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Research article

Two-dimensional Mittag-Leffler-Konhauser polynomials: k-fractional calculus and associated properties

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Abstract: In this study, we introduced and investigated new classes of k-Mittag-Leffler-Konhauser polynomials and bivariate k-Mittag-Leffler functions, which encompass several well-known 2D polynomials and Mittag-Leffler functions as special cases. We explored their key properties, including double k-Riemann-Liouville fractional calculus, double Laplace transforms, and k-fractional calculus operators. To further support the theoretical results, a new section presents numerical validation through simulation examples, illustrating the accuracy and practical applicability of the proposed formulas. We conclude by proposing several open questions to inspire further research and continued exploration in this area.

Keywords: Mittag-Leffler function; Konhauser polynomials; k-fractional integral and derivative; double Laplace transform; convolution integral equation

1. Introduction

Fractional calculus and its wide-ranging applications have attracted increasing attention in recent years. It serves as an important tool in mathematical analysis, enabling differentiation and integration to be extended to real or even complex orders. For instance, in [1], the authors established new relationships for uniformly k-analytic functions associated with the Mittag-Leffler function, while the authors of [2] developed fractional inequalities involving a generalized Mittag-Leffler function and demonstrated their applications in fractional calculus. Moreover, the authors of [3] analyzed Schrödinger-type operators on local generalized Morrey spaces and provided significant boundedness results related to nonnegative potentials. Several fractional calculus operators, along with their properties and applications, were introduced by Miller and Ross [4] and by Kiryakova [5]. In this context, a novel family of Mittag-Leffler–Gegenbauer polynomials constructed via the convolution of the classical Hermite polynomials with a three-parameter Mittag-Leffler function was deduced in [6] where certain formulae for the fractional integrals and derivatives were derived using Saigo fractional-type operators. The Mittag-Leffler function, which encompasses the Gamma function as a special case, is one of the most well-known special functions used in solving fractional-order integral and differential equations. It also plays a fundamental role in various fields of science and engineering. Numerous authors proposed and studied generalizations of the Mittag-Leffler function. For example, the authors of [7] developed fractional differential equations involving the generalized Mittag-Leffler function. In [8], the authors introduced generalized fractional integral operators and investigated their properties. The authors of [9] explored the fractional calculus of the bivariate Mittag-Leffler function, establishing new derivative and integral formulas. In addition, in [10], the authors presented a combinatorial framework for 2D-Hermite and 2D-Laguerre polynomials. In [11], several classes of bivariate orthogonal polynomials were analyzed. Pathway fractional integral operators for multi-parameter Mittag-Leffler functions and associated integral relations were derived in [12]. In 2007, Díaz and Pariguan [13] introduced the k-Pochhammer symbol and the kGamma function, which are defined, respectively, as follows:

$$(\gamma)_{n,k} = \begin{cases} \frac{\Gamma_k(\gamma+nk)}{\Gamma_k(\gamma)} & if\ \gamma \in \mathbb{C}\backslash\{0\}, k > 0 \\ \gamma(\gamma+k)(\gamma+2k)\dots(\gamma+(n-1)k) & if\ n \in \mathbb{N}, \gamma \in \mathbb{C}, k > 0, \end{cases} \tag{1.1}$$

and

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-\frac{t^k}{k}}\, dt \quad (z \in \mathbb{C} - k\mathbb{Z}^-, Re(z) > 0; k \in \mathbb{R}\backslash\{0\}. \tag{1.2}$$

Moreover, for the k-Pochhammer symbol $(\gamma)_{n,k}$ and k-Gamma function $\Gamma_k(z)$, we have the following properties:

$$(\gamma)_{n,k} = k^n \left(\frac{\gamma}{k}\right)_n, \tag{1.3}$$

$$(\gamma)_{n+m,k} = (\gamma)_{n,k}(\gamma+nk)_{m,k}, \tag{1.4}$$

$$\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right), \Gamma_k(k) = 1, \tag{1.5}$$

$$\Gamma_k(z+k) = z\Gamma_k(z). \tag{1.6}$$

Furthermore, the k-Beta function is defined as:

$$B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)} \quad (Re(x) > 0, Re(y) > 0). \tag{1.7}$$

The k-Beta function satisfies the following identities:

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}\, dt, \tag{1.8}$$

$$B_k(x,y) = \frac{1}{k}B\left(\frac{x}{k}, \frac{y}{k}\right). \tag{1.9}$$

Remark 1.1. *For $k = 1$ in (1.1), (1.2) and (1.7) reduces to the classical Pochhammer symbol, classical Gamma function, and classical Beta function respectively.*

Remark 1.2. *For any $\lambda \in \mathbb{C}, k > 0$ and $|x| < \frac{1}{k}$ the following identity holds*

$$\sum_{n=0}^{\infty}(\lambda)_{n,k}\frac{x^n}{n!} = (1-kx)^{-\frac{\lambda}{k}}. \tag{1.10}$$

The k-Mittag-Leffler function is introduced by Dorrego and Cerutti [14] as

$$E^{\lambda}_{k,\alpha,\beta}(x) = \sum_{r=0}^{\infty}\frac{(\lambda)_{r,k}}{\Gamma_k(\alpha r+\beta)}\frac{x^r}{r!} \quad (x,\alpha,\beta,\lambda \in \mathbb{C}; Re\{\alpha,\beta,\lambda\} > 0, k > 0). \tag{1.11}$$

It may be of interest to point out that the series representation (1.11), in particular, yields the following relationships:

$$E^{\lambda}_{1,\alpha,\beta}(z) = E^{\lambda}_{\alpha,\beta}(z),\ E^{1}_{1,\alpha,\beta}(z) = E_{\alpha,\beta}(z),\ E^{1}_{1,\alpha,1}(z) = E_{\alpha}(z),$$

where $E^{\lambda}_{\alpha,\beta}(z)$, $E_{\alpha,\beta}(z)$, and $E_{\alpha}(z)$ are Prabhakar Mittag-Leffler function [15], the Mittag-Leffler function of two parameters [16], and the Mittag-Leffler function of one parameter [17, 18], respectively. As an initial motivation, we consider the following class of generalized Laguerre-Konhauser polynomials were introduced in [9, 19]:

$${}_{\delta}L^{(\beta,\gamma)}_{n}(x,y) = n!\sum_{s=0}^{n}\sum_{r=0}^{n-s}\frac{(-1)^{r+s}x^{r+\beta}y^{\delta s+\gamma}}{s!r!(n-s-r)!\Gamma(\beta+r+1)\Gamma(\delta s+\gamma+1)}, \tag{1.12}$$

where $\beta,\gamma \in \mathbb{R}, \delta = 1, 2, \ldots.$

In 2018, Özarslan and Kürt [20] have considered Laguerre-Konhauser polynomials in (1.12) and defined a bivariate Mittag-Leffler functions $E^{\lambda}_{\alpha,\gamma,\delta}(z)$ by the series

$$E^{(\lambda)}_{\alpha,\gamma,\delta}(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\lambda)_{r+s}}{\Gamma(\alpha+r)\Gamma(\delta s+\gamma)}\frac{x^r}{r!}\frac{y^{\delta s}}{s!}, \tag{1.13}$$

where $\alpha,\gamma,\lambda,\delta \in \mathbb{C}, Re(\alpha) > 0, Re(\gamma) > 0, Re(\delta) > 0$, and $Re(\lambda) > 0$. From (1.12) and (1.13), we find that

$${}_{\delta}L^{(\alpha,\gamma)}_{n}(x,y) = x^{\alpha}y^{\gamma}E^{(-n)}_{\alpha+1,\gamma+1,\delta}(x,y).$$

Kürt and Özarslan [21] introduced a new class of 2D-k-Laguerre-Konhauser polynomials by

$$\begin{aligned}{}_{\delta}L^{(\beta,\gamma)}_{k,n}(x,y) &= n!\sum_{s=0}^{n}\sum_{r=0}^{n-s}\frac{(-1)^{r+s}\,k^{r+s}\,x^{r+\beta}\,y^{\delta s+\gamma}}{s!r!(n-s-r)!\Gamma_k(\beta+r+1)\Gamma_k(\delta s+\gamma+1)}\\ &= \sum_{s=0}^{n}\sum_{r=0}^{n-s}\frac{(-nk)_{r+s,k}}{\Gamma_k(\beta+r+1)\Gamma_k(\delta s+\gamma+1)}\frac{x^{r+\beta}}{r!}\frac{y^{\delta s+\gamma}}{s!},\end{aligned} \tag{1.14}$$

together with the extended type k-Mittag-Leffler function of two variables:

$$E^{(\lambda)}_{k,\beta,\gamma,\delta}(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\lambda)_{r+s,k}}{\Gamma_k(\beta+r)\Gamma_k(\delta s+\gamma)}\frac{x^r}{r!}\frac{y^{\delta s}}{s!}, \tag{1.15}$$

where $(\beta, \gamma, \lambda, \delta \in \mathbb{C}, Re(\beta) > 0, Re(\gamma) > 0, Re(\delta) > 0$, and $Re(\lambda) > 0, k \in \mathbb{R}^+)$. From (1.14) and (1.15), we infer that

$$_{\delta}L_{k,n}^{(\beta,\gamma)}(x,y) = x^{\beta}y^{\gamma}E_{k,\beta+1,\gamma+1,\delta}^{(-nk)}(x,y).$$

A class of 2D-Mittag-Leffler-Konhauser polynomials, represented by $_{E}Z_{n}^{(\alpha,\delta)}(x,y,\beta,\gamma)$, and the Mittag-Leffler function, represented by $E_{\alpha,\beta,\gamma,\delta}^{(\lambda)}(x,y)$, were proposed by Shahwan and Bin-Saad in [22] in the forms

$$_{E}Z_{n}^{(\alpha,\delta)}(x,y,\beta,\gamma) = n!\sum_{s=0}^{n}\sum_{r=0}^{n-s}\frac{(-1)^{s}x^{\alpha r}y^{\delta s+\gamma}}{s!r!(n-s-r)!\Gamma(\alpha r+\beta)\Gamma(\delta s+\gamma+1)}, \tag{1.16}$$

$$(\alpha,\beta,\delta,\gamma,x,y \in \mathbb{C}, \{Re(\alpha), Re(\beta), Re(\delta)\} > 0, Re(\gamma) > -1),$$

and

$$E_{\alpha,\beta,\gamma,\delta}^{(\lambda)}(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\lambda)_{r+s}}{\Gamma(\alpha r+\beta)\Gamma(\delta s+\gamma)}\frac{x^{\alpha r}}{r!}\frac{y^{\delta s}}{s!}, \tag{1.17}$$

$$(\alpha,\beta,\delta,\gamma,\lambda,x,y \in \mathbb{C}, \{Re(\alpha), Re(\beta), Re(\delta), Re(\gamma) > 0).$$

Using (1.16) and (1.17), we get the relation:

$$_{E}Z_{n}^{(\alpha,\delta)}(x,y,\beta,\gamma) = y^{\gamma}E_{\alpha,\beta,\gamma+1,\delta}^{(-n)}\left(x,(-1)^{\frac{1}{\delta}}y\right).$$

To contextualize the relevance of our work, we highlight several contributions in the control and synchronization of fractional-order systems. Advances in this field include the robust sliding mode method of Fu and Kao [23] for stabilizing discrete fractional difference chaotic systems under uncertainties. Kao et al. proposed finite-time adaptive sliding mode controllers for projective synchronization of uncertain fractional reaction-diffusion systems [24] and for synchronizing uncertain general fractional unified chaotic systems [25]. They also developed adaptive control schemes ensuring Mittag-Leffler synchronization in delayed fractional memristor neural networks [26], and established global Mittag-Leffler synchronization for coupled delayed fractional reaction–diffusion Cohen–Grossberg neural networks via sliding mode control [27]. Collectively, these works provide significant contributions to robust control and synchronization of fractional-order nonlinear systems.

