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Research article

The source number of a Tychonoff space in its compactifications

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Abstract: The source number $so(X)$ of a Tychonoff space X denotes the minimal cardinal τ such that X has an open source $\mathcal{S}$ in some compactification bX for which $|\mathcal{S}| \leq \tau$. In this paper some properties of source numbers are studied. Among other things, we showed that the equalities $w(X) = nw(X)so(X)$ and $w(X) = so(X)Nag(X)iw(X)$ hold for every Tychonoff space X, where $w(X)$, $nw(X)$, $iw(X)$, $Nag(X)$ denote the weight, network weight, i-weight, and Nagami number of a space X, respectively. Some corollaries about the two statements were obtained.

Keywords: source number; Nagami number; weight; network weight; compactification

1. Introduction and preliminaries

It is well known that topological spaces can be obtained from various kinds of methods. These methods include the one of constructing spaces by means of special subspaces of some classes of topological spaces. For example, p-spaces and Čech-complete spaces are important subspaces of the class of compact spaces. A subspace X of Z, obtained from some sources of X in Z, was also studied, and some important topological spaces such as Lindelöf Σ-spaces and s-spaces are generated in this way.

Suppose that $\mathcal{S}$ is a family of subsets of a space X. Let S_δ denote the family of all sets that can be represented as the intersection of some nonempty subfamily of $\mathcal{S}$, and $S_{\delta,\sigma}$ denote the family of all sets that can be represented as the union of some subfamily of S_δ. $\mathcal{S}$ is called a source for a subspace Y in X if $Y \in S_{\delta,\sigma}$ [1–3]. Clearly, every subspace Y of X may have one or more sources in X. Among them, open and closed sources are especially important and interesting. A source $\mathcal{S}$ for Y in X is called open (closed) if every member of $\mathcal{S}$ is an open (a closed) subset of the space X. Open sources and closed sources are closely related, for if γ is an open source for a space Y in X, then the family $\{X \setminus O : O \in \gamma\}$ will be a closed source for the space $X \setminus Y$ in X, and the converse is also true. Every space can be obtained from a larger space by open sources, and as well as by closed sources.

Let X be a subspace of a space Z. Recall that the source number $so(X, Z)$ of X in Z is the minimal infinite cardinal τ such that X has an open source $\mathcal{S}$ in Z for which $|\mathcal{S}| \leq \tau$ [4]. Given a space X, the cardinal $so(X, \cdot)$ is a location invariant and it depends on the extension of X, where a space Y is called an extension of X provided that Y contains X as a subspace. That is, for any two distinct spaces Z, H containing X, $so(X, Z)$ may be different from $so(X, H)$. For example, for every space X, if $Z = X \cup \{X\}$ is a T_1 space, then $so(X, Z) = \omega$. However, there are many spaces X and their extensions H such that the cardinal $so(X, H)$ is uncountable. The situation will be different if we restrict the extension of X to the class of all compactifications of X. Indeed, the spaces X with countable source number $so(X, bX)$ in some compactification bX were systematically investigated in [1], and these spaces are called s-spaces. The class of s-spaces is very wide and many classical objects, such as Lindelöf p-spaces, Čech-complete spaces, as well as metrizable spaces with weight $\leq 2^\omega$ [5], are all in this class. In [1] it was proved that if a space X has countable source number in some compactification bX, then $so(X, cX)$ is countable for any compactification cX of X. Equivalently, an s-space X has a countable open source in each of its compactifications. With a similar proof, one can generalize this property to a space Y with uncountable source number in some compactification bY, and we will present its proof in the next part for the sake of completeness. So, in the light of the above argument and also for the purpose of concision, the notation $so(X, bX)$, the source number of X in some compactification bX, can be replaced by $so(X)$ and we will call it the source number of X. Notice that herein $so(X)$ denotes the source number of a space X in some compactification but not in any other extensions of X, and the sources being used are open sources.

As a dual cardinal invariant of the source number, the Nagami number $Nag(X)$ [6] of a space X denotes the minimal infinite cardinal τ such that X has a closed source $\mathcal{F}$ in the Stone-Čech compactification βX of X for which $|\mathcal{F}| \leq \tau$. It is known that, in the definition of $Nag(X)$, βX can be replaced by any compactification bX of X (for example, see Lemma 5.3.1 in [6]). In particular, a space with countable Nagami number is called a Lindelöf Σ-space [6–8] and it is widely used in generalized metric spaces. For the Nagami number and its uses in topological groups please refer to [6].

