Electronic Research Archive

https://www.aimspress.com/journal/era

ERA, 34(1): 209–231.
DOI: 10.3934/era.2026011
Received: 23 September 2025
Revised: 17 December 2025
Accepted: 25 December 2025
Published: 07 January 2026

Research article

Existence and multiplicity of sign-changing solutions for a discrete three-point boundary value problem

Zhi Zhang*

Guangdong Construction Polytechnic, School of Economics and Trade, Guangzhou 510440, China

* **Correspondence:** Email: zhangzhi01@gdcvi.edu.cn.

Abstract: Employing variational methods in conjunction with the technique of invariant sets of descending flow, we study the existence and multiplicity of solutions for a second-order difference equation subject to a three-point boundary condition. By imposing suitable growth and sign conditions on the nonlinearity, we establish sufficient criteria for the existence of at least three nontrivial solutions. These solutions are characterized by their nodal properties: one positive, one negative, and one sign-changing solution, in addition to the trivial solution. Our results generalize and extend previous work on discrete boundary value problems (BVPs), notably encompassing and broadening known results for Robin boundary conditions. Furthermore, two demonstrative examples are provided not only to validate the theoretical results but also to illustrate the applicability of the problem in modeling phenomenological processes.

Keywords: discrete boundary value problem; sign-changing solution; variational method; invariant sets of descending flow; multiple solutions; three-point boundary condition

1. Introduction

Given an integer $N \geq 2$, let the discrete segment $\{1, 2, \cdots, N\}$ be denoted by $[1, N]$. We are interested in exploring multiple nontrivial solutions of the following discrete three-point boundary value problem (BVP)

$$\begin{cases} \Delta^2 x(n-1) + f(n, x(n)) = 0, & n \in [1, N], \\ x(0) = 0, \quad x(N+1) = \alpha x(\beta), \end{cases} \tag{1.1}$$

where $\alpha \in \mathbf{R}$, $1 \leq \beta \leq N$ is an integer and the nonlinearity $f(n, x) : \mathbf{Z} \times \mathbf{R} \to \mathbf{R}$ is continuous with respect to x. $\Delta x(n) = x(n+1) - x(n)$ represents the forward difference operator, and $\Delta^2 x(n-1) = x(n+1) - 2x(n) + x(n-1)$.

With the rapid advancement of modern digital computing, simulations have become an effective way to reveal the behavior of complex systems, thereby stimulating growing interest in discrete problems. Consequently, a substantial body of research has been devoted to difference equations, as evidenced by [1–3]. Discrete BVPs, which arise from practical applications, have a long-standing research history dating back to at least 1968 [4]. In addition to their theoretical significance, these problems have attracted considerable attention due to their applications in diverse fields such as computer science, mathematical biology, control systems, and economics. For a comprehensive overview, we refer the reader to [1] or [5] and the references therein.

Discrete BVPs have been extensively studied via various methods, including topological approaches, upper and lower solutions, fixed-point theory, and critical point theory, to establish the existence, multiplicity, and uniqueness of solutions; see [6–12] and numerous subsequent works. In particular, the study of three-point BVPs has seen notable progress. For instance, using a cone-theoretic fixed-point theorem, the authors [13] investigated positive solutions for the following three-point BVP:

$$\begin{cases} \Delta^2 u(n) + a(n)f(u(n)) = 0, & n \in [1, T], \\ u(0) = 0, \quad u(T+1) = \beta u(\xi). \end{cases} \tag{1.2}$$

Despite these advances, certain aspects of discrete three-point BVPs still need to be addressed. Most existing results not only rely on fixed-point theory, topological methods, or other analytical techniques [6, 8, 14], but also primarily focus on positive solutions. Moreover, the nonlinear term $f(n, x)$ often captures complex interactions, and the boundary condition $x(N+1) = \alpha x(\beta)$ introduces nonlocality, which can model feedback or control mechanisms. Thus, (1.1) provides a versatile framework for discrete systems with multi-point constraints. For example, let $f(n, x(n)) = V(n)x(n) + \gamma|x(n)|^2 x(n)$. Then (1.1) becomes

$$\begin{cases} \Delta^2 x(n-1) + V(n)x(n) + \gamma|x(n)|^2 x(n) = 0, & n \in [1, N], \\ x(0) = 0, \quad x(N+1) = \alpha x(\beta). \end{cases} \tag{1.3}$$

Here, the term $\gamma|x(n)|^2 x(n)$ represents nonlinear interactions (e.g., in Bose-Einstein condensates). This discrete Schrödinger equation models wave functions $x(n)$ in a quantum system, with the boundary condition simulating a nonlocal constraint or measurement effect. If $f(n, x(n)) = \lambda \tanh(x(n))$, we obtain

$$\Delta^2 x(n-1) + \lambda \tanh(x(n)) = 0, \qquad n \in [1, N].$$

Combined with the boundary condition $x(N+1) = \alpha x(\beta)$, this can describe the activation states $x(n)$ of neurons in a discrete neural network with a sigmoidal activation function $\tanh(x(n))$. The boundary condition models a feedback loop where the output at $N+1$ depends on the state of a neuron at β.

The existence of sign-changing solutions often reflects complex behaviors, which are relevant in real-world applications. For example, sign-changing solutions correspond to state transitions in dynamic systems. In Bose-Einstein condensates (our Example 4.1's physical background), sign changes in the wave function $x(n)$ indicate phase transitions between superfluid and normal states [15]. Therefore, investigating the existence and multiplicity of sign-changing solutions for (1.1) is of significant interest. Moreover, critical point theory has proven to be a powerful tool for studying discrete problems. For example, discrete Laplacian problems were studied in [16], positive solutions were dealt with in [17–19], existence and multiplicity were demonstrated in [20, 21] and homoclinic solutions and periodic solutions were obtained in [22–25] and [26, 27], respectively. In particular, variational methods combined with the technique of invariant sets of descending flow have shown great promise in handling sign-changing solutions [28].

Motivated by these observations, we study the existence of multiple solutions, involving sign-changing, positive, negative, and trivial solutions, for (1.1) by variational methods and invariant sets of descending flow. To the best of our knowledge, this is the first work to address sign-changing solutions for discrete three-point BVPs via this approach. Sign-changing solutions are especially important as they often correspond to complex dynamical behaviors such as bifurcations or multi-stability in applied contexts. As usual, a solution $x = \{x(n)\}_{n=0}^{N+1}$ of (1.1) is positive if $x(n) > 0$ for all $n \in [1, N]$; negative if $x(n) < 0$ for all $n \in [1, N]$; and sign-changing if there exist $i, j \in [1, N]$ such that $x(i) \cdot x(j) < 0$.

Let $\frac{\partial F(n,x)}{\partial x} = f(n, x)$ and define

$$\alpha_1 = \max\{-1, -\frac{N+1}{\beta[N(N+1-\beta)-1]}\}.$$

We impose the following fundamental assumptions:
(A) $\alpha_1 < \alpha \leq 1$, and all eigenvalues λ_i ($1 \leq i \leq N$) of the matrix A (defined in (2.6)) are real numbers.
(F_1) $f(n, x)$ is continuous in x, and

$$\lim_{x \to 0} \frac{f(n, x)}{x} = 0, \qquad n \in [1, N]. \tag{1.4}$$

(F_2) There exist constants $\delta_1 > 0$ and $\mu > \frac{8}{\lambda_1}$ such that

$$0 < \mu F(n, x) \leq x \cdot f(n, x), \qquad |x| \geq \delta_1, \quad n \in [1, N]. \tag{1.5}$$

(F_3) There exist constants $s > 2$ and $C > 0$ such that

$$|f(n, x)| \leq C\left(1 + |x|^{s-1}\right), \quad \forall (n, x) \in [1, N] \times \mathbf{R}. \tag{1.6}$$

(F_4) There exist constants $\rho > 0$ and $\theta > 2$ such that

$$0 < \theta F(n, x) \leq x \cdot f(n, x), \qquad |x| \geq \rho, \quad n \in [1, N]. \tag{1.7}$$

(F_5) $\max_{n \in [1,N]} \limsup_{x \to 0} \frac{f(n,x)}{x} < \lambda_1$.
(F_6) $\min_{n \in [1,N]} \liminf_{x \to +\infty} \frac{f(n,x)}{x} > \lambda_N$.

In the following, we give the remark to explain the physical/mathematical meaning of each assumption.