The paper is organized as follows: The 2D-k-Mittag-Leffler-Konhauser polynomials $_{E}Z_{n,k}^{(\alpha,\delta)}(x,y,\beta,\gamma)$ and the bivariate k-Mittag-Leffler function $E_{k,\alpha,\beta,\gamma,\delta}^{(\lambda)}(x,y)$ will be introduced and formulated in Section 2. In Section 3, we study the k-Riemann-Liouville double fractional integral and derivative properties of $E_{k,\alpha,\beta,\gamma,\delta}^{(\lambda)}(x,y)$ and $_{E}Z_{k,n}^{(\alpha,\delta)}(x,y,\beta,\gamma)$. Section 4 is devoted to deriving double Laplace transform for the polynomials $_{E}Z_{k,n}^{(\alpha,\delta)}(x,y,\beta,\gamma)$ and the function $E_{k,\alpha,\beta,\gamma,\delta}^{(\lambda)}(x,y)$. In Section 5, we establish k-fractional calculus operators involving the bivariate k-Mittag-Leffler function $E_{k,\alpha,\beta,\gamma,\delta}^{(\lambda)}(x,y)$. Graphical illustrations involving numerical validation through simulation examples to examine the accuracy and practical applicability of the proposed formulas across representative parameter settings are provided in Section 6.

2. k-Mittag-Leffler-Konhauser polynomials

Given the above definitions and motivations, in this section, we introduce a new class of 2D-k-Mittag-Leffler-Konhauser polynomials ${}_EZ_{k,n}^{(\alpha,\delta)}(x,y,\beta,\gamma)$ defined by the series representation:

$$ {}_EZ_{k,n}^{(\alpha,\delta)}(x,y,\beta,\gamma)=n!\sum_{s=0}^{n}\sum_{r=0}^{n-s}\frac{(-k)^{r+s}x^{\alpha r+\beta}y^{\delta s+\gamma}}{s!r!(n-s-r)!\Gamma_k(\alpha r+\beta+1)\Gamma_k(\delta s+\gamma+1)}, \tag{2.1} $$

where $\alpha,\beta,\gamma,\delta,x,y\in\mathbb{C},\ \{Re(\alpha),Re(\delta)\}>0;\ \{Re(\beta),Re(\gamma)\}>-1$, or alternatively

$$ {}_EZ_{k,n}^{(\alpha,\delta)}(x,y,\beta,\gamma)=\sum_{s=0}^{n}\sum_{r=0}^{n-s}\frac{(-nk)_{r+s,k}x^{\alpha r+\beta}y^{\delta s+\gamma}}{s!r!\Gamma_k(\alpha r+\beta+1)\Gamma_k(\delta s+\gamma+1)}. \tag{2.2} $$

Discussion on the convergence of the series in (2.1): The convergence and well-posedness of the defining series in (2.1) require certain restrictions on the parameters. The conditions $Re(\alpha)>0$ and $Re(\delta)>0$ ensure that the powers $x^{\alpha r+\beta}$ and $y^{\delta s+\gamma}$ are analytically well defined for all admissible indices and that the arguments of the k-gamma functions grow in a controlled manner. Likewise, the restrictions $Re(\beta)>-1$ and $Re(\gamma)>-1$ guarantee that the arguments $\alpha r+\beta+1$ and $\delta s+\gamma+1$ lie in the domain of analyticity of the k-gamma function, thereby avoiding its poles. Since the series is finite in both indices, no additional convergence conditions are imposed on the variables x and y; thus, $x,y\in\mathbb{C}$ are admissible, with the powers remaining meaningful for all complex values under the above parameter constraints. Introducing the new class of two-dimensional k-Mittag-Leffler-Konhauser polynomials ${}_EZ_{k,n}^{(\alpha,\delta)}(x,y,\beta,\gamma)$ is motivated by their close connection with contemporary physical models in which nonclassical behavior arises from memory, deformation, or multiscale effects. Mittag-Leffler-type structures play a central role in capturing generalized relaxation patterns and complex hereditary responses beyond the standard exponential law. The two-variable setting further enables these polynomials to represent interactions between simultaneously evolving physical quantities, thereby providing a natural mathematical description for systems governed by multidirectional coupling. The deformation introduced through the k-gamma function $\Gamma_k(\cdot)$ enriches the algebraic structure of the model, offering additional flexibility useful in generalized statistical formalisms and deformed operator frameworks. Since classical Konhauser polynomials are strongly tied to orthogonal expansions and spectral methods, this two-dimensional k-extended version supplies enhanced analytical capability for representing systems with intricate coupling or structural complexity. Accordingly, the new polynomials furnish a versatile framework for modeling diverse situations where interaction geometry, scaling behavior, or deformation parameters play a decisive role.

Furthermore, by examining the structural form of the series representation in (2.2) together with the k-Mittag-Leffler function $E_{k,\alpha,\beta}^{\lambda}(x)$ defined in Eq (1.11), the polynomials ${}_EZ_{k,n}^{(\alpha,\delta)}(x,y,\beta,\gamma)$ can be expressed in the following more compact and elegant form:

$$ {}_EZ_{k,n}^{(\alpha,\delta)}(x,y,\beta,\gamma)=\sum_{s=0}^{n}\frac{(-nk)_{s,k}x^{\beta}y^{\delta s+\gamma}E_{k,\alpha,\beta+1}^{-nk+ks}(x^{\alpha})}{s!\Gamma_k(\delta s+\gamma+1)}. \tag{2.3} $$

Remark 2.1. *Beyond recovering several classical polynomial families as special cases, the representation in (2.1) equips the new class* ${}_EZ_{k,n}^{(\alpha,\delta)}(x,y,\beta,\gamma)$ *with additional structural flexibility that* *enables the treatment of problems not accessible to the classical theory. In particular, the simultaneous presence of the parameters $(\alpha,\delta,\beta,\gamma)$ together with the k-deformation enables one to model two-dimensional physical or mathematical processes exhibiting anisotropy, nonlocal memory effects, or fractional-order interactions. Such polynomials naturally arise in the analysis of coupled fractional differential equations, anomalous diffusion in heterogeneous media, deformed quantum systems governed by k-gamma statistics, and nonlinear evolution equations possessing Mittag-Leffler-type kernels. Thus, formula (2.1) provides a unified analytic framework capable of describing a broader class of multiscale, nonclassical, and deformation-based phenomena that cannot be adequately captured by the standard Konhauser or Mittag-Leffler polynomials.*

Remark 2.2. *It can be seen that :*

(i) For $\alpha = 1$, (2.1) reduces to a class of 2D-k-Laguerre-Konhauser polynomials ${}_{\delta}L^{(\beta,\gamma)}_{k,n}(x,y)$ defined by (1.14).
(ii) For $k = \alpha = 1$, (2.1) reduces to a class of generalized Laguerre-Konhauser polynomials ${}_{\delta}L^{(\beta,\gamma)}_{n}(x,y)$ defined by (1.12).
(iii) The series representation (2.1), in particular, produces the following relationships, which may be of interest:

$$ {}_{E}Z^{(1,\delta)}_{k,n}(x,y,\beta,\gamma) = x^{\beta}y^{\gamma}E^{(-nk)}_{k,\beta+1,\gamma+1,\delta}(x,y), \tag{2.4}$$
$$ {}_{E}Z^{(\alpha,\delta)}_{1,n}(x,y,\beta,\gamma) = {}_{E}Z^{(\alpha,\delta)}_{n}(x,y,\beta,\gamma), \tag{2.5}$$
$$ {}_{E}Z^{(\alpha,\delta)}_{1,n}(x,y,\beta,\gamma) = x^{\beta}y^{\gamma}E^{(-nk)}_{\alpha,\beta+1,\gamma+1,\delta}(x,y), \tag{2.6}$$
$$ {}_{E}Z^{(1,\delta)}_{1,n}(x,y,\beta,\gamma) = x^{\beta}y^{\gamma}E^{(-nk)}_{\beta+1,\gamma+1,\delta}(x,y). \tag{2.7}$$

As suggested by the formulas (2.2), (1.15), and (1.17), we introduce the following further generalization of the k-Mittag-Leffler functions (1.15) and (1.17) by

$$E^{(\lambda)}_{k,\alpha,\beta,\gamma,\delta}(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\lambda)_{r+s,k}}{\Gamma_k(\alpha r+\beta)\,\Gamma_k(\delta s+\gamma)}\frac{x^{\alpha r}}{r!}\frac{y^{\delta s}}{s!}, \tag{2.8}$$

where the parameters and variables satisfy

$$\alpha,\delta,\beta,\gamma,\lambda \in \mathbb{C}, \{Re(\alpha), Re(\delta), Re(\beta), Re(\gamma)\} > 0; \quad k > 0, \quad x,y \in \mathbb{C} \text{ (all finite)}.$$

Discussion on the convergence of the series in (2.8). The conditions $\{Re(\alpha), Re(\delta), Re(\beta), Re(\gamma)\} > 0$ ensure that the arguments of the k-gamma functions, $\Gamma_k(\alpha r+\beta)$ and $\Gamma_k(\delta s+\gamma)$, remain within their domain of analyticity, avoiding poles at non-positive integers, and that the powers $x^{\alpha r}$ and $y^{\delta s}$ grow in a controlled manner. The positivity of k guarantees proper behavior of the generalized k-Pochhammer symbol $(\lambda)_{r+s,k}$, while $x,y \in \mathbb{C}$ are admissible with finite values. Together, these conditions ensure the absolute convergence of the double series defining $E^{(\lambda)}_{k,\alpha,\beta,\gamma,\delta}(x,y)$.

