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Research article

Continuous dependence result for the higher-order conduction within the second gradient of type III

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Abstract: This study investigated the spatial behavior and structural stability of solutions for a higher-order heat conduction model within the second gradient theory of type III. By constructing a tailored energy functional and employing integral-differential inequality techniques, we derived a continuous dependence estimate for the solution with respect to the elastic coefficient μ. Our results demonstrate that the energy not only decays exponentially with respect to the spatial variable z, but also diminishes as the parameter μ tends to zero. This work extends the understanding of structural stability in unbounded domains and underscores the robustness of the model under perturbations of material parameters.

Keywords: integral-differential inequality; structural stability; sontinuous dependence; second gradient of type III

1. Introduction

The concepts of structural stability and continuous dependence occupy a central role in the analysis of partial differential equations (PDEs), critically underpinning the reliability and robustness of mathematical models across numerous scientific and engineering disciplines. The significance of continuous dependence results cannot be overstated, as they play a crucial role in quantifying the influence of uncertainties and errors that are inherently present in numerical computations and physical measurements. A comprehensive understanding of how these errors impact the solutions is of utmost importance for ensuring the accuracy and dependability of mathematical models.

While significant contributions to the theory of stability and continuous dependence have been made by numerous researchers. The monograph by Ames and Straughan [1] provides an extensive array of references that underscore the significance of continuous dependence and convergence in relation to modifications in the model structure. Other results may be found in Celebi et al. [2, 3], Franchi and Straughan [4], Harfash [5], Kaloni and Guo [6], showing that a substantial portion of the existing literature focuses on bounded domains. For example, [7] studied the stability for the primitive equations, [8] studied the stability for the Darcy equation, and [9] studied the stability for double-diffusion equations in a bounded domain. Other than [10], the investigation of these properties in unbounded domains, particularly strip-like regions, remains a relatively less explored area. Unbounded strip-like domains are relevant in several practical contexts, including: long structural components in engineering (e.g., beams, plates, or thermal conductors extending far in one direction) and semi-infinite media in thermal or mechanical systems where boundary effects are localized but the domain extends far away. Understanding the spatial decay and stability of solutions in unbounded domains is crucial for ensuring the reliability and predictive accuracy of such models in real-world applications. Recently, [11] studied the spatial decay estimates for the equations in unbounded domains. This paper aims to address this gap by studying the structural stability of solutions within an unbounded strip, a problem that presents unique challenges, chief among them being the management of energy estimates associated with the biharmonic operator. This necessitates the development of novel analytical approaches.

One of the major challenges in this study lies in managing the energy relationships associated with the biharmonic operator within such domains. This necessitates the development and application of innovative approaches and techniques. We are convinced that a detailed mathematical analysis of these equations will shed light on their practical applicability in the fields of physics and engineering. Type I heat conduction models the heat flux being proportional to the gradient of temperature. However, it leads to a physically unrealistic phenomenon: an infinite signal speed of paradox where a sudden temperature or thermal disturbance changes at certain points and will be felt instantly everywhere. In order to overcome this paradox, type II and type III heat conductions arise, where the theory of type II does not allow the dissipation of the energy (the evolution equation without a damping term). For the type III model, the heat flux is assumed to be propotional to the thermal displacement gradient. The type III model is the type II model with an additional dissipation and can be understood by the evolution equation with a kind of damping mechanism. Iesan has developed distinct theories that incorporate second-gradient effects into the heat equation. The authors studied the second gradient theory for thermoelasticity in [12], for thermoviscoelasticity in [13, 14], and for microtemperatures in [15]. In [16], the authors studied analytically a higher-order equation in a semi-infinite cylinder. The spatial energy decay was proved to be of exponential type. They also considered the application to some heat conduction problems. Our analysis is set within the framework of the second gradient theory of type III. We continue the work of [16], and we study the structural stability on the same equation. The governing evolution equation for the thermal displacement u such that $\dot{u} = \theta$ is given by (see [16]):

$$c\ddot{u} = \mu\Delta u + \mu_1\Delta\dot{u} - d_0\Delta^2 u - d_1\Delta^2\dot{u}, \tag{1.1}$$

where c denotes the inertia coefficient, which reflects the material's thermal capacity or inertia, quantifying its ability to store thermal energy. The coefficient μ is the elastic coefficient, associated with thermal diffusivity, characterizing the rate at which heat diffuses through the material under steady-state conditions. The time-dependent elastic coefficient μ_1 accounts for variations in thermal diffusivity over time, while d_0 and d_1 are diffusion coefficients related to higher-order thermal effects, capturing non-Fourier heat conduction behaviors such as wave-like heat propagation. The operators Δ and Δ^2 represent the Laplacian and biharmonic operators, respectively, describing second-order and fourth-order spatial derivatives. These operators reflect the diffusion of heat and account for dispersive or non-local thermal interactions.

This model is defined on the unbounded domain Ω_0, which is defined by

$$\Omega_0 := \{(x_1, x_2) \mid x_1 > 0, 0 < x_2 < h\},$$

where h is a constant. We now define a notation

$$L_z = \{(x_1, x_2) \mid x_1 = z \geq 0, 0 \leq x_2 \leq h\},$$

where z represents the variable in the x_1 direction. The initial and boundary conditions are

$$u(x_1, 0, t) = 0 \quad x_1 > 0, t > 0, \tag{1.2}$$

$$u(x_1, h, t) = 0 \quad x_1 > 0, t > 0, \tag{1.3}$$

$$u(0, x_2, t) = g_1(x_2, t) \quad 0 \leq x_2 \leq h, t > 0, \tag{1.4}$$

$$u_{,1}(0, x_2, t) = g_2(x_2, t) \quad 0 \leq x_2 \leq h, t > 0, \tag{1.5}$$

and

$$u(x_1, x_2, 0) = 0 \quad 0 \leq x_2 \leq h, x_1 > 0. \tag{1.6}$$

We impose the standard asymptotic condition that u and its derivatives decay to zero at infinity:

$$u(x_1, x_2, t), u_{,\alpha}(x_1, x_2, t), u_{,\alpha\beta}(x_1, x_2, t) \to 0 \quad as \quad z \to \infty. \tag{1.7}$$

Equation (1.1) provides a comprehensive model for heat conduction in materials exhibiting complex microstructural and non-Fourier thermal behaviors. As noted in [12], the well-posedness of the associated boundary-initial-value problem has been established, forming a solid foundation for the current stability analysis.

A crucial step toward our continuous dependence result is the derivation of spatial decay estimates. The study of spatial decay has been extensively developed over the past decades, significantly extending the classical Saint-Venant principle. A parallel line of inquiry focuses on the Phragmén-Lindelöf alternative principle, which characterizes the spatial behavior of solutions to equations like the biharmonic without presupposing decay, stating that the energy must either grow or decay exponentially at infinity. In [17], the authors studied the spatial property for the Laplace equation. Similar work may be found in [18] and [19]. Similar results may be found in [20], [21], and [22] for the thermoelastic equations and the Boussinesq equation. Liu et al. investigated the spatial properties for different equations in [23]. In [24] and [25], the authors studied the structural stability for fluid flows. In [26], the authors studied the structural stability for the interfacing problem. For recent advances on the spatial decay for biharmonic equations, we refer the reader to [27].

