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Research article

Generalized Caputo fractional differential equations with nonlocal conditions

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Abstract: This paper investigates the existence and uniqueness of solutions for fractional differential equations with nonlocal conditions. The results are derived using measure theory, Monch's fixed point theorem, and semigroup theory. An illustrative example is also presented, involving a nonlocal fractional evolution problem in $L^2((0,\pi),\mathbb{R})$. This example demonstrates the effect of the ψ-Caputo derivative on the mild solution in comparison with the standard Caputo derivative, highlighting the influence of non-uniform time scaling and memory effects.

Keywords: semigroup; Mönch's fixed point theorem; ψ-fractional derivative; evolution equations

1. Introduction

Fractional calculus is a specialized branch of mathematics that extends classical calculus integer-order derivatives and integrals. In recent years, it has gained considerable attention beyond to its diverse applications in modeling natural phenomena physics across multiple disciplines, including engineering, finance, and biology. This growing interest well documented in numerous references [1–4].

An investigation by Derbazi et al., published in 2020 [5], examines the uniqueness and existence of solutions for fractional differential equations subject to nonlocal and fractional integral boundary conditions denoted as follows:

$$\begin{cases} cD^{\alpha}_{0+}\omega(t) = f(t,\omega(t), cD^{\beta}_{0+}\omega(t)),\ t\in[0,1],\ \alpha\in(0,1);\\ cD^{\alpha}_{0+}\omega(0) = aI^{\gamma+1}_{0+}\omega(\eta_1), 0<\eta_1<1;\\ cD^{\beta}_{1-}\omega(1) = bI^{\gamma+2}_{0+}\omega(\eta_2), 0<\eta_2<1;\\ \omega(0) = g(\omega). \end{cases} \tag{1.1}$$

This subject was addressed by Cui et al. [6], who introduced a delay term. Specifically, they investigated the uniqueness and existence of solutions for the following problem:

$$\begin{cases} cD^{\alpha}_{0+}\omega(t) = f(t, \omega(t), cD^{\beta}_{0+}\omega(t)), t \in [0, 1]; \\ cD^{\alpha}_{0+}\omega(0) = aI^{\gamma}_{0+}\omega(\eta); \\ \omega(t) = \psi(t), t \in [-\tau, 0]; \\ b\,\omega(1) + c\,\omega'(1) = g(\omega); \\ \omega(0) = 0. \end{cases} \tag{1.2}$$

Inspired by these developments, the objective of this article is to investigate the existence of solutions to a nonlinear fractional evolution problem (FEP) driven by a generalized Caputo derivative with respect to a function ψ. This formulation incorporates both a semilinear operator and a nonlocal perturbation. The abstract framework enables the treatment of a broad class of FDEs under unified assumptions. Specifically, we investigate the existence of solutions to a nonlinear fractional evolution problem (FEP, for short), formulated as follows

$$\begin{cases} {}_cD^{\alpha,\psi}_{0^+}\omega(t) - A\omega(t) = f\left(t, \omega(t), cD^{\beta}_{0+}\omega(t)\right), t \in [0, a],\ a > 0; \\ \omega(0) = \omega_0 + \Phi(\omega), \end{cases} \tag{1.3}$$

where ${}_cD^{\alpha,\psi}_{0^+}$ means the ψ-Caputo derivative operator of fractional order $\alpha \in (0, 1)$ and cD^{β}_{0+} is the particular derivative of the latter when ψ is the identity, A generates a one-parameter analytic family $\{T_t\}_{t\geq 0}$ of bounded linear operators on a Banach space $(\mathbb{X}, \|.\|)$, $\omega_0 \in \mathbb{X}$, $\Phi : \mathbb{X} \to \mathbb{X}$ and $f : [0, a] \times \mathbb{X}^2 \to \mathbb{X}$ is a function. The core objective of our study is to establish new existence and uniqueness results or a class of fractional evolution equations with nonlocal conditions, a topic with rich theoretical and practical significance. Given that, on the one hand, in most problems with generalized or non-local boundary conditions, the ψ-Caputo framework provides a more suitable analytical fit; on the other hand, the integral representation of ψ-Caputo derivatives naturally interacts with nonlocal terms, enabling the effective application of fixed point techniques (such as Mönch theorem). The ψ-Caputo fractional derivative generalizes the classical Caputo derivative by replacing the usual time variable t with a transformed time scale $\psi(t)$. Physically, this transformation allows the model to incorporate nonuniform time flows, heterogeneous media, and memory effects with nonstandard kernels, which frequently arise in real-world phenomena [7–9]. Recent developments in generalized fractional derivatives and their applications have also been extensively studied [10, 11], providing valuable context and motivation for the present work. The general Caputo-Katugampola fractional derivative and its numerical applications have been recently explored [12–15], further motivating the development of generalized fractional frameworks. The originality of this work lies in establishing new existence and uniqueness results for a class of fractional differential equations involving the ψ-Caputo derivative combined with nonlocal boundary conditions. Our approach combines Mönch's fixed point theorem, semigroup theory, and measure techniques, which together provide a robust framework for handling nonlinear and memory-dependent dynamics in Banach spaces. They are not only mathematically elegant but also particularly well suited to applied contexts involving anomalous diffusion and nonlocal constraints.

The structure of the article is organized as follows: Section 2 reviews notations and several well-known results related to fractional calculus. Section 3 establishes criteria for the existence of solutions to the proposed problem. Section 4 presents an illustrative example that demonstrates the practical applications of the theoretical results.