Let us recall some other cardinal invariants used in this paper.

The weight $w(X)$ of a space X is the minimal infinite cardinal τ such that X has a base $\mathcal{B}$ for which $|\mathcal{B}| \leq \tau$.

The network weight $nw(X)$ of a space X is the minimal infinite cardinal τ such that X has a network $\mathcal{N}$ for which $|\mathcal{N}| \leq \tau$. A family $\mathcal{N}$ of nonempty subsets of X being a network of X means that, for every point $x \in X$ and every neighbourhood U of x, there exists a member F of $\mathcal{N}$ satisfying $x \in F \subset U$.

The i-weight of a space X denoted by $iw(X)$ is defined as the minimal infinite cardinal τ such that there exists a continuous bijection of X onto a space of weight $\leq \tau$.

This paper is motivated by a series of nice properties of s-spaces established in [1], and some results about s-spaces in [1] are generalized here to spaces with arbitrary source numbers. Among other things, we show that the equalities $w(X) = nw(X)so(X)$ and $w(X) = so(X)Nag(X)iw(X)$ hold for every Tychonoff space X, and some corollaries about the two statements are obtained.

In this article, a "space" always stays for a "Tychonoff topological space". By a remainder of a space X we mean the subspace $bX \setminus X$ of a compactification bX of X. For a subset A of a space X, $\overline{A}^X$ stands for the closure of A in X.

For other notations and concepts appearing in this paper please refer to [9].

2. Main results

The Nagami number $Nag(X)$ of a space X is defined by a closed source of X in some compactification bX. In fact, the next more general statement shows that $Nag(X)$ can also be defined by a closed source of X in every compact space Z that contains X as a subspace.

Lemma 2.1. *Let X be a subspace of a compact space Z and τ be an infinite cardinal. Then $Nag(X) \leq \tau$ if and only if X has a closed source γ in Z such that $|\gamma| \leq \tau$.*

Proof. Sufficiency. Let bX be the closure of X in Z and γ be a closed source of X in Z. Then, as a closed subspace of Z, bX is compact and hence is a compactification of X. Clearly, the family $\zeta = \{F \cap bX : F \in \gamma\}$ is a closed source of X in bX and $|\zeta| \leq \tau$. Hence, $Nag(X) \leq \tau$.

Necessity. Since $Nag(X) \leq \tau$, X has a closed source ξ in its compactification bX with $|\xi| \leq \tau$, where bX is the closure of X in Z. It is easy to see that ξ is also a closed source for X in Z. Hence, X has a closed source in Z whose cardinality $\leq \tau$. □

The following statement shows that the definition $so(X)$ does not depend on the choice of the compactification bX of X.

Proposition 2.2. *Suppose that bX and cX are two compactifications of a space X. Then $so(X, bX) = so(X, cX)$.*

Proof. Let $\kappa = so(X, bX)$, γ be an open source of X in bX with $|\gamma| \leq \kappa$ and βX be the Stone-Čech compactification of X. Then the identity mapping $i : X \to X$ can be extended to a perfect mapping g of βX onto bX. Clearly, the restriction $g|_{\beta X \setminus X} : \beta X \setminus X \to bX \setminus X$ is also perfect and $g(\beta X \setminus X) = bX \setminus X$.

Since $\kappa = so(X, bX)$, $bX \setminus X$ has a closed source in bX whose cardinality $\leq \kappa$. By Lemma 2.1, $Nag(bX \setminus X) \leq \kappa$. We claim that $Nag(bX \setminus X) = \kappa$. Otherwise, using Lemma 2.1 once more, $bX \setminus X$ would have a closed source η in bX such that $|\eta| < \kappa$. It follows that X has an open source in bX whose cardinality $< \kappa$, which is a contradiction. Hence, $Nag(bX \setminus X) = \kappa$. By Proposition 5.3.6 in [6], a perfect mapping preserves the Nagami number in both directions. So $Nag(\beta X \setminus X) = \kappa$, and hence $Nag(cX \setminus X) = \kappa$, since $cX \setminus X$ is the image under another perfect mapping from $\beta X \setminus X$ onto $cX \setminus X$. This implies that X has an open source in cX of cardinality $\leq \kappa$. Hence, $so(X, cX) \leq \kappa = so(X, bX)$. Similarly, $so(X, bX) \leq so(X, cX)$. Therefore, $so(X, bX) = so(X, cX)$. □

The definition of $so(X)$ of a space X can be complemented as follows:

Proposition 2.3. *Let bX be a compactification of a space X and $Y = bX \setminus X$. Then the following statements hold:*

1) $so(X) = Nag(Y)$;

2) $so(Y) \leq Nag(X)$;

3) $X = P \cup J$, where $Nag(P) = so(Y)$ (if P is not empty) and J is locally compact.