The primary objective of this work is to investigate the continuous dependence of the solution to the problem defined by (1.1)–(1.7) on the elastic coefficient μ. The main challenge lies in effectively handling the energy structure associated with the biharmonic operator in an unbounded strip. The paper is structured as follows: Section 2 derives the essential spatial decay estimates for the solution. Based on these estimates, Section 3 establishes the continuous dependence result on the parameter μ.

In the present paper, the comma stands for partial differentiation, and $,k$ stands for partial differentiation with respect to the direction x_k. Hence, the symbol $\theta_{,i}$ represents $\frac{\partial\theta}{\partial x_i}$, and the symbol $\dot{\theta}$ represents $\frac{\partial\theta}{\partial t}$. In the whole paper, repeated Greek subscripts represent summation from 1 to 2, i.e., $\theta_{,\alpha}\theta_{,\alpha} = \sum_{\alpha=1}^{2} \frac{\partial\theta}{\partial x_\alpha}\frac{\partial\theta}{\partial x_\alpha}$.

2. Formulation of the problem

This paper is devoted to establishing the continuous dependence of the solution on the coefficient μ. To this end, let u denote the solution of the problem described by Eq (1.1) with the parameter $\mu = \mu_0$, subject to the initial and boundary conditions (1.2)–(1.7). Similarly, let v be the solution corresponding to the parameter $\mu = \tilde{\mu_0}$, under the same set of initial and boundary conditions. We define the difference in the parameters and the corresponding difference in the solutions as:

$$w = u - v, \mu = \mu_0 - \tilde{\mu_0}. \tag{2.1}$$

The function w then satisfies the following evolution equation, derived by subtracting the respective equations for u and v:

$$c\ddot{w} = \tilde{\mu_0}\Delta w + \mu\Delta u + \mu_1\Delta\dot{w} - d_0\Delta^2 w - d_1\Delta^2\dot{w}. \tag{2.2}$$

The initial and boundary conditions for the difference function w are homogeneous:

$$w(x_1, 0, t) = 0 \quad x_1 > 0, t > 0, \tag{2.3}$$

$$w(x_1, h, t) = 0 \quad x_1 > 0, t > 0, \tag{2.4}$$

$$w(0, x_2, t) = 0 \quad 0 \le x_2 \le h, t > 0, \tag{2.5}$$

$$w_{,1}(0, x_2, t) = 0 \quad 0 \le x_2 \le h, t > 0, \tag{2.6}$$

and

$$w(x_1, x_2, 0) = 0 \quad 0 \le x_2 \le h, x_1 > 0. \tag{2.7}$$

Furthermore, we impose the standard asymptotic condition that w and its relevant spatial derivatives decay to zero at infinity:

$$w(x_1, x_2, t), w_{,\alpha}(x_1, x_2, t), w_{,\alpha\beta}(x_1, x_2, t) \to 0 \quad as \quad z \to \infty. \tag{2.8}$$

The primary objective of the subsequent analysis is to rigorously demonstrate how the solution depends continuously on the coefficient μ, specifically by showing that the energy of the difference w can be controlled by the difference in the parameter μ.

3. Structural stability result

To establish the desired continuous dependence result of the solution, we construct an energy functional $F(z,t)$ that captures the spatial and temporal evolution of the system's energy.

Proposition 3.1: Suppose that w constitutes the classical solution to the initial-boundary value problems described by Eqs (2.2)–(2.8). Subsequently, we define a function

$$\begin{aligned} F(z,t) = &\frac{C}{2}\int_z^\infty \int_{L_\xi} \dot{w}^2 dA|_{\eta=t} + \frac{\tilde{\mu_0}}{2}\int_z^\infty \int_{L_\xi} w_{,\alpha} w_{,\alpha} dA|_{\eta=t} \\ &+ \mu_1 \int_0^t \int_z^\infty \int_{L_\xi} \dot{w}_{,\alpha}\dot{w}_{,\alpha} \mathrm{d}A\mathrm{d}\eta + \frac{d_0}{2}\int_z^\infty \int_{L_\xi} w_{,\alpha\beta} w_{,\alpha\beta} dA|_{\eta=t} \\ &+ d_1 \int_0^t \int_z^\infty \int_{L_\xi} \dot{w}_{,\alpha\beta}\dot{w}_{,\alpha\beta} \mathrm{d}A\mathrm{d}\eta. \end{aligned} \tag{3.1}$$

$F(z,t)$ can also be formulated as (or expressed in the form of)

$$\begin{aligned} F(z,t) = &d_1 \frac{\partial}{\partial z}\int_0^t \int_{L_z} \dot{w}\dot{w}_{,\beta\beta}\, \mathrm{d}x_2 \mathrm{d}\eta + d_0 \frac{\partial}{\partial z}\int_0^t \int_{L_z} \dot{w} w_{,\beta\beta}\, \mathrm{d}x_2 \mathrm{d}\eta \\ &- d_1 \int_0^t \int_{L_z} \dot{w}_{,1}\dot{w}_{,\beta\beta}\, \mathrm{d}x_2 \mathrm{d}\eta - d_0 \int_0^t \int_{L_z} \dot{w}_{,1} w_{,\beta\beta}\, \mathrm{d}x_2 \mathrm{d}\eta \\ &- \mu_0 \int_0^t \int_{L_z} \dot{w} w_{,1}\, \mathrm{d}x_2 \mathrm{d}\eta - \mu_1 \int_0^t \int_{L_z} \dot{w}\dot{w}_{,1}\, \mathrm{d}x_2 \mathrm{d}\eta \\ &- d_0 \int_0^t \int_{L_z} \dot{w}_{,\alpha} w_{,1\alpha}\, \mathrm{d}x_2 \mathrm{d}\eta - d_1 \int_0^t \int_{L_z} \dot{w}_{,\alpha}\dot{w}_{,1\alpha}\, \mathrm{d}x_2 \mathrm{d}\eta \\ &- \mu \int_0^t \int_{L_z} \dot{w} u_{,1}\, \mathrm{d}x_2 \mathrm{d}\eta - \mu \int_0^t \int_z^\infty \int_{L_\xi} \dot{w}_{,\alpha} u_{,\alpha}\, \mathrm{d}A\mathrm{d}\eta. \end{aligned} \tag{3.2}$$

