K}, \quad \forall \boldsymbol{u},\boldsymbol{v} \in K \times K, a.e.\ t \in [0,T];$$

(iii) there exists $\theta > 0$ such that

$$b(t,\boldsymbol{v},\boldsymbol{v}) \ge \theta \|\boldsymbol{v}\|^2_{K\times K}, \quad \forall \boldsymbol{v} \in K \times K, a.e.\ t \in [0,T].$$

Then there exists a unique solution $\boldsymbol{v}$ of (2.4),(2.5) such that

$$\boldsymbol{v} \in C([0,T];H) \times C([0,T];H) \cap L^2(0,T;K) \times L^2(0,T;K),$$

$$\boldsymbol{v}' \in L^2(0,T;K') \times L^2(0,T;K').$$

Now, we shift our focus to the variational problem below. Consider a Banach subspace $W \times W$ of $H \times H$, where the embedding $W \times W$ into $H \times H$ is continuous. Let $F : W \times W \to H \times H$ be a given function. We assume that the initial-value problem

$$\begin{cases} \frac{d\boldsymbol{v}}{dt}(t) = F(\boldsymbol{v}(t)), \quad t > 0, \\ \boldsymbol{v}(0) = \boldsymbol{v_0}, \end{cases}$$

is well posed for every $\boldsymbol{v_0} \in H \times H$, with $\boldsymbol{v}(t) \in W \times W$, $\forall t \ge 0$, and the mapping $S(t)$: $\boldsymbol{v_0} \in H \times H \to \boldsymbol{v}(t) \in H \times H$ enjoys the properties (2.6), (2.7):

$$\begin{cases} S(t+s) = S(t) \cdot S(s), \ \forall s,t \ge 0, \\ S(0) = I \text{ (unit mapping)} \end{cases} \tag{2.6}$$

$$S(t) \text{ is a nonlinear and continuous operator from } H \times H \text{ into itself.} \tag{2.7}$$

We also assume that F is Fréchet differential from $W \times W$ into $H \times H$, with differential F', and that the associated linear initial-value problem

$$\begin{cases} \frac{d\boldsymbol{V}}{dt}(t) = F'(S(t)\boldsymbol{v_0}) \cdot \boldsymbol{V}(t), \\ \boldsymbol{V}(0) = \boldsymbol{\xi}, \end{cases} \tag{2.8}$$

is well - posed for every $\boldsymbol{v_0}, \boldsymbol{\xi} \in H \times H$. Finally, we assume that $S(t)$ is uniformly differentiable in $H \times H$, i.e., for every $\boldsymbol{v} \in H \times H$, there exists a linear operator $L(\boldsymbol{v}) \in \mathcal{L}(H \times H)$ and

$$\sup_{\substack{\boldsymbol{v},\boldsymbol{u} \in H\times H \\ 0<\|\boldsymbol{v}-\boldsymbol{u}\|\leq\varepsilon}} \frac{\|S\boldsymbol{v} - S\boldsymbol{u} - L(\boldsymbol{v})(\boldsymbol{v}-\boldsymbol{u})\|}{\|\boldsymbol{v}-\boldsymbol{u}\|} \to 0 \quad \text{as } \varepsilon \to 0.$$

Additionally, it is necessary to acquire the following conditions for the operator $L(\boldsymbol{v})$:

$$\sup_{\boldsymbol{v}\in H\times H} \|L(\boldsymbol{v})\|_{\mathcal{L}(H\times H)} < +\infty,$$

and

$$\sup_{\boldsymbol{v}\in H\times H} \omega_d(L(\boldsymbol{v})) < 1 \text{ for some } d > 0,$$

where $\omega_d(L)$ represents the d-dimensional volume expansion rate:

- $\omega_d(L) < 1$: d-dimensional volume is compressed.
- $\omega_d(L) > 1$: d-dimensional volume expands.
- $\omega_d(L) = 1$: volume is preserved.

Now, for $\boldsymbol{v_0}$ fixed in $H \times H$, let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_m$ be m unit elements of $H \times H$ and let $\boldsymbol{V}_1, \boldsymbol{V}_2, \ldots, \boldsymbol{V}_m$ be the corresponding solutions of (2.8). Then, in view of studying the ratio

$$\begin{aligned} &\frac{|L(t,\boldsymbol{v_0})\boldsymbol{\xi}_1 \wedge L(t,\boldsymbol{v_0})\boldsymbol{\xi}_2 \wedge \cdots \wedge L(t,\boldsymbol{v_0})\boldsymbol{\xi}_m|_{\wedge^m H\times H}}{|\boldsymbol{\xi}_1 \wedge \boldsymbol{\xi}_2 \wedge \cdots \wedge \boldsymbol{\xi}_m|_{\wedge^m H\times H}} \\ =&|L(t,\boldsymbol{v_0})\boldsymbol{\xi}_1 \wedge L(t,\boldsymbol{v_0})\boldsymbol{\xi}_2 \wedge \cdots \wedge L(t,\boldsymbol{v_0})\boldsymbol{\xi}_m|_{\wedge^m H\times H}, \end{aligned}$$