Remark 1.1. **(A)** *ensures H is well-posed and the boundary condition is non-degenerate.*
(F_1) *models "weak nonlinearity near the origin" (e.g., small displacements in mechanical systems, where linear effects dominate).*
(F_2)/(F_4) *indicates "superlinear growth at infinity" (e.g., strong nonlinear interactions in Bose-Einstein condensates, where $f(n, x) \propto x^3$).*
(F_3) *is a "growth bound" condition to ensure the energy functional $I(x)$ is Fréchet-differentiable.*

(F_5) *means "sublinearity near the origin relative to the smallest eigenvalue" and ensures the functional* $I(x)$ *has a local minimum near the trivial solution.*
(F_6) *implies "superlinearity at infinity relative to the largest eigenvalue" and ensures* $I(x) \to -\infty$ *as* $\|x\| \to \infty$*, which guarantees non-trivial critical points.*

Now we state our main results as follows:

Theorem 1.1. *Under assumptions* (**A**) *and* (F_1)–(F_3)*, the BVP (1.1) admits at least four solutions: one trivial solution, one positive solution, one negative solution, and one sign-changing solution.*

Theorem 1.2. *Suppose that* (**A**)*,* (F_3)*,* (F_5)*, and* (F_6) *are validated. Then (1.1) has at least four solutions: one trivial solution, one positive solution, one negative solution, and one sign-changing solution.*

Corollary 1.3. *Assume that* (**A**)*, and* (F_3)–(F_5) *are fulfilled. The BVP (1.1) possesses at least four solutions: one trivial solution, one positive solution, one negative solution, and one sign-changing solution.*

Proof. According to Theorem 1.2, it suffices to show (F_6) is true. In fact, if (F_4) holds,

$$\frac{f(n,x)}{F(n,x)} \geq \frac{\theta}{x}, \qquad \forall n \in [1,N], \quad |x| \geq \rho. \tag{1.8}$$

Integrating both sides of (1.8), it follows that there exists constant $c > 0$ such that

$$F(n,x) \geq c|x|^{\theta}, \qquad \forall n \in [1,N], \quad |x| \geq \rho,$$

which implies that

$$|f(n,x)| \geq c\theta|x|^{\theta-1}, \qquad \forall n \in [1,N], \quad |x| \geq \rho.$$

Since $\theta > 2$, it yields that

$$\lim_{x\to+\infty} \frac{f(n,x)}{x} = \lim_{x\to+\infty} \frac{|f(n,x)|}{|x|} \geq \lim_{x\to+\infty} \frac{c\theta|x|^{\theta-1}}{|x|} = \lim_{x\to+\infty} c\theta|x|^{\theta-2} = +\infty.$$

Therefore, (F_6) in Theorem 1.2 is valid. The proof is completed. □

Corollary 1.4. *If* (F_5) *and* (F_6) *in Theorem 1.2 are replaced by* (F_1) *and*
(F_7) $\lim\limits_{x\to+\infty} \frac{f(n,x)}{x} = +\infty$.
Then the conclusions in Theorem 1.2 still hold.

Theorem 1.5. *Assume* (**A**)*,* (**F_1**)*,* (**F_3**)*, and either*
(**F_8**) $\lim\limits_{|x|\to+\infty} [xf(n,x) - 2F(n,x)] = +\infty$ *for all* $n \in [1,N]$ *or*
(**F_9**) $\lim\limits_{|x|\to+\infty} [xf(n,x) - 2F(n,x)] = -\infty$ *for all* $n \in [1,N]$.
Then the BVP (1.1) attains at least four solutions: one trivial solution, one positive solution, one negative solution, and one sign-changing solution.

Remark 1.2. *Our theorems directly extend the results of [13] by proving sign-changing solutions, which were not addressed in [13]. Moreover, when* $\alpha = 1$ *and* $\beta = N$*, the BVP (1.1) reduces to a Robin boundary value problem studied in [29]. Thus, our work generalizes [29] to non-Robin three-point constraints, covering more practical scenarios (e.g., nonlocal resource feedback).*

The rest of this paper is organized as follows: Section 2 presents the variational framework and preliminary lemmas. Section 3 contains the detailed proofs of Theorems 1.1, 1.2, and 1.5. Section 4 provides two illustrative examples, and Section 5 concludes the paper.

2. Mathematical framework

Let X be a real Hilbert space and $I \in C^1(X, \mathbf{R})$ a continuously Fréchet-differentiable functional defined on X. To facilitate the proofs of our main results, we recall the following definitions and a key theorem:

Definition 2.1. *[30] $\{x^{(j)}\}_{j\in\mathbf{N}} \subset X$ is called a Palais-Smale (PS) sequence if $I(x^{(j)})$ is bounded and $I'(x^{(j)}) \to 0$ as $j \to \infty$. The functional I is said to satisfy the PS condition if every PS sequence admits a convergent subsequence.*

Definition 2.2. *[31] The functional I satisfies the Cerami condition at level ($(C)_c$ condition) if every sequence $\{x^{(j)}\}_{j\in\mathbf{N}} \subset X$ such that*

$$I'(x^{(j)}) \to c \quad \text{and} \quad (1 + \|x^{(j)}\|)I'(x^{(j)}) \to 0 \quad \text{as} \quad j \to \infty$$

has a convergent subsequence. If this holds for all $c \in \mathbf{R}$, we say I fulfills the Cerami condition.

Theorem 2.1. *[32, Theorem 3.2] Let open convex sets $D_1, D_2 \subset X$ such that $D_1 \bigcap D_2 \neq \emptyset$. Suppose I satisfies the PS condition on X, and $I'(x) = x - S(x)$ for some completely continuous operator $S : X \to X$ such that*

$$S(\partial D_1) \subset D_1 \quad \text{and} \quad S(\partial D_2) \subset D_2.$$

Assume that there is an $h : [0, 1] \to X$ such that

$$h(0) \in D_1 \setminus D_2, \quad h(1) \in D_2 \setminus D_1, \quad \sup_{t\in[0,1]} I(h(t)) < \inf_{x\in\overline{D_1} \bigcap \overline{D_2}} I(x).$$

Then I admits at least four critical points, x_1, x_2, x_3, and x_4 satisfying

$$x_1 \in D_2 \setminus \overline{D_1}, \quad x_2 \in X \setminus (\overline{D_1} \bigcup \overline{D_2}), \quad x_3 \in D_1 \bigcap D_2, \quad \text{and} \quad x_4 \in D_1 \setminus \overline{D_2}.$$

Remark 2.1. *As pointed out by X. Liu and J. Liu [33, Theorem 5.1], the conclusion of Theorem 2.1 remains valid if the PS condition is replaced by the weaker Cerami condition.*

Let $H = \{x : [0, N + 1] \to \mathbf{R} | x(0) = 0, x(N + 1) = \alpha x(\beta)\}$. Define a new inner product

$$\langle x, y\rangle = \sum_{n=1}^{N} \Delta x(n - 1)\Delta y(n - 1) + \sum_{n=1}^{N} x(n)y(n), \qquad x, y \in H,$$

which induces the norm $\|\cdot\|_H$ in the following form:

$$\|x\|_H = \left(\sum_{n=1}^{N} |\Delta x(n - 1)|^2 + \sum_{n=1}^{N} |x(n)|^2 \right)^{\frac{1}{2}}, \quad \forall x \in H.$$

Denote $E = \{x | x^T = (x(1), x(2), \cdots, x(N)) \in \mathbf{R}^N\}$, which is equipped with the usual inner product $(\cdot,\cdot)$ and the usual norm $\|\cdot\|$ as

$$(x, y) = \sum_{n=1}^{N} x(n)y(n) \quad \text{and} \quad \|x\| = \left(\sum_{n=1}^{N} |x(n)|^2\right)^{\frac{1}{2}}, \qquad \forall x, y \in E,$$

respectively. Then $(E, (\cdot,\cdot))$ is an N-dimensional Hilbert space and isomorphic to H. Here and hereafter, we deem $x \in H$ as an extension of $x \in E$. Namely, we imply that $x^T = (x(0), x(1), x(2), \cdots, x(N), x(N+1)) \in H$, and we say $x^T = (x(1), x(2), \cdots, x(N)) \in E$.

Given $1 < r < +\infty$, for any $x \in E$, define another norm

$$\|x\|_r = \left(\sum_{n=1}^{N} |x(n)|^r\right)^{\frac{1}{r}}, \qquad \forall x \in E.$$

Then $\|x\| = \|x\|_2$. Similar to [28], the discrete Hölder inequality gives

$$\|x\|_r = N^{\frac{2-r}{2r}} \|x\|. \tag{2.1}$$

Define the energy function $I(x) : E \to \mathbf{R}$ as

$$I(x) = \frac{1}{2} \sum_{n=1}^{N} |\Delta x(n-1)|^2 - \sum_{n=1}^{N} F(n, x(n)) \tag{2.2}$$

with $x(0) = 0$ and $x(N+1) = \alpha x(\beta)$. As usual, $x \in E$ is a critical point of functional $I(x)$ on E meaning that $(I'(x), y) = 0$ for all $y \in E$. To obtain the solutions of (1.1), we have

Lemma 2.1. *Assume that $z \in E$ is a critical point of the functional I on E. Then $z = \{z(n)\}_{n=0}^{N+1}$ solves the BVP (1.1).*

Proof. It follows from the continuity of f that $I \in C^1(E, \mathbf{R})$. Then direct computation gives that the derivative of I is

$$(I'(x), y) = \sum_{n=1}^{N} [(\Delta x(n-1), \Delta y(n-1)) - (f(n, x(n)), y(n))], \quad \forall x, y \in E. \tag{2.3}$$