Proof: Multiplying $(2.2)_1$ by $\dot{w}$ and integrating, we can get

$$
\begin{aligned}
0 &= \int_0^t \int_z^\infty \int_{L_\xi} \dot{w}(c\ddot{w} - \tilde{\mu}_0 w_{,\alpha\alpha} - \mu u_{,\alpha\alpha} - \mu_1 \dot{w}_{,\alpha\alpha} + d_0 w_{,\alpha\alpha\beta\beta} + d_1 \dot{w}_{,\alpha\alpha\beta\beta})\mathrm{d}A\mathrm{d}\eta \\
&= \frac{C}{2} \int_z^\infty \int_{L_\xi} \dot{w}^2 dA|_{\eta=t} + \tilde{\mu}_0 \int_0^t \int_z^\infty \int_{L_\xi} \dot{w}_{,\alpha} w_{,\alpha} \mathrm{d}A\mathrm{d}\eta + \tilde{\mu}_0 \int_0^t \int_{L_z} \dot{w} w_{,1} \mathrm{d}x_2 \mathrm{d}\eta \\
&+ \mu_1 \int_0^t \int_z^\infty \int_{L_\xi} \dot{w}_{,\alpha} \dot{w}_{,\alpha} \mathrm{d}A\mathrm{d}\eta + \mu_1 \int_0^t \int_{L_z} \dot{w}\dot{w}_{,1} \mathrm{d}x_2 \mathrm{d}\eta \\
&- d_0 \int_0^t \int_z^\infty \int_{L_\xi} \dot{w}_{,\alpha} \dot{w}_{,\alpha\beta\beta} \mathrm{d}A\mathrm{d}\eta - d_0 \int_0^t \int_{L_z} \dot{w} w_{,1\beta\beta} \mathrm{d}x_2 \mathrm{d}\eta \\
&- d_1 \int_0^t \int_z^\infty \int_{L_\xi} \dot{w}_{,\alpha} \dot{w}_{,\alpha\beta\beta} \mathrm{d}A\mathrm{d}\eta - d_1 \int_0^t \int_{L_z} \dot{w}\dot{w}_{,1\beta\beta} \mathrm{d}x_2 \mathrm{d}\eta \\
&+ \mu \int_0^t \int_z^\infty \int_{L_\xi} \dot{w}_{,\alpha} u_{,\alpha} \mathrm{d}A\mathrm{d}\eta + \mu \int_0^t \int_{L_z} \dot{w} u_{,1} \mathrm{d}x_2 \mathrm{d}\eta \\
&= \frac{C}{2} \int_z^\infty \int_{L_\xi} \dot{w}^2 dA|_{\eta=t} + \frac{\tilde{\mu}_0}{2} \int_z^\infty \int_{L_\xi} w_{,\alpha} w_{,\alpha} dA|_{\eta=t} + \tilde{\mu}_0 \int_0^t \int_{L_z} \dot{w} w_{,1} \mathrm{d}x_2 \mathrm{d}\eta \\
&+ \mu_1 \int_0^t \int_z^\infty \int_{L_\xi} \dot{w}_{\alpha} \dot{w}_{,\alpha} \mathrm{d}A\mathrm{d}\eta + \mu_1 \int_0^t \int_{L_z} \dot{w}\dot{w}_{,1} \mathrm{d}x_2 \mathrm{d}\eta \\
&+ d_0 \int_0^t \int_z^\infty \int_{L_\xi} \dot{w}_{,\alpha\beta} w_{,\alpha\beta} \mathrm{d}A\mathrm{d}\eta + d_0 \int_0^t \int_{L_z} \dot{w}_{,\alpha} w_{,1\alpha} \mathrm{d}x_2 \mathrm{d}\eta \\
&- d_0 \int_0^t \int_{L_z} \dot{w} w_{,1\beta\beta} \mathrm{d}x_2 \mathrm{d}\eta + d_1 \int_0^t \int_z^\infty \int_{L_\xi} \dot{w}_{,\alpha\beta} \dot{w}_{,\alpha\beta} \mathrm{d}A\mathrm{d}\eta \\
&+ d_1 \int_0^t \int_{L_z} \dot{w}_{,\alpha} \dot{w}_{,\alpha 1} \mathrm{d}x_2 \mathrm{d}\eta - d_1 \int_0^t \int_{L_z} \dot{w}\dot{w}_{,1\beta\beta} \mathrm{d}x_2 \mathrm{d}\eta \\
&+ \mu \int_0^t \int_z^\infty \int_{L_\xi} \dot{w}_{,\alpha} u_{,\alpha} \mathrm{d}A\mathrm{d}\eta + \mu \int_0^t \int_{L_z} \dot{w} u_{,1} \mathrm{d}x_2 \mathrm{d}\eta.
\end{aligned}
\tag{3.3}
$$

From (3.3), we obtain

$$
\begin{aligned}
&\frac{C}{2}\int_z^\infty\int_{L_\xi}\dot{w}^2dA|_{\eta=t}+\frac{\tilde{\mu_0}}{2}\int_z^\infty\int_{L_\xi}w_{,\alpha}w_{,\alpha}dA|_{\eta=t}\\
&+\mu_1\int_0^t\int_z^\infty\int_{L_\xi}\dot{w}_{,\alpha}\dot{w}_{,\alpha}\mathrm{d}A\mathrm{d}\eta+\frac{d_0}{2}\int_z^\infty\int_{L_\xi}w_{,\alpha\beta}w_{,\alpha\beta}dA|_{\eta=t}\\
&+d_1\int_0^t\int_z^\infty\int_{L_\xi}\dot{w}_{,\alpha\beta}\dot{w}_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta\\
&=-\tilde{\mu_0}\int_0^t\int_{L_z}\dot{w}w_{,1}\mathrm{d}x_2\mathrm{d}\eta-\mu_1\int_0^t\int_{L_z}\dot{w}\dot{w}_{,1}\mathrm{d}x_2\mathrm{d}\eta\\
&-d_0\int_0^t\int_{L_z}\dot{w}_{,\alpha}w_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta+d_0\int_0^t\int_{L_z}\dot{w}\dot{w}_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta\\
&-d_1\int_0^t\int_{L_z}\dot{w}_{,\alpha}\dot{w}_{,\alpha1}\mathrm{d}x_2\mathrm{d}\eta+d_1\int_0^t\int_{L_z}\dot{w}\dot{w}_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta\\
&-\mu\int_0^t\int_z^\infty\int_{L_\xi}\dot{w}_{,\alpha}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta-\mu\int_0^t\int_{L_z}\dot{w}u_{,1}\mathrm{d}x_2\mathrm{d}\eta\\
&=d_1\frac{\partial}{\partial z}\int_0^t\int_{L_z}\dot{w}\dot{w}_{,\beta\beta}\,\mathrm{d}x_2\mathrm{d}\eta+d_0\frac{\partial}{\partial z}\int_0^t\int_{L_z}\dot{w}w_{,\beta\beta}\,\mathrm{d}x_2\mathrm{d}\eta\\
&-d_1\int_0^t\int_{L_z}\dot{w}_{,1}\dot{w}_{,\beta\beta}\,\mathrm{d}x_2\mathrm{d}\eta-d_0\int_0^t\int_{L_z}\dot{w}_{,1}w_{,\beta\beta}\,\mathrm{d}x_2\mathrm{d}\eta\\
&-\mu_0\int_0^t\int_{L_z}\dot{w}w_{,1}\,\mathrm{d}x_2\mathrm{d}\eta-\mu_1\int_0^t\int_{L_z}\dot{w}\dot{w}_{,1}\,\mathrm{d}x_2\mathrm{d}\eta\\
&-d_0\int_0^t\int_{L_z}\dot{w}_{,\alpha}w_{,1\alpha}\,\mathrm{d}x_2\mathrm{d}\eta-d_1\int_0^t\int_{L_z}\dot{w}_{,\alpha}\dot{w}_{,1\alpha}\,\mathrm{d}x_2\mathrm{d}\eta\\
&-\mu\int_0^t\int_{L_z}\dot{w}u_{,1}\,\mathrm{d}x_2\mathrm{d}\eta-\mu\int_0^t\int_z^\infty\int_{L_\xi}\dot{w}_{,\alpha}u_{,\alpha}\,\mathrm{d}A\mathrm{d}\eta.
\end{aligned}\tag{3.4}
$$