where "$\wedge$" represents the exterior product, and we want to seek an evolution equation satisfied by

$$\begin{aligned} &|L(t,\boldsymbol{v_0})\boldsymbol{\xi}_1 \wedge L(t,\boldsymbol{v_0})\boldsymbol{\xi}_2 \wedge \cdots \wedge L(t,\boldsymbol{v_0})\boldsymbol{\xi}_m|_{\wedge^m H\times H} \\ =&|\boldsymbol{V}_1(t) \wedge \boldsymbol{V}_2(t) \cdots \wedge \boldsymbol{V}_m(t)|_{\wedge^m H\times H}. \end{aligned}$$

By time differentiation:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|\boldsymbol{V}_1(t) \wedge \boldsymbol{V}_2(t) \wedge \cdots \wedge \boldsymbol{V}_m(t)\right\|^2_{\wedge^m H\times H} \\ &= \Big(\frac{d}{dt}(\boldsymbol{V}_1(t) \wedge \cdots \wedge \boldsymbol{V}_m(t)), \boldsymbol{V}_1(t) \wedge \cdots \wedge \boldsymbol{V}_m(t)\Big)_{\wedge^m H\times H} \\ &= (\boldsymbol{V}_1'(t) \wedge \cdots \wedge \boldsymbol{V}_m(t), \boldsymbol{V}_1(t) \wedge \cdots \wedge \boldsymbol{V}_m(t))_{\wedge^m H\times H} \\ &\quad + \cdots + (\boldsymbol{V}_1(t) \wedge \cdots \wedge \boldsymbol{V}_m'(t), \boldsymbol{V}_1(t) \wedge \cdots \wedge \boldsymbol{V}_m(t))_{\wedge^m H\times H} \\ &= (F'(\boldsymbol{v}(t))\boldsymbol{V}_1(t) \wedge \cdots \wedge \boldsymbol{V}_m(t), \boldsymbol{V}_1(t) \wedge \cdots \wedge \boldsymbol{V}_m(t))_{\wedge^m H\times H} \\ &\quad + \cdots + (\boldsymbol{V}_1(t) \wedge \cdots \wedge F'(\boldsymbol{v}(t))\boldsymbol{V}_m(t), \boldsymbol{V}_1(t) \wedge \cdots \wedge \boldsymbol{V}_m(t))_{\wedge^m H\times H} \\ &= (F'(\boldsymbol{v}(t))_m(\boldsymbol{V}_1(t), \ldots, \boldsymbol{V}_m(t)), \boldsymbol{V}_1(t) \wedge \cdots \wedge \boldsymbol{V}_m(t))_{\wedge^m H\times H} \\ &= \left\|\boldsymbol{V}_1(t) \wedge \cdots \wedge \boldsymbol{V}_m(t)\right\|^2_{\wedge^m H\times H} \operatorname{Tr}(F'(\boldsymbol{v}(t)) \circ Q_m), \end{aligned}$$

where $Q_m = Q_m(t, \boldsymbol{v_0}; \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_m)$ is the orthogonal projector in $H \times H$ onto the space spanned by $\boldsymbol{V}_1(t), \ldots, \boldsymbol{V}_m(t)$. Therefore,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{V}_1(t) \wedge \boldsymbol{V}_2(t) \wedge \cdots \wedge \boldsymbol{V}_m(t)\|^2_{\wedge^m H\times H} \\
&= \|\boldsymbol{V}_1(t) \wedge \boldsymbol{V}_2(t) \wedge \cdots \wedge \boldsymbol{V}_m(t)\|^2_{\wedge^m H\times H} \operatorname{Tr}\left(F'(\boldsymbol{v}) \circ Q_m\right), \\
&\frac{d}{dt}\|\boldsymbol{V}_1(t) \wedge \boldsymbol{V}_2(t) \wedge \cdots \wedge \boldsymbol{V}_m(t)\|_{\wedge^m H\times H} \\
&= \|\boldsymbol{V}_1(t) \wedge \boldsymbol{V}_2(t) \wedge \cdots \wedge \boldsymbol{V}_m(t)\|_{\wedge^m H\times H} \operatorname{Tr}\left(F'(\boldsymbol{v}) \circ Q_m\right).
\end{aligned}$$

