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Research article

Blowup for the fractional KGS system with nongauge nonlinearities

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Abstract: In this paper, we investigate the finite-time blowup phenomenon in the fractional Klein-Gordon-Schrödinger (FKGS) system featuring nongauge-invariant power-type nonlinearities on $\mathbb{R}^n$. The system models interactions between nucleon and meson fields, augmented by fractional Laplacian operators and nonlinear terms that disrupt conservation laws. By introducing a novel test function tailored to address the difficulties posed by mixed fractional operators and the absence of energy conservation, we established sufficient conditions for finite-time blowup under specific initial data constraints. Our analysis revealed that solutions fail to exist globally when the nonlinear exponents and initial energy satisfy critical inequalities, with the lifespan bounded by a power-law dependence on the problem parameters.

Keywords: FKGS system; test function; weak solution; blowup

1. Introduction

In this paper, we are devoted to the analysis of finite-time blowup for the following fractional Klein-Gordon-Schrödinger (FKGS) system with nongauge-invariant power-type nonlinearities

$$\begin{cases} i\partial_t u + (-\triangle)^{\frac{\alpha}{2}} u = uv + \lambda|u|^p, \\ \partial_{tt} v + \Omega_m v + (-\triangle)^{\frac{\beta}{2}} \partial_t v = |u|^2 + \mu|v|^q, \\ (u, v, \partial_t v)(0, x) = (u_0, v_0, v_1), \end{cases} \tag{1.1}$$

where $(u, v) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C} \times \mathbb{R}$, $\Omega_m = m^2 - \triangle$, $m \geq 0$, $\alpha \in (0, 2]$, $\beta \in (0, 1]$, $\lambda \in \mathbb{C} \setminus \{0\}$, $\mu \in \mathbb{R} \setminus \{0\}$, and $\triangle$ denotes the Laplacian in $\mathbb{R}^n$. Here, $(-\triangle)^{\frac{\alpha}{2}} := \mathscr{F}^{-1}|\omega|^{\alpha}\mathscr{F}$, where $\mathscr{F}$ is the Fourier transform, as given by

$$(\mathscr{F} f)(\omega) = \hat{f}(\omega) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\omega} f(x) dx.$$

System (1.1) models the interaction between a scalar nucleon field u and a scalar meson field v. The power nonlinearities $|u|^p$ and $|v|^q$, as opposed to the more common gauge-invariant forms $|u|^{p-1}u$ and $|v|^{q-1}v$, is physically motivated by scenarios where solutions tend to nonzero constants at spatial infinity, such as in Bose-Einstein condensation [1–3]. This structural difference leads to the breakdown of mass and energy conservation, significantly altering the dynamics. The use of fractional Laplacians $(-\Delta)^{\alpha/2}$ and $(-\Delta)^{\beta/2}$ is motivated by mathematical generality and physical realism. These operators naturally model nonlocal diffusion and anomalous transport, arising in contexts such as Lévy flights, long-range interactions, and materials with complex microstructures [4–6]. In dispersive equations, fractional models have attracted considerable interest. The fractional Schr̈odinger equation was introduced by Laskin [7], while fractional wave equations and their damped variants have been studied in [8–12]. However, coupled systems with mixed fractional orders, such as (1.1), which combines a fractional Schrödinger-type equation with a fractionally damped Klein-Gordon equation remain largely unexplored, especially in the presence of nongauge nonlinearities.

Over the past decades, significant effort has been devoted to the classical (non-fractional) KGS system [13–17]. For the KGS system with gauge-invariant power nonlinearities, the existence of solutions was established by Hayashi [18] and Wang [19], while global well-posedness were studied by Shi and Wang [20–22], and the blowup under the assumption of huge negative energy was shown in [23].

Recall the scalar case, and the Schrödinger equation

$$i\partial_t\phi - \triangle\phi = g(\phi) \quad (x,t) \in \mathbb{R}^n \times \mathbb{R} \tag{1.2}$$

is known to admit global L^2 solutions for $g(\phi) = \mu|\phi|^{p-1}\phi$ with $1 < p < 1 + \frac{4}{n}$ and $\mu > 0$ [24]. In contrast, for $g(\phi) = \lambda|\phi|^p$ with $p \le 1 + \frac{4}{n}$ and $\lambda \in \mathbb{C} \setminus \{0\}$, finite-time blowup occurs even for small initial data [25–27]. These results motivate us for blowup for the coupled system (1.1).

The main difficulty in analyzing (1.1) lies in the combination of nongauge nonlinearities and mixed fractional operators, which together invalidate standard tools such as conservation laws and scaling arguments. In particular, the contradiction method used in [25, 26, 28, 29], which relies on taking the time $T \to \infty$, fails in our setting, especially when $m > 0$ [30].