Let $z \in E$ be a critical point of I. Then

$$(I'(z), y) = 0, \qquad \forall y \in E. \tag{2.4}$$

Remember integer $1 \le \beta \le N$ and write $[1, N] = [1, \beta - 1] \cup [\beta, N] \triangleq \Omega_1 \cup \Omega_2$. By (2.4), for any $w \in E$ and $i = 1, 2$, we have

$$\begin{aligned}
0 &= \sum_{n \in \Omega_i} [(\Delta z(n-1)\Delta w(n-1) - f(n, z(n)), w(n))] \\
&= \sum_{n \in \Omega_i} [-(\Delta^2 z(n-1), w(n)) - (f(n, z(n)), w(n))] \\
&= -\sum_{n \in \Omega_i} (\Delta^2 z(n-1) - f(n, z(n)), w(n)),
\end{aligned}$$

which ensures that

$$\sum_{n=1}^{N}(\Delta^2 z(n-1) - f(n, z(n)), w(n)) = 0. \tag{2.5}$$

For the arbitrariness of w, (2.5) means that

$$\Delta^2 z(n-1) - f(n, z(n)) = 0, \qquad \forall n \in [1, N].$$

That is, z solves the BVP (1.1). Consequently, the critical point x of $I(x)$ corresponds to the solution $x = \{x(n)\}_{n=0}^{N+1}$ of the BVP (1.1). □

In view of Lemma 2.1, we devote to seek critical points of $I(x)$ to solve the BVP (1.1). To facilitate the estimation of I, we write the matrix $A = (a_{ij})_{N\times N}$ as

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -\alpha & \cdots & 0 & -1 & 2 \end{pmatrix}_{N\times N}. \tag{2.6}$$

Then $I(x)$, defined by (2.2), is reformed as

$$I(x) = \frac{1}{2}x^T Ax - \sum_{n=1}^{N} F(n, x(n)). \tag{2.7}$$

As to the matrix A, defined by (2.6), we have

Lemma 2.2. *Let λ_i, $i \in [1, N]$, be the eigenvalue of the matrix A. For any $i \in [1, N]$, if $|\alpha| \leq 1$ such that $\lambda_i \in \mathbf{R}$, then A is nonsingular and $\lambda_i > 0$.*

Proof. According to [34], $|\alpha| < \frac{(N+1)}{\beta}$ guarantees that A is nonsingular. Recall that $|\alpha| \leq 1$ and $1 \leq \beta \leq N$, we get that A is nonsingular.

Notice the fact that if matrix $A = (a_{ij})_{N\times N}$ is nonsingular and satisfies

$$a_{ii} \geq \sum_{j=1, j\neq i}^{N} |a_{ij}|, \qquad i = 1, 2, \cdots, N,$$

then $\lambda_i > 0$. From the expression of A, it follows that

$$\text{if} \quad i = 1, 2, \cdots, N-1, \quad \text{then} \quad a_{ii} = 2 \quad \text{and} \quad a_{ii} \geq \sum_{j=1, j\neq i}^{N} |a_{ij}|;$$

$$\text{if} \quad i = N, \quad \text{then} \quad a_{NN} = 2 \quad \text{and} \quad \sum_{j=1, j\neq N}^{N} |a_{Nj}| = |-\alpha| + 1 = |\alpha| + 1.$$

Hence, $|\alpha| \leq 1$ guarantees

$$a_{ii} \geq \sum_{j=1, j\neq i}^{N} |a_{ij}|, \qquad i = 1, 2, \cdots, N,$$

which implies that all eigenvalues $\lambda_i s$ $(1 \le i \le N)$ of nonsingular matrix A are positive. Without loss of generality, let the positive eigenvalues $\lambda_i s$ $(1 \le i \le N)$ be denoted by

$$0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_N. \tag{2.8}$$

The verification of Lemma 2.2 is completed. □

Aim to meet with $I'(x) = x - S(x)$ for $x \in E$. We set out to find a completely continuous operator S. For $\gamma(n) : [1, N] \to \mathbf{R}$, study the following BVP:

$$\begin{cases} -\Delta^2 x(n-1) = \gamma(n), & n \in [1, N], \\ x(0) = 0, \quad x(N+1) = \alpha x(\beta). \end{cases} \tag{2.9}$$

Evidently, (2.9) can be transformed into linear algebra equation

$$Ax = \gamma.$$

Notice the fact, demonstrated in Lemma 2.2, that A is a nonsingular matrix. Then

$$x = A^{-1}\gamma \tag{2.10}$$

solves (2.9). To construct S, we are to obtain the unique solution of (2.9) by using the Green function $G(n, s)$. Denote

$$g(n, s) = \begin{cases} \dfrac{s(N+1-n)}{N+1}, & 1 \le s \le n \le N, \\ \dfrac{n(N+1-s)}{N+1}, & 1 \le n \le s \le N. \end{cases}$$

In [35], by the method of considering the inversive matrix of the corresponding vector matrix equation, the authors presented results on the Green function as the following:

Lemma 2.3. *[35] If*

$$-\frac{N+1}{\beta[N(N+1-\beta)-1]} < \alpha < \frac{N+1}{\beta}, \tag{2.11}$$

then

$$G(n, s) = g_{ns} + \frac{n\alpha}{N+1-\alpha\beta} g_{\beta s} > 0, \quad n, s \in [1, N] \tag{2.12}$$

serves as the Green function of the BVP (1.1).

Immediately, by Lemma 2.3, we get that

$$x(n) = \sum_{n=1}^{N} G(n, s)\gamma(s) \tag{2.13}$$

is the unique solution of the BVP (2.9). Let $K : E \to E$ be expressed by

$$(Kx)(n) = \sum_{n=1}^{N} G(n, s)x(s), \quad x \in E, \quad n \in [1, N]. \tag{2.14}$$

Then by (2.10), (2.13) and (2.14), we deduce that

$$K = A^{-1}.$$

Define an operator $S : E \to E$ as

$$S = Kf. \tag{2.15}$$

In the following, we verify that S is a completely continuous operator. Actually, the definitions of S and K mean that we can transform S into

$$(Sx)(n) = \sum_{n=1}^{N} G(n,s)f(s,x(s)), \quad x \in E, \quad n \in [1,N]. \tag{2.16}$$

In view of Lemma 2.3, there is $M > 0$ such that $M = \max_{n,s\in[1,N]}\{G(n,s)\}$. Then

$$\|Sy - Sz\| \leq M \sum_{n=1}^{N} |f(s,y(s)) - f(s,z(s))|, \qquad \forall y,z \in E, \quad s \in [1,N].$$

Thus, S is continuous for the continuity of $f(n,x)$ in x.

Next, we show that S is completely continuous. Let $\tilde{E} \subset E$ be bounded. Since E is finite-dimensional, it suffices to verify that $S(\tilde{E})$ is bounded in E to prove that $S(\tilde{E})$ is relatively compact in E. For any $y \in \tilde{E}$, there exists a constant $C_1 > 0$ such that $\|y\| \leq C_1$. Hence, $|y(n)| \leq C_1$. Moreover, $f(n,x)$ is continuous in x, indicating that there is a constant $C_2 > 0$ such that

$$|f(n,y(n))| \leq C_2, \quad \forall y \in \tilde{E}, \quad n \in [1,N]. \tag{2.17}$$

Combining (2.16) with (2.17), we achieve that

$$\|Sy\| \leq M \sum_{n=1}^{N} |f(s,y(s))| \leq C_2MN, \qquad \forall y \in \tilde{E},$$

which manifests the boundedness of $S(\tilde{E})$. Subsequently, S, defined by (2.15), is completely continuous.

Denote $\alpha_1 = \max\{-1, -\frac{N+1}{\beta[N(N+1-\beta)-1]}\}$. In light of the above observations, we draw a conclusion as follows.

Lemma 2.4. *If $\alpha_1 < \alpha \leq 1$ such that all eigenvalues of matrix A are real numbers, then $x = \{x(n)\}_{n=0}^{N+1} \in E$ is a solution of the BVP (1.1) if and only if x is a fixed point of the completely continuous operator S.*

Remark 2.2. *Illustrated by Lemmas 2.1 and 2.4, we conclude that the fixed points of the operator S, the solutions of the BVP (1.1), and the critical points of I are equivalent.*

Lemma 2.5. *Let S be defined by (2.15). Then, for all $x \in E$, there holds $I'(x) = x - S(x)$.*

Proof. Since $\frac{\partial F(n,x)}{\partial x} = f(n,x)$, by Lagrange's mean value theorem, there is $\kappa(n) \in (0,1)$ such that

$$F(n,(x+y)(n)) - F(n,x(n)) = f(n,x(n)+\kappa(n)y(n)), \quad \forall x,y \in E, \quad n \in [1,N].$$

Thus,

$$I(x+y)-I(x)=\sum_{n=1}^{N}[\Delta x(n-1)\Delta y(n-1)-f(n,x(n)+\kappa(n)y(n))y(n)]+\frac{1}{2}\sum_{n=1}^{N}|\Delta y(n-1)|^2,$$

and

$$\begin{aligned}
&I(x+y)-I(x)-\langle x,y\rangle+\sum_{n=1}^{N}[(f(n,x(n))+x(n))y(n)]\\
=&\sum_{n=1}^{N}[\Delta x(n-1)\Delta y(n-1)-f(n,x(n)+\kappa(n)y(n))y(n)]+\frac{1}{2}\sum_{n=1}^{N}|\Delta y(n-1)|^2\\
&-\sum_{n=1}^{N}\Delta x(n-1)\Delta y(n-1)-\sum_{n=1}^{N}x(n)y(n)+\sum_{n=1}^{N}[(f(n,x(n))+x(n))y(n)]\\
=&\sum_{n=1}^{N}\{[f(n,x(n))-f(n,x(n)+\kappa(n)y(n))]y(n)\}+\frac{1}{2}\sum_{n=1}^{N}|\Delta y(n-1)|^2\\
=&\sum_{n=1}^{N}\{[f(n,x(n))-f(n,x(n)+\kappa(n)y(n))]y(n)\}+\frac{1}{2}\|y\|_H^2-\frac{1}{2}\|y\|^2.
\end{aligned}$$