Integral over t, we derive:

$$\begin{aligned}
&|\boldsymbol{V}_1(t) \wedge \boldsymbol{V}_2(t) \wedge \cdots \wedge \boldsymbol{V}_m(t)|_{\wedge^m H\times H} \\
=&|\boldsymbol{\xi}_1 \wedge \cdots \wedge \boldsymbol{\xi}_m|_{\wedge^m H\times H} \exp\left(\int_0^t \operatorname{Tr}(F'(S(\tau)\boldsymbol{v_0}) \circ Q_m(\tau))d\tau\right).
\end{aligned}$$

Since

$$\omega_m(L(t, \boldsymbol{v_0})) = \sup_{\substack{\boldsymbol{\xi}_i \in H\times H \\ \|\boldsymbol{\xi}_i\| \le 1 \\ i=1,\ldots,m}} \left|\boldsymbol{V}_1(t) \wedge \boldsymbol{V}_2(t) \wedge \cdots \wedge \boldsymbol{V}_m(t)\right|_{\wedge^m H\times H}$$

and $\left|\boldsymbol{\xi}_1 \wedge \cdots \wedge \boldsymbol{\xi}_m\right|_{\wedge^m H\times H} \le 1$, we derive

$$\omega_m(L(t, \boldsymbol{v_0})) \le \sup_{\substack{\boldsymbol{\xi}_i \in H\times H \\ \|\boldsymbol{\xi}_i\| \le 1 \\ i=1,\ldots,m}} \exp\left(\int_0^t \operatorname{Tr}(F'(S(\tau)\boldsymbol{v_0}) \circ Q_m(\tau))d\tau\right). \tag{2.9}$$

Therefore, it is useful to give the following quantities:

$$q_m(t) = \sup_{\boldsymbol{v_0} \in X} \sup_{\substack{\boldsymbol{\xi}_i \in H\times H \\ \|\boldsymbol{\xi}_i\| \le 1 \\ i=1,\ldots,m}} \frac{1}{t}\int_0^t \operatorname{Tr}(F'(S(\tau)\boldsymbol{v_0}) \circ Q_m(\tau))d\tau, \tag{2.10}$$

$$q_m = \limsup_{t\to\infty} q_m(t),$$

where, as above, $Q_m(\tau) = Q_m(\tau, \boldsymbol{v_0}; \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_m)$, and $\boldsymbol{v_0} \in \mathbb{X}$ is a functional-invariant set.

Infer from (2.9), (2.10):

$$\overline{\omega}_m(t) = \sup_{\boldsymbol{v_0} \in X} \omega_m(L(t, \boldsymbol{v_0})) \le \exp(tq_m(t)), \tag{2.11}$$

or alternatively,

$$\overline{\omega}_m(t)^{1/t} \le \exp(q_m(t)), \quad \frac{1}{t}\log\overline{\omega}_m(t) \le q_m(t). \tag{2.12}$$

Then, when $m \to \infty$, we can get the upper bound of the uniform Lyapunov numbers and Lyapunov exponents for $\mathbb{X}$.

$$\Lambda_1\Lambda_2\cdots\Lambda_m \le \exp(q_m). \tag{2.13}$$

$$\mu_1 + \cdots + \mu_m \le q_m. \tag{2.14}$$

As a result, we can conclude with

Proposition 2.1. *If for some m and some $t_0 > 0$,*

$$q_m(t) \le -\delta < 0, \quad \forall t \ge t_0,$$

then the volume element $\left|\boldsymbol{V}_1(t) \wedge \boldsymbol{V}_2(t) \wedge \cdots \wedge \boldsymbol{V}_m(t)\right|_{\wedge^m H\times H}$ *exhibits exponential decay as* $t \to \infty$, *uniformly for* $\boldsymbol{v_0} \in \mathbb{X}$, $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_m \in H \times H$,

$$\left|\boldsymbol{V}_1(t) \wedge \boldsymbol{V}_2(t) \wedge \cdots \wedge \boldsymbol{V}_m(t)\right|_{\wedge^m H\times H} \le C \exp(-\delta t).$$

If $\mathbb{X}$ is a functional-invariant set for the semigroup $S(t)$ and

$$q_m < 0$$

for some m, then $\Pi_m = \Lambda_1 \cdots \Lambda_m < 1$, $\mu_1 + \cdots + \mu_m < 0$.

Finally, we give the two lemmas needed for the following proof of the main theorem.