We now state the main result. Let $H^s(\mathbb{R}^n)$ be the usual Sobolev space and the symbol $(\cdot,\cdot)$ denote the L^2 inner product. We write $h \lesssim g$ when there exists a positive constant C such that $h \le Cg$, and denote $h \sim g$ if both $g \lesssim h$ and $h \lesssim g$. Let $(d)_+ = \max\{d, 0\}$ for any $d \in \mathbb{R}$, and set $\frac{1}{(d)_+} = \infty$, when $d \le 0$. For a given positive time duration T, we hereby define the function space X_T as follows

$$\begin{aligned} X_T := \{\psi \in C([0,\infty); H^2(\mathbb{R}^n)) \cap C^1([0,\infty); H^\beta(\mathbb{R}^n)) \cap C^2([0,\infty); L^2(\mathbb{R}^n)) \\ : \operatorname{supp}\psi \subset [0,T) \times \mathbb{R}^n, \psi \text{ is } \mathbb{R}\text{-value}\}. \end{aligned} \tag{1.3}$$

Definition 1.1. *Let $u_0, v_1 \in L^2(\mathbb{R}^n), v_0 \in H^\beta(\mathbb{R}^n)$ and $T > 0$. We define (u,v) is a weak solution to Eq (1.1) on $[0,T)$, if $(u,v) \in L^1_{loc}(0,T; L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)) \cap L^1_{loc}(0,T; L^{2p}(\mathbb{R}^n)\times L^{2q}(\mathbb{R}^n))$ and the identities*

$$\begin{cases} \displaystyle\int_0^T (u, i\partial_t\varphi + (-\triangle)^{\frac{\alpha}{2}}\varphi)dt = \int_0^T (uv + \lambda|u|^p, \varphi)dt + i(u_0, \varphi_0), \\ \displaystyle\int_0^T (v, \partial_{tt}\varphi + m^2\varphi - \triangle\varphi + (-\triangle)^{\frac{\beta}{2}}\partial_t\varphi)dt = \int_0^T (|u|^2 + \mu|v|^q, \varphi)dt + (v_1, \varphi_0) \\ \qquad\qquad - (v_0, \varphi_1) + ((-\triangle)^{\frac{\beta}{2}}v_0, \varphi_0), \end{cases} \tag{1.4}$$

hold for any $\varphi \in X_T$, *where* $\varphi_0 = \varphi(0,x), \varphi_1 = \partial_t\varphi(0,x)$.

Proposition 1.2. *Let us fix* (p,q) *in* (1.1)*, such that*

$$1 < p < 1 + \frac{2\alpha}{n}, \quad 1 < q < 1 + \frac{2(1+\beta)}{(n-2(1+\beta))_+}, \tag{1.5}$$

for any data

$$(u_0, v_0, v_1) \in L^2(\mathbb{R}^n) \times H^\beta(\mathbb{R}^n) \times H^{\beta-1}(\mathbb{R}^n), \tag{1.6}$$

there exists a local weak solution (u,v) *defined by* (1.4).

The local existence of a weak solution to (1.1), under the conditions in Proposition 1.2, follows from a standard fixed-point argument applied to its Duhamel formulation. The requisite linear estimates for the fractional Schrödinger and damped wave operators are established in [4, 24, 31]. Combined with the control of the nongauge nonlinearities afforded by the exponent range (1.5), these estimates ensure the solution map is a contraction on a suitable function space over a small time interval, as justified by the well-posedness theory for mixed-order systems in [9].

Theorem 1.3. *Let* $v_1, u_0 \in L^2(\mathbb{R}^n), v_0 \in H^\beta(\mathbb{R}^n)$ *and* $T > 0$. *Let* (u,v) *denote the weak solution derived from Proposition 1.2 to* (1.1) *on [0,T). Moreover, we suppose that* $q > \frac{p}{p-1}, \mu > 0, -\Re\lambda - \frac{1}{p} > 0$ *and the initial data satisfies*

$$(\Im u_0 + v_1 + (-\triangle)^{\frac{\beta}{2}} v_0) \geq \begin{cases} \mu|x|^{-l}, & |x| \leq 1, \\ 0, & |x| > 1, \end{cases} \qquad v_0 \geq 0, \tag{1.7}$$

where $\mu > 0, 0 < l < \zeta - 1$, *where* $\zeta = \max\{\frac{2q}{q-1}, \frac{\alpha p}{p-1}\}$. *There exists* μ_0 *such that* (1.1) *does not have a global weak solution when* $\mu > \mu_0$. *Moreover, for any* $\mu \in [\mu_0, \infty)$*, the lifespan of the weak solution to* (1.1) *has the following upper estimate*

$$T \lesssim \mu^{\frac{-1}{(\zeta-l-1)}}. \tag{1.8}$$

While our proof also employs a contradiction argument in principle, it fundamentally diverges from the methodology established in [25–27]. The core distinction lies in the treatment of the time variable. The approach in the cited literature crucially relies on taking the limit as $T \to \infty$ to derive a contradiction against global existence. This strategy, however, is unavailable in our setting, primarily due to the lack of conservation laws and the presence of mixed fractional derivatives in system (1.1). To this end, we introduce a novel test function that is carefully tailored to accommodate the mixed fractional orders and the absence of conservation laws. This enables us to derive the contradiction directly on the finite time interval $[0,T)$, thereby establishing the blowup result for (1.1) for any $m > 0$, under constraints on the initial data alone.

The paper is organized as follows. In Section 2, we collect the necessary preliminaries, including the definition of the fractional Laplacian and the construction of a novel test function. A pointwise control property for this test function is established, which plays a crucial role in the subsequent analysis. Section 3 is devoted to the proof of Theorem 1.3, verifying the finite-time blowup of weak solutions and deriving the upper bound for the lifespan.