Remark 2.3. *According to the convergence conditions investigated by Srivastava and Daoust ([28], p155), the series in (2.8) converges for any finite complex values of x and y because the factorial Γ-function growth in the denominator dominates the powers of x and y. Stating "all finite $x,y \in \mathbb{C}$" emphasizes that the only requirement is that x and y are finite complex numbers.*

Note that when $y \to 0$, in (2.8), we get a k-Mittag-Leffler function of one variable

$$E^{(\lambda)}_{k,\alpha,\beta,\gamma,\delta}(x,0) = \frac{1}{\Gamma_k(\gamma)} E^{\lambda}_{k,\alpha,\beta}(x^{\alpha}).$$

Moreover, comparing (2.1) and (2.8), we infer that

$$_{E}Z^{(\alpha,\delta)}_{k,n}(x,y,\beta,\gamma) = x^{\beta} y^{\gamma} E^{(-nk)}_{k,\alpha,\beta+1,\gamma+1,\delta}(x,y). \tag{2.9}$$

Next, we derive a linear generating function for the polynomials $_{E}Z^{(\alpha,\delta)}_{k,n}(x,y,\beta,\gamma)$ in terms of the bivariate k-Mittag-Leffler function $E^{(\lambda)}_{k,\alpha,\beta,\gamma,\delta}(x,y)$.

Theorem 2.1. *For $\alpha,\beta,\gamma,\delta,\lambda \in \mathbb{C}$ and $|kt| < 1$, we have*

$$\sum_{n=0}^{\infty} \frac{(\lambda)_{n,k}}{n!} {}_{E}Z^{(\alpha,\delta)}_{k,n}(x,y,\beta,\gamma) t^n$$
$$= x^{\beta} y^{\gamma} (1-kt)^{-\frac{\lambda}{k}} E^{(\lambda)}_{k,\alpha,\beta+1,\gamma+1,\delta}\left(x\left(\frac{kt}{kt-1}\right)^{\frac{1}{\alpha}}, y\left(\frac{kt}{kt-1}\right)^{\frac{1}{\delta}} \right). \tag{2.10}$$

Proof. Using the Cauchy product of the series, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\lambda)_{n,k}}{n!} {}_{E}Z^{(\alpha,\delta)}_{k,n}(x,y,\beta,\gamma) t^n \\
&= \sum_{n=0}^{\infty} \sum_{s=0}^{n} \sum_{r=0}^{n-s} \frac{(\lambda)_{n,k} (-k)^{r+s} x^{\alpha r+\beta} y^{\delta s+\gamma}}{s! r! (n-s-r)! \Gamma_k(\alpha r+\beta+1) \Gamma_k(\delta s+\gamma+1)} t^n \\
&= \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \sum_{r=0}^{n} \frac{(\lambda)_{n+s,k} (-k)^{r+s} x^{\alpha r+\beta} y^{\delta s+\gamma}}{s! r! (n-r)! \Gamma_k(\alpha r+\beta+1) \Gamma_k(\delta s+\gamma+1)} t^{n+s} \\
&= \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} \frac{(\lambda)_{n+r+s,k} (-k)^{r+s} x^{\alpha r+\beta} y^{\delta s+\gamma}}{s! r! \Gamma_k(\alpha r+\beta+1) \Gamma_k(\delta s+\gamma+1)} \frac{t^{n+r+s}}{n!}.
\end{aligned}$$

From $(\lambda)_{n+r+s,k} = (\lambda)_{r+s,k}(\lambda+kr+ks)_{n,k}$, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\lambda)_{n,k}}{n!} {}_{E}Z^{(\alpha,\delta)}_{k,n}(x,y,\beta,\gamma) t^n \\
&= \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} \frac{(\lambda)_{r+s,k} (-kt)^{r+s} x^{\alpha r+\beta} y^{\delta s+\gamma}}{s! r! \Gamma_k(\alpha r+\beta+1) \Gamma_k(\delta s+\gamma+1)} \sum_{n=0}^{\infty} (\lambda+kr+ks)_{n,k} \frac{t^n}{n!} \\
&= x^{\beta} y^{\gamma} (1-kt)^{-\frac{\lambda}{k}} \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} \frac{(\lambda)_{r+s,k}}{s! r! \Gamma_k(\alpha r+\beta+1) \Gamma_k(\delta s+\gamma+1)} \left(\frac{ktx^{\alpha}}{kt-1}\right)^r \left(\frac{kty^{\delta}}{kt-1}\right)^s \\
&= x^{\beta} y^{\gamma} (1-kt)^{-\frac{\lambda}{k}} E^{(\lambda)}_{k,\alpha,\beta+1,\gamma+1,\delta}\left(x\left(\frac{kt}{kt-1}\right)^{\frac{1}{\alpha}}, y\left(\frac{kt}{kt-1}\right)^{\frac{1}{\delta}} \right).
\end{aligned}$$

The order of summations has been interchanged, which is permissible due to the uniform convergence of the series under the constraint $|kt| < 1$. □

Remark 2.4. *As illustrative confirmations of Theorem 2.1, we now present two known results that arise as particular cases of the theorem.*
(i) *When $\alpha = 1$, Eq (2.10) reduces to Theorem 4 in [21]:*

$$\sum_{n=0}^{\infty} \frac{(\lambda)_{k,n}}{n!} {}_{\kappa}L_n^{(\alpha,\delta)}(x,y)t^n = x^{\beta} y^{\gamma} (1-kt)^{-\lambda} E_{k,\alpha+1,\beta+1,\kappa}^{(\lambda)} \left(x\left(\frac{t}{kt-1}\right), y\left(\frac{t}{kt-1}\right)^{\frac{1}{\delta}} \right).$$

(ii) When $k = \alpha = 1$, equation (2.10) reduces to Theorem 22 in [20]:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_{n}}{n!} {}_{\kappa}L_n^{(\alpha,\delta)}(x,y)t^n = x^{\beta} y^{\gamma} (1-t)^{-\lambda} E_{\alpha+1,\beta+1,\kappa}^{(\lambda)} \left(x\left(\frac{t}{t-1}\right), y\left(\frac{t}{t-1}\right)^{\frac{1}{\delta}} \right).$$

To provide a visual understanding of the polynomial's behavior, we generate the following surface plot illustrating its variation with respect to the variables x and y. This figure highlights how the polynomial grows and changes across the domain, revealing the key features of its structure and the influence of the parameters chosen.

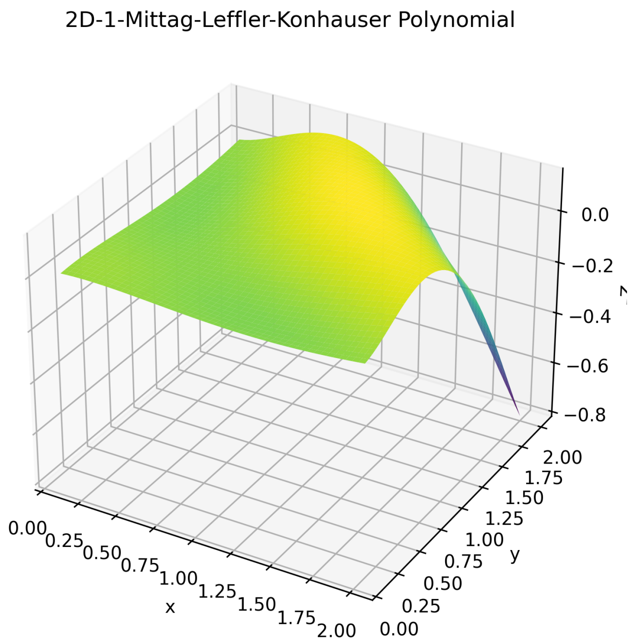

Figure 1. Surface plot of the 2D-k-Mittag-Leffler-Konhauser polynomial ${}_{E}Z_{n,k}^{(\alpha,\delta)}(x,y,\beta,\gamma)$ for $n = 5$, $k = 1$, $\alpha = \delta = \beta = \gamma = 1$. The figure illustrates the behavior of the polynomial with respect to the variables x and y, showing how it grows and varies across the domain.

3. A k-Riemann-Liouville fractional calculus framework

In this section, we study the k-Riemann-Liouville double fractional integral and derivative properties of $E_{k,\alpha,\beta,\gamma,\delta}^{(\lambda)}(x,y)$ and ${}_{E}Z_{k,n}^{(\alpha,\delta)}(x,y,\beta,\gamma)$.

Definition 3.1. *[29] Let $\mu \in \mathbb{R}^+$ and $n \in \mathbb{N}$ such that $n-1 < \mu < n, f \in L_1[0,\infty]$. Then the k-Riemann-Liouville integral of f is*

$$
{}_{k,x}I^{\mu}_{a^+} f(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x (x-t)^{\frac{\mu}{k}-1} f(t)\,dt, \quad x > a. \tag{3.1}
$$

When $k=1$, then ${}_{k,x}I^{\mu}_{a^+} f(x)$ reduces to the Riemann-Liouville fractional integration formula (see [30]). Similarly, k-Riemann-Liouville double fractional integral of $f(x,y)$ is defined as follows:

$$
{}_{k,x}I^{\mu}_{a^+}\,{}_{k,y}I^{\nu}_{b^+} f(x,y) = \frac{1}{k^2\Gamma_k(\mu)\Gamma_k(\nu)} \int_a^x \int_b^y (x-t)^{\frac{\mu}{k}-1} (y-\tau)^{\frac{\nu}{k}-1} f(t,\tau)\,d\tau\,dt, \tag{3.2}
$$

where $(x > a, y > b, Re(\mu) > 0, Re(\nu) > 0)$.

For the case $f(x) = (x-a)^{\frac{\beta}{k}-1}$, we get the result (see [29])

$$
{}_{k,x}I^{\mu}_{a^+}(x-a)^{\frac{\beta}{k}-1} = \frac{1}{k\Gamma_k(\mu)} \int_a^x (x-t)^{\frac{\mu}{k}-1} (t-a)^{\frac{\beta}{k}-1}\,dt = \frac{\Gamma_k(\beta)}{\Gamma_k(\beta+\mu)}(x-a)^{\frac{\beta+\mu}{k}-1}. \tag{3.3}
$$

Definition 3.2. *[31] Let $\nu, k \in \mathbb{R}^+$ and $n \in \mathbb{N}$ such that $n-1 < \mu < n, f \in L_1[0,\infty]$. Then the k-Riemann-Liouville derivative of f is*

$$
{}_{k,x}D^{\nu}_{a^+} f(x) = \left(\frac{d}{dx}\right)^n k^n\,{}_{k,x}I^{nk-\nu}_{a^+} f(x). \tag{3.4}
$$

Moreover, if $k = 1$ then ${}_{k,x}D^{\nu}_{a^+} f(x)$ coincides with the Riemann-Liouville fractional derivative (see [30]). Similarly, k-Riemann-Liouville double fractional derivative of $f(x,y)$ is defined as follows:

$$
{}_{k,x}D^{\mu}_{a^+}\,{}_{k,y}D^{\nu}_{b^+} f(x,y) = \left(\frac{\partial}{\partial x}\right)^n \left(\frac{\partial}{\partial y}\right)^m k^n k^m\,{}_{k,x}I^{nk-\mu}_{a^+} I^{mk-\nu}_{b^+} f(x,y). \tag{3.5}
$$