2. Preliminaries

We now present several definitions, notations, and properties that will be used consistently throughout this article. Let $C([0,a],\mathbb{R})$ denote the Banach space of all continuous functions g with the norm defined as follows:

$$\|g\|_\infty = \max_{0\le t\le a} |g(t)|,$$

Definition 2.1. *[16] Consider a strictly increasing function $\psi \in C^n([0,a],\mathbb{R})$ and $g \in L^1((0,a),\mathbb{R})$. The fractional integral operator ψ of order α acting on g is defined by the following expression*

$$I_{0^+}^{\alpha,\psi} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\psi'(s)}{(\psi(t)-\psi(s))^{1-\alpha}} g(s)ds. \tag{2.1}$$

Remark 2.2. *If we assume that ψ is a logarithmic function or the identity, then Eq* (2.1) *reduces to the Hadamard and Riemann-Liouville fractional integral operators, respectively.*

Definition 2.3. *[16] Let ψ be as in Definition 2.1, $\alpha > 0$, and $g \in C^{n-1}([0,a],\mathbb{R})$. The ψ-Caputo fractional derivative operator of order α acting on g is defined by the following formula*

$${}_cD_{0^+}^{\alpha,\psi} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{\psi'(s)}{(\psi(t)-\psi(s))^{\alpha-n+1}} g_\psi^{[n]}(s)ds, \tag{2.2}$$

where $g_\psi^{[n]}(t) = (\frac{1}{\psi'(t)}\frac{d}{dt})^n g(t)$, $n = [\alpha]+1$ and $[\alpha]$ is the integer part of α. In particular, if $0 < \alpha < 1$, then

$${}_cD_{0^+}^{\alpha,\psi} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(\psi(t)-\psi(s))^{1-\alpha}} g'(s)ds$$

and

$${}_cD_{0^+}^{\alpha,\psi} g(t) = I_{0^+}^{1-\alpha,\psi}(\frac{g'(t)}{\psi'(t)}).$$

Definition 2.4. *[17] Consider a function $\omega : [0,a] \to \mathbb{X}$. The integral transform of Laplace in the generalized sense of ω is expressed as*

$$\mathcal{L}_\psi\{\omega(t)\}(s) = \int_0^\infty \psi'(\tau)\exp[-s(\psi(t)-\psi(0))]\omega(\tau)d\tau.$$

Lemma 2.5. *[17] Let $\alpha > 0$ and let ω be a continuous function on the interval $[0,t]$ with $\psi(t)$-exponential order. In this case, we have the following assertions*

1. $\mathcal{L}_\psi\{I_{0^+}^{\alpha,\psi}\omega(t)\}(s) = s^{-\alpha}\mathcal{L}\omega(s)$.
2. $\mathcal{L}_\psi\{{}_cD_{0^+}^{\alpha,\psi}\omega(t)\}(s) = s^\alpha[\mathcal{L}_\psi\{\omega(t)\} - \sum_{k=0}^{n-1} s^{-k-1}\omega^{(k)}(0)]$, *such that* $n = [\alpha]+1$.

Definition 2.6. *[18] For any $t \in [0, \infty)$, the Wright function W_α (named in honor of the eminent British mathematician E. Maitland Wright) is defined by the following expression*

$$W_\alpha(t) = \frac{1}{\pi}\sum_{m=1}^{\infty}(-1)^{n-1}\frac{\Gamma(\alpha n + 1)}{n!}(\psi(t) - \psi(0))^{-\alpha n-1}\sin(n\pi\alpha)$$

is called the one-sided stable probability density, and its Laplace transform is given by the inverse of an exponential of a power, as follows:

$$\int_0^\infty \psi'(t)\, W_\alpha(t)\exp\{-\lambda(\psi(t) - \psi(0))\}dt = \exp(-\lambda^\alpha).$$

Lemma 2.7. *[19] Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces, and let H and G be non-empty, bounded subsets of $\mathbb{X}$. A measure of noncompactness, denoted by μ, is defined on $\mathbb{X}$. The following properties are satisfied:*

1. *μ is a monotonically increasing function.*
2. *$\mu(G) = 0$ characterizes the pre-compactness of the set G.*
3. *$\mu(G) = \mu(\operatorname{conv} G) = \mu(\overline{G})$, where $\overline{G}$ and $\operatorname{conv} G$ represent the closure and convex hull of G, respectively.*
4. *$\mu(H + G) \le \mu(H) + \mu(G)$, with $H + G$ denotes the Minkowski sum of H and G.*
5. *If the mapping $\mathcal{F} : D(\mathcal{F}) \subset \mathbb{X} \to \mathbb{Y}$ is Lipschitz continuous with the constant ℓ. Then, for each bounded subset G in $D(\mathcal{F})$, we have $\mu(Q(G)) \le \ell\mu(G)$.*
6. *$\mu(G) = \inf\{d(G, H), H \subset \mathbb{X},\ \text{be pre-compact}\}$, where, $d(G, H)$ denotes the Hausdorff distance between the two subsets G and H in $\mathbb{X}$.*
7. *If $\{G_m\}_{m=1}^\infty$ is a decreasing sequence and $\lim_{m\to\infty}\mu(G_m) = 0$, then $\cap_{m=1}^\infty G_m \neq \emptyset$ and compact in the Banach space $\mathbb{X}$, where G_m are bounded non-empty closed subsets of $\mathbb{X}$*

Lemma 2.8. *[19] If the set B_r is bounded. Then, for any given $\varepsilon > 0$, there exists a sequence $\{\omega_m\}_{m=1}^\infty$ contained within B_r with*

$$\mu(B_r) \le 2\mu(\{\omega_m\}_{m=1}^\infty) + \varepsilon.$$

3. Results

We employ the generalized Laplace transform to derive the integral representation of the solution to FEP (1.3).