2. Preliminaries

Definition 2.1. *[32] Let $\alpha \in (0,2)$. Let X denote a suitable set of functions defined over $\mathbb{R}^n$. Then, the fractional operator $(-\triangle)^{\frac{\alpha}{2}}$ in $\mathbb{R}^n$ is characterized as a nonlocal operator, which is defined by*

$$(-\triangle)^{\frac{\alpha}{2}} : u \in X \to (-\triangle)^{\frac{\alpha}{2}} u := C_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+\alpha}} dy, \tag{2.1}$$

where P.V. stand for Cauchy principal value, $C_{n,\alpha}$ is a normalization constant.

Lemma 2.2. *Let $\lceil x \rfloor = \sqrt[\kappa]{1 + (|x|-1)^\kappa}$, and $\gamma = \min\{\alpha, \beta\}$, where κ is a constant more than two. Let $\rho(x)$ be a continuous piecewise function, defined as follows*

$$\rho(x) = \begin{cases} 1 & |x| \le 1 \\ \lceil x \rfloor^{-n-\gamma} & |x| \ge 1 \end{cases}. \tag{2.2}$$

Then for any $x \in \mathbb{R}^n, \rho \in C^2(\mathbb{R}^n)$, and there exists a positive constant $D_{n,\alpha}$, such that $(-\triangle)^{\frac{\alpha}{2}}\rho$ satisfies

$$|(-\triangle)^{\frac{\alpha}{2}}\rho(x)| \le D_{n,\alpha}\rho(x), \tag{2.3}$$

where $D_{n,\alpha}$ depends on n and α.

Proof. First, we denote $\hbar := |x|$ and consider the radial derivatives of $\rho(x)$

$$\nabla\rho(x) = \frac{x}{\hbar}\rho'(\hbar) = \begin{cases} 0, & \hbar \le 1 \\ -(n+\gamma)(\hbar-1)^{\kappa-1}\frac{x}{\hbar}\lceil x \rfloor^{-n-\gamma-\kappa}, & \hbar \ge 1 \end{cases} \tag{2.4}$$

and

$$\begin{aligned} \triangle\rho(x) &= \frac{n-1}{\hbar}\rho'(\hbar) + \rho''(\hbar) \\ &= \begin{cases} 0, & \hbar \le 1 \\ -(n+\gamma)(\hbar-1)^{\kappa-1}\frac{n-1}{\hbar}\lceil x \rfloor^{-n-\gamma-\kappa} & \\ -(\kappa-1)(n+\gamma)(\hbar-1)^{\kappa-2}\lceil x \rfloor^{-n-\gamma-\kappa} & \\ +(n+\gamma)(n+\gamma+3)(\hbar-1)^{2\kappa-2}\lceil x \rfloor^{-n-\gamma-2\kappa}, & \hbar \ge 1 \end{cases}. \end{aligned} \tag{2.5}$$

It can be readily ascertained that $\rho(x) \in C^2(\mathbb{R}^n)$ and $|\triangle\rho(x)| \le \lceil x \rfloor^{-n-\gamma}$. Hence, we consider only $0 < \alpha < 2$. In addition, from the above computation, we obtain $\|\rho\|_{L^\infty}, \|\partial_x^2\rho\|_{L^\infty} < \infty$, which enables us to remove the principle value of integral at the origin. Indeed, according to Definition 2.1 of fractional Laplacian, yields by variable transformation

$$\begin{aligned} (-\triangle)^{\frac{\alpha}{2}}\rho(x) &= -\frac{C_{n,\alpha}}{2} P.V. \int_{\mathbb{R}^n} \frac{\rho(x+y) + \rho(x-y) - 2\rho(x)}{|y|^{n+\alpha}} dy \\ &= -\frac{C_{n,\alpha}}{2} \lim_{\sigma \to 0+} \int_{\sigma \le |y| \le 1} \frac{\rho(x+y) + \rho(x-y) - 2\rho(x)}{|y|^{n+\alpha}} dy \\ &= -\frac{C_{n,\alpha}}{2} \int_{|y| \ge 1} \frac{\rho(x+y) + \rho(x-y) - 2\rho(x)}{|y|^{n+\alpha}} dy. \end{aligned} \tag{2.6}$$

Performing a second-order Taylor expansion for ρ, we obtain

$$\frac{\rho(x+y)+\rho(x-y)-2\rho(x)}{|y|^{n+\alpha}} \lesssim \frac{\|\partial_x^2\rho\|_{L^\infty}}{|y|^{n+\alpha-2}} \tag{2.7}$$

by these estimates and $\alpha \in (0,2)$, we remove the principal value of the integral at the origin to get

$$(-\triangle)^{\frac{\alpha}{2}}\rho(x) = -\frac{C_{n,\alpha}}{2}\int_{\mathbb{R}^n}\frac{\rho(x+y)+\rho(x-y)-2\rho(x)}{|y|^{n+\alpha}}dy. \tag{2.8}$$