Proof. 1) follows from the proof of Proposition 2.2.

To prove 2), we consider two cases.

Case 2a). Y is compact. It is immediate that $so(Y) \leq Nag(X)$, since in this case $so(Y) = \omega$.

Case 2b). Y is not compact. Let Z be the closure of Y in bX. It follows from 1) that $so(Y) = Nag(Z \setminus Y)$. Since $Z \setminus Y = Z \cap X$, $Z \setminus Y$ is a closed subspace of X. Hence, $Nag(Z \setminus Y) \leq Nag(X)$, which implies that $so(Y) \leq Nag(X)$.

3) follows from the proof of 2) if one takes $P = Z \cap X$ and $J = X \setminus P$. □

Proposition 2.4. *For any space X and its open (respectively, closed, or G_δ-)subspace Y, $so(Y) \leq so(X)$.*

Proof. Let bX be a compactification of X and γ be an open source for X in bX such that $|\gamma| \leq so(X)$. Denote $so(X)$ by κ.

Since every open subspace of X is a G_δ-subspace, we show the case for G_δ-subspaces. For a G_δ-subspace G of X, take a sequence $\{U_n : n \in \omega\}$ of open subsets of X such that $G = \cap\{U_n : n \in \omega\}$. For each $n \in \omega$, choose an open subset V_n of bX such that $V_n \cap X = U_n$. Put $\xi = \{V_n \cap \overline{G}^{bX} : n \in \omega\} \cup \{O \cap \overline{G}^{bX} : O \in \gamma\}$. It is not difficult to check that ξ is an open source for G in its compactification $\overline{G}^{bX}$ with cardinality $\leq \kappa$.

For a closed subspace Z of X, put $\eta = \{O \cap \overline{Z}^{bX} : O \in \gamma\}$. Clearly, η is a required source. □

This proposition shows that, for many subspaces of a space X, their source numbers don't increase. However, this phenomenon needn't hold for any subspaces. Indeed, for any space Z, each of its compactifications bZ is an s-space and hence has countable source number.

Proposition 2.5. *For any dense subspace X of Z, $so(X, Z) \leq w(X)$. In particular, $so(X) \leq w(X)$.*

Proof. Let $\mathcal{B}$ be a base of X such that $|\mathcal{B}| \leq w(X)$. For every $O \in \mathcal{B}$, take an open subset U_O of Z such that $U_O \cap Z = O$ and put $\gamma = \{U_O : O \in \mathcal{B}\}$. Then γ is an open source for X in Z. Indeed, for any two points $x \in X, y \in Z \setminus X$, pick an open neighbourhood W of x in Z such that $\overline{W}^Z \subset Z \setminus \{y\}$. Choose a member O of $\mathcal{B}$ such that $x \in O \subset W$. Notice that X is dense in Z, so $x \in U_O \subset \overline{U_O}^Z = \overline{O}^Z \subset Z \setminus \{y\}$. Obviously, $|\gamma| \leq w(X)$. Therefore, $so(X, Z) \leq w(X)$. □

The source number alone cannot bound the weight of a space. To see this, one only needs to take an uncountable discrete space, which is obviously an s-space, so its weight is larger than its source number. This simple example also shows that even a space with a hereditary source number τ could still have a larger weight. Later we will show that network weight is the cardinal invariant which can bridge the distance between source number and weight.

Proposition 2.6. *Let κ be an infinite cardinal and a space X be the union of κ many of its dense (respectively, open) subspaces X_α, $\alpha < \kappa$. Then $so(X) \leq \kappa \cdot sup\{so(X_\alpha) : \alpha < \kappa\}$.*

Proof. Let bX be a compactification of X and suppose that each X_α, $\alpha < \kappa$, is dense in X. Then bX is a compactification of the space X_α as well, since X_α is dense in X and hence dense in bX. For $\alpha < \kappa$, choose an open source $\mathcal{S}_\alpha$ for X_α in bX such that $|\mathcal{S}_\alpha| \leq so(X_\alpha)$, and let $\mathcal{S} = \bigcup_{\alpha<\kappa} \mathcal{S}_\alpha$. Then it is easy to verify that $\mathcal{S}$ is an open source for X in bX and $|\mathcal{S}| \leq \kappa \cdot sup\{so(X_\alpha) : \alpha < \kappa\}$.