It follows from the continuity of $f(n,x)$ in x that

$$\lim_{|y|\to 0}[f(n,x)-f(n,x+\kappa(n)y)]=0,$$

which implies that

$$\lim_{|y|\to 0}\{I(x+y)-I(x)-\langle x,y\rangle+\sum_{n=1}^{N}[(f(n,x(n))+x(n))y(n)]\}=0,\qquad \forall x,y\in E.$$

Therefore, we get that

$$\langle I'(x),y\rangle=\langle x,y\rangle-\sum_{n=1}^{N}[f(n,x(n))+x(n)]y(n),\qquad \forall x,y\in E. \tag{2.18}$$

At the same time, for any $x,y\in E$, similar to the proof of Lemma 2.1, there holds

$$\begin{aligned}
\langle x-Sx,y\rangle=&\langle x,y\rangle-\langle Sx,y\rangle\\
=&\langle x,y\rangle-\sum_{n=1}^{N}[\Delta(Sx)(n-1)\Delta y(n-1)+(Sx)(n)y(n)]\\
=&\langle x,y\rangle-\sum_{n\in\Omega_1}[\Delta(Sx)(n-1)\Delta y(n-1)+(Sx)(n)y(n)]\\
&-\sum_{n\in\Omega_2}[\Delta(Sx)(n-1)\Delta y(n-1)+(Sx)(n)y(n)]\\
=&\langle x,y\rangle-\sum_{n=1}^{N}[-\Delta^2(Sx)(n-1)y(n)+(Sx)(n)y(n)]\\
=&\langle x,y\rangle-\sum_{n=1}^{N}[f(n,x(n))y(n)+x(n)y(n)].
\end{aligned}\tag{2.19}$$

Owing to (2.18) and (2.19), we derive that

$$\langle I'(x), y\rangle = \langle x - Sx, y\rangle. \tag{2.20}$$

Since y is arbitrary, (2.20) ensures that

$$I'(x) = x - Sx, \qquad \forall x \in E.$$

This has completed the verification of Lemma 2.5. □

3. Proofs of main results

Drawing upon the preliminary framework established in Section 2, we now proceed to present the detailed proofs of our main results from Theorems 1.1, 1.2, and 1.5, using Theorem 2.1. We first verify that the functional I satisfies the requisite compactness condition.

Lemma 3.1. *Assume that* **(A)** *and* **(F_2)** *hold. Then $I(x)$ satisfies the PS condition.*

Proof. Suppose $\{x^{(k)}\}_{k\in\mathbf{N}} \subset E$ is a PS sequence. Namely, there is a constant $M_1 > 0$ such that $|J(x^{(k)})| \leq M_1$ for $k \in \mathbf{N}$ and $J(x^{(k)}) \to 0$ as $k \to \infty$. We verify that $\{x^{(k)}\}$ possesses a convergent subsequence. For the finite-dimensioness of E, it is sufficient to demonstrate that $\{x^{(k)}\}$ is bounded.

For any $x \in E$, in view of (2.7), (2.8), and (2.3), we obtain that

$$I(x) = \frac{1}{2}x^T Ax - \sum_{n=1}^{N} F(n, x(n)) \geq \frac{\lambda_1}{2}\|x\|^2 - \sum_{n=1}^{N} F(n, x(n)), \tag{3.1}$$

and

$$\begin{aligned}
(I'(x), x) &= \sum_{n=1}^{N}\left[|\Delta x(n-1)|^2 - f(n, x(n))x(n)\right]\\
&= \sum_{n=1}^{N}\left[|x(n) - x(n-1)|^2 - f(n, x(n))x(n)\right]\\
&\leq 4\|x\|^2 - \sum_{n=1}^{N}\left[f(n, x(n))x(n)\right].
\end{aligned} \tag{3.2}$$

Set

$$\Omega_3 = \{n \in [1, N] \| x^{(k)}(n)| \geq \delta_1\}.$$

On account of (3.1) and (3.2), it yields that

$$
\begin{aligned}
M_1 \geq & I(x^{(k)}) = I(x^{(k)}) - \frac{1}{\mu}(I'(x^{(k)}), x^{(k)}) + \frac{1}{\mu}(I'(x^{(k)}), x^{(k)}) \\
= & \frac{1}{\mu}(I'(x^{(k)}), x^{(k)}) + \frac{1}{2}x^{(k)T}Ax^{(k)} - \sum_{n=1}^{N} F(n, x^{(k)}(n)) \\
& - \frac{1}{\mu}\sum_{n=1}^{N}\left(\Delta^2 x^{(k)}(n-1) - f(n, x^{(k)}(n))x^{(k)}(n)\right) \\
\geq & \frac{1}{\mu}(I'(x^{(k)}), x^{(k)}) + (\frac{\lambda_1}{2} - \frac{4}{\mu})\|x^{(k)}\|^2 + \sum_{n=1}^{N}\left[\frac{1}{\mu}(f(n, x^{(k)}(n))x^{(k)}(n)) - F(n, x^{(k)}(n))\right] \\
= & \frac{1}{\mu}(I'(x^{(k)}), x^{(k)}) + (\frac{\lambda_1}{2} - \frac{4}{\mu})\|x^{(k)}\|^2 + \sum_{n\in\Omega_3}\left[\frac{1}{\mu}(f(n, x^{(k)}(n))x^{(k)}(n)) - F(n, x^{(k)}(n))\right] \\
& + \sum_{n\in[1,N]\backslash\Omega_3}\left[\frac{1}{\mu}(f(n, x^{(k)}(n))x^{(k)}(n)) - F(n, x^{(k)}(n))\right].
\end{aligned} \tag{3.3}
$$

By (F_2), it follows that

$$\sum_{n\in\Omega_3}\left[\frac{1}{\mu}(f(n, x^{(k)}(n)), x^{(k)}(n)) - F(n, x^{(k)}(n))\right] > 0.$$

Since both $F(n,x)$ and $f(n,x)$ are continuous in x, there exists some constant $\tilde{M}$ independently of k such that

$$\sum_{n\in[1,N]\backslash\Omega_3}\left[\frac{1}{\mu}(f(n, x^{(k)}(n)), x^{(k)}(n)) - F(n, x^{(k)}(n))\right] = \tilde{M}.$$

For $\frac{\lambda_1}{2} - \frac{4}{\mu} > 0$, (3.3) ensures that $\{x^{(k)}\}_{k\in\mathbf{N}}$ is bounded in E. The proof is finished. □

Lemma 3.2. *Suppose that* **(A)** *and* (F_6) *are fulfilled. Then* $I(x)$ *validates the PS condition.*

Proof. Owing to (F_6), there exist constants $\hat{\epsilon} > 0$ and $\delta_2 > 0$ such that

$$f(n,x) \geq \lambda_N(1+\hat{\epsilon})x, \qquad \forall n \in [1,N], \quad |x| > \delta_2,$$

which implies that

$$F(n,x) \geq \frac{\lambda_N(1+\hat{\epsilon})}{2}x^2, \qquad \forall n \in [1,N], \quad |x| > \delta_2. \tag{3.4}$$

Notice that $f(n,x)$ is continuous in x, and $\frac{\partial F(n,x)}{\partial x} = f(n,x)$. Then $F(n,x)$ is continuous in x for all $x \in E$. Thus, there exists a constant $C_1 > 0$ such that

$$F(n,x) \geq \frac{\lambda_N(1+\hat{\epsilon})}{2}x^2 - C_1, \qquad \forall n \in [1,N], \quad 0 \leq x \leq \delta_2. \tag{3.5}$$

By (3.4) and (3.5), we derive that

$$F(n,x) \geq \frac{\lambda_N(1+\hat{\epsilon})}{2}x^2 - C_1, \qquad \forall n \in [1,N], \quad x \geq 0. \tag{3.6}$$

Suppose $\{x^{(i)}\}_{i\in\mathbf{N}} \subset E$ such that $I(x^{(i)})$ is bounded and $I(x^{(i)}) \to 0$ as $i \to \infty$. Then there exists a constant $M_2 > 0$ such that $|I(x^{(i)})| \le M_2$ for $i \in \mathbf{N}$. Recall (2.7) and apply (3.6). We obtain that

$$\begin{aligned} -M_2 \le & I(x^{(i)}) = \frac{1}{2}x^{(i)}A(x^{(i)})^T - \sum_{n=1}^{N} F(n, x^{(i)}(n)) \\ \le & \frac{\lambda_N}{2}\|x^{(i)}\|^2 - \sum_{n=1}^{N}\left[\frac{\lambda_N(1+\hat{\epsilon})}{2}(x^{(i)})^2 - C_1\right] \\ = & -\frac{\lambda_N\hat{\epsilon}}{2}\|x^{(i)}\|^2 + C_1 N. \end{aligned}$$