To establish the intended conclusion, we bifurcate the problem into two distinct scenarios: $|x| \le 2$ and $|x| \ge 2$. For $|x| \le 2$, we partition the integral domain into the ensuing two distinct components

$$\Omega_1 = \{(x,y) : |y| \le 1\}, \Omega_2 = \{(x,y) : |y| \ge 1\}.$$

On Ω_1 and Ω_2, there are the following estimates, respectively

$$\int_{\Omega_1}\frac{\rho(x+y)+\rho(x-y)-2\rho(x)}{|y|^{n+\alpha}}dy \lesssim \|\partial_x^2\rho\|_{L^\infty}\int_{\Omega_1}\frac{1}{|y|^{n+\alpha-2}} \lesssim 1, \tag{2.9}$$

$$\int_{\Omega_2}\frac{\rho(x+y)+\rho(x-y)-2\rho(x)}{|y|^{n+\alpha}}dy \lesssim \|\rho\|_{L^\infty}\int_{\Omega_2}\frac{1}{|y|^{n+\alpha}} \lesssim 1. \tag{2.10}$$

For $|x| \ge 2$, we divide the integral domain into the following three segments:

$$\Omega_3 = \{(x,y) : |y| \ge 2|x|\}, \Omega_4 = \{(x,y) : \frac{1}{2}|x| \le |y| \le 2|x|\}, \Omega_5 = \{(x,y) : |y| \le \frac{1}{2}|x|\}.$$

In fact, there is $\lceil x \rfloor \sim |x| \sim |x|-1$ for every $|x| \in [2,\infty)$.
On Ω_3, we noticed that $|x \pm y| \ge |x| \ge 2$. with the help of the monotonicity of ρ, we have $\rho(x \pm y) \le \rho(x) = \lceil x \rfloor^{-n-\gamma}$. Hence, we obtain

$$\begin{aligned}\int_{\Omega_3}\frac{\rho(x+y)+\rho(x-y)-2\rho(x)}{|y|^{n+\alpha}}dy &\lesssim 4\rho(x)\int_{|y|\ge 2|x|}\frac{1}{|y|^{n+\alpha}}dy\\ &\lesssim \lceil x \rfloor^{-n-\gamma}\int_{|y|\ge 2|x|}\frac{1}{|y|^{1+\alpha}}d|y|\\ &\lesssim \lceil x \rfloor^{-n-\gamma}|x|^{-\alpha}\\ &\lesssim \lceil x \rfloor^{-n-\gamma-\alpha}.\end{aligned} \tag{2.11}$$

On Ω_4, by $|y| \sim |x|$ and $\Omega_4 \in \{y \in \mathbb{R}^n : |x \pm y| \le 3|x|\}$, we have

$$\begin{aligned}&\int_{\Omega_4}\frac{\rho(x+y)+\rho(x-y)-2\rho(x)}{|y|^{n+\alpha}}dy\\ &\lesssim |x|^{-n-\alpha}\Big(\int_{|x+y|\le 3|x|}\rho(x+y)dy + \int_{|x-y|\le 3|x|}\rho(x-y)dy + 2\rho(x)\int_{\frac{1}{2}|x|\le|y|\le 2|x|}1dy\Big)\\ &\lesssim \lceil x \rfloor^{-n-\alpha}\Big(\int_{|x+y|\le 3|x|}\rho(x+y)dy + \lceil x \rfloor^{-n-\gamma}|x|^n\Big)\\ &\lesssim \lceil x \rfloor^{-n-\alpha}(|x|^{-\gamma}+1)\\ &\lesssim \lceil x \rfloor^{-n-\gamma},\end{aligned} \tag{2.12}$$

here we used the following estimate

$$\begin{aligned}\int_{|x-y|\le 3|x|}\rho(x-y)dy &= \int_{|x+y|\le 3|x|}\rho(x+y)dy\\ &= \int_{2\le|x+y|\le 3|x|}\rho(x+y)dy+\int_{|x+y|\le 2}\rho(x+y)dy\\ &\lesssim \int_2^{3|x|}(1+(\hbar-1)^{\kappa})^{-(n+\gamma)/\kappa}\hbar^{n-1}d\hbar+\int_0^2\hbar^{n-1}d\hbar\\ &\lesssim \int_2^{3|x|}\hbar^{-\gamma-1}d\hbar+\int_0^2\hbar^{n-1}d\hbar\\ &\lesssim 1.\end{aligned} \tag{2.13}$$

Next, we treat the intergral on the third domain Ω_5. Upon conducting a second-order Taylor expansion for ρ we can deduce:

$$\int_{\Omega_5}\frac{\rho(x+y)+\rho(x-y)-2\rho(x)}{|y|^{n+\alpha}}dy\lesssim\int_{\Omega_5}\max_{\delta\in[0,1]}\partial_x^2\rho(x\pm\delta y)\frac{1}{|y|^{n+\alpha-2}}dy. \tag{2.14}$$