Theorem 3.1. *The FEP* (1.3) *can be represented by the integral equation formulated by*

$$\omega(t) = S_{\alpha,\psi}(t)(\omega_0 + \Phi(\omega)) + \int_0^t R_{\alpha,\psi}(\psi(t) - \psi(s)) \times f\Big(s, \omega(s), cD^\beta_{0+}\omega(s)\Big)\psi'(s)ds.$$

where

$$R_{\alpha,\psi}(t) = t^{\alpha-1}\int_0^\infty \alpha(\psi(s) - \psi(0))\, W_\alpha(s)\, T_{t^\alpha(\psi(s)-\psi(0))}\, \psi'(s)\, ds.$$

and

$$S_{\alpha,\psi}(t) = I^{1-\alpha,\psi}_{0^+} R_{\alpha,\psi}(t),$$

Proof. **Step 1.** Assume that ω is a solution of problem (1.3). By applying the ψ-fractional integral $I^{\alpha,\psi}_{0^+}$ to both sides of Eq (1.3), we obtain the following result

$$I^{\alpha,\psi}_{0^+}\ {}_cD^{\alpha,\psi}_{0^+}\omega(t) = I^{\alpha,\psi}_{0^+}[Av(t) + f\left(t, \omega(t), cD^\beta_{0+}\omega(t)\right)].$$

Utilizing the relation between the fractional derivative and the integral, see [16], we obtain

$$\omega(t) - \omega(0) = I_{0^+}^{\alpha,\psi}[Av(t) + f(t, \omega(t), cD_{0+}^{\beta}\omega(t)ds)],$$

by the initial data, it follows that

$$\omega(t) = \omega_0 + \Phi(\omega) + I_{0^+}^{\alpha,\psi}[Av(t) + f(t, \omega(t), cD_{0+}^{\beta}\omega(s))], \text{ for all } t \in [0, a]. \tag{3.1}$$

Conversely, it is obvious that if ω satisfies the last integral equation, then (1.3) holds.

Step 2. Let $\lambda > 0$. Utilizing the generalized Laplace transforms on the integral Eq (3.1) and employing Lemma 2.5, we yield

$$\widehat{\omega}(\lambda) = \frac{1}{\lambda}(\omega_0 + \Phi(\omega)) + \frac{1}{\lambda^{\alpha}}(A\widehat{\omega}(\lambda) + \widehat{f}(\lambda)),$$

where $\widehat{\omega}(\lambda) = \displaystyle\int_0^{\infty} \exp(-\lambda(\psi(t) - \psi(0)))\psi'(t)\omega(t)\,dt$, and

$$\widehat{f}(\lambda) = \int_0^{\infty} \psi'(t)\exp(-\lambda(\psi(t) - \psi(0)))f(t, \omega(t), cD_{0+}^{\beta}\omega(t))\,dt,$$

therefore,

$$\widehat{\omega}(\lambda) = \lambda^{\alpha-1}(\lambda^{\alpha}I - A)^{-1}(\omega_0 + \Phi(\omega)) + (\lambda^{\alpha}I - A)^{-1}\widehat{f}(\lambda).$$

Hence,

$$\widehat{\omega}(\lambda) = \lambda^{\alpha-1}\int_0^{+\infty} \exp(-\lambda^{\alpha}s)T(s)(\omega_0 + \Phi(\omega))ds + \int_0^{+\infty} \exp(-\lambda^{\alpha}s)T(s)\widehat{f}(\lambda)ds.$$

Choosing $s = \tau^{\alpha}$ with $\tau = \psi(t) - \psi(0)$, we get

$$\widehat{\omega}(\lambda) = \alpha\lambda^{\alpha-1}\int_0^{+\infty} \tau^{\alpha-1}\exp(-(\lambda\tau)^{\alpha})T(\tau^{\alpha})(\omega_0 + \Phi(\omega))d\tau + \alpha\int_0^{+\infty} \tau^{\alpha-1}\exp(-(\lambda\tau)^{\alpha})T(\tau^{\alpha})\widehat{f}(\lambda)d\tau := I_1 + I_2,$$

where I_1 and I_2 are given by

$$\begin{aligned}
I_1 &= \lambda\int_0^{+\infty} \frac{d}{dt}[\exp(-\lambda^{\alpha}(\psi(t) - \psi(0))^{\alpha})]T((\psi(t) - \psi(0))^{\alpha})(\omega_0 + \Phi(\omega))\,dt \\
&= \alpha\lambda^{\alpha-1}\int_0^{+\infty}\int_0^{+\infty} \Big[\exp(-\lambda(\psi(t) - \psi(0))(\psi(\rho) - \psi(0)))W_{\alpha}(\rho)\psi'(\rho) \\
&\times T((\psi(t) - \psi(0))^{\alpha})(\omega_0 + \Phi(\omega))(\psi(t) - \psi(0))^{\alpha-1}\psi'(t)\Big]dtd\rho \\
&= \alpha\lambda^{\alpha-1}\iint_0^{+\infty} \Big[\exp(-\lambda(\psi(t) - \psi(0)))T\left(\frac{\psi(t) - \psi(0)}{\psi(\rho) - \psi(0)}\right)^{\alpha}\frac{(\psi(t) - \psi(0))^{\alpha-1}}{(\psi(\rho) - \psi(0))^{\alpha}}W_{\alpha}(\rho)(\omega_0 + \Phi(\omega))\psi'(\rho)\psi'(t))\Big]dtd\rho \\
&= \lambda^{\alpha-1}\int_0^{+\infty} \Big[\exp(-\lambda(\psi(t) - \psi(0))) \\
&\times \int_0^{+\infty} \Big(\alpha W_{\alpha}(\rho)T\left(\frac{\psi(t) - \psi(0)}{\psi(\rho) - \psi(0)}\right)^{\alpha}\frac{(\psi(t) - \psi(0))^{\alpha-1}}{(\psi(\rho) - \psi(0))^{\alpha}}(\omega_0 + \Phi(\omega))\psi'(\rho)d\rho\Big)\psi'(t)\Big]dt \\
&= \lambda^{\alpha-1}\mathcal{L}_{\psi}(\mathcal{R}_{\alpha,\psi}(\psi(t) - \psi(0)))(\omega_0 + \Phi(\omega)),
\end{aligned}$$