We define

$$
\begin{aligned}
F(z,t)=&\frac{C}{2}\int_z^\infty\int_{L_\xi}\dot{w}^2dA|_{\eta=t}+\frac{\tilde{\mu_0}}{2}\int_z^\infty\int_{L_\xi}w_{,\alpha}w_{,\alpha}dA|_{\eta=t}\\
&+\mu_1\int_0^t\int_z^\infty\int_{L_\xi}\dot{w}_{,\alpha}\dot{w}_{,\alpha}\mathrm{d}A\mathrm{d}\eta+\frac{d_0}{2}\int_z^\infty\int_{L_\xi}w_{,\alpha\beta}w_{,\alpha\beta}dA|_{\eta=t}\\
&+d_1\int_0^t\int_z^\infty\int_{L_\xi}\dot{w}_{,\alpha\beta}\dot{w}_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta.
\end{aligned}\tag{3.5}
$$

Inserting (3.5) into (3.4), we can also get

$$
\begin{aligned}
F(z,t) = & -\tilde{\mu_0}\int_0^t\int_{L_z}\dot{w}w_{,1}\mathrm{d}x_2\mathrm{d}\eta-\mu_1\int_0^t\int_{L_z}\dot{w}\dot{w}_{,1}\mathrm{d}x_2\mathrm{d}\eta \\
& -d_0\int_0^t\int_{L_z}\dot{w}_{,\alpha}w_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta+d_0\int_0^t\int_{L_z}\dot{w}\dot{w}_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta \\
& -d_1\int_0^t\int_{L_z}\dot{w}_{,\alpha}\dot{w}_{,\alpha 1}\mathrm{d}x_2\mathrm{d}\eta+d_1\int_0^t\int_{L_z}\dot{w}\dot{w}_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta \\
& -\mu\int_0^t\int_z^\infty\int_{L_\xi}\dot{w}_{,\alpha}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta-\mu\int_0^t\int_{L_z}\dot{w}u_{,1}\mathrm{d}x_2\mathrm{d}\eta \\
= & \, d_1\frac{\partial}{\partial z}\int_0^t\int_{L_z}\dot{w}\dot{w}_{,\beta\beta}\,\mathrm{d}x_2\mathrm{d}\eta+d_0\frac{\partial}{\partial z}\int_0^t\int_{L_z}\dot{w}w_{,\beta\beta}\,\mathrm{d}x_2\mathrm{d}\eta \\
& -d_1\int_0^t\int_{L_z}\dot{w}_{,1}\dot{w}_{,\beta\beta}\,\mathrm{d}x_2\mathrm{d}\eta-d_0\int_0^t\int_{L_z}\dot{w}_{,1}w_{,\beta\beta}\,\mathrm{d}x_2\mathrm{d}\eta \\
& -\mu_0\int_0^t\int_{L_z}\dot{w}w_{,1}\,\mathrm{d}x_2\mathrm{d}\eta-\mu_1\int_0^t\int_{L_z}\dot{w}\dot{w}_{,1}\,\mathrm{d}x_2\mathrm{d}\eta \\
& -d_0\int_0^t\int_{L_z}\dot{w}_{,\alpha}w_{,1\alpha}\,\mathrm{d}x_2\mathrm{d}\eta-d_1\int_0^t\int_{L_z}\dot{w}_{,\alpha}\dot{w}_{,1\alpha}\,\mathrm{d}x_2\mathrm{d}\eta \\
& -\mu\int_0^t\int_{L_z}\dot{w}u_{,1}\,\mathrm{d}x_2\mathrm{d}\eta-\mu\int_0^t\int_z^\infty\int_{L_\xi}\dot{w}_{,\alpha}u_{,\alpha}\,\mathrm{d}A\mathrm{d}\eta.
\end{aligned}
\tag{3.6}
$$

Proposition 3.2: For the function $F(z,t)$ defined in (3.1) and (3.2), we can obtain the following inequality:

$$
\int_z^\infty F(\xi,t)\mathrm{d}\xi \leq k_1(t)\left(-\frac{\partial F(z,t)}{\partial z}\right)+k_2(t)F(z,t)+\mu^2\tilde{k_3}(t)e^{-k_4 z},
\tag{3.7}
$$

where $k_1(t)=\frac{\frac{h^2}{\pi^2}(d_1+d_0)}{\mu_1}+\frac{d_1+\frac{4d_0t^2}{\pi^2}}{d_1}$ and $k_2(t)=\frac{2d_1+2d_0+\frac{h^2}{\pi^2}\mu_0+\frac{4t^2}{\pi^2}\mu_0+\frac{h^2}{\pi^2}\mu_1+\mu_1+\frac{h^2}{\pi^2}}{\mu_1}+d_1+\frac{4d_0t^2}{\pi^2}$.

Proof: An integration of (3.2) with respect to z gives

$$
\begin{aligned}
\int_z^\infty F(\xi,t)d\xi = & \, d_1\int_0^t\int_{L_z}\dot{w}\dot{w}_{,\beta\beta}\mathrm{d}x_2\mathrm{d}\eta+d_0\int_0^t\int_{L_z}\dot{w}w_{,\beta\beta}\mathrm{d}x_2\mathrm{d}\eta \\
& -d_1\int_0^t\int_z^\infty\int_{L_\xi}\dot{w}_{,1}\dot{w}_{,\beta\beta}\mathrm{d}A\mathrm{d}\eta-d_0\int_0^t\int_z^\infty\int_{L_\xi}\dot{w}_{,1}w_{,\beta\beta}\mathrm{d}A\mathrm{d}\eta \\
& -\mu_0\int_0^t\int_z^\infty\int_{L_\xi}\dot{w}w_{,1}\mathrm{d}A\mathrm{d}\eta-\mu_1\int_0^t\int_z^\infty\int_{L_\xi}\dot{w}\dot{w}_{,1}\mathrm{d}A\mathrm{d}\eta \\
& -d_0\int_0^t\int_z^\infty\int_{L_\xi}\dot{w}_{,\alpha}w_{,1\alpha}\mathrm{d}A\mathrm{d}\eta-d_1\int_0^t\int_z^\infty\int_{L_\xi}\dot{w}_{,\alpha}\dot{w}_{,1\alpha}\mathrm{d}A\mathrm{d}\eta \\
& -\mu\int_0^t\int_z^\infty\int_{L_\xi}\dot{w}u_{,1}\mathrm{d}A\mathrm{d}\eta-\mu\int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)\dot{w}_{,\alpha}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta.
\end{aligned}
\tag{3.8}
$$