Next we consider the estimate of $\partial_x^2\rho(x\pm\delta y)$. Let $\hbar=|x+\delta y|,\delta\in[0,1]$, we obtain

$$\begin{aligned}\left|\partial_x^2\rho(x\pm\delta y)\right| &\lesssim \begin{cases}0 & \hbar\le 1,\\ \frac{(\hbar-1)^{\kappa-1}}{\hbar}\lceil x\rfloor^{-n-\gamma-\kappa}+(\hbar-1)^{\kappa-2}\lceil x\rfloor^{-n-\gamma-\kappa} & \\ +(\hbar-1)^{2\kappa-2}\lceil x\rfloor^{-n-\gamma-2\kappa} & \hbar\ge 1,\end{cases}\\ &\lesssim \begin{cases}0 & \hbar\leqslant 1,\\ \lceil x\rfloor^{-n-\gamma-2} & \hbar\geqslant 1.\end{cases}\end{aligned} \tag{2.15}$$

from $|x\pm\delta y|\ge|x|-\delta|y|\ge|x|-|x|/2\ge 1$ and the above estimate, we obtain:

$$|\partial_x^2\rho(x\pm\delta y)|\lesssim\lceil x\pm\delta y\rfloor^{-n-\gamma-2}.$$

If $|x\pm\delta y|\ge 2$, the $\lceil x\pm\delta y\rfloor\sim|x\pm\delta y|\ge|x|/2\gtrsim\lceil x\rfloor$, which implies

$$\lceil x\pm\delta y\rfloor^{-n-\gamma-2}\lesssim\lceil x\rfloor^{-n-\gamma-2}.$$

If $|x\pm\delta y|\le 2$, the $|x|/2\le|x\pm\delta y|\le 2$, which illustrates $2\le|x|\le 4$. Therefore, we may have $\lceil x\pm\delta y\rfloor\sim|x|\sim 1$ and $\lceil x\pm\delta y\rfloor^{-n-\gamma-2}\sim\lceil x\rfloor^{-n-\gamma-2}$, Then the following holds for all $\delta\in[0,1]$

$$|\partial_x^2\rho(x\pm\delta y)|\lesssim\lceil x\rfloor^{-n-\gamma-2}.$$

By (2.14), we have

$$\begin{aligned}\int_{\Omega_5}\frac{|\rho(x+y)+\rho(x-y)-2\rho(x)|}{|y|^{n+\alpha}}dy &\lesssim \lceil x\rfloor^{-n-\gamma-2}\int_{\Omega_5}\frac{1}{|y|^{n+\alpha-2}}dy\\ &\lesssim \lceil x\rfloor^{-n-\gamma-2}\int_0^{|x|/2}\frac{1}{|y|^{\alpha-1}}d|y|\\ &\lesssim \lceil x\rfloor^{-n-\gamma-2}|x|^{-\alpha+2}\\ &\lesssim \lceil x\rfloor^{-n-\gamma-\alpha}.\end{aligned} \tag{2.16}$$

Combining (2.9)–(2.12) and (2.16), we conclude that

$$|(-\triangle)^{\frac{\alpha}{2}}\rho| \leq \lceil x \rfloor^{-n-\gamma} \tag{2.17}$$

for all $x \in \mathbb{R}^n$, which completes our proof.

Lemma 2.3. *Let $0 < \alpha \leq 2$ and ρ be a smooth function satisfying $\partial_x^2 \varrho \in L^\infty(\mathbb{R}^n)$. For any $\tau > 0$, Let $\varrho_\tau(x) = \varrho(x/\tau)$. Then, for all $x \in \mathbb{R}^n$, the following identity can be obtained by direct calculation*

$$(-\triangle)^{\frac{\alpha}{2}}\varrho_\tau(x) = \tau^{-\alpha}((-\triangle)^{\frac{\alpha}{2}}\varrho)(x/\tau).$$

Proof. We prove the scaling identity for the fractional Laplacian by considering two cases separately.

Case 1: $0 < \alpha < 2$ For $0 < \alpha < 2$, we employ the singular integral representation of the fractional Laplacian

$$(-\triangle)^{\frac{\alpha}{2}} f(x) = C_{n,\alpha} \lim_{\epsilon \to 0^+} \int_{|y-x|>\epsilon} \frac{f(x) - f(y)}{|x-y|^{n+\alpha}} dy,$$

where $C_{n,\alpha}$ is a normalization constant.

Let $f(x) = \rho_\tau(x) = \rho(x/\tau)$. Substituting into the definition

$$(-\triangle)^{\frac{\alpha}{2}}\rho_\tau(x) = C_{n,\alpha} \lim_{\epsilon \to 0^+} \int_{|y-x|>\epsilon} \frac{\rho(x/\tau) - \rho(y/\tau)}{|x-y|^{n+\alpha}} dy.$$

Make the change of variables $\tilde{x} = x/\tau$, $\tilde{y} = y/\tau$. The condition $|y - x| > \epsilon$ becomes $|\tilde{y} - \tilde{x}| > \epsilon/\tau$. We obtain

$$(-\triangle)^{\frac{\alpha}{2}}\rho_\tau(x) = C_{n,\alpha} \lim_{\epsilon \to 0^+} \int_{|\tilde{y}-\tilde{x}|>\epsilon/\tau} \frac{\rho(\tilde{x}) - \rho(\tilde{y})}{\tau^{n+\alpha}|\tilde{x}-\tilde{y}|^{n+\alpha}} \tau^n d\tilde{y}.$$

Simplifying the powers of τ

$$(-\triangle)^{\frac{\alpha}{2}}\rho_\tau(x) = \tau^{-\alpha} C_{n,\alpha} \lim_{\epsilon \to 0^+} \int_{|\tilde{y}-\tilde{x}|>\epsilon/\tau} \frac{\rho(\tilde{x}) - \rho(\tilde{y})}{|\tilde{x}-\tilde{y}|^{n+\alpha}} d\tilde{y}.$$