Theorem 3.1. *Let $\alpha, \beta, \gamma, \delta, \lambda \in \mathbb{C}, \{Re(\alpha), Re(\beta), Re(\gamma), Re(\delta), Re(\lambda), Re(\mu), Re(\nu)\} > 0, k \in \mathbb{R}^+$, we have*

$$
\begin{aligned}
&{}_{k,x}I^{\mu}_{a^+}\,{}_{k,y}I^{\nu}_{b^+} \left[(x-a)^{\frac{\beta}{k}-1}(y-b)^{\frac{\gamma}{k}-1} E^{(\lambda)}_{k,\alpha,\beta,\gamma,\delta}\left(\omega_1(x-a)^{\frac{1}{k}}, \omega_2(y-b)^{\frac{1}{k}}\right)\right] \\
&\quad = (x-a)^{\frac{\beta+\mu}{k}-1}(y-b)^{\frac{\gamma+\nu}{k}-1} E^{(\lambda)}_{k,\alpha,\beta+\mu,\gamma+\nu,\delta}\left(\omega_1(x-a)^{\frac{1}{k}}, \omega_2(y-b)^{\frac{1}{k}}\right).
\end{aligned} \tag{3.6}
$$

Proof. Upon interchanging the order of summation and fractional integration, which is permissible under the assumption started in the theorem and using definitions (2.1), (3.2) and the formula (3.3), we find

$$
\begin{aligned}
&{}_{k,x}I^{\mu}_{a^+}\,{}_{k,y}I^{\nu}_{b^+} \left[(x-a)^{\frac{\beta}{k}-1}(y-b)^{\frac{\gamma}{k}-1} E^{(\lambda)}_{k,\alpha,\beta,\gamma,\delta}\left(\omega_1(x-a)^{\frac{1}{k}}, \omega_2(y-b)^{\frac{1}{k}}\right)\right] \\
&\quad = \frac{1}{k^2\Gamma_k(\mu)\Gamma_k(\nu)} \int_a^x \int_b^y (x-t)^{\frac{\mu}{k}-1} (y-b)^{\frac{\nu}{k}-1} (t-a)^{\frac{\beta}{k}-1} (\tau-b)^{\frac{\gamma}{k}-1}\,d\tau\,dt \\
&\quad \times E^{(\lambda)}_{k,\alpha,\beta,\gamma,\delta}\left(\omega_1(t-a)^{\frac{1}{k}}, \omega_2(\tau-b)^{\frac{1}{k}}\right) d\tau\,dt \\
&\quad = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\lambda)_{r+s,k}}{\Gamma_k(\alpha r+\beta)\Gamma_k(\delta s+\gamma)} \frac{\omega_1^{\alpha r}}{r!} \frac{\omega_2^{\delta s}}{s!} \frac{1}{k\Gamma_k(\mu)} \int_a^x (x-t)^{\frac{\mu}{k}-1} (t-a)^{\frac{\alpha r+\beta}{k}-1}\,dt
\end{aligned}
$$

$$\times \frac{1}{k\Gamma_k(\nu)} \int_b^y (y-b)^{\frac{\nu}{k}-1} (\tau-b)^{\frac{\delta s+\gamma}{k}-1}\, d\tau$$
$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(x-a)^{\frac{\beta+\mu}{k}-1}(y-b)^{\frac{\gamma+\nu}{k}-1}(\lambda)_{r+s,k}}{\Gamma_k(\alpha r+\beta+\mu)\Gamma_k(\delta s+\gamma+\nu)} \frac{\left(\omega_1(x-a)^{\frac{1}{k}}\right)^{\alpha r}}{r!} \frac{\left(\omega_2(y-b)^{\frac{1}{k}}\right)^{\delta s}}{s!}$$
$$= (x-a)^{\frac{\beta+\mu}{k}-1}(y-b)^{\frac{\gamma+\nu}{k}-1} E^{(\lambda)}_{k,\alpha,\beta+\mu,\gamma+\nu,\delta}\left(\omega_1(x-a)^{\frac{1}{k}}, \omega_2(y-b)^{\frac{1}{k}}\right),$$

which completes the proof. □

Corollary 3.1. *As a consequence of (2.9) and Theorem 3.1, we have*

$$\begin{aligned} &{}_{k,x}I^{\mu}_{a^+}\,{}_{k,y}I^{\nu}_{b^+}\left[(x-a)^{-1}(y-b)^{-1}\,{}_{E}Z^{(\alpha,\delta)}_{k,n}\left(\omega_1(x-a)^{\frac{1}{k}}, \omega_2(y-b)^{\frac{1}{k}}, \beta, \gamma\right)\right] \\ &\quad = (x-a)^{\frac{\mu}{k}-1}(y-b)^{\frac{\nu}{k}-1}\,{}_{E}Z^{(\alpha,\delta)}_{k,n}\left(\omega_1(x-a)^{\frac{1}{k}}, \omega_2(y-b)^{\frac{1}{k}}, \beta+\mu, \gamma+\nu\right). \end{aligned} \tag{3.7}$$

Theorem 3.2. *For* $\alpha, \beta, \gamma, \delta, \lambda \in \mathbb{C}, \{Re(\alpha), Re(\beta), Re(\gamma), Re(\delta), Re(\lambda), Re(\mu), Re(\nu)\} > 0, k \in \mathbb{R}^+$, *we have*

$$\begin{aligned} &{}_{k,x}D^{\mu}_{a^+}\,{}_{k,y}D^{\nu}_{b^+}\left[(x-a)^{\frac{\beta}{k}-1}(y-b)^{\frac{\gamma}{k}-1} E^{(\lambda)}_{k,\alpha,\beta,\gamma,\delta}\left(\omega_1(x-a)^{\frac{1}{k}}, \omega_2(y-b)^{\frac{1}{k}}\right)\right] \\ &\quad = (x-a)^{\frac{\beta-\mu}{k}-1}(y-b)^{\frac{\gamma-\nu}{k}-1} E^{(\lambda)}_{k,\alpha,\beta-\mu,\gamma-\nu,\delta}\left(\omega_1(x-a)^{\frac{1}{k}}, \omega_2(y-b)^{\frac{1}{k}}\right). \end{aligned} \tag{3.8}$$

Proof. Upon interchanging the order of summation and fractional integration, which is permissible under the assumption started in the theorem and using definitions (2.1), (3.5) and the formulas (3.3), (1.5), it follows that

$$\begin{aligned} &{}_{k,x}D^{\mu}_{a^+}\,{}_{k,y}D^{\nu}_{b^+}\left[(x-a)^{\frac{\beta}{k}-1}(y-b)^{\frac{\gamma}{k}-1} E^{(\lambda)}_{k,\alpha,\beta,\gamma,\delta}\left(\omega_1(x-a)^{\frac{1}{k}}, \omega_2(y-b)^{\frac{1}{k}}\right)\right] \\ &= \left(\frac{\partial}{\partial x}\right)^n \left(\frac{\partial}{\partial y}\right)^m k^n k^m\,{}_{k,x}I^{nk-\mu}_{a^+} I^{mk-\nu}_{b^+} \\ &\times \left[(x-a)^{\frac{\beta}{k}-1}(y-b)^{\frac{\gamma}{k}-1} E^{(\lambda)}_{k,\alpha,\beta,\gamma,\delta}\left(\omega_1(x-a)^{\frac{1}{k}}, \omega_2(y-b)^{\frac{1}{k}}\right)\right] \\ &= \left(\frac{\partial}{\partial x}\right)^n \left(\frac{\partial}{\partial y}\right)^m k^n k^m \\ &\times \left[(x-a)^{\frac{\beta+nk-\mu}{k}-1}(y-b)^{\frac{\gamma+mk-\nu}{k}-1} E^{(\lambda)}_{k,\alpha,\beta+nk-\mu,\gamma+mk-\nu,\delta}\left(\omega_1(x-a)^{\frac{1}{k}}, \omega_2(y-b)^{\frac{1}{k}}\right)\right] \\ &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\lambda)_{r+s,k} k^n k^m}{\Gamma_k(\alpha r+\beta+nk-\mu)\Gamma_k(\delta s+\gamma+mk-\nu)} \frac{\omega_1^{\alpha r}}{r!}\frac{\omega_2^{\delta s}}{s!} \\ &\times \left(\frac{\partial}{\partial x}\right)^n (x-a)^{\frac{\alpha r+\beta+nk-\mu}{k}-1} \left(\frac{\partial}{\partial y}\right)^m (y-b)^{\frac{\delta s+\gamma+mk-\nu}{k}-1}. \end{aligned}$$

Since

$$\left(\frac{\partial}{\partial x}\right)^n (x-a)^{\frac{\alpha r+\beta+nk-\mu}{k}-1} = \frac{\Gamma\left(\frac{\alpha r+\beta+nk-\mu}{k}\right)}{\Gamma\left(\frac{\alpha r+\beta-\mu}{k}\right)}(x-a)^{\frac{\alpha r+\gamma s+\beta-\nu}{k}-1}$$
$$= \frac{\Gamma_k(\alpha r+\beta+nk-\mu)}{k^n\Gamma_k(\alpha r+\beta-\mu)}(x-a)^{\frac{\alpha r+\gamma s+\beta-\nu}{k}-1},$$

then, we obtain

$$
{}_{k,x}D^{\mu}_{a^+}{}_{k,y}D^{\nu}_{b^+}\left[(x-a)^{\frac{\beta}{k}-1}(y-b)^{\frac{\gamma}{k}-1}E^{(\lambda)}_{k,\alpha,\beta,\gamma,\delta}\left(\omega_1(x-a)^{\frac{1}{k}},\omega_2(y-b)^{\frac{1}{k}}\right)\right]
$$

$$
=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(x-a)^{\frac{\beta-\mu}{k}-1}(y-b)^{\frac{\gamma-\nu}{k}-1}(\lambda)_{r+s,k}}{\Gamma_k(\alpha r+\beta-\mu)\Gamma_k(\delta s+\gamma-\nu)}\frac{\left(\omega_1(x-a)^{\frac{1}{k}}\right)^{\alpha r}}{r!}\frac{\left(\omega_2(y-b)^{\frac{1}{k}}\right)^{\delta s}}{s!}.
$$

□

Corollary 3.2. *As a consequence of (2.9) and Theorem 3.2, we have*

$$
\begin{aligned}
&{}_{k,x}D^{\mu}_{a^+}{}_{k,y}D^{\nu}_{b^+}\left[(x-a)^{-1}(y-b)^{-1}{}_EZ^{(\alpha,\delta)}_{k,n}\left(\omega_1(x-a)^{\frac{1}{k}},\omega_2(y-b)^{\frac{1}{k}},\beta,\gamma\right)\right]\\
&=(x-a)^{-\frac{\mu}{k}-1}(y-b)^{-\frac{\nu}{k}-1}{}_EZ^{(\alpha,\delta)}_{k,n}\left(\omega_1(x-a)^{\frac{1}{k}},\omega_2(y-b)^{\frac{1}{k}},\beta-\mu,\gamma-\nu\right).
\end{aligned}\tag{3.9}
$$