Namely,

$$\frac{\lambda_N\hat{\epsilon}}{2}\|x^{(i)}\|^2 \le M_2 + C_1 N, \tag{3.7}$$

which guarantees the boundedness of $\{x^{(i)}\}_{i\in\mathbf{N}}$. Considering E is an N-dimensional space, the boundedness of $\{x^{(i)}\}_{i\in\mathbf{N}}$ validates that $I(x)$ satisfies the PS condition. □

Aiming to apply Theorem 2.1 for the proofs of our main results, the following lemma is essential. For the sake of convenience, we introduce a few notations. For $x \in E$, denote the distance in E with respect to $\|\cdot\|_H$ by $\operatorname{dist}_H(x, \pm\Omega) = \inf_{\varphi\in\pm\Omega}\|x - \varphi\|_H$. Let

$$\Omega = \{x \in E : x \ge 0\} \quad \text{and} \quad -\Omega = \{x \in E : x \le 0\}$$

represent the positive and negative convex cones in E, respectively. For $\varepsilon > 0$, write

$$D_\varepsilon^+ = \{x \in E : \operatorname{dist}_H(x, \Omega) < \varepsilon\}, \quad D_\varepsilon^- = \{x \in E : \operatorname{dist}_H(x, -\Omega) < \varepsilon\}.$$

Evidently, both $D_\varepsilon^+ \subset E$ and $D_\varepsilon^- \subset E$ are open with $D_\varepsilon^+ \bigcap D_\varepsilon^- \ne \emptyset$. Further, functions which belong to $E \setminus (\overline{D_\varepsilon^+} \bigcup \overline{D_\varepsilon^-})$ are sign-changing.

Lemma 3.3. *If* **(A)**, **(F₁)**, *and* **(F₃)** *hold, then there exists* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$

$$S(\partial D_\varepsilon^-) \subset D_\varepsilon^- \quad \text{and} \quad S(\partial D_\varepsilon^+) \subset D_\varepsilon^+.$$

Further, a critical point $x \in D_\varepsilon^-$ *(*$x \in D_\varepsilon^+$*) implies that* x *is the negative (positive) solution of the BVP (1.1).*

Proof. For conciseness, we show the case of D_ε^- at length. The other case can be demonstrated in the same manner and is therefore left out.

For any $\delta > 0$, the continuity of $f(n, x)$ in x and $(\mathbf{F_1})$ and $(\mathbf{F_3})$ indicate that there exists $C_\delta > 0$ such that

$$|f(n, x)| \le \delta|x| + C_\delta|x|^{s-1}, \quad \forall(n, x) \in [1, N] \times \mathbf{R}. \tag{3.8}$$

Then by **(A)** and the definition of $\|\cdot\|_H$, we have

$$\lambda_1\|x\|^2 \le \|x\|_H^2 = \sum_{n=1}^{N}[|\Delta^2 x(n-1)|^2 + |x(n)|^2] = x^T A x + \|x\|^2 \le (\lambda_N + 1)\|x\|^2, \quad \forall x \in E.$$

Hence,

$$\sqrt{\lambda_1}\|x\| \leq \|x\|_H \leq \sqrt{\lambda_N + 1}\|x\|. \tag{3.9}$$

For each $x \in E$, define $x^+ = \max\{x, 0\}$ and $x^- = \min\{x, 0\}$. By the definition of $\mathrm{dist}_H(x, \pm\Omega)$ and (3.9), we infer that

$$\|x^+\| = \inf_{\varphi\in-\Omega} \|x - \varphi\| \leq \frac{1}{\sqrt{\lambda_1}} \inf_{\varphi\in-\Omega} \|x - \varphi\|_H = \frac{1}{\sqrt{\lambda_1}}\mathrm{dist}_H(x, -\Omega). \tag{3.10}$$

Denote $y = S(x) \in E$. It follows that $y^+ = y - y^-$ and $y^- \in -\Omega$. Then

$$\mathrm{dist}_H(y, -\Omega) = \inf_{\varphi\in-\Omega} \|y - \varphi\|_H \leq \|y - y^-\|_H = \|y^+\|_H. \tag{3.11}$$

Write $\delta = \frac{D}{4}$ with $D = \min\{\sqrt{\lambda_1}, \lambda_1\}$ and take (2.1) and (2.19) into consideration. Then by Hölder inequality and (3.8)–(3.11), we get

$$\begin{aligned}
&\mathrm{dist}_H(y, -\Omega)\|y^+\|_H \leq \langle y^+, y^+\rangle = \langle S(x^+), y^+\rangle = \sum_{n=1}^{N}[f(n, x^+(n)), y^+(n)] \\
&\leq \sum_{n=1}^{N}[(\delta|x^+(n)| + C_\delta|x^+(n)|^{s-1}, y^+(n))] \\
&\leq \delta\left(\sum_{n=1}^{N}|x^+(n)|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{n=1}^{N}|y^+(n)|^2\right)^{\frac{1}{2}} + C_\delta\left(\sum_{n=1}^{N}|x^+(n)|^{s-1\cdot\frac{s}{s-1}}\right)^{\frac{s-1}{s}} \cdot \left(\sum_{n=1}^{N}|y^+(n)|^s\right)^{\frac{1}{s}} \\
&= \delta\|x^+\| \cdot \|y^+\| + C_\delta\|x^+\|_s^{s-1} \cdot \|y^+\|_s \\
&\leq \frac{\delta}{\sqrt{\lambda_1}}\|x^+\| \cdot \|y^+\|_H + \frac{C_\delta}{\sqrt{\lambda_1}}N^{\frac{(2-s)(s-1)}{2s}}N^{\frac{2-s}{2s}}\|x^+\|^{s-1}\|y^+\|_H \\
&= \left(\frac{\delta}{\sqrt{\lambda_1}}\|x^+\| + \frac{C_\delta}{\sqrt{\lambda_1}}N^{\frac{2-s}{2}}\|x^+\|^{s-1}\right)\|y^+\|_H \\
&\leq \left(\frac{\delta}{\lambda_1}\mathrm{dist}_H(x, -\Omega) + \frac{C_\delta}{\sqrt{\lambda_1}^s}N^{\frac{2-s}{2}}(\mathrm{dist}_H(x, -\Omega))^{s-1}\right)\|y^+\|_H \\
&\leq \frac{1}{4}\mathrm{dist}_H(x, -\Omega) + C_\delta D^{-s}N^{\frac{2-s}{2}}(\mathrm{dist}_H(x, -\Omega))^{s-1},
\end{aligned}$$

which manifests that $\mathrm{dist}_H(x, -\Omega) \leq \epsilon \leq \epsilon_0$ and

$$\mathrm{dist}_H(S(x), -\Omega) \leq \frac{1}{2}\mathrm{dist}_H(x, -\Omega) < \epsilon, \tag{3.12}$$

for $0 < \epsilon_0 < \left(4C_\delta D^{-s}N^{\frac{2-s}{2}}\right)^{\frac{1}{2-s}}$. Consequently, for any $x \in \partial D_\varepsilon^-$, $S(x) \in D_\varepsilon^-$. Namely, $S(\partial D_\varepsilon^-) \subset D_\varepsilon^-$.

Further, Lemma 2.5 ensures that a nontrivial critical point of $x \in D_\varepsilon^-$ satisfies $S(x) = x$. By (3.12), we attain that $x \in -\Omega \setminus \{0\}$. Therefore, x is negative and solves the BVP (1.1). □

With the help of Lemmas 3.1, 3.2, and 3.3, now we are in position to display the detailed proofs of our theorems.

Proof of Theorem 1.1. Fix $\epsilon = \frac{\lambda_1}{2}$. By ($\mathbf{F_1}$), there exists a constant $0 < \delta_3 < 1$ satisfying

$$F(n, x(n)) \leq \frac{\epsilon}{2}|x(n)|^2, \quad \forall|x| \leq \delta_3, \quad \forall n \in [1, N]. \tag{3.13}$$

Then

$$I(x) = \frac{1}{2}xAx^T - \sum_{n=1}^{N} F(n, x(n)) \geq \frac{\lambda_1}{2}\|x\|^2 - \frac{\epsilon}{2}\|x\|^2 = (\frac{\lambda_1}{2} - \frac{\epsilon}{2})\|x\|^2 = \frac{\lambda_1}{4}\|x\|^2. \tag{3.14}$$

Choose $\xi = \max\{\delta_3, \varepsilon_0\}$. Similar to (3.9), we have

$$\|x^{\pm}\| \leq \frac{1}{\sqrt{\lambda_1}}\mathrm{dist}_H(x, \mp\Omega) \leq \frac{1}{\sqrt{\lambda_1}}\xi, \quad \forall x \in \overline{D_\varepsilon^+} \bigcap \overline{D_\varepsilon^-}. \tag{3.15}$$

Based on (3.14) and (3.15), we reach the conclusion that there is a non-negative constant c_0 such that $\inf\limits_{x \in \overline{D_\varepsilon^+} \bigcap \overline{D_\varepsilon^-}} I(x) = c_0$.