Taking the limit $\epsilon \to 0^+$

$$(-\triangle)^{\frac{\alpha}{2}}\rho_\tau(x) = \tau^{-\alpha} C_{n,\alpha} \text{P.V.} \int_{\mathbb{R}^n} \frac{\rho(\tilde{x}) - \rho(\tilde{y})}{|\tilde{x}-\tilde{y}|^{n+\alpha}} d\tilde{y} = \tau^{-\alpha}(-\triangle)^{\frac{\alpha}{2}}\rho(\tilde{x}) = \tau^{-\alpha}\left((-\triangle)^{\frac{\alpha}{2}}\rho\right)(x/\tau).$$

Case 2: $\alpha = 2$ When $\alpha = 2$, the fractional Laplacian reduces to the classical regime. We compute directly using the chain rule.

Let $\rho_\tau(x) = \rho(x/\tau)$. For the first derivative

$$\partial_{x_j}\rho_\tau(x) = \frac{1}{\tau}(\partial_{x_j}\rho)(x/\tau).$$

For the second derivative

$$\partial_{x_j}^2\rho_\tau(x) = \frac{1}{\tau^2}(\partial_{x_j}^2\rho)(x/\tau).$$

Summing over all coordinates to form the Laplacian

$$\triangle\rho_\tau(x) = \sum_{j=1}^{n} \partial_{x_j}^2 \rho_\tau(x) = \frac{1}{\tau^2} \sum_{j=1}^{n} (\partial_{x_j}^2 \rho)(x/\tau) = \frac{1}{\tau^2}(\triangle\rho)(x/\tau).$$

Therefore,

$$(-\triangle)\rho_\tau(x) = -\triangle\rho_\tau(x) = -\frac{1}{\tau^2}(\triangle\rho)(x/\tau) = \frac{1}{\tau^2}(-\triangle\rho)(x/\tau).$$

Since $(-\triangle) = (-\triangle)^{2/2}$ for $\alpha = 2$, this confirms the identity

$$(-\triangle)^{\frac{\alpha}{2}}\rho_\tau(x) = \tau^{-\alpha}(-\triangle)^{\frac{\alpha}{2}}\rho(x/\tau)$$

holds for $\alpha = 2$ as well. This completes the proof for both cases.

Remark 2.4. This Condition $\partial_x^2\rho \in L^\infty(\mathbb{R}^n)$ ensures that the fractional Laplacian $(-\triangle)^{\alpha/2}\rho$ is well-defined in the principal value sense for $0 < \alpha < 2$, as it guarantees the necessary regularity and decay properties for the singular integrals to converge. For $\alpha = 2$, it ensures the classical derivatives exist in the required sense.

3. Proof of the main result

Proof. Let us initially introduce the supplementary functions as follows:

$$\eta(t) := \begin{cases} 1 - t, & 0 \le t \le 1, \\ 0, & t > 1, \end{cases} \qquad \rho_\tau(x) := \rho(x/\tau). \tag{3.1}$$

Then we define the test function

$$\varphi(t, x) = \rho_\tau(x)\eta(\frac{t}{\tau})^k, \tag{3.2}$$

for $\tau \in (0, T)$, where $k > 1$ is a large enough constant.

From (1.4) , using Young's inequality and taking the real part, we have

$$\begin{aligned} &\Im(u_0, \varphi(0, x)) + (v_1, \varphi(0, x) - (v_0, \varphi_t(0, x) + ((-\triangle)^{\frac{\beta}{2}} v_0, \varphi(0, x)) \\ &\qquad \le (\Re\lambda + \frac{1}{p})I_\tau - \widetilde{I_\tau} - \mu E_\tau + \int_0^\tau (\frac{1}{p'}|v|^{p'}, \varphi)dt \\ &\qquad - \Re\int_0^\tau (u, i\partial_t\varphi)dt - \Re\int_0^\tau (u, (-\triangle)^{\frac{\alpha}{2}}\varphi)dt + \int_0^\tau (v, \partial_{tt}\varphi)dt \\ &\qquad + \int_0^\tau (v, m^2\varphi)dt - \int_0^\tau (v, \triangle\varphi)dt + \int_0^\tau (v, (-\triangle)^{\frac{\beta}{2}}\partial_t\varphi)dt \\ &\qquad =: (\Re\lambda + \frac{1}{p})I_\tau - \widetilde{I_\tau} - \mu E_\tau + K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7, \end{aligned} \tag{3.3}$$

where

$$E_\tau := \int_0^\tau (|v|^q, \varphi)dt; \quad \widetilde{I_\tau} = \int_0^\tau (|u|^2, \varphi)dt; \quad I_\tau = \int_0^\tau (|u|^p, \varphi)dt,$$

and $\frac{1}{p}+\frac{1}{p'}=1, \frac{1}{q}+\frac{1}{q'}=1$.