where

$$\mathcal{R}_{\alpha,\psi}(\psi(t)-\psi(0))=\int_0^{+\infty}\Big[\alpha W_\alpha(\rho)T\left(\frac{\psi(t)-\psi(0)}{\psi(\rho)-\psi(0)}\right)^\alpha\frac{(\psi(t)-\psi(0))^{\alpha-1}}{(\psi(\rho)-\psi(0))^\alpha}\psi'(\rho)\Big]d\rho,$$

and W_α is the probability density, which can be found in [18, equality (4.5)]. So, we can write

$$\mathcal{R}_{\alpha,\psi}(t)=\int_0^{+\infty}\gamma_\alpha(\rho)T\left(t^\alpha(\psi(\rho)-\psi(0))\right)t^{\alpha-1}\psi'(\rho)d\rho, \tag{3.2}$$

such that $\gamma_\alpha(\rho)=(\psi(\rho)-\psi(0))^{-\frac{1}{\alpha}}W_\alpha\Big[\psi^{-1}\Big((\frac{1}{\psi(\rho)-\psi(0)})^{\frac{1}{\alpha}}+\psi(0)\Big)\Big]$. Also we have

$$\mathcal{L}_\psi\Big(\frac{(\psi(t)-\psi(0))^{-\alpha}}{\Gamma(1-\alpha)}\Big)(\lambda)=\lambda^{\alpha-1}.$$

Then,

$$\begin{aligned}I_1&=\mathcal{L}_\psi\Big(\frac{(\psi(.)-\psi(0))^{-\alpha}}{\Gamma(1-\alpha)}*R_{\alpha,\psi}(.)\Big)(t)(\omega_0+\Phi(\omega))\\&=\mathcal{L}_\psi(I_{0^+}^{1-\alpha,\psi}R_{\alpha,\psi}(t))(\omega_0+\Phi(\omega)).\end{aligned}$$

Let $\psi_t^\tau=\psi(\tau)+\psi(t)-\psi(0)$, then we have

$$\begin{aligned}I_2&=\int_0^{+\infty}\exp\left(-\lambda^\alpha\left(\psi(t)-\psi(0)\right)\right)T((\psi(t)-\psi(0)))\widehat{f}(\lambda)\psi'(t)dt\\
&=\iint_0^{+\infty}\Big[\exp\left(-\lambda^\alpha\left(\psi(t)-\psi(0)\right)\right)\times\exp\left(-\lambda(\psi(s)-\psi(0))\right)T((\psi(t)-\psi(0)))\\
&\quad\times f\Big(s,\omega(s),cD_{0+}^\beta\omega(s)\Big)\psi'(s)\psi'(t)\Big]dsdt\\
&=\iiint_0^\infty\Big[\alpha\left(\psi(t)-\psi(0)\right)^{\alpha-1}\exp\left(-\lambda(\psi(t)-\psi(0))(\psi(\rho)-\psi(0))\right)\exp\left(-\lambda(\psi(s)-\psi(0))\right)W_\alpha(\rho)\\
&\quad\times T((\psi(t)-\psi(0))^\alpha)f\Big(s,\omega(s),cD_{0+}^\beta\omega(s)\Big)\psi'(\rho)\psi'(t)\psi'(s)\Big]d\rho dsdt\\
&=\iiint_0^\infty\Big[\alpha\exp\left(-\lambda(\psi(t)+\psi(s)-2\psi(0))\right)T\left(\frac{\psi(t)-\psi(0)}{\psi(\rho)-\psi(0)}\right)^\alpha\frac{(\psi(t)-\psi(0))^{\alpha-1}}{(\psi(\rho)-\psi(0))^\alpha}W_\alpha(\rho)\\
&\quad\times f\Big(s,\omega(s),cD_{0+}^\beta\omega(s)\Big)\psi'(\rho)\psi'(t)\psi'(s)\Big]d\rho dsdt\\
&=\iiint_0^\infty\Bigg[\alpha\exp\left(-\lambda(\psi(t)-\psi(0))\right)T\left(\frac{\psi(t)-\psi(0)}{\psi(\rho)-\psi(0)}\right)^\alpha\frac{(\psi(t)-\psi(0))^{\alpha-1}}{(\psi(\rho)-\psi(0))^\alpha}W_\alpha(\rho)\\
&\quad\times f\Big(\psi^{-1}\left(\psi_t^\tau\right),\omega(\psi^{-1}\left(\psi_t^\tau\right)),cD_{0+}^\beta\omega(\psi_t^\tau)\Big)\psi'(t)\,\psi'(\rho)\psi'(\tau)\Bigg]d\rho d\tau dt.\end{aligned}$$