Lemma 2.1. *Under assumptions (2.11)-(2.14) and, if for some $n \ge 1$,*

$$\mu_1 + \cdots + \mu_{n+1} < 0,$$

then

$$\mu_{n+1} < 0, \quad \frac{\mu_1 + \cdots + \mu_n}{|\mu_{n+1}|} < 1,$$

and

1) the Hausdorff dimension of $\mathbb{X}$ is less than or equal to

$$n + \frac{(\mu_1 + \cdots + \mu_n)}{|\mu_{n+1}|};$$

2) the fractal dimension of $\mathbb{X}$ is less than or equal to

$$(n+1)\left[\max_{1\le j\le n} \frac{(\mu_1 + \cdots + \mu_j)}{|\mu_1 + \cdots + \mu_{n+1}|}\right].$$

Lemma 2.2. *Assume that the sequence of numbers $\mu_j (j \ge 1)$ satisfies the following inequalities:*

$$\mu_1 + \cdots + \mu_j \le -\alpha j^\theta + \beta, \quad \forall j \ge 1,$$

where $\alpha, \beta, \theta > 0$. Let $m \in \mathbb{N}$ be defined by

$$m - 1 < \left(\frac{2\beta}{\alpha}\right)^{\frac{1}{\theta}} \le m.$$

Then $\mu_1 + \cdots + \mu_m < 0$ and

$$\frac{(\mu_1 + \cdots + \mu_j)_+}{|\mu_1 + \cdots + \mu_m|} \le 1, \quad j = 1, \ldots, m,$$

where $(\cdot)_+$ denotes the positive part, i.e., $x_+ = \max\{x, 0\}$ for any real number x.

3. Proof of the main theorem

For simplicity, we denote $|\cdot|$ as the norm in the space $H \times H$, and $\|\cdot\|$ as the norm in space $K \times K$ throughout this section. The first variation equation corresponding to (1.1) is given by

$$\begin{cases} \dfrac{\partial \boldsymbol{V}}{\partial t} - (\lambda + i\alpha)\Delta \boldsymbol{V} + (\kappa + i\beta)\left\{|\boldsymbol{v}|^2 \boldsymbol{V} + 2\mathrm{Re}(\bar{\boldsymbol{v}}\boldsymbol{V})\right\} - \gamma \boldsymbol{V} = \boldsymbol{0} \\ \boldsymbol{V}(0) = \boldsymbol{\xi} \end{cases}, \tag{3.1}$$

and added by one of the boundary conditions (2.1)–(2.3).

If $\boldsymbol{v}$ is a solution of (1.1), then by application of Theorem 2.1, for every $\boldsymbol{\xi}$ given in $H \times H$, (3.1) possesses a unique solution $\boldsymbol{V}$ satisfying

$$\boldsymbol{V} \in L^2(0, T; K) \times L^2(0, T; K) \cap L^\infty(0, T; H) \times L^\infty(0, T; H), \ \forall T > 0.$$

Here, the semigroup $S(t)$ in $H \times H$ is Fréchet differentiable (The Fréchet differentiability of $S(t)$ can refer to [24]).

Then, the dimension of the global attractor $\mathcal{O}$ provided by (1.1) is estimated.

For $m \in \mathbb{N}$, consider $\boldsymbol{\xi} = \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_m$ m elements of $H \times H$, and the corresponding solutions $\boldsymbol{V}_1, \boldsymbol{V}_2, \ldots, \boldsymbol{V}_m$ of (3.1), where $\boldsymbol{v} = \boldsymbol{v}(\tau) = S(\tau)\boldsymbol{v_0}$ is a fixed orbit. Since

$$\begin{aligned} &|\boldsymbol{V}_1(t) \wedge \boldsymbol{V}_2(t) \wedge \cdots \wedge \boldsymbol{V}_m(t)|_{\wedge^m H \times H} \\ &\quad = \left|\boldsymbol{\xi}_1 \wedge \cdots \wedge \boldsymbol{\xi}_m\right|_{\wedge^m H \times H} \exp\left(\int_0^t \mathrm{ReTr}(F'(S(\tau)\boldsymbol{v_0}) \circ Q_m(\tau))d\tau\right), \end{aligned}$$

where $Q_m(\tau) = Q_m(\tau, \boldsymbol{v_0}; \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_m)$ is the orthonormal basis of $H \times H$ such that

$$Q_m(\tau)H \times H = \mathrm{Span}\{\boldsymbol{V}_1(\tau), \ldots, \boldsymbol{V}_m(\tau)\},$$

and since $\boldsymbol{V}_j(\tau) \in K \times K$ for almost every τ, it follows that $\boldsymbol{\varphi}_1(\tau), \ldots \boldsymbol{\varphi}_m(\tau)$ also belong to $K \times K$ for almost every τ.