We estimate each item from K_1-K_7. First, for K_1, by the Hölder's inequality, the uniform boundedness of $\eta(t)$ and the variable transformation $\tilde{x}=x/\tau$, we get

$$\begin{aligned}
|K_1| &\lesssim \int_0^\tau (|v|^{p'}, \varphi^{\frac{p'}{q}}\varphi^{\frac{q-p'}{q}})dt \\
&\lesssim (\int_0^\tau (|v|^q,\varphi)dt)^{\frac{p'}{q}}(\int_0^\tau\int_{\mathbb{R}^n}\eta(t)\rho_\tau(x)dxdt)^{\frac{q-p'}{q}} \\
&\lesssim E_\tau^{\frac{p'}{q}}\tau^{\frac{q-p'}{q}}(\int_{\mathbb{R}^n}\tau^n\rho(\tilde{x})d\tilde{x})^{\frac{q-p'}{q}} \\
&\lesssim E_\tau^{\frac{p'}{q}}\tau^{\frac{q-p'+n(q-p')}{q}}.
\end{aligned} \tag{3.4}$$

Second, we consider K_2

$$\begin{aligned}
|K_2| &\lesssim \tau^{-1}\int_0^\tau \left|\left(u,\rho_\tau(x)(\eta(\frac{t}{\tau}))^{k-1}\right)\right|dt \\
&\lesssim \tau^{-1}\int_0^\tau \left(|u|^2,\varphi\right)^{\frac{1}{2}}\left(\rho_\tau(x)\eta(\frac{t}{\tau})^k,(\eta(\frac{t}{\tau}))^{-2}\right)^{\frac{1}{2}}dt \\
&\lesssim \tau^{-1}\widetilde{I}_\tau^{\frac{1}{2}}\left(\int_0^\tau\left(\rho_\tau(x)\eta(\frac{t}{\tau})^k,(\eta(\frac{t}{\tau}))^{-2}\right)dt\right)^{1/2} \\
&\lesssim \tau^{-\frac{1}{2}}\widetilde{I}_\tau^{\frac{1}{2}}(\int_{\mathbb{R}^n}\rho_\tau(x)dx)^{\frac{1}{2}} \\
&\lesssim \tau^{\frac{n-1}{2}}\widetilde{I}_\tau^{\frac{1}{2}}.
\end{aligned} \tag{3.5}$$

Third, we consider K_3, K_4, K_5, K_6, by Lemmas 2 and 3 and Hölder's inequality

$$\begin{aligned}
|K_3| &\lesssim \int_0^\tau |(u,\eta(\frac{t}{\tau})^k(-\triangle)^{\frac{\alpha}{2}}\rho_\tau(x))|dt \\
&\lesssim \tau^{-\alpha}\int_0^\tau (|u|,\eta(\frac{t}{\tau})^k((-\triangle)^{\frac{\alpha}{2}}\rho)(\frac{x}{\tau}))dt \\
&\lesssim \tau^{-\alpha}I_\tau^{\frac{1}{p}}(\int_0^\tau\eta(\frac{t}{\tau})^kdt\int_{\mathbb{R}^n}\rho_\tau(x)dx)^{\frac{1}{p'}} \\
&\lesssim \tau^{\frac{n+1}{p'}-\alpha}I_\tau^{\frac{1}{p}},
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
|K_4| &\lesssim \tau^{-2}\int_0^\tau (|v|,(\eta(\frac{t}{\tau}))^{k-2}\rho_\tau(x))dt \\
&\lesssim \tau^{-2}\int_0^\tau (|v|,\varphi^{\frac{1}{q}})(\varphi^{\frac{1}{q'}},(\eta(\frac{t}{\tau}))^{-2})dt \\
&\lesssim \tau^{\frac{1+n}{q'}-2}E_\tau^{\frac{1}{q}},
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
|K_5| &\lesssim m^2(\int_0^\tau (|v|^q,\varphi)dt)^{\frac{1}{q}}(\int_0^\tau\int_{\mathbb{R}^n}\varphi dxdt)^{\frac{1}{q'}} \\
&\lesssim m^2\tau^{\frac{1+n}{q'}}E_\tau^{\frac{1}{q}},
\end{aligned} \tag{3.8}$$

$$
\begin{aligned}
|K_6| &\lesssim \tau^{-2}\int_0^\tau (|v|, (\eta(\frac{t}{\tau}))^k(\triangle\rho)(\frac{x}{\tau}))dt \\
&\lesssim \tau^{-2}E_\tau^{\frac{1}{q}}(\int_0^\tau \eta(\frac{t}{\tau})^k dt \int_{\mathbb{R}^n}\rho_\tau(x)dx)^{\frac{1}{q'}} \\
&\lesssim \tau^{\frac{1+n}{q'}-2}E_\tau^{\frac{1}{q}}.
\end{aligned} \tag{3.9}
$$

Finally, we consider K_7

$$
\begin{aligned}
|K_7| &\lesssim \tau^{-\beta-1}\int_0^\tau (|v|, (\eta(\frac{t}{\tau}))^{k-1}((-\triangle)^{\frac{\beta}{2}}\rho)(\frac{x}{\tau}))dt \\
&\lesssim \tau^{\frac{1+n}{q'}-\beta-1}E_\tau^{\frac{1}{q}}.
\end{aligned} \tag{3.10}
$$