For the case each X_α is open in X, let bX_α denote the closure of X_α in bX. Then bX_α is a compactification of X_α, $\alpha < \kappa$. Take an open source $\mathcal{T}_\alpha$ for X_α in bX_α such that $|\mathcal{T}_\alpha| \leq so(X_\alpha)$, and let U_α be the interior of bX_α in bX. Clearly, $X_\alpha \subseteq U_\alpha$. Put $\mathcal{S}_\alpha = \{O \cap U_\alpha : O \in \mathcal{T}_\alpha\}$. Then $\mathcal{S}_\alpha$ is also an open source for X_α in bX_α. Now let $\mathcal{S} = \bigcup_{\alpha<\kappa} \mathcal{S}_\alpha$. It is easy to see that each member of $\mathcal{S}$ is an open subset of bX and $|\mathcal{S}| \leq \kappa \cdot sup\{so(X_\alpha) : \alpha < \kappa\}$, so it suffices to show that $\mathcal{S}$ is a source for X in bX. To this end, fix two points $x \in X$ and $y \in bX \setminus X$. Then there exists $\alpha < \kappa$ such that $x \in X_\alpha$. If $y \in bX_\alpha$, then $y \in bX_\alpha \setminus X_\alpha$ and hence one can find an element $U \in \mathcal{S}_\alpha$ such that $U \cap \{x, y\} = \{x\}$. If $y \notin bX_\alpha$, then any element of $\mathcal{S}_\alpha$ containing x doesn't contain the point y. The proof is completed. □

The condition “open” or “dense” in the above proposition cannot be dropped. To see this, take a countable non-metrizable space X. Then X is the union of all its singletons. However, X is not a s-space, since every countable s-space is metrizable. This fact also shows that the condition “open” or “dense” in Proposition 2.6 cannot be replaced by that of “closed”.

Proposition 2.7. *Suppose that f is a perfect mapping of a space X onto a space Y. Then $so(Y) = so(X)$.*

Proof. Let βX and βY be the Stone-Čech compactifications of X, Y respectively. Then the mapping f can be extended to a continuous mapping g of βX onto βY. Since f is perfect, it follows that the restriction mapping $g|_{\beta X\setminus X}$ is also perfect and $g(\beta X \setminus X) = \beta Y \setminus Y$. Clearly, $Nag(\beta X \setminus X) = so(X)$ and $Nag(\beta Y \setminus Y) = so(Y)$. Since a perfect mapping preserves the Nagami number in both directions, it follows that $Nag(\beta X \setminus X) = Nag(\beta Y \setminus Y)$. Therefore, $so(Y) = so(X)$. □

Proposition 2.8. *Let κ be an infinite cardinal and suppose that a space X is the union of κ many of its closed subspaces X_α such that the family $\{X_\alpha : \alpha < \kappa\}$ is locally finite in X and the source number of each X_α is at most κ. Then $so(X) \leq \kappa$.*

Proof. Consider the sum space $\bigoplus_{\alpha<\kappa} X_\alpha$ of the family of spaces X_α. Then each X_α is an open-closed subspace of $\bigoplus_{\alpha<\kappa} X_\alpha$. Since the source number of each $X_\alpha \leq \kappa$, it follows from Proposition 2.6 that $so(\bigoplus_{\alpha<\kappa} X_\alpha) \leq \kappa$. Let $f : \bigoplus_{\alpha<\kappa} X_\alpha \to X$ be the natural mapping that restricts to the identity on X_α for each $\alpha < \kappa$. Since the family $\{X_\alpha : \alpha < \kappa\}$ is locally finite and each X_α is closed in X, it follows that f is a perfect mapping. Then, by Proposition 2.7, $so(X) \leq \kappa$. □

Corollary 2.9. *If a space X is the union of a finite family η of its closed subspaces such that the source number of each member of η is not larger than some cardinal κ, then $so(X) \leq \kappa$.*