Furthermore, since $H = \mathbb{L}^2(\Omega) = L^2(\Omega)^2$, the $\boldsymbol{\varphi}_j(\tau)$ considered as vectors of $H \times H$ are orthogonal too. As a result, the following holds:

$$\begin{aligned} \mathrm{Re}\,\mathrm{Tr}(F'(\boldsymbol{v}(\tau)) \circ Q_m(\tau)) &= \sum_{j=1}^{\infty} \mathrm{Re}\left\langle F'(\boldsymbol{v}(\tau)) \circ Q_m(\tau)\boldsymbol{\varphi}_j(\tau), \boldsymbol{\varphi}_j(\tau)\right\rangle \\ &= \sum_{j=1}^{m} \mathrm{Re}\langle F'(\boldsymbol{v}(\tau))\boldsymbol{\varphi}_j(\tau), \boldsymbol{\varphi}_j(\tau)\rangle, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $H \times H$. Temporarily omitting the variable τ, it follows that:

$$\begin{aligned} \langle F'(\boldsymbol{v})\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_j\rangle &= (\lambda + i\alpha)\langle \Delta\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_j\rangle \\ &\quad - (\kappa + i\beta)\left\{|\boldsymbol{v}|^2\langle \boldsymbol{\varphi}_j, \boldsymbol{\varphi}_j\rangle + 2\,\mathrm{Re}(\bar{\boldsymbol{v}}\boldsymbol{\varphi}_j)\langle \boldsymbol{v}, \boldsymbol{\varphi}_j\rangle\right\} + \gamma\langle \boldsymbol{\varphi}_j, \boldsymbol{\varphi}_j\rangle \\ &= -(\lambda + i\alpha)\int_\Omega \nabla\boldsymbol{\varphi}_j \cdot \nabla\bar{\boldsymbol{\varphi}}_j\, dx \end{aligned}$$

$$-(\kappa+i\beta)\int_\Omega \left\{|\boldsymbol{v}|^2\boldsymbol{\varphi}_j + 2\boldsymbol{v}\,\mathrm{Re}(\bar{\boldsymbol{v}}\boldsymbol{\varphi}_j)\right\}\cdot\bar{\boldsymbol{\varphi}}_j\,dx + \gamma\int_\Omega \boldsymbol{\varphi}_j\cdot\bar{\boldsymbol{\varphi}}_j\,dx.$$

Therefore,

$$\begin{aligned}\mathrm{Re}\langle F'(\boldsymbol{v})\boldsymbol{\varphi}_j,\boldsymbol{\varphi}_j\rangle &= -\lambda\left\|\boldsymbol{\varphi}_j\right\|^2 - \kappa\int_\Omega |\boldsymbol{v}|^2|\boldsymbol{\varphi}_j|^2\,dx \\ &\quad + 2\int_\Omega \mathrm{Re}(\bar{\boldsymbol{v}}\boldsymbol{\varphi}_j)\left\{\beta\,\mathrm{Im}(\bar{\boldsymbol{v}}\boldsymbol{\varphi}_j) - \kappa\,\mathrm{Re}(\bar{\boldsymbol{v}}\boldsymbol{\varphi}_j)\right\}dx + \gamma|\boldsymbol{\varphi}_j|^2.\end{aligned}$$

Since $|\boldsymbol{\varphi}_j| = 1$, the family $\boldsymbol{\varphi}_j$ being orthonormal, and

$$\int_\Omega \beta\,\mathrm{Re}(\bar{\boldsymbol{v}}\boldsymbol{\varphi}_j)\,\mathrm{Im}(\bar{\boldsymbol{v}}\boldsymbol{\varphi}_j)\,dx \le |\beta|\int_\Omega |\boldsymbol{v}|^2|\boldsymbol{\varphi}_j|^2\,dx,$$

it follows that

$$\langle F'(\boldsymbol{v})\boldsymbol{\varphi}_j,\boldsymbol{\varphi}_j\rangle \le -\lambda\|\boldsymbol{\varphi}_j\|^2 - (\kappa - 2|\beta|)\int_\Omega |\boldsymbol{v}|^2|\boldsymbol{\varphi}_j|^2\,dx + \gamma,$$

hence ($\kappa > 0$)

$$\sum_{j=1}^m \langle F'(\boldsymbol{v})\boldsymbol{\varphi}_j,\boldsymbol{\varphi}_j\rangle \le -\lambda\sum_{j=1}^m \|\boldsymbol{\varphi}_j\|^2 + 2|\beta|\int_\Omega |\boldsymbol{v}|^2\rho\,dx + \gamma m. \tag{3.2}$$

Here we define

$$\rho = \rho(x,\tau) = \sum_{j=1}^m |\boldsymbol{\varphi}_j(x,\tau)|^2 \quad \text{a.e. } x,\tau.$$

Refer to Corollary A.4.1 in [24], there exist two dimensionally independent constants c_1 and c_2, which depend only on n and the shape of Ω , such that

$$\int_\Omega \rho^{1+\frac{2}{n}}dx \le \frac{c_1 m}{|\Omega|^{\frac{2}{n}}} + c_2\sum_{j=1}^m \left\|\boldsymbol{\varphi}_j\right\|^2.$$