By (3.3)–(3.10) and Young's inequality, we have

$$
\begin{aligned}
&(\Im u_0, \varphi(0,x)) + (v_1, \varphi(0,x)) - (v_0, \varphi_t(0,x)) + ((-\triangle)^{\frac{\beta}{2}}v, \varphi(0,x)) \\
&\qquad \lesssim \tau^{n-1} + \tau^{1+n-\alpha p'} + \tau^{1-2q'+n} + (m^{2q'}+1)\tau^{n+1} + \tau^{1-q'+n-\beta q'}.
\end{aligned} \tag{3.11}
$$

By (1.7), we obtain

$$
\begin{aligned}
&(\Im u_0 + v_1 + (-\triangle)^{\frac{\beta}{2}}v_0, \varphi(0,x)) - (v_0, \varphi_t(0,x) \\
&\qquad = \int_{\mathbb{R}^n}(\Im u_0 + v_1 + (-\triangle)^{\frac{\beta}{2}}v_0)\rho_\tau(x)dx + \int_{\mathbb{R}^n}(v_0 k\tau^{-1})\rho_\tau(x)dx \\
&\qquad \geq \int_{|x|\leq 1}\mu|x|^{-l}\rho_\tau(x)dx \\
&\qquad \geq \mu\tau^{n-l}\int_{\tilde{x}\leq\frac{1}{\tau}}|\tilde{x}|^{-l}\rho(\tilde{x})d\tilde{x}.
\end{aligned} \tag{3.12}
$$

Let $M(\tau) := \int_{\tilde{x}\leq\frac{1}{\tau}}|\tilde{x}|^{-l}\rho(\tilde{x})d\tilde{x}$. Then, by (3.11), we have

$$
\mu \lesssim (M(\tau))^{-1}((m^{2q'}+1)\tau^{l+1} + \tau^{l+1-2q'} + \tau^{l-1} + \tau^{l+1-\alpha p'} + \tau^{l+1-q'-\beta q'}). \tag{3.13}
$$

Claim. There exists μ_0 such that when $\mu > \mu_0$, $T \leq 3$.

We assume $T > 3$ and $\mu > \mu_0$. Taking $\tau = 3$, μ satisfies

$$
\mu \lesssim (M(3))^{-1}((m^{2q'}+1)3^{l+1} + 3^{l+1-2q'} + 3^{l-1} + 3^{l+1-\alpha p'} + 3^{l+1-q'-\beta q'}) := \mu_0. \tag{3.14}
$$

This contradicts $\mu > \mu_0$ since $M(3)$ is a finite positive constant. Therefore, the Claim is true.

Let $\mu > \mu_0$. Noting that M is decreasing, which implies that $M(\tau) > M(3)$ for any $\tau < T(< 3)$. For every $\tau \in [0, T)$,

$$
\begin{aligned}
\mu &\lesssim (M(\tau))^{-1}((m^{2q'}+1)\tau^{l+1} + \tau^{l+1-2q'} + \tau^{l-1} + \tau^{l+1-\alpha p'} + \tau^{l+1-q'-\beta q'}) \\
&\lesssim (M(3)^{-1}(3^\zeta m^{2q'+1}\tau^{l+1-\zeta} + 3^{\zeta-2q'}\tau^{l+1-\zeta} + 3^{\zeta-2}\tau^{l+1-\zeta} \\
&\quad + 3^{\zeta-\alpha p'}\tau^{l+1-\zeta} + 3^{\zeta-q'-\beta q'}\tau^{l+1-\zeta}) \\
&\lesssim \tau^{l+1-\zeta}.
\end{aligned} \tag{3.15}
$$

Hence, for every $\tau \in [0, T)$, $\tau \lesssim \mu^{\frac{-1}{(\zeta-l-1)}}$. We note $l + 1 - \zeta < 0$. Let $\tau \to T$, we have

$$T \lesssim \mu^{\frac{-1}{(\zeta-l-1)}}, \tag{3.16}$$

which completes the proof.

Remark 3.1. The lifespan estimate $T \lesssim \mu^{-\frac{1}{\zeta-l-1}}$ in (3.16) is expected to be "sharp" with respect to parameter μ, which quantifies the intensity of the initial singularity. This exponent can be heuristically justified via scaling analysis. Suppose the solution near the blow-up time behaves asymptotically as $|u(t, x)| \sim t^{-a}$, $|v(t, x)| \sim t^{-b}$. By neglecting the complex coupling and focusing on the dominant nonlinearities $|u|^p$ and $|v|^q$, a formal dimensional balance in system (1.1) yields relations among the exponents a and b. The critical index $\zeta = \max\left\{\frac{2q}{q-1}, \frac{\alpha p}{p-1}\right\}$ emerges from this process, synthesizing the competing influences of the two nonlinearities and the fractional dispersion. The exponent l, which governs the spatial singularity of the initial data, subsequently enters as a correction term in the final blow-up rate $1/(\zeta - l - 1)$.

Although the proof establishes only an upper bound for the lifespan, this rate is likely optimal for the considered class of initial data. Improving the exponent would necessitate a fundamentally different proof strategy or more restrictive initial conditions. However, deriving a matching lower bound remains a challenging open problem, primarily due to the absence of conserved energy and the nonlocality of the fractional operators, which obstructs the construction of precise self-similar blow-up profiles.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

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