Integrate both sides of (1.5). Then there exist constants $c, a_1 > 0$ and $\mu > 2$ such that

$$F(n, x(n)) \geq c|x(n)|^\mu - a_1, \quad \forall n \in [1, N], \quad x \in E. \tag{3.16}$$

In conjunction with (2.7), (2.8), and (2.1), it follows that

$$I(tx) \leq \frac{\lambda_N t^2}{2}\|x\|^2 - \sum_{n=1}^{N}[c|tx(n)|^\mu - a_1] \leq \frac{\lambda_N}{2}\|tx\|^2 - c \cdot N^{\frac{2-\mu}{2\mu}}\|tx\|^\mu, \quad \forall t > 0,$$

which guarantees $I(x) \to -\infty$ as $\|x\| \to +\infty$ for $\mu > 2$. Let $E_1 = \mathrm{span}\{z_1, z_2\}$, where z_1, z_2 are the eigenvectors corresponding to the eigenvalues λ_1, λ_2 of matrix A, respectively. For any $x \in E_1$, the equivalence of $\|\cdot\|$ and $\|\cdot\|_H$ yields that $I(x) \to -\infty$ as $\|x\|_H \to +\infty$. Then for some constant $\omega > 2\varepsilon_0$ large enough, there holds $I(x) < c_0 - 1$ with $\|x\|_H = \omega$. Define $h : [0, 1] \to E_1$ as

$$h(t) = \omega\frac{\cos(\pi t)z_1 + \sin(\pi t)z_2}{\|\cos(\pi t)z_1 + \sin(\pi t)z_2\|_H}.$$

Directly, $\|h\|_H = \omega$, $h(0) = \omega\frac{z_1}{\|z_1\|_H} \in D_\varepsilon^+ \setminus D_\varepsilon^-$, and $h(1) = -\omega\frac{z_1}{\|z_1\|_H} \in D_\varepsilon^- \setminus D_\varepsilon^+$. Then

$$\sup_{t \in [0,1]} I(h(t)) < c_0 - 1 < c_0 = \inf_{x \in \overline{D_\varepsilon^+} \bigcap \overline{D_\varepsilon^-}} I(x).$$

In view of Lemmas 2.5, 3.1 and 3.3, Theorem 2.1 ensures that I admits at least four critical points: $x_1 \in E \setminus (\overline{D_\varepsilon^+} \bigcup \overline{D_\varepsilon^-})$, $x_2 \in D_\varepsilon^+ \setminus \overline{D_\varepsilon^-}$, $x_3 \in D_\varepsilon^- \setminus \overline{D_\varepsilon^+}$, and $x_4 \in D_\varepsilon^+ \bigcap D_\varepsilon^-$. Consequently, the BVP (1.1) has at least four solutions: sign-changing one x_1, positive one x_2, negative one x_3, and trivial one x_4. The proof is completed.

Proof of Theorem 1.2. Under the assumptions of Theorem 1.2, $I(x)$ satisfies the PS condition in the light of Lemma 3.2. Further, the same method as Lemma 3.3 can be used to produce results that are comparable to it.

By ($\mathbf{F_5}$), there exist constants $0 < \tilde{\epsilon} < 1$ and $\delta_4 > 0$ such that

$$|f(n, x)| \leq \lambda_1(1 - \tilde{\epsilon})|x|, \qquad \forall n \in [1, N], \quad |x| \leq \delta_4.$$

Then

$$F(n, x) \leq \frac{\lambda_1(1 - \tilde{\epsilon})}{2}|x|^2, \qquad \forall n \in [1, N], \quad |x| \leq \delta_4.$$

Jointly with (2.7), it follows that

$$I(x) \geq \frac{\lambda_1}{2}\|x\|^2 - \sum_{n=1}^{N} \frac{\lambda_1(1-\tilde{\epsilon})}{2}|x|^2 = \frac{\tilde{\epsilon}}{2}\|x\|^2. \tag{3.17}$$

Denoting $\tilde{\xi} = \max\{\delta_4, \varepsilon_0\}$, one has

$$\|x^{\pm}\| \leq \frac{1}{\sqrt{\lambda_1}}\mathrm{dist}_H(x, \mp\Omega) \leq \frac{1}{\sqrt{\lambda_1}}\tilde{\xi}, \quad \forall x \in \overline{D_\varepsilon^+} \bigcap \overline{D_\varepsilon^-}. \tag{3.18}$$

Thus, there exists $\tilde{c_0} \geq 0$ such that $\inf\limits_{x \in \overline{D_\varepsilon^+} \bigcap \overline{D_\varepsilon^-}} I(x) = \tilde{c_0}$.

Recall $(\mathbf{F_6})$. The continuity of F leads to

$$F(n, x) \geq \frac{\lambda_N(1+\hat{\epsilon})}{2}x^2 - C_1, \qquad \forall n \in [1, N], \quad x \geq 0.$$

Then

$$I(\tilde{t}x) \leq \frac{\lambda_N \tilde{t}^2}{2}\|x\|^2 - \sum_{n=1}^{N}\left[\frac{\lambda_N(1+\hat{\epsilon})}{2}(\tilde{t}x)^2 - C_1\right] = -\frac{\lambda_N \hat{\epsilon}}{2}\|\tilde{t}x\|^2 + C_1 N, \qquad \tilde{t} > 0. \tag{3.19}$$

Hence, $I(\tilde{t}x) \to -\infty$ as $\tilde{t}x \to +\infty$. Therefore, $I(x) \to -\infty$ as $\|x\| \to +\infty$.

For simplicity, we have omitted the remaining proof, which is identical to that of Theorem 1.1. This completes the proof of Theorem 1.2.

Proof of Theorem 1.5. Either $(\mathbf{F_8})$ or $(\mathbf{F_9})$ holds, so Theorem 1.5 is proved in a similar way. Here, we only display the proof of the case $(\mathbf{F_8})$.

First, we demonstrate that I satisfies the $(C)_c$ condition. Let $\{x^{(j)}\}_{j \in \mathbf{N}} \subset E$ be a $(C)_c$ sequence of I in E. That is,

$$I(x^{(j)}) \to c \qquad \text{and} \qquad (1 + \|x^{(j)}\|)I'(x^{(j)}) \to 0 \quad \text{as} \quad j \to \infty.$$

Without loss of generality, we assume that there is $R_1 > 2c + 1$ such that

$$|I(x^{(j)})| \leq \frac{R_1 - 1}{2} \quad \text{and} \quad (1 + \|x^{(j)}\|)\|I'(x^{(j)})\| < 1, \qquad j \in \mathbf{N}. \tag{3.20}$$

Since E is finite dimensional, we verify that $\{x^{(j)}\}_{j \in \mathbf{N}}$ is bounded in E to guarantee I fulfills the $(C)_c$ condition.

Using (2.2), (2.3), and (3.20), we find that

$$\begin{aligned} &\sum_{n=1}^{N}\left[f(n, x^{(j)}(n))x^{(j)}(n) - 2F(n, x^{(j)}(n)\right] = 2I(x^{(j)}) - (I'(x^{(j)}), x^{(j)}) \\ \leq& 2|I(x^{(j)})| + \|I'(u^{(j)})\|\|u^{(j)}\| \leq 2|I(x^{(j)})| + (1 + \|x^{(j)}\|)\|I'(x^{(j)})\| \\ \leq& R_1. \end{aligned} \tag{3.21}$$

We assert that $\{x^{(j)}\}$ is bounded. By contradiction, assume that there exists a subsequence of $\{x^{(j)}\}$, still denoted by $\{x^{(j)}\}$ for the sake of simplicity, and some $n_0 \in [1, N]$ such that $|x^{(j)}(n_0)| \to +\infty$ as $j \to \infty$. Then $(\mathbf{F_8})$ means that

$$f(n_0, x^{(j)}(n_0))x^{(j)}(n_0) - 2F(n_0, x^{(j)}(n_0) \to +\infty, \qquad j \to +\infty.$$

Note that $f(n,x)$ and $F(n,x)$ are both continuous in x. Then there exists a constant $\bar{M} > 0$ such that

$$f(n, x^{(j)}(n))x^{(j)}(n) - 2F(n, x^{(j)}(n)) \geq \bar{M}, \qquad n \in [1, N], \quad x \in E.$$

Subsequently,

$$\begin{aligned}
&\sum_{n=1}^{N} \left[f(n, x^{(j)}(n))x^{(j)}(n) - 2F(n, x^{(j)}(n) \right] \\
=&\sum_{n=1}^{n_0-1} \left[f(n, x^{(j)}(n))x^{(j)}(n) - 2F(n, x^{(j)}(n) \right] + \sum_{n=n_0+1}^{N} \left[f(n, x^{(j)}(n))x^{(j)}(n) + 2F(n, x^{(j)}(n) \right] \\
&+ f(n_0, x^{(j)}(n_0))x^{(j)}(n_0) - 2F(n_0, x^{(j)}(n_0) \\
\geq&(N-1)\bar{M} + f(n_0, x^{(j)}(n_0))x^{(j)}(n_0) - 2F(n_0, x^{(j)}(n_0) \\
\rightarrow& +\infty.
\end{aligned} \tag{3.22}$$

It is evident that (3.21) and (3.22) are in contradiction with each other. Therefore, $\{x^{(j)}\}$ is bounded and $I(x)$ fulfills the $(C)_c$ condition.