Hence,

$$\begin{aligned}I_2=&\int_0^\infty\exp\left(-\lambda(\psi(t)-\psi(0))\right)\Big(\int_0^t\int_0^{+\infty}\alpha T\left(\frac{\psi(t)-\psi(s)}{\psi(\rho)-\psi(0)}\right)^\alpha\frac{(\psi(t)-\psi(s))^{\alpha-1}}{(\psi(\rho)-\psi(0))^\alpha}W_\alpha(\rho)\\
&\times f\left(s,\omega(s),cD_{0+}^\beta\omega(s)\right)\psi'(\rho)\psi'(s)d\rho\,ds\Big)\psi'(t)dt.\end{aligned}$$

It follows that,

$$\widehat{\omega}(\lambda)=\mathcal{L}_\psi(I_{0^+}^{1-\alpha,\psi}R_{\alpha,\psi}(t))+\int_0^\infty\exp\left(-\lambda(\psi(t)-\psi(0))\right)\Big[\int_0^t\int_0^{+\infty}\alpha T\left(\frac{\psi(t)-\psi(s)}{\psi(\rho)-\psi(0)}\right)^\alpha\times$$

$$(\psi(t)-\psi(s))^{\alpha-1}\,(\psi(\rho)-\psi(0))^{-\alpha}\,W_\alpha(\rho)f\left(s,\omega(s),cD^{\beta}_{0+}\omega(s)\right)\psi'(\rho)\psi'(s)d\rho\,ds\Big]\psi'(t)dt.$$

By applying the inverse Laplace transform and the expression (3.2) we derive

$$\begin{aligned}\omega(t) &= I^{1-\alpha,\psi}_{0^+}R_{\alpha,\psi}(t)(\omega_0+\Phi(\omega))+\int_0^t R_{\alpha,\psi}(\psi(t)-\psi(s))\times f\Big(s,\omega(s),cD^{\beta}_{0+}\omega(s)\Big)\psi'(s)ds.\\ &= S_{\alpha,\psi}(t)(\omega_0+\Phi(\omega))+\int_0^t R_{\alpha,\psi}(\psi(t)-\psi(s))\times f\Big(s,\omega(s),cD^{\beta}_{0+}\omega(s)\Big)\psi'(s)ds,\end{aligned}$$

which completes the proof. □

From now on, we assume that A satisfies the following assumption:

H(A): A generates a strongly continuous semigroup $\{T_t\}_{t\geq 0}$ in $\mathbb{X}$, and there is a real number $M \geq 1$ with $\sup\limits_{t\in[0,\infty)} |T(t)|_{L(\mathbb{X})} \leq M$.

Proposition 3.2. *We assume that $(H(A))$ holds; then $(S_{\alpha,\psi}(t))_{t>0}$ and $(R_{\alpha,\psi}(t))_{t>0}$ are strongly continuous linear operators on $\mathbb{X}$.*

Proof. Linearity is obvious. Using the integral of $\rho \to \alpha\rho W_\alpha(\rho)\psi'(\rho)$ on the real half-line, we find, $\| R_{\alpha,\psi}(t)x \| \leq \frac{t^{\alpha-1}M}{\Gamma(1+\alpha)}\| x \|$. From the above inequality, it follows that

$$\begin{aligned}\left\| S_{\alpha,\psi}(\psi(t)-\psi(0))x \right\| &= \left\| I^{1-\alpha,\psi}_{0^+}R_{\alpha,\psi}(\psi(t)-\psi(0))x \right\|\\ &\leq \frac{MI^{1-\alpha,\psi}(\psi(t)-\psi(0))^{\alpha-1}}{\Gamma(1+\alpha)}\| x \|\\ &\leq \frac{M\Gamma(\alpha)}{\Gamma(1+\alpha)}\| x\| \leq \frac{M}{\alpha}\| x \|.\end{aligned}$$

This implies that $|S_{\alpha,\psi}(t)| \leq \frac{M}{\alpha}$. Moreover, let $x \in \mathbb{X}$ and $0 < t_1 < t_2 \leq t$, from a simple calculation, it follows that $\lim\limits_{t_1\to t_2}\left\| R_{\alpha,\psi}(t_1)x - R_{\alpha,\psi}(t_2)x \right\| = 0$ and $\lim\limits_{t_1\to t_2}\left\| S_{\alpha,\psi}(t_1)x - S_{\alpha,\psi}(t_2)x \right\| = 0$. □

Now, we introduce the following hypotheses:

(A_1) The function $\Phi : \mathbb{X} \to \mathbb{X}$ is continuous, and there exist two positive constants K and d with $\|\Phi(\omega)-\Phi(u)\| \leq K\|\omega-u\|$ and $\|\Phi(\omega)\| \leq K\|\omega\|+d$, for all $u,\ \omega \in C$.

(A_2) $f : [0,a]\times\mathbb{X}^2 \to \mathbb{X}$ is of Carathéodory type, i.e. $f(t,.,.)$ is continuous for a.e $t \in [0,a]$ and $f(.,\omega,u)$ is measurable for every $\omega,\ u$.

(A_3) (i) There exists a non-decreasing continuous function $\gamma_f : \mathbb{X} \to \mathbb{X}$ and a function $m_f \in C([0,a],\mathbb{X})$ with $\|f(t,u,\omega)\| \leq m_f\gamma_f(\|\omega\|+\|u\|)$, for every $u,\omega \in \mathbb{X}$ a.e $t \in [0,a]$.