Thanks to the Hölder and Young inequalities:

$$2|\beta|\int_\Omega |\boldsymbol{v}|^2\rho\,\mathrm{d}x \le 2|\beta|\|\boldsymbol{v}\|^2_{L^{n+2}(\Omega)\times L^{n+2}(\Omega)}\|\rho\|_{L^{\frac{n+2}{n}}(\Omega)}; \tag{3.3}$$

$$\|\rho\|_{L^1(\Omega)} = m \le |\Omega|^{\frac{2}{n+2}}\|\rho\|_{L^{\frac{n+2}{n}}(\Omega)}, \tag{3.4}$$

the right-hand side of (3.2) can be majorized as

$$-\frac{\lambda}{c_2}\|\rho\|^{\frac{n+2}{n}}_{L^{\frac{n+2}{n}}(\Omega)} + \frac{\lambda c_1}{c_2}\frac{m}{|\Omega|^{\frac{2}{n}}} + 2|\beta|\|\boldsymbol{v}\|^2_{L^{n+2}(\Omega)\times L^{n+2}(\Omega)}\|\rho\|_{L^{\frac{n+2}{n}}(\Omega)} + \gamma m.$$

Through the application of the Young inequality, we have

$$\begin{aligned}&2|\beta|\|\boldsymbol{v}\|^2_{L^{n+2}(\Omega)\times L^{n+2}(\Omega)}\|\rho\|_{L^{\frac{n+2}{n}}(\Omega)} \\ &\le \frac{\|\rho\|^{\frac{n+2}{n}}_{L^{\frac{n+2}{n}}(\Omega)}}{\frac{n+2}{n}} + \frac{\left(2|\beta|\|\boldsymbol{v}\|^2_{L^{n+2}(\Omega)\times L^{n+2}(\Omega)}\right)^{\frac{n+2}{2}}}{\frac{n+2}{2}} \\ &= \frac{n}{n+2}\|\rho\|^{\frac{n+2}{n}}_{L^{\frac{n+2}{n}}(\Omega)} + \frac{2^{\frac{2+n}{2}}|\beta|^{\frac{n+2}{2}}\|\boldsymbol{v}\|^{n+2}_{L^{n+2}(\Omega)\times L^{n+2}(\Omega)}}{n+2}.\end{aligned} \tag{3.5}$$

By setting $c_3 = \dfrac{4}{n+2}(2\lambda)^{\frac{n}{2}}$, and combining (3.3)–(3.5), we have

$$\begin{aligned}
&-\frac{\lambda}{c_2}\left\|\rho\right\|^{\frac{n+2}{n}}_{L^{\frac{n+2}{n}}(\Omega)}+\frac{\lambda c_1}{c_2}\frac{m}{\left\|\Omega\right\|^{\frac{2}{n}}}+2|\beta|\left\|\boldsymbol{v}\right\|^2_{L^{n+2}(\Omega)\times L^{n+2}(\Omega)}\left\|\rho\right\|_{L^{\frac{n+2}{n}}(\Omega)}\\
&\le-\frac{\lambda}{2c_2}\left\|\rho\right\|^{\frac{n+2}{n}}_{L^{\frac{n+2}{n}}(\Omega)}+\frac{\lambda c_1}{c_2}\frac{m}{\left\|\Omega\right\|^{\frac{2}{n}}}+c_3|\beta|^{\frac{n+2}{2}}\lambda^{-\frac{n}{2}}\left\|\boldsymbol{v}\right\|^{n+2}_{L^{n+2}(\Omega)\times L^{n+2}(\Omega)}\\
&\le-\frac{\lambda}{2c_2}\frac{m^{\frac{n+2}{n}}}{\left\|\Omega\right\|^{\frac{2}{n}}}+\frac{\lambda c_1}{c_2}\frac{m}{\left\|\Omega\right\|^{\frac{2}{n}}}+c_3|\beta|^{\frac{n+2}{2}}\lambda^{-\frac{n}{2}}\left\|\boldsymbol{v}\right\|^{n+2}_{L^{n+2}(\Omega)\times L^{n+2}(\Omega)}\\
&\le-\frac{\lambda}{4c_2}\frac{m^{\frac{n+2}{n}}}{\left\|\Omega\right\|^{\frac{2}{n}}}+2c_4\frac{\lambda}{\left\|\Omega\right\|^{\frac{2}{n}}}+c_3|\beta|^{\frac{n+2}{2}}\lambda^{-\frac{n}{2}}\left\|\boldsymbol{v}\right\|^{n+2}_{L^{n+2}(\Omega)\times L^{n+2}(\Omega)}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathrm{Re}\ \mathrm{Tr}(F'(\boldsymbol{v}(\tau))\circ Q_m(\tau)) \le -c_5\frac{\lambda}{|\Omega|^{\frac{2}{n}}}m^{\frac{n+2}{n}}+c_6\frac{\lambda}{|\Omega|^{\frac{2}{n}}}(1+&\\
\Big(\frac{|\beta|}{\lambda}\Big)^{\frac{n+2}{2}}|\Omega|^{\frac{2}{n}}\left\|\boldsymbol{v}(\tau)\right\|^{n+2}_{L^{n+2}(\Omega)\times L^{n+2}(\Omega)}\Big)&,
\end{aligned}$$

where $c_5 = \frac{1}{4c_2}$ and $c_6 = \max\{c_3, 2c_4\}$.