Proposition 2.10. *Let $X = \prod_{\alpha<\kappa} X_\alpha$ be the product space of a family of non-empty spaces X_α with $\alpha < \kappa$. Then $so(X) \leq \kappa$ if and only if, for every $\alpha < \kappa$, $so(X_\alpha) \leq \kappa$.*

Proof. Sufficiency. For each $\alpha < \kappa$, take a compactification bX_α of X_α and an open source ζ_α for X_α in bX_α such that $|\zeta_\alpha| \leq \kappa$. Put $\gamma_\alpha = \{\prod_{\beta<\kappa} U_\beta : U_\alpha \in \zeta_\alpha \text{ and } U_\beta = X_\beta, \beta \neq \alpha\}$, $\alpha < \kappa$, and $\gamma = \bigcup_{\alpha<\kappa} \gamma_\alpha$. Obviously, $|\gamma| \leq \kappa$. It remains to verify that γ is a source for $\prod_{\alpha<\kappa} X_\alpha$ in its compactification $\prod_{\alpha<\kappa} bX_\alpha$. Indeed, for any pair of points $(x_\alpha)_{\alpha<\kappa} \in \prod_{\alpha<\kappa} X_\alpha$ and $(y_\alpha)_{\alpha<\kappa} \in \prod_{\alpha<\kappa} bX_\alpha \setminus \prod_{\alpha<\kappa} X_\alpha$, there exists $\delta < \kappa$ such that $x_\delta \in X_\delta, y_\delta \in bX_\delta \setminus X_\delta$. Now one can immediately find an element U of γ_δ such that $U \cap \{(x_\alpha)_{\alpha<\kappa}, (y_\alpha)_{\alpha<\kappa}\} = \{(x_\alpha)_{\alpha<\kappa}\}$.

Necessity. Each X_α is homeomorphic to a closed subspace of the product space $\prod_{\alpha<\kappa} X_\alpha$. Therefore, by Proposition 2.4, $so(X_\alpha) \leq \kappa$. □

Theorem 2.11. *The equality $w(X) = nw(X)so(X)$ is valid for each space X.*

Proof. Let bX be a compactification of the space X and $\mathcal{N}$ be a network of X such that $|\mathcal{N}| \leq nw(X)$. Put $\zeta = \{\overline{F}^{bX} : F \in \mathcal{N}\}$ and $Y = bX \setminus X$. Then $Nag(Y) = so(X)$. Choose a closed source η of Y in bX such that $|\eta| \leq Nag(Y)$, and let γ denote the family of subsets of bX consisting of all the finite intersections of the family $\zeta \cup \eta$. Clearly, every member of γ is a closed subset of bX and $|\gamma| \leq nw(X)so(X)$.

Claim 1: For any two distinct points $x, z \in X$, there exist two disjoint members of γ one of which contains x and the other contains z.

Indeed, take in X two neighbourhoods U and V of x and z, respectively, such that the intersection of their closures in bX is empty. Since $\mathcal{N}$ is a network of X, there exists $F, H \in \mathcal{N}$ such that $x \in F \subset U$ and $z \in H \subset V$. Clearly, the closures of F, H in bX belong to γ and their intersection is empty.

Claim 2: For any pair of points $x \in X, y \in Y$, one can find two disjoint members of γ containing x and y respectively.

Suppose the contrary. Then every member of γ that contains x has nonempty intersection with any member of γ containing y. Let $\mathcal{F}_x = \{F \in \gamma : x \in F\}$, $\mathcal{F}_y = \{F \in \gamma : y \in F\}$ and $\mathcal{F} = \mathcal{F}_x \cup \mathcal{F}_y$. Clearly, the set $\mathcal{F}$ has the finite intersection property. Then it follows from the compactness of bX that the intersection of all members of $\mathcal{F}$ is nonempty, and take a point a from this intersection. If $a \in X$, then it implies that each element of $\mathcal{F}_y$ must contain a. This contradicts the fact that η is a closed source for Y in bX. Now we consider the case $a \in Y$. Clearly, x is different from a and each element of $\mathcal{F}_x$ contains a. Select a closed neighbourhood W of x in bX such that W does not contain a. Then one can take some $D \in \mathcal{N}$ such that $x \in D \subset W$. Obviously, the closure of D in bX is a member of $\mathcal{F}_x$ and doesn't contain the point a. This is a contradiction, too. The proof of the claim is completed.