(ii) There is an integrable function $\eta : [0,a] \to [0,+\infty)$ such that for almost every $t \in [0,a]$, $\mu(f(t,D_1,D_2)) \leq \eta(t)\left[\sup\limits_{-\infty<t\leq 0}\mu(D_1(t))+\mu(D_2)\right]$, where μ represents the Hausdorff measure of noncompactness, and $D_1, D_2 \subset \mathbb{X}$ are bounded subsets. Additionally, we have $\int_0^t \eta(s)ds \leq \kappa^*$.

(A_4) (i) The function $h(.,.,\omega) : \Delta \to \mathbb{X}$ is measurable for every $\omega \in \mathbb{X}$, and $h(t,s,.) : \mathbb{X} \to \mathbb{X}$ is a continuous function for all $(t,s) \in \Delta$. Additionally, there is a function $u : \Delta \to \mathbb{R}^+$ with $\|h(t,s,\omega)\| \leq u(t,s)\|\omega\|$ such that $\sup\limits_{t\in[0,a]}\int_0^t u(t,s)ds := u^* < \infty$, for every $\omega \in \mathbb{X}$.

(ii) For any $0 \leq s \leq t \leq a$ and bounded set $D_1 \subset X$, there is a functions $\gamma : \Delta \to R^+$ with $\mu(h(t, s, D_1)) \leq \gamma(s, t)\mu(D_1)$ such that $\sup_{t\in[0,a]} \int_0^t \gamma(s, t)ds := \gamma^*$.

(A_5) For any $t \in [0, a]$, the function $\varrho(t, .)$ is measurable on $[0, t]$ and $\varrho_s(t) = \sup\{|\varrho(t, s)|,\ 0 \leq s \leq t\}$ is bounded on $[0, a]$ and $l = \sup_{t\in[0,a]} \varrho_s(t)$.

Theorem 3.3. *Under the assumptions* (H_A) *and* $(A_1) - (A_5)$*, if the inequality*

$$M\left(\frac{K}{\alpha} + \frac{2\kappa}{\Gamma(1+\alpha)}(\psi(a) - \psi(0))^\alpha \left[1 + 2l\gamma^*\right]\right) < 1$$

holds, then there is at least one solution for problem (1.3).

Proof. We define a mapping $F : C \longrightarrow C$ by

$$F\omega(t) = S_{\alpha,\psi}(t)(\Phi(\omega) + \omega_0) + \int_0^t R_{\alpha,\psi}(\psi(t) - \psi(s)) f\Big(s, \omega(s), cD^\beta_{0+}\omega(s)\Big)\psi'(s)ds.$$

First, we will show that F is continuous. Let $\{\omega_m\}$ be a sequence that converges to ω in $C([0, a], \mathbb{X})$. Then, by (A_2), when m tends to infinity, we have the following

$$f\Big(s, \omega_m(s), cD^\beta_{0+}\omega(s)\Big) \to f\Big(s, \omega(s), cD^\beta_{0+}\omega(s)\Big),$$

for every $s \in [0, a]$. To make the expressions easier to handle, we state that

$$\begin{aligned} K_{f_n} &= f\Big(s, \omega_m(s), cD^\beta_{0+}\omega(s)\Big),\\ K_f &= f\Big(s, \omega(s), cD^\beta_{0+}\omega(s)\Big). \end{aligned}$$

For any $t \in [0, a]$, we have

$$\begin{aligned} \|F\omega_m(t) - F\omega(t)\| &= \|S_{\alpha,\psi}(t)(\omega_0 + \Phi(\omega_m)) - S_{\alpha,\psi}(t)(\omega_0 + \Phi(\omega)) + \int_0^t R_{\alpha,\psi}(\psi(t) - \psi(s)) \times \left[K_{f_n} - K_f\right]\psi'(s)ds\| \\ &\leq \|S_{\alpha,\psi}(t)(\Phi(\omega_m)) - \Phi(\omega)))\| + \int_0^t \|R_{\alpha,\psi}(\psi(t) - \psi(s))\|\psi'(s)\|K_{f_n} - K_f\|ds \\ &\leq \frac{MK}{\alpha}\|\omega_m - \omega\| + \frac{M}{\Gamma(1+\alpha)}\int_0^t (\psi(t) - \psi(s))^{\alpha-1}\psi'(s)\|K_{f_n} - K_f\|ds. \end{aligned}$$

Thus, we infer that $\|F\omega_m - F\omega\|_\infty \to 0$ as $n \to \infty$. Consequently, the mapping F is continuous on $C([0, a], \mathbb{X})$. Now, we have to prove that F is bounded. We claim that $F(B_r) \subset B_r$ for any $\omega \in B_r$ and $t \in [0, a]$, we have

$$\begin{aligned} \|F\omega(t)\| &\leq \frac{M}{\alpha}(\|\omega_0\| + K\|\omega\| + d) + \frac{M}{\Gamma(1+\alpha)}\int_0^t (\psi(t) - \psi(s))^{\alpha-1}\|f\Big(s, \omega(s), \int_0^s \varrho(s, t)h(s, t, \omega(t))dt\Big)\|\psi'(s)ds \\ &\leq \frac{M}{\alpha}(\|\omega_0\| + K\|\omega\| + d)\frac{M(\psi(b) - \psi(0))^\alpha}{\Gamma(1+\alpha)}\Big[m_f\sigma_f(\|\omega\| + l\int_0^s u(s, \tau)dt\|\omega\|)\Big] \\ &\leq \frac{M}{\alpha}(\|\omega_0\| + Kr + d) + \frac{M(\psi(a) - \psi(0))^\alpha}{\Gamma(1+\alpha)}m_f\sigma_f r(1 + lu^*) \\ &\leq r. \end{aligned}$$