Assuming that $\boldsymbol{v_0}$ belong to the global attractor $\mathcal{O}$, the quantity

$$q_m = \limsup_{t\to\infty}\sup_{\substack{\xi_i\in H\times H\\|\xi_i|\le1\\i=1,\dots,m}}\frac{1}{t}\int_0^t \mathrm{Re}\ \mathrm{Tr}(F'(\boldsymbol{v}(\tau))\circ Q_m(\tau))d\tau$$

can be majorized as follows:

$$q_m \le -\kappa_1 m^{\frac{n+2}{n}}+\kappa_2, \tag{3.6}$$

where

$$\kappa_1 = c_5\frac{\lambda}{|\Omega|^{\frac{2}{n}}},\quad \kappa_2 = c_6\frac{\lambda}{|\Omega|^{\frac{2}{n}}}\Big(1+\Big(\frac{|\beta|}{\lambda}\Big)^{\frac{n+2}{2}}|\Omega|^{\frac{2}{n}}\delta\Big),$$

and

$$\delta = \limsup_{t\to\infty}\sup_{\boldsymbol{v_0}\in\mathcal{O}}\frac{1}{t}\int_0^t\left\|S(\tau)\boldsymbol{v_0}\right\|^{n+2}_{L^{n+2}(\Omega)\times L^{n+2}(\Omega)}dx.$$

Here, κ_1, κ_2 depend on the parameter $\lambda, \kappa, \beta, m, n$, and the volume of the region $|\Omega|$.

From (3.6) and (2.14), we infer the following bound on the uniform Lyapunov exponents μ_j, $j \in N$, associated with $\mathcal{O}$:

$$\mu_1+\cdots+\mu_j \le q_j \le -\kappa_1 j^{\frac{n+2}{n}}+\kappa_2,\quad \forall j\in N.$$

By Lemma 2.2, let m satisfy

$$m-1 < \left(\frac{2\kappa_2}{\kappa_1}\right)^{\frac{n}{n+2}} \le m,$$

then $\mu_1+\mu_2+\cdots+\mu_m < 0$ and

$$\frac{(\mu_1+\cdots+\mu_j)_+}{|\mu_1+\cdots+\mu_m|} \le 1,\quad \forall j = 1,\dots,m-1.$$

Finally, by applying Proposition 2.1 and Lemma 2.1, we establish the proof of Theorem 1.1.

4. Conclusions

In this work, we have established an explicit upper bound for the dimension of the global attractor associated with the four-dimensional cubic complex Ginzburg-Landau system by employing the Lyapunov exponent method. The analysis relies on the first variational equation, semigroup estimates, and a carefully constructed functional inequality that enables rigorous control of volume distortion in phase space. As a consequence, the Hausdorff and fractal dimensions of the global attractor are shown to be finite and bounded above in terms of the system parameters.

Compared with existing studies on lower-dimensional or fractional-order complex Ginzburg-Landau systems, our results extend the theory to the critical four-dimensional case, where the absence of compact Sobolev embeddings and the presence of strong nonlinear couplings pose significant analytical challenges. This demonstrates that the Lyapunov exponent method remains a powerful tool for addressing the geometric complexity of attractors in high-dimensional dissipative systems.

The present findings contribute to a deeper understanding of spatio-temporal chaos in high-dimensional complex Ginzburg-Landau dynamics, and provide a rigorous foundation for future investigations of related models. Possible extensions include the study of exponential attractors, stochastic perturbations, and systems with more general nonlinearities or boundary conditions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

References

1. V. L. Ginzburg, L. D. Landau, On the theory of superconductivity, *Zh. Eksp. Teor. Fiz.*, **20** (1950), 1064.
2. B. Guo, Y. Li, M. Jiang, *Ginzburg-Landau Equations*, Science Press, 2020.
3. G. Iooss, Y. Demay, A. Mielke, Theory of steady Ginzburg-Landau equation in hydrodynamic stability problems, *Eur. J. Mech. B/Fluids*, **8** (1989), 229–268.