Integrating (3.1) with respect to z, we can also get

$$\begin{aligned}\int_z^\infty F(\xi,t)d\xi = \frac{1}{2}\int_z^\infty\int_{L_z}(\xi-z)\dot{w}^2\mathrm{d}A\Big|_{\eta=t} + \frac{\tilde{\mu}_0}{2}\int_z^\infty\int_{L_z}(\xi-z)w_{,\alpha}w_{,\alpha}\mathrm{d}A\Big|_{\eta=t}\\ + \mu_1\int_0^t\int_z^\infty\int_{L_z}(\xi-z)\dot{w}_{,\alpha}\dot{w}_{,\alpha}\mathrm{d}A\mathrm{d}\eta + \frac{d_0}{2}\int_z^\infty\int_{L_z}(\xi-z)w_{,\alpha\beta}w_{,\alpha\beta}\mathrm{d}A\Big|_{\eta=t}\\ + d_1\int_0^t\int_z^\infty\int_{L_z}(\xi-z)\dot{w}_{,\alpha\beta}\dot{w}_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta.\end{aligned} \tag{3.9}$$

We want to give a bound for (3.8). Using Schwarz's inequality, we can obtain

$$\begin{aligned}\left|\mu\int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)\dot{w}_{,\alpha}u_{,\alpha}dAd\eta\right| \le \frac{\mu_1}{2}\int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)\dot{w}_{,\alpha}\dot{w}_{,\alpha}dAd\eta\\ + \frac{\mu^2}{2\mu_1}\int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)u_{,\alpha}u_{,\alpha}dAd\eta.\end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10), we obtain

$$\begin{aligned}\left|\mu\int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)\dot{w}_{,\alpha}u_{,\alpha}dAd\eta\right| \le \frac{1}{2}\int_z^\infty F(\xi,t)d\xi\\ + \frac{\mu^2}{2\mu_1}\int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)u_{,\alpha}u_{,\alpha}dAd\eta.\end{aligned} \tag{3.11}$$

Inserting (3.11) into (3.8) and using (3.9), we obtain

$$\begin{aligned}\int_z^\infty F(\xi,t)d\xi \le 2d_1\int_0^t\int_{L_z}w\dot{w}_{,\beta\beta}\mathrm{d}x_2\mathrm{d}\eta + 2d_0\int_0^t\int_{L_z}\dot{w}w_{,\beta\beta}\mathrm{d}x_2\mathrm{d}\eta\\ - 2d_1\int_0^t\int_z^\infty\int_{L_\xi}\dot{w}_{,1}\dot{w}_{,\beta\beta}\mathrm{d}A\mathrm{d}\eta - 2d_0\int_0^t\int_z^\infty\int_{L_\xi}\dot{w}_{,1}w_{,\beta\beta}\mathrm{d}A\mathrm{d}\eta\\ - 2\mu_0\int_0^t\int_z^\infty\int_{L_\xi}\dot{w}w_{,1}\mathrm{d}A\mathrm{d}\eta - 2\mu_1\int_0^t\int_z^\infty\int_{L_\xi}\dot{w}\dot{w}_{,1}\mathrm{d}A\mathrm{d}\eta\\ - 2d_0\int_0^t\int_z^\infty\int_{L_\xi}\dot{w}_{,\alpha}w_{,1\alpha}\mathrm{d}A\mathrm{d}\eta - 2d_1\int_0^t\int_z^\infty\int_{L_\xi}\dot{w}_{,\alpha}\dot{w}_{,1\alpha}\mathrm{d}A\mathrm{d}\eta\\ - 2\mu\int_0^t\int_z^\infty\int_{L_\xi}\dot{w}u_{,1}\mathrm{d}A\mathrm{d}\eta + \frac{\mu^2}{\mu_1}\int_z^\infty\int_{L_\xi}(\xi-z)u_{,\alpha}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta.\end{aligned} \tag{3.12}$$

Differentiating (3.1) with respect to z, we have

$$\begin{aligned}-\frac{\partial F(z,t)}{\partial z} = \frac{C}{2}\int_{L_z}\dot{w}^2dx_2|_{\eta=t} + \frac{\tilde{\mu}_0}{2}\int_{L_z}w_{,\alpha}w_{,\alpha}dx_2|_{\eta=t}\\ + \mu_1\int_0^t\int_{L_z}\dot{w}_{,\alpha}\dot{w}_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta + \frac{d_0}{2}\int_{L_z}w_{,\alpha\beta}w_{,\alpha\beta}dx_2|_{\eta=t}\\ + d_1\int_0^t\int_{L_z}\dot{w}_{,\alpha\beta}\dot{w}_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta.\end{aligned} \tag{3.13}$$

The following Poincar*é* inequality will be used(see [18]):

$$\int_{L_z} u^2 \mathrm{d}x_2 \le \frac{h^2}{\pi^2} \int_{L_z} u_{,2}^2 \mathrm{d}x_2. \tag{3.14}$$

We also use the following Wirtinger inequality (see (2.7) in [28]),

$$\int_0^t \int_{L_z} u^2 \mathrm{d}x_2 \mathrm{d}\eta \le \frac{4t^2}{\pi^2} \int_0^t \int_{L_z} u_{,2}^2 \mathrm{d}x_2 \mathrm{d}\eta. \tag{3.15}$$

Using Schwarz's inequality, the Poincar*é* inequality (3.14), and Wirtinger's inequality (3.15) in (3.12), we obtain

$$\begin{aligned}
\int_z^\infty F(\xi,t)d\xi \le{}& \frac{h^2}{\pi^2} d_1 \int_0^t \int_{L_z} \dot{w}_{,\alpha}\dot{w}_{,\alpha} \mathrm{d}x_2 \mathrm{d}\eta + d_1 \int_0^t \int_{L_z} \dot{w}_{,\alpha\beta}\dot{w}_{,\alpha\beta} \mathrm{d}x_2 \mathrm{d}\eta \\
&+ \frac{h^2 d_0}{\pi^2} \int_0^t \int_{L_z} \dot{w}_{,\alpha}\dot{w}_{,\alpha} \mathrm{d}x_2 \mathrm{d}\eta + \frac{4t^2 d_0}{\pi^2} \int_0^t \int_{L_z} \dot{w}_{,\alpha\beta}\dot{w}_{,\alpha\beta} \mathrm{d}x_2 \mathrm{d}\eta \\
&+ d_1 \int_0^t \int_z^\infty \int_{L_\xi} \dot{w}_{,\alpha}\dot{w}_{,\alpha} \mathrm{d}A \mathrm{d}\eta + d_1 \int_0^t \int_z^\infty \int_{L_\xi} \dot{w}_{,\alpha\beta}\dot{w}_{,\alpha\beta} \mathrm{d}A \mathrm{d}\eta \\
&+ d_0 \int_0^t \int_z^\infty \int_{L_\xi} \dot{w}_{,\alpha}\dot{w}_{,\alpha} \mathrm{d}A \mathrm{d}\eta + \frac{4t^2 d_0}{\pi^2} \int_0^t \int_z^\infty \int_{L_\xi} \dot{w}_{,\alpha\beta}\dot{w}_{,\alpha\beta} \mathrm{d}A \mathrm{d}\eta \\
&+ \frac{h^2}{\pi^2}\mu_0 \int_0^t \int_z^\infty \int_{L_\xi} \dot{w}_{,\alpha}\dot{w}_{,\alpha} \mathrm{d}A \mathrm{d}\eta + \frac{4t^2 \mu_0}{\pi^2} \int_0^t \int_z^\infty \int_{L_\xi} \dot{w}_{,\alpha}\dot{w}_{,\alpha} \mathrm{d}A \mathrm{d}\eta \\
&+ \frac{h^2}{\pi^2}\mu_1 \int_0^t \int_z^\infty \int_{L_\xi} \dot{w}_{,\alpha}\dot{w}_{,\alpha} \mathrm{d}A \mathrm{d}\eta + \mu_1 \int_0^t \int_z^\infty \int_{L_\xi} \dot{w}_{,\alpha}\dot{w}_{,\alpha} \mathrm{d}A \mathrm{d}\eta \\
&+ d_0 \int_0^t \int_z^\infty \int_{L_\xi} \dot{w}_{,\alpha}\dot{w}_{,\alpha} \mathrm{d}A \mathrm{d}\eta + \frac{4t^2 d_0}{\pi^2} \int_0^t \int_z^\infty \int_{L_\xi} \dot{w}_{,\alpha\beta}\dot{w}_{,\alpha\beta} \mathrm{d}A \mathrm{d}\eta \\
&+ d_1 \int_0^t \int_z^\infty \int_{L_\xi} \dot{w}_{,\alpha}\dot{w}_{,\alpha} \mathrm{d}A \mathrm{d}\eta + d_1 \int_0^t \int_z^\infty \int_{L_\xi} \dot{w}_{,\alpha\beta}\dot{w}_{,\alpha\beta} \mathrm{d}A \mathrm{d}\eta \\
&+ \frac{h^2}{\pi^2} \int_0^t \int_z^\infty \int_{L_\xi} \dot{w}_{,\alpha}\dot{w}_{,\alpha} \mathrm{d}A \mathrm{d}\eta + \frac{\mu^2}{\mu_1} \int_0^t \int_z^\infty \int_{L_\xi} (\xi - z) u_{,\alpha}u_{,\alpha} dA d\eta \\
&+ \mu^2 \int_0^t \int_z^\infty \int_{L_\xi} u_{,1}^2 dA d\eta.
\end{aligned} \tag{3.16}$$