We claim that $F(B_r)$ is equicontinuous. For every $t_1, t_2 \in [0, a]$ with $t_1 \leq t_2$, we have the following

$\|F\omega(t_2) - F\omega(t_1)\|$

$$
\begin{aligned}
&\leq \|S_{\alpha,\psi}(t_2)-S_{\alpha,\psi}(t_1)\|(\|\omega_0\|+K\|\omega\|+d)+\Big\|\int_0^{t_1}(R_{\alpha,\psi}(\psi(t_2)-\psi(s))-R_{\alpha,\psi}(\psi(t_1)-\psi(s)))\times f\Big(s,\omega(s),cD^{\beta}_{0+}\omega(s)\Big)\psi'(s)ds\\
&\quad+\int_{t_1}^{\theta_2}R_{\alpha,\psi}(\psi(t_2)-\psi(s))\times f\Big(s,\omega(s),cD^{\beta}_{0+}\omega(s)\Big)\psi'(s)ds\Big\|\\
&\leq \|S_{\alpha,\psi}(t_2)-S_{\alpha,\psi}(t_1)\|(\|\omega_0\|+K\|\omega\|+d)+\int_0^{t_1}\Big\|R_{\alpha,\psi}(\psi(t_2)-\psi(s))-R_{\alpha,\psi}(\psi(t_1)-\psi(s))\Big\|\times m_f\sigma_f r(1+lu^*)\psi'(s)ds\\
&\quad+\int_{t_1}^{t_2}\Big\|R_{\alpha,\psi}(\psi(t_2)-\psi(s))\Big\|m_f\sigma_f r(1+lu^*)\psi'(s)ds.
\end{aligned}
$$

At this point, we will check the hypotheses of the Mönch theorem, see [20, Theorem 2.10]. Let $\mathcal{U}\subset B_r$ be countable such that $\mathcal{U}\subset\overline{\text{conv}}(0\cup F(\mathcal{U}))$. We demonstrate that $\mu(\mathcal{U})=0$, where μ denotes the Hausdorff noncompactness measure. Without loss of generality, we may suppose $\mathcal{U}=\{u\}_{k=1}^{\infty}$, and it can be readily verified that $\mathcal{U}$ is equicontinuous and bounded. For any $t\in[0,a]$, we have

$$
\begin{aligned}
F\omega(t)&=S_{\alpha,\psi}(t)(\Phi(\omega)+\omega_0)+\int_0^t R_{\alpha,\psi}(\psi(t)-\psi(s))f\Big(s,\omega(s),cD^{\beta}_{0+}\omega(s)\Big)\psi'(s)ds\\
&=\mathit{\Gamma}_1\omega(t)+\mathit{\Gamma}_2\omega(t),
\end{aligned}
$$

where $\mathit{\Gamma}_1\omega(t)=S_{\alpha,\psi}(t)(\Phi(\omega)+\omega_0)$ and

$$
\mathit{\Gamma}_2\omega(t)=\int_0^t R_{\alpha,\psi}(\psi(t)-\psi(s))f\Big(s,\omega(s),cD^{\beta}_{0+}\omega(s)\Big)\psi'(s)ds.
$$

Moreover, the mapping $\mathit{\Gamma}_1:\mathcal{U}\to C([0,a],\mathbb{X})$ is Lipschitz continuous with the constant KM_α, as indicated by condition (A_1). Indeed, for any $\omega,u\in\mathcal{U}$, we have the following:

$$
\begin{aligned}
\|\mathit{\Gamma}_1\omega(t)-\mathit{\Gamma}_1u(t)\|&\leq\sup_{t\in[0,b]}\|S_{\alpha,\psi}(t)\|\,\|\Phi(\omega)-\Phi(u)\|\\
&\leq\frac{MK}{\alpha}\|\omega-u\|_\infty.
\end{aligned}
$$

So, from Lemma 2.8 and hypotheses $(A_3)(ii)$, $(A_4)(ii)$, we have

$$
\begin{aligned}
\mu(\{F\omega_m\}_{m=1}^{\infty})&\leq\mu(\{\mathit{\Gamma}_1\omega_m\}_{m=1}^{\infty})+\mu(\{\mathit{\Gamma}_2\omega_m\}_{m=1}^{\infty})\\
&\leq\frac{MK}{\alpha}\mu(\{\omega_m\}_{m=1}^{\infty})+\mu\Big(\int_0^t R_{\alpha,\psi}(\psi(t)-\psi(s))\times f(s,\omega(s),cD^{\beta}_{0+}\omega(s))\psi'(s)ds\Big)\\
&\leq\frac{MK}{\alpha}\mu(\{\omega_m\}_{m=1}^{\infty})+\frac{2M}{\Gamma(1+\alpha)}\int_0^t\mu((\psi(t)-\psi(s))^{\alpha-1}\times f(s,\omega(s),cD^{\beta}_{0+}\omega(s))\psi'(s)ds\\
&\leq\frac{MK}{\alpha}\mu(\{\omega_m\}_{m=1}^{\infty})+\frac{2M\kappa^*}{\Gamma(1+\alpha)}(\psi(a)-\psi(0))^{\alpha}[1+2l\gamma^*]\times\sup_{0<s\leq a}\mu_C(\mathcal{U}(s))\\
&\leq M\Big(\frac{K}{\alpha}+\frac{2\kappa^*}{\Gamma(1+\alpha)}(\psi(a)-\psi(0))^{\alpha}[1+2l\gamma^*]\Big)\sup_{0<s\leq a}\mu_C(\mathcal{U}(s)).
\end{aligned}
$$