For any disjoint pair of sets $F, H \in \gamma$, we take a continuous real-valued function $f_{F,H} : bX \to [0, 1]$ such that $f_{F,H}(F) \subseteq \{0\}$ and $f_{F,H}(H) \subseteq \{1\}$. This can be done, since F, H are disjoint closed subsets of the normal space bX. Denote the set of all such functions by ξ. Now consider the diagonal product φ of the set ξ, that is, φ is a mapping from the space bX to the Tychonoff product space $[0, 1]^{\xi}$ such that $\varphi(x) = (f(x))_{f\in\xi}$. Clearly, φ is continuous. Since bX is compact and $[0, 1]^{\xi}$ is Hausdorff, φ is a perfect mapping. From Claim 2, it follows that the intersection of the sets $\varphi(X)$ and $\varphi(Y)$ is empty, so the equation $X = \varphi^{-1}(\varphi(X))$ holds. From Claim 1, it follows that the restriction mapping $\varphi|_X : X \to \varphi(X)$ is injective and hence bijection. Further, $\varphi|_X$ is perfect. Therefore, $\varphi|_X : X \to \varphi(X)$ is a homeomorphism. $\varphi(X)$ is a subspace of $[0, 1]^{\xi}$, so $w(\varphi(X)) \leq |\xi| \leq |\gamma| \leq nw(X)so(X)$. Hence, $w(X) \leq nw(X)so(X)$. However, the converse of the inequality is obvious. Therefore, $w(X) = nw(X)so(X)$. □

Corollary 2.12. *For any space X, the following statements hold:*

1) if $nw(X) < w(X)$, then $so(X) = w(X)$;

2) if $so(X) \leq nw(X)$, then $w(X) = nw(X)$;

3) if $so(X) < w(X)$, then $w(X) = nw(X)$;

4) if $so(X) = nw(X)$, then $so(X) = w(X) = nw(X)$.

Corollary 2.13. *[1] For an s-space X, $w(X) = nw(X)$.*

Corollary 2.14. *Let κ be the source number of a space X. If X is covered by κ many of its subspaces X_α such that $w(X_\alpha) \leq \kappa$ for each $\alpha < \kappa$, then $w(X) \leq \kappa$ as well.*

Proof. Since $w(X_\alpha) \leq \kappa$, it follows that $nw(X_\alpha) \leq \kappa$ for each $\alpha < \kappa$. Hence, $nw(X) = nw(\bigcup_{\alpha<\kappa} X_\alpha) \leq \kappa$. Therefore, by Theorem 2.11, $w(X) = nw(X)so(X) \leq \kappa$. □

Theorem 2.15. *For every space X, there exists a perfect mapping $f : X \to Z$ of the space X onto a space Z such that $w(Z) \leq so(X)Nag(X)$.*

Proof. Let $\kappa = so(X)Nag(X)$ and bX be a compactification of the space X. Take an open source ζ and a closed source η for X in bX, respectively, such that $|\zeta| \leq \kappa$ and $|\eta| \leq \kappa$. Let $Y = bX \setminus X$ and $\xi = \{bX \setminus O : O \in \zeta\}$. Since ζ is an open source for X in bX, it follows that ξ is a closed source for the space Y in bX. Let γ be the family of subsets of bX consisting of all finite intersections of members of the set $\eta \cup \xi$. Clearly, every member of γ is closed in bX and $|\gamma| \leq \kappa$.

Claim: For any pair of points $x \in X$, $y \in Y$, there exists F, $H \in \gamma$ such that $x \in F$, $y \in H$ and $F \cap H = \emptyset$.

Indeed, to find such F and H, put $\mathcal{F}_x = \{T \in \gamma : x \in T\}$ and $\mathcal{F}_y = \{T \in \gamma : y \in T\}$. Since η and ξ are closed sources for X and Y in bX respectively, it follows that the intersection $\bigcap \mathcal{F}_x$ of all members of $\mathcal{F}_x$ is nonempty and contained in X, and the intersection $\bigcap \mathcal{F}_y$ of all members of $\mathcal{F}_y$ is a nonempty subset of Y. Fix two disjoint open subsets U, V of bX such that $\bigcap \mathcal{F}_x \subset U$ and $\bigcap \mathcal{F}_y \subset V$. Then, from the compactness of bX, one can find a finite family $\mu \subset \mathcal{F}_x$ and a finite family $\nu \subset \mathcal{F}_y$, respectively, such that $\bigcap \mu \subset U$ and $\bigcap \nu \subset V$. Put $F = \bigcap \mu$ and $H = \bigcap \nu$. Then $F \cap H = \emptyset$. Clearly, $F, H \in \gamma$, since the intersection of any finite family of members of γ also belongs to γ. This finishes the proof of the claim.