The remainder of the proof is similar to that of Theorem 1.1 and will not be covered again here. We complete the verification of Theorem 1.5.

4. Two examples of practical applications

The BVP given in (1.1) serves as a mathematical model for diverse phenomena arising in fields where discrete systems subject to multi-point constraints are encountered. In this section, we present two concrete examples to not only validate our obtained theoretical results but also to demonstrate the applicability of the BVP (1.1) in practical contexts.

Example 4.1. (Biological population model)*:*
Let $\frac{a(n)x(n)}{1+x^2(n)}$ *represent a saturation effect and* $b(n)x(n)$ *account for linear growth. Consider the following biological population model:*

$$\begin{cases} \Delta^2 x(n-1) + \dfrac{a(n)x(n)}{1+x^2(n)} + b(n)x(n) = 0, & n \in [1, N], \\ x(0) = 0, \quad x(N+1) = \alpha x(\beta), \end{cases} \tag{4.1}$$

where constant α *satisfies* (**A**) *and* $0 < a(n) + b(n) < \lambda_1$*,* $b(n) > \lambda_N$*.*

Equation (4.1) is applied to model population dynamics with density-dependent growth (saturation effect $\frac{a(n)x(n)}{1+x^2(n)}$) and Allee effects $b(n)x(n)$. The three-point boundary $x(N+1) = \alpha x(\beta)$ might reflect a feedback mechanism where the population at the endpoint relays on a midpoint measurement β (e.g., resource allocation based on a central observation).

In (4.1), $f(n, x(n)) = \frac{a(n)x(n)}{1+x^2(n)} + b(n)x(n)$. Direct calculations yield that

$$\lim_{x\to 0} \frac{f(n,x)}{x} = a(n) + b(n) < \lambda_1 \quad \text{and} \quad \lim_{x\to +\infty} \frac{f(n,x)}{x} = b(n) > \lambda_N, \qquad \forall n \in [1, N],$$

which indicate that ($\mathbf{F_5}$) and ($\mathbf{F_6}$) are satisfied. Further, ($\mathbf{F_3}$) holds. By Theorem 1.2, the BVP (4.1) admits at least four solutions that are composed of a sign-changing one, a positive one, a negative one, and a trivial one.

To make the results more convenient to see, take $N = 4$, $a(n) = -3.3$, $b(n) = 3.6$, $\alpha = 0.2$, $\beta = 2$. Then $\alpha_1 = \max\{-\frac{5}{22}, -1\} = -\frac{5}{22}$ satisfies $\alpha_1 < \alpha < 1$. Therefore, (4.1) becomes

$$\begin{cases} \Delta^2 x(n-1) - \dfrac{3.3x(n)}{1+x^2(n)} + 3.6x(n) = 0, & n \in [1,4], \\ x(0) = 0, \quad x(5) = 0.2x(2). \end{cases} \tag{4.2}$$

Hence,

$$\bar{A} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & -0.2 & -1 & 2 \end{pmatrix},$$

and the eigenvalues of $\bar{A}$ are

$$\lambda_1 = 0.3393, \quad \lambda_2 = 1.4242, \quad \lambda_3 = 2.6657, \quad \text{and} \quad \lambda_4 = 3.5708.$$

Thus, $\lim\limits_{x\to 0} \frac{f(n,x)}{x} = 0.3 < \lambda_1$ and $\lim\limits_{x\to +\infty} \frac{f(n,x)}{x} = 3.6 > \lambda_4$.

Computations give that the BVP (4.2) has 13 real solutions involving 1 trivial, 1 positive, 1 negative and 10 sign-changing. For a more direct presentation, we list a few as: trivial solution (0, 0, 0, 0, 0, 0), sign-changing solutions (0, −0.7966, 0.8237, 0.3016, −0.3940, 0.1647) and (0, 1.8910, 1.6358, −1.1488, −1.4320, 0.3272), negative solution (0, −0.0875, −0.0689, −0.1160, −0.1233, −0.0138), and positive solution (0, 0.0875, 0.0689, 0.1160, 0.1233, 0.0138).

Remark 4.1. *In this model, sign-changing and negative solutions are interpreted as non-physical but mathematically valid configurations that may represent transient or unstable states in population dynamics.*

Example 4.2. (Mechanical oscillation model: sign-changing solutions correspond to vibrational modes with phase reversals, relevant in structural dynamics)
To model discrete oscillations in a mechanical system (e.g., a discretized beam or lattice structure) with nonlinear restoring forces, where the term $\frac{a(n)x^2(n)}{1+x(n)}$ introduces a nonlinear stiffness that increases with displacement and the boundary condition $x(N+1) = \alpha x(\beta)$ simulates a pinned joint or control mechanism at a specific node β, we consider the BVP (1.1) in the form of

$$\begin{cases} \Delta^2 x(n-1) + \dfrac{a(n)x^2(n)}{1+x(n)} = 0, & n \in [1,N], \\ x(0) = 0, \quad x(N+1) = \alpha x(\beta), \end{cases} \tag{4.3}$$

where α satisfies the assumption (**A**) *and $a(n)$ is bounded on* $[1, N]$.

Here $f(n,x) = \frac{a(n)x^2}{1+x}$. Then

$$F(n,x) = \frac{a(n)x^2}{2} - a(n)(x - \ln|1+x|).$$

For any $n \in [1, N]$, there has

$$\lim_{|x|\to+\infty} [xf(n,x) - 2F(n,x)] = \lim_{|x|\to+\infty} \left[\frac{a(n)x^3}{1+x} - a(n)x^2 - 2a(n)(x - \ln|1+x|) \right] = \infty.$$

Since $a(n)$ is bounded for all $n \in [1, N]$, it follows that

$$\lim_{x\to 0} \frac{f(n,x)}{x} = \lim_{x\to 0} \frac{a(n)x^2}{(1+x)x} = \lim_{x\to 0} \frac{a(n)x}{(1+x)} = 0, \qquad n \in [1, N].$$

Thus, all assumptions in Theorem 1.5 are validated. Consequently, Theorem 1.5 guarantees that the BVP (4.3) possesses at least four solutions: one is trivial, one is sign-changing, one is positive and one is negative.

For clarity, choose $N = 4$, $\beta = 1$, $\alpha = -0.3$, and $a(n) = 0.5$. Then $\alpha_1 = -\frac{1}{3}$ and $\alpha_1 < \alpha < 1$. Excluding duplicate and non-satisfactory solutions, by Matlab, the BVP

$$\begin{cases} \Delta^2 x(n-1) + \dfrac{x^2(n)}{2(1+x(n))} = 0, \qquad n \in [1,4], \\ x(0) = 0, \quad x(5) = -0.3x(1) \end{cases} \tag{4.4}$$

has 16 real number solutions, which are exhibited by the following Table 1:

Table 1. Solutions $x = \{x(n)\}_{n=0}^5$ for the BVP (4.4).

♯	$x(0)$	$x(1)$	$x(2)$	$x(3)$	$x(4)$	$x(5)$	character
1	0	0	0	0	0	0	trivial
2	0	0.0563	0.6366	0.5128	0.3021	−0.0169	positive
3	0	−2.2036	−0.5154	−0.7895	−2.5445	0.6611	negative
4	0	−1.4446	−0.3315	−0.4137	−0.6417	0.4334	negative
5	0	−1.2947	−3.1416	−0.8373	−0.6876	0.3884	negative
6	0	−7.2299	−0.9380	−8.0367	−10.5460	2.1690	negative
7	0	−0.5726	−0.4750	−0.6898	−1.6714	0.1718	negative
8	0	−0.2041	−0.7303	−1.7193	−0.6535	0.0612	negative
9	0	−0.3060	−3.2054	−0.8760	−1.6412	0.0918	negative
10	0	−1.7049	−3.2352	−0.8939	−2.3187	0.5115	negative
11	0	−0.7695	−0.8429	−3.1048	−3.0767	0.2309	negative
12	0	−1.4544	−0.7390	−1.7856	−0.8028	0.4363	negative
13	0	−0.8317	8.7378	4.8176	−1.0974	0.2495	sign-changing
14	0	−3.0687	9.0044	4.9522	−1.1601	0.9206	sign-changing
15	0	−1.4843	2.8563	1.7985	0.1628	0.4453	sign-changing
16	0	0.3944	−3.0901	−0.8059	−0.1941	−0.1183	sign-changing

.