Using (3.13) and (3.8) in (3.16), we obtain

$$\begin{aligned}
\int_z^\infty F(\xi,t)d\xi \le{}& k_1(t)\left(-\frac{\partial F(z,t)}{\partial z}\right) + k_2(t)F(z,t) \\
&+ \frac{\mu^2}{\mu_1} \int_0^t \int_z^\infty \int_{L_\xi} (\xi - z) u_{,\alpha}u_{,\alpha} dA d\eta \\
&+ \mu^2 \int_0^t \int_z^\infty \int_{L_\xi} u_{,1}^2 dA d\eta,
\end{aligned} \tag{3.17}$$

with $k_1(t) = \frac{\frac{h^2}{\pi^2}(d_1+d_0)}{\mu_1} + \frac{d_1+\frac{4d_0t^2}{\pi^2}}{d_1}$ and $k_2(t) = \frac{2d_1+2d_0+\frac{h^2}{\pi^2}\mu_0+\frac{4t^2}{\pi^2}\mu_0+\frac{h^2}{\pi^2}\mu_1+\mu_1+\frac{h^2}{\pi^2}}{\mu_1} + d_1 + \frac{4d_0t^2}{\pi^2}$.

In the following study of structural stability, it is necessary to use the estimates for $\int_0^t\int_z^\infty\int_{L_\xi} u_{,\alpha\beta}u_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta$, the study of which belongs to the domain of spatial decay estimates. The results for spatial decay estimates have already been established in [27]. The equation in this paper is similar to that in [27], with the core terms being identical and only minor differences in a few terms. Following the method from [27], we can similarly obtain the estimates for the bounds of $\int_0^t\int_z^\infty\int_{L_\xi} u_{,\alpha\beta}u_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta$. In this paper, we omit the derivation here. Following the same procedure as deriving the main result (3.17) in [27], we can also get the decay estimates:

$$\int_0^t\int_z^\infty\int_{L_\xi} u_{,\alpha\beta}u_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta \le k_3(t)e^{-k_4 z}, \tag{3.18}$$

where $k_3(t)$ is a positive function and k_4 is a positive constant.

Using the Poincaré inequality, we have

$$\begin{aligned}\int_0^t\int_z^\infty\int_{L_\xi} u_{,\alpha}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta &\le \frac{h^2}{\pi^2}\int_0^t\int_z^\infty\int_{L_\xi} u_{,\alpha\beta}u_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta\\ &\le \frac{h^2}{\pi^2}k_3(t)e^{-k_4 z}.\end{aligned} \tag{3.19}$$

We can also get

$$\begin{aligned}\frac{1}{\mu_1}\int_0^t\int_z^\infty\int_{L_\xi}(\xi-z)u_{,\alpha}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta+\int_0^t\int_z^\infty\int_{L_\xi}u_{,1}^2\mathrm{d}A\mathrm{d}\eta &\le \frac{h^2}{\pi^2}k_3(t)\int_z^\infty e^{-k_4\xi}\mathrm{d}\xi+\frac{h^2}{\pi^2}k_3(t)e^{-k_4 z}\\ &= \frac{h^2k_3(t)}{\mu_1k_4\pi^2}e^{-k_4 z}+\frac{h^2}{\pi^2}k_3(t)e^{-k_4 z}\\ &= \tilde{k_3}(t)e^{-k_4 z},\end{aligned} \tag{3.20}$$

with $\tilde{k_3}(t) = \frac{h^2k_3(t)}{\mu_1k_4\pi^2} + \frac{h^2}{\pi^2}k_3(t)$.

Inserting (3.20) into (3.17), we obtain

$$\int_z^\infty F(\xi,t)\mathrm{d}\xi \le k_1(t)\left(-\frac{\partial F(z,t)}{\partial z}\right) + k_2(t)F(z,t) + \mu^2\tilde{k_3}(t)e^{-k_4 z}. \tag{3.21}$$

Inequality (3.21) is the desired result (3.7).

In the following discussions, we will define two functions

$$H(z,t) = e^{-\frac{k_2(t)}{k_1(t)}z}F(z,t) \tag{3.22}$$

and

$$M(z,t) = H(z,t) + r(t)\int_z^\infty e^{\frac{k_2(t)}{k_1(t)}(\xi-z)}H(\xi,t)d\xi, \tag{3.23}$$

where $r(t)$ is a positive function to be determined later.

Proposition 3.3: For the function $M(z,t)$ defined in (3.23), we can obtain the following estimates:

$$\frac{\partial M(z,t)}{\partial z} + r_0(t)M(z,t) \le \frac{\widetilde{k_3}(t)}{k_1(t)}\mu^2 e^{-k_5(t)z}, \tag{3.24}$$

where $r_0(t) = \frac{\frac{k_2(t)}{k_1(t)} + \sqrt{\left(\frac{k_2(t)}{k_1(t)}\right)^2 + \frac{4}{k_1(t)}}}{2}$ and $k_5(t) = k_4 + \frac{k_2(t)}{k_1(t)}$.