For such a pair of disjoint $F, H \in \gamma$, take a continuous real-valued function $g_{F,H}$ of bX to the closed interval $[0, 1]$ such that $g_{F,H}(F) \subseteq \{0\}$ and $g_{F,H}(H) \subseteq \{1\}$. Let λ be the set of all functions obtained in this way. Clearly, $|\lambda| \leq \kappa$. Take the diagonal product φ of the set λ. Then φ is a perfect mapping from bX to the Tychonoff product space $[0, 1]^{\lambda}$ of $|\lambda|$ many of $[0, 1]$. From the choice of λ one can see that the set λ separates the points of X from the points of Y. Hence, $X = \varphi^{-1}(\varphi(X))$. Let $Z = \varphi(X)$ and f be the restriction of φ to X. Then $f : X \to Z$ is a perfect mapping. Obviously, $w(Z) \leq \kappa$. The proof is finished. □

A space that is a perfect preimage of a separable metrizable space is called a Lindelöf p-space [10], so for a space with countable source number and countable Nagami number, we have the following:

Corollary 2.16. *[1] A space X is a Lindelöf p-space if and only if it is a Lindelöf Σ-space and an s-space.*

Theorem 2.17. *Every space X satisfies the equality $w(X) = so(X)Nag(X)iw(X)$.*

Proof. By Proposition 5.3.15 in [6], the equality $nw(X) = Nag(X)iw(X)$ holds for any Tychonoff space X. Then the conclusion immediately follows from Theorem 2.11. □

Corollary 2.18. *For every space X, if $so(X)Nag(X) \leq iw(X)$ or $so(X)Nag(X) < w(X)$, then $w(X) = iw(X)$.*

Remark 2.19. *For the case $so(X)Nag(X) < w(X)$, the above corollary can also be proved by use of Theorem 2.15. Now let us give its short proof. Indeed, by Theorem 2.15, there exist a space Z and a perfect mapping $f : X \to Z$ of X onto Z such that $w(Z) \leq so(X)Nag(X)$. Since $so(X)Nag(X) < w(X)$, it follows that $w(Z) < w(X)$. Further, take a continuous bijection $g : X \to Y$ of X onto a space Y such that $w(Y) = iw(X)$. Let $h : X \to Z \times Y$ be the diagonal product of the mappings f and g. Since f is perfect and g is a condensation, it follows that f is an embedding of X onto a closed subspace of $Z \times Y$, so the mapping $h : X \to h(X)$ is a homeomorphism. Hence, $w(X) = w(h(X))$. However, $w(h(X)) \leq w(Z \times Y) = w(Z)w(Y) = w(Z)iw(X) \leq w(X)$. It implies that $w(X) = w(Z)iw(X)$. Since $w(Z) < w(X)$, it follows that $w(X) = iw(X)$.*

Corollary 2.20. *For every space X, if $so(X) = Nag(X) = \omega$, then $w(X) = iw(X)$.*

Remark 2.21. *It may take place that only one of the three cardinal invariants in the right side of the equation $w(X) = so(X)Nag(X)iw(X)$ happens to coincide with $w(X)$. We present such three simple spaces.*

a) Let $X = \omega \cup \{p\}$, *where* X *is a subspace* $\beta\omega$ *of the Stone-Čech compactification of the discrete space* ω *and* $p \in \beta\omega \setminus \omega$. *Since* $Nag(X)iw(X) \leq nw(X)$, $Nag(X) = iw(X) = \omega$. *So* $so(X) = w(X) > \omega$.

b) Let X *be a discrete space with cardinality* 2^ω. *Then* $so(X) = iw(X) = \omega$. *Hence,* $Nag(X) = w(X) = 2^\omega$. *Further, a nontrivial space for this case is the Michael line* Y. *Clearly,* $iw(Y) = \omega$. *It is also an s-space [4]. Hence,* $Nag(Y) = w(Y) > \omega$.

c) For any Lindelöf p-space (especially any compact space) X *that is not metrizable,* $w(X) = iw(X) > \omega$ *and* $Nag(X) = so(X) = \omega$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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