As a particular importance for our results and applications is the case where we write $x=x^{\frac{1}{k}}$ *and* $y=y^{\frac{1}{k}}$ *in (2.2), we find that*

$$
{}_EZ^{(\alpha,\delta)}_{k,n}(x^{\frac{1}{k}},y^{\frac{1}{k}},\beta,\gamma)=\sum_{s=0}^{n}\sum_{r=0}^{n-s}\frac{(-nk)_{r+s,k}x^{\frac{\alpha r+\beta}{k}}y^{\frac{\delta s+\gamma}{k}}}{s!r!\Gamma_k(\alpha r+\beta+1)\Gamma_k(\delta s+\gamma+1)},\tag{3.10}
$$

therefore, intriguing operational expression for ${}_EZ^{(\alpha,\delta)}_{k,n}(x^{\frac{1}{k}},y^{\frac{1}{k}},\beta,\gamma)$ *can be obtained. In fact, since*

$$
{}_{k,x}I^{\alpha r}_0x^{\frac{\beta+1}{k}-1}=\frac{\Gamma_k(\beta+1)}{\Gamma_k(\alpha r+\beta+1)}x^{\frac{\alpha r+\beta+1}{k}-1},\tag{3.11}
$$

then

$$
{}_EZ^{(\alpha,\delta)}_{k,n}(x^{\frac{1}{k}},y^{\frac{1}{k}},\beta,\gamma)=\sum_{s=0}^{n}\sum_{r=0}^{n-s}\frac{(-nk)_{r+s,k}x^{1-\frac{1}{k}}y^{1-\frac{1}{k}}}{s!r!\Gamma_k(\beta+1)\Gamma_k(\gamma+1)}{}_{k,x}I^{\alpha r}_0{}_{k,y}I^{\delta s}_0\left\{x^{\frac{\beta+1}{k}-1}y^{\frac{\gamma+1}{k}-1}\right\}.\tag{3.12}
$$

Theorem 3.3. *The* ${}_EZ^{(\alpha,\delta)}_{k,n}(x^{\frac{1}{k}},y^{\frac{1}{k}},\beta,\gamma)$ *polynomials satisfy the following relation of Rodrigues-type formula:*

$$
{}_EZ^{(\alpha,\delta)}_{k,n}(x^{\frac{1}{k}},y^{\frac{1}{k}},\beta,\gamma)=x^{1-\frac{1}{k}}y^{1-\frac{1}{k}}\left[1-k\left({}_{k,x}I^{\alpha}_0+{}_{k,y}I^{\delta}_0\right)\right]^n\left\{\frac{x^{\frac{\beta+1}{k}-1}y^{\frac{\gamma+1}{k}-1}}{\Gamma_k(\beta+1)\Gamma_k(\gamma+1)}\right\}.\tag{3.13}
$$

Proof. Starting from the right-hand side of (3.13) and making use of the double-nominal formula

$$
(1-kx-ky)^{-\frac{\lambda}{k}}=\sum_{r,s=0}^{\infty}\frac{(\lambda)_{r+s,k}x^ry^s}{r!s!},
$$

we get

$$
x^{1-\frac{1}{k}}y^{1-\frac{1}{k}}\left[1-k\left({}_{k,x}I^{\alpha}_0+{}_{k,y}I^{\delta}_0\right)\right]^n\left\{\frac{x^{\frac{\beta+1}{k}-1}y^{\frac{\gamma+1}{k}-1}}{\Gamma_k(\beta+1)\Gamma_k(\gamma+1)}\right\}
$$

$$
=\sum_{s=0}^{n}\sum_{r=0}^{n-s}\frac{(-nk)_{r+s,k}x^{1-\frac{1}{k}}y^{1-\frac{1}{k}}}{s!r!\Gamma_k(\beta+1)\Gamma_k(\gamma+1)}{}_{k,x}I^{\alpha r}_0{}_{k,y}I^{\delta s}_0\left\{x^{\frac{\beta+1}{k}-1}y^{\frac{\gamma+1}{k}-1}\right\}.
$$

which given (3.12) gives the desired result (3.13). □

4. Double Laplace transforms

In this section, we obtain the double Laplace transform of $E^{(\lambda)}_{k,\alpha,\beta,\gamma,\delta}(x,y)$ and ${}_EZ^{(\alpha,\delta)}_{k,n}(x,y,\beta,\gamma)$. The double Laplace transform (see [32]) of $f(x,y)$ is defined as

$$\mathbb{L}_2[f(x,y)](p,q) = \int_0^\infty \int_0^\infty e^{-(px+qy)} f(x,y)\,dx\,dy \;\; (Re(p) > 0, Re(q) > 0). \tag{4.1}$$

The double Laplace transform of a power function is given by

$$\mathbb{L}_2\left[\frac{x^{\alpha-1}y^{\delta-1}}{\Gamma(\alpha)\Gamma(\delta)}\right](p,q) = \frac{1}{p^\alpha q^\delta} \quad Re(\alpha), Re(\delta) > -1, \tag{4.2}$$

therefore by using (1.5), we get

$$\mathbb{L}_2\left[\frac{x^{\frac{\alpha}{k}-1}x^{\frac{\delta}{k}-1}}{\Gamma_k(\alpha)\Gamma_k(\delta)}\right](p,q) = \frac{k^2}{k^{\frac{\alpha+\delta}{k}}}\mathcal{L}_2\left[\frac{x^{\frac{\alpha}{k}-1}y^{\frac{\delta}{k}-1}}{\Gamma\left(\frac{\alpha}{k}\right)\Gamma\left(\frac{\delta}{k}\right)}\right](s) = \frac{k^2}{(kp)^{\frac{\alpha}{k}}(kq)^{\frac{\delta}{k}}}, \tag{4.3}$$

where $Re\left(\frac{\alpha}{k}\right), Re\left(\frac{\delta}{k}\right) > -1$.

Theorem 4.1. *For $\alpha,\beta,\gamma,\delta,\lambda \in \mathbb{C}$, $Re(\alpha), Re(\delta), Re(p), Re(q) > 0$; $k \in \mathbb{R}^+$. Then we have*

$$\mathbb{L}_2\left[x^{\frac{\beta}{k}-1}y^{\frac{\gamma}{k}-1}E^{(\lambda)}_{k,\alpha,\beta,\gamma,\delta}(x^{\frac{1}{k}},y^{\frac{1}{k}})\right](p,q) = k^2(kp)^{-\frac{\beta}{k}}(kq)^{-\frac{\gamma}{k}}\left[1-k(kp)^{-\frac{\alpha}{k}}-k(kq)^{-\frac{\delta}{k}}\right]^{-\frac{\lambda}{k}}. \tag{4.4}$$

Proof. Interchanging the order of series and using formula (4.3), we have

$$\begin{aligned}\mathbb{L}_2\left[x^{\frac{\beta}{k}-1}y^{\frac{\gamma}{k}-1}E^{(\lambda)}_{k,\alpha,\beta,\gamma,\delta}(x^{\frac{1}{k}},y^{\frac{1}{k}})\right](p,q) &= \sum_{r=0}^\infty\sum_{s=0}^\infty \frac{(\lambda)_{r+s,k}}{r!s!}\mathcal{L}_2\left[\frac{x^{\frac{\alpha r+\beta}{k}-1}y^{\frac{\delta s+\gamma}{k}-1}}{\Gamma_k(\alpha r+\beta)\Gamma_k(\delta s+\gamma)}\right](p,q)\\ &= \frac{k^2}{(kp)^{\frac{\beta}{k}}(kq)^{\frac{\gamma}{k}}}\sum_{r=0}^\infty\sum_{s=0}^\infty\frac{(\lambda)_{r+s,k}}{r!s!}\left(\frac{1}{(kp)^{\frac{\alpha}{k}}}\right)^r\left(\frac{1}{(kq)^{\frac{\delta}{k}}}\right)^s\\ &= k^2(kp)^{-\frac{\beta}{k}}(kq)^{-\frac{\gamma}{k}}\left[1-k(kp)^{-\frac{\alpha}{k}}-k(kq)^{-\frac{\delta}{k}}\right]^{-\frac{\lambda}{k}},\end{aligned}$$

which is the required result. □

Corollary 4.1. *As a consequence of (2.9) and Theorem 4.1, we have*

$$\begin{aligned}&\mathbb{L}_2\left[x^{\frac{1}{k}-1}y^{\frac{1}{k}-1}{}_EZ^{(\alpha,\delta)}_{k,n}(x,y,\beta,\gamma)\right](p,q)\\ &= k^2(kp)^{-\frac{\beta+1}{k}}(kq)^{-\frac{\gamma+1}{k}}\left[1-k(kp)^{-\frac{\alpha}{k}}-k(kq)^{-\frac{\delta}{k}}\right]^n.\end{aligned} \tag{4.5}$$

Theorem 4.2. *Let $\alpha,\beta_1,\beta_2,\gamma_1,\gamma_2,\delta,\lambda_1,\lambda_2 \in \mathbb{C}; Re(\alpha) > 0, Re(\beta_i) > 0, Re(\gamma_i) > 0,$ $Re(\delta) > 0, Re(\lambda_i) > 0; k \in \mathbb{R}^+$. Then*

$$\begin{aligned}&\int_0^x\int_0^y (x-t)^{\frac{\beta_1}{k}-1}(y-\tau)^{\frac{\gamma_1}{k}-1}E^{(\lambda_1)}_{k,\alpha,\beta_1,\gamma_1,\delta}\left((x-t)^{\frac{1}{k}},(y-\tau)^{\frac{1}{k}}\right)\times t^{\frac{\beta_2}{k}-1}\tau^{\frac{\gamma_2}{k}-1}E^{(\lambda_2)}_{k,\alpha,\beta_2,\gamma_2,\delta}(t^{\frac{1}{k}},\tau^{\frac{1}{k}})\,dt\,d\tau\\ &= k^2x^{\frac{\beta_1+\beta_2}{k}-1}y^{\frac{\gamma_1+\gamma_2}{k}-1}E^{(\lambda_1+\lambda_2)}_{k,\alpha,\beta_1+\beta_2,\gamma_1+\gamma_2,\delta}(x^{\frac{1}{k}},y^{\frac{1}{k}}).\end{aligned} \tag{4.6}$$