Remark 4.2. *Our examples are the nonlinear algebraic systems, where the total number of solutions (counting multiplicities) is consistent with the fundamental theorem of algebra. For Examples 4.1 and 4.2, they are four variable systems and have infinitely many complex solutions (since nonlinear systems with complex variables do not have a finite count like polynomial systems). However, most complex* *solutions have nonzero imaginary parts, which are not physically meaningful in the modeled scenarios. Further, for complex solutions with real and imaginary parts, the real parts do not exhibit consistent positive/negative/sign-changing properties (due to interference from imaginary parts), so they do not correspond to the solution types defined in the paper.*

5. Conclusions

In this paper, we study the discrete three-point BVP (1.1) by employing the variational methods in conjunction with the method of invariant sets of descending flow. The approach bridges critical point theory with discrete dynamical systems, providing a powerful framework for analyzing nonlinear difference equations subject to nonlocal boundary conditions. We establish several sufficient conditions, Theorems 1.1, 1.2, 1.5, and Corollaries 1.3 and 1.4 to guarantee the existence of at least three nontrivial solutions (one positive, one negative, and one sign-changing) along with one trivial solution. Our results generalize and improve multiplicity of previous studies on discrete BVPs, such as the Robin boundary value problem (when $\alpha = 1$ and $\beta = N$), by addressing more general three-point constraints. The identification of sign-changing solutions is particularly noteworthy, as such solutions often correspond to complex dynamical behaviors like bifurcations or multi-stability in applied contexts (e.g., population dynamics, network systems). In Section 4, both numerical examples not only confirm the validity of the established theorems but also show the practical significance of our findings. The two examples validate sign-changing solutions, which are crucial in bistable systems (e.g., neural networks, phase transitions). This work underscores the power of variational methods in discrete systems and opens new avenues for studying nonlinear phenomena in finite-dimensional systems. Focusing on extending these results to more general forms of the nonlinear function $f(n, x)$ or more general types of discrete problems such as higher-order or partial difference equations, and exploring applications in emerging fields such as discrete neural networks or quantum computing lattices will be important and meaningful. They will be our future work.

Use of AI tools declaration

The author declare he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors sincerely thank the editor and anonymous reviewers for their helpful comments. There is no funding support of this work.

Conflict of interest

The author declares there is no conflicts of interest.

References

1. Y. Zhou, H. Cao, Y. Xiao, *Difference Equations and Their Applications*, Science Press, 2014.

2. Y. T. Shi, J. S. Yu, Wolbachia infection enhancing and decaying domains in mosquito population based on discrete models, *J. Biol. Dyn.*, **14** (2020), 679–695. https://doi.org/10.1080/17513758.2020.1805035
3. Y. H. Long, X. F. Pang, Q. Q. Zhang, Codimension-one and codimension-two bifurcations of a discrete Leslie-Gower type predator-prey model, *Discrete Contin. Dyn. Syst. Ser. B*, **30** (2025), 1357–1389. https://doi.org/10.3934/dcdsb.2024132
4. A. Losota, A discrete boundary value problem, *Ann. Polon. Math.*, **20** (1968), 183–190. https://doi.org/10.4064/ap-20-2-183-190
5. W. G. Kelly, A. C. Peterson, *Difference Equations: An Introduction with Applications*, Academic Press, 1991.
6. G. Zhang, R. Medina, Three-point boundary value problems for difference equations, *Comput. Math. Appl.*, **48** (2004), 1791–1799. https://doi.org/10.1016/j.camwa.2004.09.002
7. J. Henderson, R. Luca, Existence of positive solutions for a system of semipositone coupled discrete boundary value problems, *J. Differ. Equations Appl.*, **25** (2019), 516–541. https://doi.org/10.1080/10236198.2019.1585831
8. R. Y. Ma, Positive solutions for second-order three-point boundary value problems, *Appl. Math. Lett.*, **14** (2001), 1–5. https://doi.org/10.1016/S0893-9659(00)00102-6
9. J. H. Kuang, L. J. Kong, Positive solutions for a class of singular discrete Dirichlet problems with a parameter, *Appl. Math. Lett.*, **109** (2020), 106548. https://doi.org/10.1016/j.aml.2020.106548
10. S. Heidarkhani, G. A. Afrouzi, A. Salari, G. Caristi, Discrete fourth-order boundary value problems with four parameters, *Appl. Math. Comput.*, **346** (2019), 167–182. https://doi.org/10.1016/j.amc.2018.10.039
11. F. M. Atici, A. Cabada, Existence and uniqueness results for discrete second-order periodic boundary value problems, *Comput. Math. Appl.*, **45** (2003), 1417–1427. https://doi.org/10.1016/S0898-1221(03)00097-X
12. J. S. Yu, Z. M. Guo, On boundary value problems for a discrete generalized Emden-Fowler equation, *J. Differ. Equations*, **231** (2006), 18–31. https://doi.org/10.1016/j.jde.2006.08.011
13. R. Ma, Y. Raffoul, Positive solutions of three-point nonlinear discrete second order boundary value problem, *J. Differ. Equations Appl.*, **10** (2004), 129–138. https://doi.org/10.1080/10236190310001 14323
14. R. Y. Ma, Positive solutions of a nonlinear three-point boundary-value problem, *Electron. J. Differ. Equations*, **34** (1999), 1–8.
15. C. J. Pethick, H. Smith, *Bose-Einstein Condensation in Dilute Gases*, Godin Lyttle Press, 2008. https://doi.org/10.1017/CBO9780511802850
16. F. Xiong, Y. H. Xia, Large constant-sign solutions for bvps of the partial difference equation involving (p, q)-Laplacian, *Math. Methods Appl. Sci.*, **48** (2025), 8864–8869. https://doi.org/10.1002/mma.10759
17. F. Xiong, W. T. Huang, Y. H. Xia, Positive solutions of partial discrete Kirchhoff type problems, *Alex. Eng. J.*, **91** (2024), 503–509. https://doi.org/10.1016/j.aej.2024.01.069

18. F. Xiong, W. T. Huang, Positive solutions for Dirichlet BVP of PDE involving ϕ_p-Laplacian, *Fractal Fractional*, **8** (2024), 130. https://doi.org/10.3390/fractalfract8030130
19. R. P. Agarwal, K. Perera, D. O'Regan, Multiple positive solutions of singular and nonsingular discrete problems via variational methods, *Nonlinear Anal.*, **58** (2004), 69–73. https://doi.org/10.1016/j.na.2004.11.012
20. H. H. Liang, P. X. Weng, Existence and multiple solutions for a second-order difference boundary value problem via critical point theory, *J. Math. Anal. Appl.*, **326** (2007), 511–520. https://doi.org/10.1016/j.jmaa.2006.03.017
21. Y. H. Long, Nontrivial solutions of discrete Kirchhoff type problems via Morse theory, *Adv. Nonlinear Anal.*, **11** (2022), 1352–1364. https://doi.org/10.1515/anona-2022-0251
22. Y. H. Long, On homoclinic solutions of nonlinear Laplacian partial difference equations with a parameter, *Discrete Contin. Dyn. Syst. Ser. S*, **17** (2024), 2489–2510. https://doi.org/10.3934/dcdss.2024005
23. Y. H. Long, Q. Q. Zhang, Homoclinic and ground state solutions of partial difference equations with parametric potentials, *Commun. Pure Appl. Anal.*, **24** (2025), 2106–2129. https://doi.org/10.3934/cpaa.202507
24. S. Li, Y. H. Long, On homoclinic solutions for a class of fourth-order ϕ_c-Laplacian partial difference equations, *J. Differ. Equations Appl.*, forthcoming.
25. Y. H. Long, S. Li, Existence and multiplicity of homoclinic solutions for ϕ_c-Laplacian parametric partial difference equations, *Math. Methods Appl. Sci.*, **48** (2025), 14222–14233. https://doi.org/10.1002/mma.11172
26. D. Li, Y. H. Long, Existence and nonexistence of periodic solutions for a class of fourth-order partial difference equations, *J. Math.*, **2025** (2025), 2982321. https://doi.org/10.1155/jom/2982321
27. D. Li, Y. H. Long, On periodic solutions of second-order partial difference equations involving p-Laplacian, *Commun. Anal. Mech.*, **17** (2025), 128–144. https://doi.org/10.3934/cam.2025006
28. Y. H. Long, Existence of multiple and sign-changing solutions for a second-order nonlinear functional difference equation with periodic coefficients, *J. Differ. Equations Appl.*, **26** (2020), 966–986. https://doi.org/10.1080/10236198.2020.1804557
29. Y. H. Long, B. L. Zeng, Multiple and sign-changing solutions for discrete Robin boundary value problem with parameter dependence, *Open Math.*, **15** (2017), 1549–1557. https://doi.org/10.1515/math-2017-0129
30. J. Mawhin, M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer, Berlin, 1989.
31. P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, in *Proceedings of the CBMS Regional Conference Series in Mathematics*, 1986.
32. Z. L. Liu, J. X. Sun, Invariant sets of descending flow in critical point theory with applications to nonlinear differential equations, *J. Differ. Equations*, **172** (2001), 257–299. https://doi.org/10.1006/jdeq.2000.3867
33. X. Q. Liu, J. Q. Liu, On a boundary value problem in the half-space, *J. Differ. Equations*, **250** (2011), 2099–2142. https://doi.org/10.1016/j.jde.2010.11.001

34. G. Zhang, L. Xu, X. F. Li, H. Y. Cui, Y. Q. Du, Y. Wang, *Partial Difference Equation and Its Applications*, Science Press, 2018.

35. H. Yuan, G. Zhang, H. L. Zhao, Existence of positive solutions for a discrete three-point boundary value problem, *Discrete Dyn. Nat. Soc.*, **2007** (2007), 49293. https://doi.org/10.1155/2007/49293

AIMS Press