Proof: We can rewrite (3.7) as

$$\frac{\partial F(z,t)}{\partial z} + \frac{1}{k_1(t)}\int_z^{\infty} F(\xi,t)d\xi \le \frac{k_2(t)}{k_1(t)}F(z,t) + \frac{\widetilde{k_3}(t)}{k_1(t)}\mu^2 e^{-k_4 z}. \tag{3.25}$$

Differentiating (3.23) with respect to z, we have

$$\begin{aligned}\frac{\partial M(z,t)}{\partial z} &= \frac{\partial H(z,t)}{\partial z} - r(t)\frac{k_2(t)}{k_1(t)}\int_z^{\infty} e^{-\frac{k_2(t)}{k_1(t)}(\xi-z)}H(\xi,t)d\xi - rH(z,t)\\ &= -\frac{k_2(t)}{k_1(t)}e^{-\frac{k_2(t)}{k_1(t)}z}F(z,t) + e^{-\frac{k_2(t)}{k_1(t)}z}\frac{\partial F(z,t)}{\partial z}\\ &\quad - r(t)\frac{k_2(t)}{k_1(t)}\int_z^{\infty} e^{-\frac{k_2(t)}{k_1(t)}z}F(\xi,t)d\xi - r(t)e^{-\frac{k_2(t)}{k_1(t)}z}F(z,t).\end{aligned} \tag{3.26}$$

We can easily get

$$\begin{aligned}\frac{\partial M(z,t)}{\partial z} + r(t)M(z,t) = &-\frac{k_2(t)}{k_1(t)}e^{-\frac{k_2(t)}{k_1(t)}z}F(z,t) + e^{-\frac{k_2(t)}{k_1(t)}z}\frac{\partial F(z,t)}{\partial z}\\ &- r(t)\frac{k_2(t)}{k_1(t)}\int_z^{\infty} e^{-\frac{k_2(t)}{k_1(t)}z}F(\xi,t)d\xi - r(t)e^{-\frac{k_2(t)}{k_1(t)}z}F(z,t)\\ &+ r(t)e^{-\frac{k_2(t)}{k_1(t)}z}F(z,t) + r^2(t)\int_z^{\infty} e^{-\frac{k_2(t)}{k_1(t)}z}F(\xi,t)d\xi.\end{aligned} \tag{3.27}$$

From (3.7), we have

$$\begin{aligned}-\frac{k_2(t)}{k_1(t)}e^{-\frac{k_2(t)}{k_1(t)}z}F(z,t) + e^{-\frac{k_2(t)}{k_1(t)}z}\frac{\partial F(z,t)}{\partial z} \le &-\frac{1}{k_1(t)}\int_z^{\infty} e^{-\frac{k_2(t)}{k_1(t)}z}F(\xi,t)d\xi\\ &+ \frac{\widetilde{k_3}(t)}{k_1(t)}\mu^2 e^{-k_5(t)z},\end{aligned} \tag{3.28}$$

with $k_5(t) = k_4 + \frac{k_2(t)}{k_1(t)}$.

Inserting (3.28) into (3.27), we have

$$\frac{\partial M(z,t)}{\partial z} + r(t)M(z,t) \le \left(r^2(t) - \frac{k_2(t)}{k_1(t)}r(t) - \frac{1}{k_1(t)}\right)\int_z^{\infty} e^{-\frac{k_2(t)}{k_1(t)}z}F(\xi,t)d\xi + \frac{\widetilde{k_3}(t)}{k_1(t)}\mu^2 e^{-k_5(t)z}. \tag{3.29}$$

In order to delete the term $\int_z^{\infty} e^{-\frac{k_2(t)}{k_1(t)}z}F(\xi,t)d\xi$ in (3.29), we choose suitable $r(t)$ that satisfies $r^2(t) - \frac{k_2(t)}{k_1(t)}r(t) - \frac{1}{k_1(t)} = 0$.

If we choose $r(t) = r_0(t) = \frac{\frac{k_2(t)}{k_1(t)} + \sqrt{\left(\frac{k_2(t)}{k_1(t)}\right)^2 + \frac{4}{k_1(t)}}}{2}$, we have

$$\frac{\partial M(z,t)}{\partial z} + r_0(t)M(z,t) \leq \frac{\widetilde{k_3}(t)}{k_1(t)}\mu^2 e^{-k_5(t)z}. \tag{3.30}$$

Proposition 3.4: For the energy function $F(z,t)$ defined in (3.1), we have the following estimates: For any fixed t_0, if $r_0(t_0) - k_5(t_0) \neq 0$, the energy satisfies the following estimates:

$$\begin{aligned} F(z,t_0) \leq & \left[M(0,t_0) - \frac{1}{r_0(t_0) - k_5(t_0)}\frac{\widetilde{k_3}(t_0)}{k_1(t_0)}\mu^2\right] e^{-\left(r_0(t_0) - \frac{k_2(t_0)}{k_1(t_0)}\right)z} \\ & + \frac{1}{r_0(t_0) - k_5(t_0)}\frac{\widetilde{k_3}(t_0)}{k_1(t_0)}\mu^2 e^{-k_4 z}. \end{aligned} \tag{3.31}$$

If there exist a t_0 such that $r_0(t_0) - k_5(t_0) = 0$, the energy satisfies the following estimates:

$$F(z,t_0) \leq M(0,t_0)e^{-\left(r_0(t_0) - \frac{k_2(t_0)}{k_1(t_0)}\right)z} + \frac{\widetilde{k_3}(t_0)}{k_1(t_0)}\mu^2 e^{-\left(r_0(t_0) - \frac{k_2(t_0)}{k_1(t_0)}\right)z}. \tag{3.32}$$

Proof: For any fixed t_0, if $r_0(t_0) - k_5(t_0) \neq 0$, an integration of (3.24) gives

$$\begin{aligned} M(z,t_0) \leq & \left[M(0,t_0) - \frac{1}{r_0(t_0) - k_5(t_0)}\frac{\widetilde{k_3}(t_0)}{k_1(t_0)}\mu^2\right] e^{-r_0(t_0)z} \\ & + \frac{1}{r_0(t_0) - k_5(t_0)}\frac{\widetilde{k_3}(t_0)}{k_1(t_0)}\mu^2 e^{-k_5(t_0)z}. \end{aligned} \tag{3.33}$$

From (3.22) and (3.23), we have

$$\begin{aligned} F(z,t_0) \leq & \left[M(0,t_0) - \frac{1}{r_0(t_0) - k_5(t_0)}\frac{\widetilde{k_3}(t_0)}{k_1(t_0)}\mu^2\right] e^{-\left(r_0(t_0) - \frac{k_2(t_0)}{k_1(t_0)}\right)z} \\ & + \frac{1}{r_0(t_0) - k_5(t_0)}\frac{\widetilde{k_3}(t_0)}{k_1(t_0)}\mu^2 e^{-k_4 z}. \end{aligned} \tag{3.34}$$

If there exist a t_0 such that $r_0(t_0) - k_5(t_0) = 0$, an integration of (3.24) gives

$$M(z,t_0) \leq M(0,t_0)e^{-r_0(t_0)z} + \frac{\widetilde{k_3}(t_0)}{k_1(t_0)}\mu^2 e^{-r_0(t_0)z}. \tag{3.35}$$

From (3.22) and (3.23), we obtain

$$F(z,t_0) \leq M(0,t_0)e^{-\left(r_0(t_0) - \frac{k_2(t_0)}{k_1(t_0)}\right)z} + \frac{\widetilde{k_3}(t_0)}{k_1(t_0)}\mu^2 e^{-\left(r_0(t_0) - \frac{k_2(t_0)}{k_1(t_0)}\right)z}. \tag{3.36}$$

Proposition 3.5: For the function $M(0,t_0)$, we have the following estimate:

$$M(0,t_0) \leq \mu^2 k_6(t_0), \tag{3.37}$$

with $k_6(t_0)=\left(\frac{h^2}{\pi^2}k_3(t_0)+\frac{h^2}{\mu_1\pi^2}r_0(t_0)k_2(t_0)k_3(t_0)+\tilde{k_3}(t_0)\right)$.