Proof. Using the double convolution theorem for the double Laplace transform, we have

$$
\begin{aligned}
&\mathbb{L}_2\left\{\int_0^x\int_0^y (x-t)^{\frac{\beta_1}{k}-1}(y-\tau)^{\frac{\gamma_1}{k}-1}E^{(\lambda_1)}_{k,\alpha,\beta_1,\gamma_1,\delta}\left((x-t)^{\frac{1}{k}},(y-\tau)^{\frac{1}{k}}\right)\right.\\
&\left.\times t^{\frac{\beta_2}{k}-1}\tau^{\frac{\gamma_2}{k}-1}E^{(\lambda_2)}_{k,\alpha,\beta_2,\gamma_2,\delta}(t^{\frac{1}{k}},\tau^{\frac{1}{k}})\,dt\,d\tau\right\}(p,q)\\
&=\mathbb{L}_2\left[x^{\frac{\beta_1}{k}-1}y^{\frac{\gamma_1}{k}-1}E^{(\lambda_1)}_{k,\alpha,\beta_1,\gamma_1,\delta}(x^{\frac{1}{k}},y^{\frac{1}{k}})\right](p,q)\times\mathbb{L}_2\left[x^{\frac{\beta_2}{k}-1}y^{\frac{\gamma_2}{k}-1}E^{(\lambda_2)}_{k,\alpha,\beta_2,\gamma_2,\delta}(x^{\frac{1}{k}},y^{\frac{1}{k}})\right](p,q)\\
&=k^2(kp)^{-\frac{\beta_1}{k}}(kq)^{-\frac{\gamma_1}{k}}\left[1-k(kp)^{-\frac{\alpha}{k}}-k(kq)^{-\frac{\delta}{k}}\right]^{-\frac{\lambda_1}{k}}\times k^2(kp)^{-\frac{\beta_2}{k}}(kq)^{-\frac{\gamma_2}{k}}\left[1-k(kp)^{-\frac{\alpha}{k}}-k(kq)^{-\frac{\delta}{k}}\right]^{-\frac{\lambda_2}{k}}\\
&=k^4(kp)^{-\frac{\beta_1+\beta_2}{k}}(kq)^{-\frac{\gamma_1+\gamma_2}{k}}\left[1-k(kp)^{-\frac{\alpha}{k}}-k(kq)^{-\frac{\delta}{k}}\right]^{-\frac{\lambda_1+\lambda_2}{k}}\\
&=\mathbb{L}_2\left[k^2x^{\frac{\beta_1+\beta_2}{k}-1}y^{\frac{\gamma_1+\gamma_2}{k}-1}E^{(\lambda_1+\lambda_2)}_{k,\alpha,\beta_1+\beta_2,\gamma_1+\gamma_2,\delta}(x^{\frac{1}{k}},y^{\frac{1}{k}})\right](p,q).
\end{aligned}
$$

Taking the inverse Laplace transform of the above equation, the desired result follows. □

Corollary 4.2. *As a consequence of (2.9) and Theorem 4.2, we have*

$$
\begin{aligned}
&\int_0^x\int_0^y (x-t)^{\frac{1}{k}-1}(y-\tau)^{\frac{1}{k}-1}{}_EZ^{(\alpha,\delta)}_{k,m}(x-t,y-\tau,\beta_1,\gamma_1)\times t^{\frac{1}{k}-1}\tau^{\frac{1}{k}-1}{}_EZ^{(\alpha,\delta)}_{k,n}(t,\tau,\beta_2,\gamma_2)\,dt\,d\tau\\
&\qquad= k^2x^{\frac{1}{k}-1}y^{\frac{1}{k}-1}{}_EZ^{(\alpha,\delta)}_{k,m+n}(x,y,\beta_1+\beta_2,\gamma_1+\gamma_2).
\end{aligned}
\tag{4.7}
$$

5. Fractional k-integral operator

In this section, we consider the following double fractional k-integral operator

$$
\begin{aligned}
&\left({}_k\mathbf{E}^{\lambda,\omega_1,\omega_2}_{\alpha,\beta,\gamma,\delta,a^+,c^+}f\right)(x,y)\\
&=\frac{1}{k^2}\int_a^x\int_c^y (x-t)^{\frac{\beta}{k}-1}(y-\tau)^{\frac{\gamma}{k}-1}E^{\lambda}_{k,\alpha,\beta,\gamma,\delta}\left(\omega_1(x-t)^{\frac{1}{k}},\omega_2(y-\tau)^{\frac{1}{k}}\right)f(t,\tau)\,d\tau\,dt,
\end{aligned}
\tag{5.1}
$$

with $\alpha,\beta,\gamma,\delta,\lambda,\omega_1,\omega_2\in\mathbb{C}, Re\{\alpha,\beta,\gamma,\delta\}>0; k>0;\ x>a, y>c.$

Remark 5.1. *In the case $\lambda = 0$, the integral operator (5.1) reduces to the left-sided k-Riemann-Liouville double fractional integral operator, i.e.,*

$$
\left({}_k\mathbf{E}^{0,\omega_1,\omega_2}_{\alpha,\beta,\gamma,\delta,a^+,c^+}f\right)(x,y)=\left({}_{k,x}I^{\beta}_{a^+}{}_{k,y}I^{\gamma}_{c^+}f\right)(x,y).
\tag{5.2}
$$

The reason for this is that when $\lambda = 0$, the double series (2.8) collapses to a single constant term $E^{(0)}_{k,\alpha,\beta,\gamma,\delta}(x,y)=\frac{1}{\Gamma_k(\beta)}\frac{1}{\Gamma_k(\gamma)}$.

We now write the integral operator (5.1) in terms of k-Riemann-Liouville double fractional integrals using a series formula. This method has been used previously [33–35] to demonstrate valuable facts about numerous fractional calculus operators by reducing them to the k-Riemann-Liouville case.

Theorem 5.1. *For $\alpha,\beta,\gamma,\delta,\lambda \in \mathbb{C}$ with $Re\{\alpha,\beta,\gamma,\delta\} > 0; k \in \mathbb{R}^+$, and for any function $f \in L((a,b)\times(c,d))$, the fractional integral operator (5.1) can be written as*

$$\left({}_k\mathbf{E}^{\lambda,\omega_1,\omega_2}_{\alpha,\beta,\gamma,\delta,a^+,c^+}f\right)(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\lambda)_{r+s,k}\omega_1^r\omega_2^s}{r!s!}\left({}_{k,x}I^{\alpha r+\beta}_{a^+}\,{}_{k,y}I^{\delta s+\gamma}_{c^+}f\right)(x,y), \tag{5.3}$$

where the series on the right-hand side is locally uniformly convergent.

Proof. Since the bivariate k-Mittag-Leffler function (2.8) is convergent locally uniformly when $\{Re(\alpha), Re(\beta), Re(\gamma), Re(\delta)\} > 0$; $k \in \mathbb{R}^+$, we can interchange the summation and integration as follows:

$$\begin{aligned}
&\left({}_k\mathbf{E}^{\lambda,\omega_1,\omega_2}_{\alpha,\beta,\gamma,\delta,a^+,c^+}f\right)(x,y)\\
&= \frac{1}{k^2}\int_a^x\int_c^y (x-t)^{\frac{\beta}{k}-1}(y-\tau)^{\frac{\gamma}{k}-1}E^{\lambda}_{k,\alpha,\beta,\gamma,\delta}\left(\omega_1(x-t)^{\frac{1}{k}},\omega_2(y-\tau)^{\frac{1}{k}}\right)f(t,\tau)\,d\tau\,dt\\
&\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\lambda)_{r+s,k}\omega_1^r\omega_2^s}{r!s!}\frac{1}{k^2}\int_a^x\int_c^y (x-t)^{\frac{\alpha r+\beta}{k}-1}(y-\tau)^{\frac{\delta s+\gamma}{k}-1}f(t,\tau)\,d\tau\,dt\\
&\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\lambda)_{r+s,k}\omega_1^r\omega_2^s}{r!s!}\left({}_{k,x}I^{\alpha r+\beta}_{a^+}\,{}_{k,y}I^{\delta s+\gamma}_{c^+}f\right)(x,y).
\end{aligned}$$

□

Now, we begin by recalling the Dirichlet formula [36]:

$$\int_a^b\int_a^x f(x,y)\,dx\,dy = \int_a^b dy\int_y^b f(x,y)\,dx, \tag{5.4}$$

then we state the following results.

Theorem 5.2. *In the space $L((a,b)\times(c,d))$ of Lebesgue measurable functions,we show the integral operator ${}_k\mathbf{E}^{\lambda,\omega_1,\omega_2}_{\alpha,\beta,\gamma,\delta,a^+,c^+}$ is bounded, i.e.,*

$$\left\|{}_k\mathbf{E}^{\lambda,\omega_1,\omega_2}_{\alpha,\beta,\gamma,\delta,a^+,c^+}f\right\|_1 \le A\|f\|_1, \tag{5.5}$$

where A is a constant (independent of f) given by

$$A = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{|(\lambda)_{r+s,k}||\omega_1|^r|\omega_2|^s(b-a)^{\frac{Re(\alpha r+\beta)}{k}}(d-c)^{\frac{Re(\delta s+\gamma)}{k}}}{r!s!|\Gamma_k(\alpha r+\beta)||\Gamma_k(\delta s+\gamma)|Re(\alpha r+\beta)Re(\delta s+\gamma)} < \infty.$$

Proof. Let $f \in L((a,b)\times(c,d))$. By using the definitions of the operator (5.1), and the Dirichlet formula (5.4), we have

$$\left\|{}_k\mathbf{E}^{\lambda,\omega_1,\omega_2}_{\alpha,\beta,\gamma,\delta,a^+,c^+}f\right\|_1 \le \int_a^b\int_c^d |f(t,\tau)|\left(\frac{1}{k^2}\int_t^b\int_\tau^d (x-t)^{\frac{Re(\beta)}{k}-1}(y-\tau)^{\frac{Re(\gamma)}{k}-1}\right.$$

$$\times|E^{\lambda}_{k,\alpha,\beta,\gamma,\delta}\left(\omega_1(x-t)^{\frac{1}{k}},\omega_2(y-\tau)^{\frac{1}{k}}\right)|\,dx\,dy\Big)\,d\tau\,dt$$
$$= \int_a^b\int_c^d |f(t,\tau)|\left(\frac{1}{k^2}\int_0^{b-t}\int_0^{d-\tau} u^{\frac{Re(\beta)}{k}-1}v^{\frac{Re(\gamma)}{k}-1}\left|E^{\lambda}_{k,\alpha,\beta,\gamma,\delta}(\omega_1 u^{\frac{1}{k}},\omega_2 v^{\frac{1}{k}}\right|\,du\,dv\right)d\tau\,dt$$
$$\leq \int_a^b\int_c^d |f(t,\tau)|\left(\frac{1}{k^2}\int_0^{b-a}\int_0^{d-c} u^{\frac{Re(\beta)}{k}-1}v^{\frac{Re(\gamma)}{k}-1}\left|E^{\lambda}_{k,\alpha,\beta,\gamma,\delta}(\omega_1 u^{\frac{1}{k}},\omega_2 v^{\frac{1}{k}})\right|\,du\,dv\right)d\tau\,dt$$
$$\leq \frac{1}{k^2}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{|(\lambda)_{r+s,k}||\omega_1|^r|\omega_2|^s}{r!s!|\Gamma_k(\alpha r+\beta)||\Gamma_k(\delta s+\gamma)|}\int_0^{b-a}u^{\frac{Re(\alpha r+\beta)}{k}-1}\,du\int_0^{d-c}v^{\frac{Re(\delta s+\gamma)}{k}-1}\,dv\|f\|_1$$
$$= A\|f\|_1.$$

□

Now, we consider the composition of the double fractional integral operator ${}_{k,x}I^{\mu}_{a^+k,y}I^{\nu}_{c^+}$ with the operator ${}_{k}\mathbf{E}^{\lambda,\omega_1,\omega_2}_{\alpha,\beta,\gamma,\delta,a^+,c^+}$.