4. A. Zafar, M. Shakeel, A. Ali, H. Rezazadeh, A. Bekir, Analytical study of complex Ginzburg-Landau equation arising in nonlinear optics, *J. Nonlinear Opt. Phys. Mater.*, **32** (2023), 2350010. https://doi.org/10.1142/S0218863523500108

5. A. T. Olanipekun, A. A. Abioye, K. E. Oluwabunmi, Time-dependent Ginzburg-Landau equation modelling of electron beam additive manufactured titanium alloy, *Leonardo Electron. J. Pract. Technol.*, **32** (2018), 93–102.

6. A. M. Lyapunov, The general problem of the stability of motion, *Int. J. Control*, **55** (1992), 531–534. https://doi.org/10.1080/00207179208934253

7. N. Kuznetsov, V. Reitmann, Lyapunov dimension for dynamical systems in Euclidean spaces, in *Attractor Dimension Estimates for Dynamical Systems: Theory and Computation*, **38** (2021), 257–305. https://doi.org/10.1007/978-3-030-50987-3_6

8. L. Young, Mathematical theory of lyapunov exponents, *J. Phys. A: Math. Theor.*, **46** (2013), 254001. https://doi.org/10.1088/1751-8113/46/25/254001

9. G. A. Leonov, V. A. Boichenko, Lyapunov's direct method in the estimation of the hausdorff dimension of attractors, *Acta Appl. Math.*, **26** (1992), 1–60. https://doi.org/10.1007/BF00046607

10. B. Guo, B. Wang, Finite dimensional behaviour for the derivative Ginzburg-Landau equation in two spatial dimensions, *Phys. D Nonlinear Phenom.*, **89** (1995), 83–99. https://doi.org/10.1016/0167-2789(95)00216-2

11. B. Guo, B. Wang, Exponential attractors for the generalized Ginzburg-Landau equation, *Acta Math. Sinica*, **16** (2000), 515–526. https://doi.org/10.1007/s101140000064

12. H. Lu, S. Lü, Z. Feng, Asymptotic dynamics of 2D fractional complex ginzburg-landau equation, *Int. J. Bifurcation Chaos*, **23** (2013), 1350202. https://doi.org/10.1142/S0218127413502027

13. Q. Zhang, Y. Li, M. Su, The local and global existence of solutions for a time fractional complex Ginzburg-Landau equation, *J. Math. Anal. Appl.*, **469** (2019), 16–43. https://doi.org/10.1016/j.jmaa.2018.08.008

14. C. Zhao, S. Zhou, Limit behavior of global attractors for the complex Ginzburg-Landau equation on infinite lattices, *Appl. Math. Lett.*, **21** (2008), 628–635. https://doi.org/10.1016/j.aml.2007.07.016

15. G. Du, Z. Zhu, C. Zhao, The existence of exponential attractor for discrete Ginzburg-Landau equation, *Discrete Dyn. Nat. Soc.*, **2015** (2015), 217608. https://doi.org/10.1155/2015/217608

16. S. V. Zelik, Attractors. Then and now, *Russ. Math. Surv.*, **78** (2023), 635–777. https://doi.org/10.4213/rm10095e

17. X. Cheng, C. Guo, J. Zheng, Y. Zheng, Global well-posedness of the energy-critical complex Ginzburg-Landau equation in exterior domains, *Calc. Var. Partial Differ. Equations*, **64** (2025), 298. https://doi.org/10.1007/s00526-025-03165-5

18. X. Cheng, Y. Zheng, Threshold solutions of the energy-critical complex Ginzburg-Landau equation, preprint, arXiv:2402.18347.

19. L. Yang, Q. Huang, Z. Dai, Global attractor of nonlinear optical fiber materials with two cores, *J. Hunan Univ. Nat. Sci.*, **36** (2009), 59–62.

20. N. Maaroufi, Invariance and computation of the extended fractal dimension for the attractor of CGL on ℝ, *Chaos Solitons Fractals*, **82** (2016), 87–96. https://doi.org/10.1016/j.chaos.2015.10.037
21. F. Zhou, C. Sun, J. Cheng, Dynamics for the complex Ginzburg-Landau equation on non-cylindrical domains II: The monotone case, *J. Math. Phys.*, **59** (2018), 022703. https://doi.org/10.1063/1.5024214
22. C. Tone, F. Tone, Approximation of the long-time dynamics of the dynamical system generated by the Ginzburg-Landau equation, *Commun. Math. Sci*, **39** (2023), 501–522. https://doi.org/10.4208/cmr.2023-0003
23. F. Zhou, C. Sun, Dynamics for the complex Ginzburg-Landau equation on non-cylindrical domains I: The diffeomorphism case, *Discrete Contin. Dyn. Syst. Ser.-B*, **21** (2016), 3767–3792. https://doi.org/10.3934/dcdsb.2016120
24. R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, 2nd edition, Springer, 1997. https://doi.org/10.1007/978-1-4612-0645-3