Proof: From the definition of $M(z,t)$ defined in (3.23), we can easily get

$$\begin{aligned} M(0,t_0) &= H(0,t_0)+r_0(t_0)\int_0^\infty e^{\frac{k_2(t_0)}{k_1(t_0)}\xi}H(\xi,t)d\xi \\ &= F(0,t_0)+r_0(t_0)\int_0^\infty F(\xi,t_0)d\xi. \end{aligned} \tag{3.38}$$

From the definition of $F(z,t)$ in (3.1) and (3.2), we have

$$\begin{aligned} F(0,t_0) &= \frac{C}{2}\int_0^\infty\int_{L_\xi}\dot{w}^2dA\Big|_{\eta=t_0}+\frac{\widetilde{\mu_0}}{2}\int_0^\infty\int_{L_\xi}w_{,\alpha}w_{,\alpha}dA\Big|_{\eta=t_0} \\ &+\mu_1\int_0^{t_0}\int_0^\infty\int_{L_\xi}\dot{w}_{,\alpha}\dot{w}_{,\alpha}dAd\eta+\frac{d_0}{2}\int_0^\infty\int_{L_\xi}w_{,\alpha\beta}w_{,\alpha\beta}dA\Big|_{\eta=t_0} \\ &+d_1\int_0^{t_0}\int_0^\infty\int_{L_\xi}\dot{w}_{,\alpha\beta}\dot{w}_{,\alpha\beta}dAd\eta \\ &= -\mu\int_0^{t_0}\int_0^\infty\int_{L_\xi}\dot{w}_{,\alpha}u_{,\alpha}dAd\eta. \end{aligned} \tag{3.39}$$

Since we know

$$\begin{aligned} \left|\mu\int_0^{t_0}\int_0^\infty\int_{L_\xi}\dot{w}_{,\alpha}u_{,\alpha}dAd\eta\right| &\le \frac{\mu_1}{2}\int_0^{t_0}\int_0^\infty\int_{L_\xi}\dot{w}_{,\alpha}\dot{w}_{,\alpha}dAd\eta \\ &+\frac{\mu^2}{2\mu_1}\int_0^{t_0}\int_0^\infty\int_{L_\xi}u_{,\alpha}u_{,\alpha}dAd\eta, \end{aligned} \tag{3.40}$$

we thus obtain

$$F(0,t_0)\le\frac{\mu^2}{\mu_1}\int_0^{t_0}\int_0^\infty\int_{L_\xi}u_{,\alpha}u_{,\alpha}dAd\eta. \tag{3.41}$$

Using (3.19), we obtain

$$F(0,t_0)\le\frac{\mu^2}{\mu_1}\frac{h^2}{\pi^2}k_3(t_0). \tag{3.42}$$

From (3.7), we have

$$\int_0^\infty F(\xi,t_0)\mathrm{d}\xi\le k_1(t_0)\left(-\frac{\partial F(0,t_0)}{\partial z}\right)+k_2(t_0)F(0,t_0)+\mu^2\tilde{k_3}(t_0). \tag{3.43}$$

From (3.13), we have

$$-\frac{\partial F(0,t_0)}{\partial z}=0, \tag{3.44}$$

we thus obtain

$$\int_0^\infty F(\xi,t_0)\mathrm{d}\xi\le k_2(t_0)F(0,t_0)+\mu^2\tilde{k_3}(t_0). \tag{3.45}$$

Inserting (3.42) and (3.45) into (3.38), we obtain

$$M(0,t_0) \leq \mu^2 \left(\frac{h^2}{\pi^2} k_3(t_0) + \frac{h^2}{\mu_1 \pi^2} r_0(t_0) k_2(t_0) k_3(t_0) + \tilde{k_3}(t_0) \right). \tag{3.46}$$

Combining the results of Propositions 2.4 and 2.5, we can obtain the main results of this paper:

Theorem 3.1: For the energy function $F(z,t)$ defined in (3.1), we establish the following important estimates: Given any fixed t_0, provided that $r_0(t_0) - k_5(t_0) \neq 0$, we obtain

$$F(z,t_0) \leq \mu^2 \left(k_7(t_0) e^{-\left(r_0(t_0) - \frac{k_2(t_0)}{k_1(t_0)}\right)z} + k_8(t_0) e^{-k_4 z} \right). \tag{3.47}$$

In the critical case where there exists a t_0 such that $r_0(t_0) - k_5(t_0) = 0$, we notably obtain

$$F(z,t_0) \leq \mu^2 \left(k_6(t_0) + \frac{\widetilde{k_3}(t_0)}{k_1(t_0)} \right) e^{-\left(r_0(t_0) - \frac{k_2(t_0)}{k_1(t_0)}\right)z}, \tag{3.48}$$

with $k_7(t_0) = k_6(t_0) - \frac{1}{r_0(t_0)-k_5(t_0)} \frac{\widetilde{k_3}(t_0)}{k_1(t_0)}$, $k_8(t_0) = \frac{1}{r_0(t_0)-k_5(t_0)} \frac{\widetilde{k_3}(t_0)}{k_1(t_0)}$.

Inequalities (3.47) and (3.48) not only demonstrate exponential decay with respect to the variable z, but also indicate that the amplitude-related terms in these inequalities tend to zero as the parameter μ approaches zero.

4. Conclusions

In this paper, we have investigated the structural stability and continuous dependence of solutions for a higher-order heat conduction model within the framework of the second gradient theory of type III. By introducing a carefully constructed energy functional and employing integral-differential inequality techniques, we have rigorously derived estimates that demonstrate how the solution depends continuously on the elastic coefficient μ. Our main results, encapsulated in inequalities (3.44) and (3.45), show that the energy functional $F(z,t)$ not only decays exponentially with respect to the spatial variable z, but also diminishes as the parameter μ tends to zero. This confirms the robustness of the model under perturbations in material parameters and validates the reliability of the mathematical formulation in unbounded strip-like domains.

The methodology developed in this study, particularly the use of integral-differential inequalities and energy estimates, can be extended to analyze the continuous dependence of solutions on other model coefficients. Moreover, this work lays a foundation for future investigations into the convergence of solutions, which remains a challenging direction due to the loss of certain terms in the energy structure and the complexity of the associated inequalities.

In summary, this research contributes to the broader understanding of structural stability in unbounded domains and reinforces the applicability of second-gradient type III models in capturing non-classical thermal behaviors. Future work will focus on extending these results to study the convergence properties of solutions under varying physical parameters. This paper primarily focuses on the theoretical investigation of structural stability, and the results establish a theoretical foundation for subsequent numerical simulations. In later stages, we will endeavor to represent the findings of this study through numerical simulations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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