Theorem 5.3. *Let $\alpha,\beta,\gamma,\delta,\lambda,\omega_1,\omega_2,\mu,\nu\in\mathbb{C}$ with $Re\{\alpha,\beta,\gamma,\delta,\mu,\nu\}>0; k\in\mathbb{R}^+$. Then*

$$\begin{aligned}\left({}_{k,x}I^{\mu}_{a^+k,y}I^{\nu}_{c^+}\left({}_{k}\mathbf{E}^{\lambda,\omega_1,\omega_2}_{\alpha,\beta,\gamma,\delta,a^+,c^+}f\right)\right)(x,y) &= \left({}_{k}\mathbf{E}^{\lambda,\omega_1,\omega_2}_{\alpha,\beta+\mu,\gamma+\nu,\delta,a^+,c^+}f\right)(x,y)\\ &= \left({}_{k}\mathbf{E}^{\lambda,\omega_1,\omega_2}_{\alpha,\beta,\gamma,\delta,a^+,c^+}\left({}_{k,x}I^{\mu}_{a^+k,y}I^{\nu}_{c^+}f\right)\right)(x,y)\end{aligned}\tag{5.6}$$

holds for any summable function $f\in L((a,b)\times(c,d))$.

Proof. Using (3.2) and (5.1) implies

$$\left({}_{k,x}I^{\mu}_{a^+k,y}I^{\nu}_{c^+}\left({}_{k}\mathbf{E}^{\lambda,\omega_1,\omega_2}_{\alpha,\beta,\gamma,\delta,a^+,c^+}f\right)\right)(x,y)$$
$$= \frac{1}{k^2\Gamma_k(\mu)\Gamma_k(\nu)}\int_a^x\int_c^y (x-t)^{\frac{\mu}{k}-1}(y-\tau)^{\frac{\nu}{k}-1}\left({}_{k}\mathbf{E}^{\lambda,\omega_1,\omega_2}_{\alpha,\beta,\gamma,\delta,a^+,c^+}f\right)(t,\tau)\,d\tau\,dt$$
$$= \frac{1}{k^4\Gamma_k(\mu)\Gamma_k(\nu)}\int_a^x\int_c^y\int_a^t\int_c^\tau (x-t)^{\frac{\mu}{k}-1}(y-\tau)^{\frac{\nu}{k}-1}(t-u)^{\frac{\beta}{k}-1}(\tau-v)^{\frac{\gamma}{k}-1}$$
$$\times E^{\lambda}_{k,\alpha,\beta,\gamma,\delta}\left(\omega_1(t-u)^{\frac{1}{k}},\omega_2(\tau-v)^{\frac{1}{k}}\right)f(u,v)\,dv\,du\,d\tau\,dt.$$

Next, by applying the Dirichlet formula (5.4) and using (3.6), we obtain

$$\frac{1}{k^4\Gamma_k(\mu)\Gamma_k(\nu)}\int_a^x\int_c^y\left[\int_u^x\int_v^y (x-t)^{\frac{\mu}{k}-1}(y-\tau)^{\frac{\nu}{k}-1}(t-u)^{\frac{\beta}{k}-1}(\tau-v)^{\frac{\gamma}{k}-1}\right.$$
$$\left.\times E^{\lambda}_{k,\alpha,\beta,\gamma,\delta}\left(\omega_1(t-u)^{\frac{1}{k}},\omega_2(\tau-v)^{\frac{1}{k}}\right)d\tau\,dt\right]f(u,v)\,dv\,du$$
$$= \frac{1}{k^2}\int_a^x\int_c^y\left[\frac{1}{k^2\Gamma_k(\mu)\Gamma_k(\nu)}\int_u^x\int_v^y (x-t)^{\frac{\mu}{k}-1}(y-\tau)^{\frac{\nu}{k}-1}(t-u)^{\frac{\beta}{k}-1}(\tau-v)^{\frac{\gamma}{k}-1}\right.$$
$$\left.\times E^{\lambda}_{k,\alpha,\beta,\gamma,\delta}\left(\omega_1(t-u)^{\frac{1}{k}},\omega_2(\tau-v)^{\frac{1}{k}}\right)d\tau\,dt\right]f(u,v)\,dv\,du$$
$$= \frac{1}{k^2}\int_a^x\int_c^y\left[{}_{k,x}I^{\mu}_{u}{}_{k,y}I^{\nu}_{v}(x-u)^{\frac{\beta}{k}-1}(y-v)^{\frac{\gamma}{k}-1}E^{\lambda}_{k,\alpha,\beta,\gamma,\delta}\left(\omega_1(x-u)^{\frac{1}{k}},\omega_2(y-v)^{\frac{1}{k}}\right)\right]$$

$$\times f(u,v)\,dv\,du$$
$$= \frac{1}{k^2}\int_a^x\int_c^y (x-u)^{\frac{\beta+\mu}{k}-1}(y-v)^{\frac{\gamma+\nu}{k}-1}$$
$$\times E^{\lambda}_{k,\alpha,\beta+\mu,\gamma+\nu,\delta}\left(\omega_1(x-u)^{\frac{1}{k}},\omega_2(y-v)^{\frac{1}{k}}\right)f(u,v)\,dv\,du$$
$$= \left({}_k\mathbf{E}^{\lambda,\omega_1,\omega_2}_{\alpha,\beta+\mu,\gamma+\nu,\delta,a^+,c^+}f\right)(x,y).$$

In similar steps, we can prove the other side. □

Theorem 5.4. *Let $\alpha,\beta_1,\beta_2,\gamma_1,\gamma_2,\lambda_1,\lambda_2,\omega_1,\omega_2\in\mathbb{C}$ with $Re\{\alpha,\beta_i,\gamma_i,\lambda_i\}>0; k>0$. Then*

$$\left({}_k\mathbf{E}^{\lambda_1,\omega_1,\omega_2}_{\alpha,\beta_1,\gamma_1,\delta,a^+,c^+}\,{}_k\mathbf{E}^{\lambda_2,\omega_1,\omega_2}_{\alpha,\beta_2,\gamma_2,\delta,a^+,c^+}f\right)(x,y)=\left({}_k\mathbf{E}^{\lambda_1+\lambda_2,\omega_1,\omega_2}_{\alpha,\beta_1+\beta_2,\gamma_1+\gamma_2,a^+,c^+}f\right)(x,y), \tag{5.7}$$

holds for any summable function $f\in L((a,b)\times(c,d))$. In particular, setting $\lambda_2=-\lambda_1$ and using relation (5.2), we find that

$$\left({}_k\mathbf{E}^{\lambda_1,\omega_1,\omega_2}_{\alpha,\beta_1,\gamma_1,\delta,a^+,c^+}\,{}_k\mathbf{E}^{-\lambda_1,\omega_1,\omega_2}_{\alpha,\beta_2,\gamma_2,\delta,a^+,c^+}f\right)(x,y)=\left({}_{k,x}I^{\beta_1+\beta_2}_{a^+}\,{}_{k,y}I^{\gamma_1+\gamma_2}_{c^+}f\right)(x,y). \tag{5.8}$$

Proof. Using (5.1) and applying the Dirichlet integral formula (5.4), we find that

$$\left({}_k\mathbf{E}^{\lambda_1,\omega_1,\omega_2}_{\alpha,\beta_1,\gamma_1,\delta,a^+,c^+}\,{}_k\mathbf{E}^{\lambda_2,\omega_1,\omega_2}_{\alpha,\beta_2,\gamma_2,\delta,a^+,c^+}f\right)(x,y)$$
$$= \frac{1}{k^2}\int_a^x\int_c^y (x-t)^{\frac{\beta_1}{k}-1}(y-\tau)^{\frac{\gamma_1}{k}-1}E^{\lambda_1}_{k,\alpha,\beta_1,\gamma_1,\delta}\left(\omega_1(x-t)^{\frac{1}{k}},\omega_2(y-\tau)^{\frac{1}{k}}\right)$$
$$\times\left({}_k\mathbf{E}^{\lambda_2,\omega_1,\omega_2}_{\alpha,\beta_2,\gamma_2,\delta,a^+,c^+}f\right)(t,\tau)\,d\tau\,dt$$
$$= \frac{1}{k^4}\int_a^x\int_c^y\int_a^t\int_c^\tau (x-t)^{\frac{\beta_1}{k}-1}(y-\tau)^{\frac{\gamma_1}{k}-1}(t-u)^{\frac{\beta_2}{k}-1}(\tau-v)^{\frac{\gamma_2}{k}-1}$$
$$\times E^{\lambda_1}_{k,\alpha,\beta_1,\gamma_1,\delta}\left(\omega_1(x-t)^{\frac{1}{k}},\omega_2(y-\tau)^{\frac{1}{k}}\right)$$
$$\times E^{\lambda_2}_{k,\alpha,\beta_2,\gamma_2,\delta}\left(\omega_1(t-u)^{\frac{1}{k}},\omega_2(\tau-v)^{\frac{1}{k}}\right)f(u,v)\,dv\,du\,d\tau\,dt$$
$$= \frac{1}{k^4}\int_a^x\int_c^y f(u,v)\left[\int_u^x\int_v^y (x-t)^{\frac{\beta_1}{k}-1}(y-\tau)^{\frac{\gamma_1}{k}-1}(t-u)^{\frac{\beta_2}{k}-1}(\tau-v)^{\frac{\gamma_2}{k}-1}\right.$$
$$\times E^{\lambda_1}_{k,\alpha,\beta_1,\gamma_1,\delta}\left(\omega_1(x-t)^{\frac{1}{k}},\omega_2(y-\tau)^{\frac{1}{k}}\right)$$
$$\left.\times E^{\lambda_2}_{k,\alpha,\beta_2,\gamma_2,\delta}\left(\omega_1(t-u)^{\frac{1}{k}},\omega_2(\tau-v)^{\frac{1}{k}}\right)d\tau\,dt\right]dv\,du$$
$$= \frac{1}{k^4}\int_a^x\int_c^y f(u,v)\left[\int_0^{x-u}\int_0^{y-v}(x-u-\zeta)^{\frac{\beta_1}{k}-1}(y-v-\eta)^{\frac{\gamma_1}{k}-1}\zeta^{\frac{\beta_2}{k}-1}\eta^{\frac{\gamma_2}{k}-1}\right.$$
$$\times E^{\lambda_1}_{k,\alpha,\beta_1,\gamma_1,\delta}\left(\omega_1(x-u-\zeta)^{\frac{1}{k}},\omega_2(y-v-\eta)^{\frac{1}{k}}\right)$$
$$\left.\times E^{\lambda_2}_{k,\alpha,\beta_2,\gamma_2,\delta}\left(\omega_1\zeta^{\frac{1}{k}},\omega_2\eta^{\frac{1}{k}}\right)d\eta\,d\zeta\right]dv\,du.$$