From above, we have

$$
\begin{aligned}
\mu_C(F(\mathcal{U}))&\leq M\Big[\frac{K}{\alpha}+\frac{2\kappa^*}{\Gamma(1+\alpha)}(\psi(a)-\psi(0))^{\alpha}[1+2l\gamma^*]\Big]\mu_C(\mathcal{U})\\
&\leq\varpi^*\mu_C(\mathcal{U}),
\end{aligned}
$$

where

$$
\varpi^*=M\Big[\frac{K}{\alpha}+\frac{2\kappa^*}{\Gamma(1+\alpha)}(\psi(a)-\psi(0))^{\alpha}[1+2l\gamma^*]\Big].
$$

Thus,

$$\mu_C(\mathcal{U}) \leq \mu_C((\overline{conv\{0\}}) \cup F(\mathcal{U})) = \mu_C(F(\mathcal{U})) \leq \varpi^* \mu_C(\mathcal{U}),$$

This implies $\mu_C(\mathcal{U}) = 0$. Therefore, by applying [20, Theorem 2.3], we find a fixed point ω of F in B_r. This fixed point serves as a solution to the FEP (1.3).

□

4. Application

In this section, let $\mathbb{X} = L^2((0,\pi),\mathbb{R})$ and A is an operator defined by $A\omega = \omega''$ with the domain

$$D(A) = \{\omega \in \mathbb{X} \quad \omega'' \in \mathbb{X}, \quad \omega(0) = \omega(\pi) = 0\}.$$

It's a well-known fact that A represents a bounded linear operator of a compact semigroup $\{T_t\}_{t\geq 0}$ on $\mathbb{X}$, with $\|T(t)\| \leq \exp(-t)$ for every $\tau \geq 0$. Now, consider a nonlocal FEP

$$\begin{cases} {}^C D_{0^+}^{\alpha,\psi} \omega(\tau,t) = \frac{\partial^2}{\partial t^2}\omega(\tau,t) + \int_0^\tau h(\tau,\varsigma,\omega(\varsigma,t))d\varsigma + F\left(\tau,\omega(\tau,t)\right), \\ \omega(0,w) + \sum\limits_{j=1}^{n} c_j u(\tau_j,t) = \omega_0(t), \quad t \in [0,\pi] \end{cases} \tag{4.1}$$

with $\{\tau_k\}_{k=0}^n$ is a finite increasing sequence of elements of real numbers, $\omega_0 \in \mathbb{X}$, and the function F is a continuous function defined on the domain $[0,a] \times \mathbb{R}$ and taking values in $\mathbb{R}$.

To express problem (4.1) in the abstract form of problem (1.3), we assume that

(i) The function $f : [0,a] \times \mathbb{X} \to \mathbb{X}$ is given by

$$f(\tau,\omega)(t) = \int_0^\tau h(\tau,\varsigma,\omega(\varsigma,t))\,d\varsigma + F\left(\tau,\omega(\tau,t)\right)$$

for $t \in [0,\pi]$ and $\tau \in [0,b]$.

(ii) The function $\gamma : C([0,b],\mathbb{X}) \to \mathbb{X}$ is continuous and given by

$$\gamma(\omega)(t) = \omega_0(t) - \sum_{j=1}^{n} c_j u(\tau_j,t),$$

where $\tau_j \in [0,a]$ and $t \in [0,\pi]$. Additionally, $\omega(\tau)(t) = \omega(\tau,t)$ for $\tau \geq 0$ and $t \in [0,\pi]$.

With these adaptations, an element $\omega \in C(\mathbb{X})$ is called a mild solution of (4.1) if $\omega(\cdot)$ satisfies the mild solution concept associated with the corresponding abstract problem (1.3).

To illustrate the advantages of the ψ-Caputo derivative, we choose

$$\psi(t) = t^p, \quad 0 < p \leq 1.$$

This choice allows the model to incorporate non-uniform time scaling and memory effects. For $p = 1$, the standard Caputo derivative is recovered, enabling direct comparison. We now define the application operator as follows:

$$f(\tau,\omega)(t) = \int_0^\tau h(\tau,\varsigma,\omega(\varsigma,t))\,d\varsigma + F(\tau,\omega(\tau,t)), \quad t \in [0,\pi],\ \tau \in [0,a],$$

and

$$\gamma(\omega)(t) = \omega_0(t) - \sum_{j=1}^{n} c_j u(\tau_j, t), \quad t \in [0, \pi].$$

Then, problem (4.1) can be rewritten in the abstract form of (1.3):

$${}^C D_{0^+}^{\alpha,\psi} \omega(\tau) = A\omega(\tau) + f(\tau, \omega(\tau)), \quad \omega(0) = \gamma(\omega).$$

Mild solution and comparison. An element $\omega \in C([0, a], \mathbb{X})$ is a mild solution if

$$\omega(\tau) = S_{\alpha,\psi}(\tau)\gamma(\omega) + \int_0^\tau R_{\alpha,\psi}(\psi(\tau) - \psi(s)) f(s, \omega(s)) \psi'(s) ds,$$

where $S_{\alpha,\psi}$ and $R_{\alpha,\psi}$ are the solution operators defined in Section 3. For comparison with the standard Caputo derivative: Setting $p = 1$ gives $\psi(t) = t$, which reduces the problem to the classical Caputo case. Comparing the solution operators for $p < 1$ and $p = 1$ demonstrates how the ψ-Caputo derivative affects the solution profile, capturing generalized memory and nonlocal effects not seen in the standard case.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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