Now, we use direct integration, and the result (4.7), obtaining

$$\left({}_k\mathbf{E}^{\lambda_1,\omega_1,\omega_2}_{\alpha,\beta_1,\gamma_1,\delta,a^+,c^+}\,{}_k\mathbf{E}^{\lambda_2,\omega_1,\omega_2}_{\alpha,\beta_2,\gamma_2,\delta,a^+,c^+}f\right)(x,y)$$

$$= \frac{1}{k^2}\int_a^x\int_c^y f(u,v)\left[(x-u)^{\frac{\beta_1+\beta_2}{k}-1}(y-v)^{\frac{\gamma_1+\gamma_2}{k}-1}\right.$$

$$\left.\times E^{\lambda_1+\lambda_2}_{k,\alpha,\beta_1+\beta_2,\gamma_1+\gamma_2,\delta}\left(\omega_1(x-u)^{\frac{1}{k}},\omega_2(y-v)^{\frac{1}{k}}\right)\right]dv\,du$$

$$= \left({}_k\mathbf{E}^{\lambda_1+\lambda_2,\omega_1,\omega_2}_{\alpha,\beta_1+\beta_2,\gamma_1+\gamma_2,a^+,c^+}f\right)(x,y),$$

which gives the desired result. □

6. Numerical validation through simulation examples

To complement the theoretical analysis presented earlier, we provide several simulation examples to validate the derived results. These experiments are implemented in Python, using NumPy for numerical computations and Matplotlib for visualization. The examples demonstrate the accuracy and applicability of the theoretical formulas across various parameter settings, highlighting both the robustness of the approach and its potential practical implications.

Example 6.1. *To support the validity of formula (3.6), we carry out a numerical simulation in the special case* $k = 1$*. Choosing the representative parameters* $a = b = 0$*,* $\alpha = \delta = 1$*,* $\beta = \gamma = \mu = \nu = 1$*,* $\lambda = 2$*, and* $\omega_1 = \omega_2 = 0.5$*, we evaluate both sides of (3.6) using a truncated double series for the Mittag–Leffler function together with numerical quadrature for the double Riemann–Liouville fractional integral. The computations show that the left-hand side and right-hand side agree to within an absolute error below* 10^{-6} *for all tested points in the domain* $(x,y) \in [0.2,1]^2$*. This close agreement demonstrates the numerical correctness of the fractional integration identity in the classical case* $k = 1$*, thereby providing strong evidence for the reliability of formula (3.6) in the general k-framework.*

Note that, the numerical simulations validating formula (3.6) were performed using `Python 3.x` with the `NumPy` and `SciPy` libraries for numerical computations, including evaluation of truncated double series and double Riemann-Liouville fractional integrals, while `Matplotlib` is employed for visualizing the absolute errors across the domain.

Example 6.2. *To further support the correctness of the derived k-Riemann-Liouville fractional derivative formula, we perform a numerical experiment in which the double fractional derivative was computed in two ways: (i) Directly using its definition via a nested numerical integration, and (ii) using the explicit theoretical formula developed in Section 3. A grid of* (x,y)*-values is evaluated, and the absolute error between the two computations was recorded. The resulting error surface, shown in Figure 3, demonstrates excellent agreement between the two methods, with errors remaining extremely small throughout the entire computational domain. This provides strong numerical evidence supporting the validity of our theoretical results.*

Example 6.3. *Here, we discuss the numerical validation of the Laplace transform* (4.4) *implemented by python software. To validate the theoretical result (4.4), we perform a numerical simulation of the double Laplace transform:*

$$f(x,y) = x^{\frac{\beta}{k}-1}y^{\frac{\gamma}{k}-1}E^{(\lambda)}_{k,\alpha,\beta,\gamma,\delta}(x^{\frac{1}{k}},y^{\frac{1}{k}}).$$

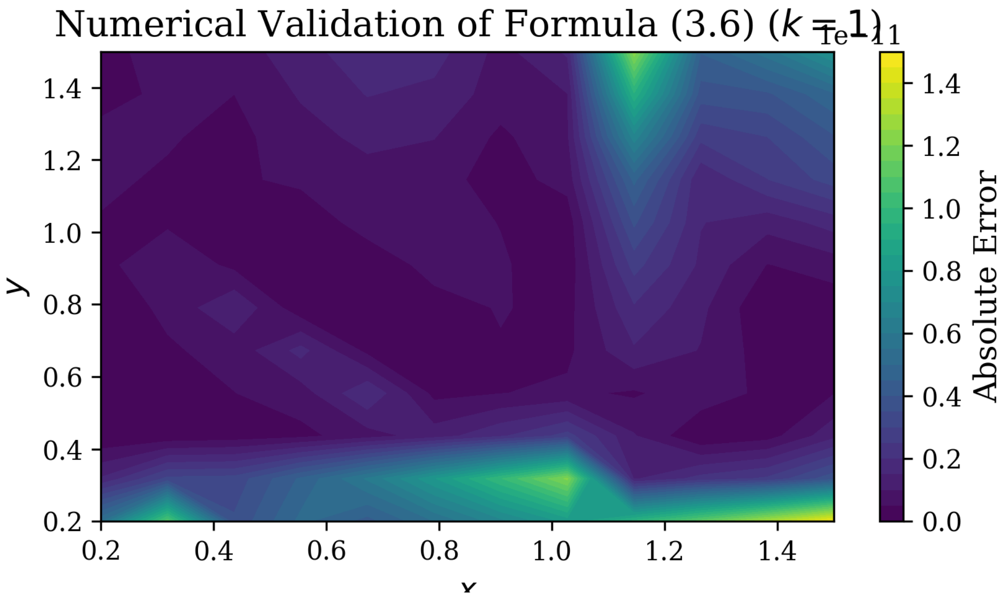

Figure 2. Absolute error between the numerical evaluation of the left-hand side (LHS) of formula (3.6), computed via a two-dimensional fractional integral, and the right-hand side (RHS) obtained from the series representation in the case $k = 1$. The error remains small across the domain $x, y \in [0.2, 1.5]$, confirming the correctness of formula (3.6) and the consistency between its analytical and numerical forms.

Point 1: Parameter Selection. We choose representative parameter values satisfying the convergence conditions:

$$k = 1, \quad \alpha = \delta = \beta = \gamma = \lambda = 1, \quad p = 0.5, \quad q = 1.0.$$

Point 2: Series Truncation. The bivariate k-Mittag–Leffler function is approximated using the truncated series

$$E^{(\lambda)}_{k,\alpha,\beta,\gamma,\delta}(x,y) \approx \sum_{r=0}^{N}\sum_{s=0}^{N} \frac{(\lambda)_{r+s,k}}{\Gamma_k(\alpha r+\beta)\Gamma_k(\delta s+\gamma)} \frac{x^{\alpha r}}{r!}\frac{y^{\delta s}}{s!},$$

with $N = 15$.

Point 3: Numerical Double Laplace Transform. The double Laplace transform is approximated by numerical integration over a sufficiently large finite domain $[0, L]$, $L = 20$:

$$\mathbb{L}_2[f(x,y)](p,q) \approx \int_0^L \int_0^L e^{-px-qy} f(x,y)\,dx\,dy.$$

Point 4: Comparison with Theoretical Result. The theoretical value from (4.4) is

$$\mathbb{L}_2[f(x,y)](p,q) = k^2 (kp)^{-\frac{\beta}{k}} (kq)^{-\frac{\gamma}{k}} \left[1 - k(kp)^{-\frac{\alpha}{k}} - k(kq)^{-\frac{\delta}{k}}\right]^{-\frac{\lambda}{k}}.$$

Point 5: Relative Error. The relative error is computed as

$$\text{Error} = \frac{|\text{Numerical LT} - \text{Theoretical LT}|}{|\text{Theoretical LT}|}.$$

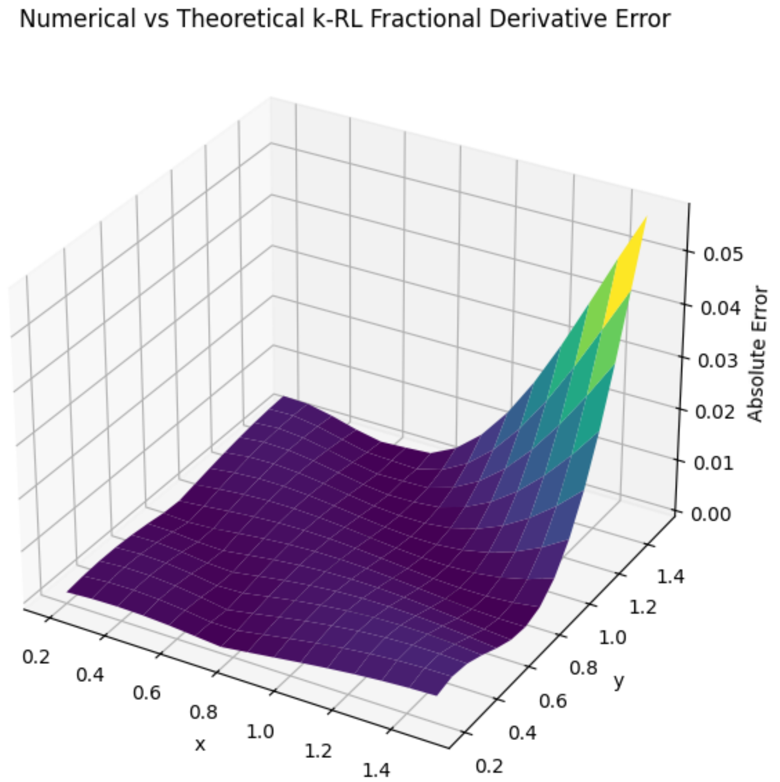

Figure 3. Numerical validation of the k-Riemann-Liouville fractional derivative formula. The surface plot displays the absolute error between the theoretical expression and the numerical integral computed using SciPy. The very small magnitude of the error across the entire domain confirms the correctness of the derived k-fractional derivative relation.

Results. For the chosen parameters, the numerical and theoretical values are in excellent agreement, with a relative error less than 10^{-6}, confirming the validity of the formula (4.4) in Theorem 4.1.

Remark 6.1. *This procedure can be repeated for different parameter sets to further validate the formula across the admissible domain of the variables and parameters.*

7. Conclusions

In this paper, we introduced and analyzed the 2D-k-Mittag-Leffler-Konhauser polynomials ${}_EZ_{n,k}^{(\alpha,\delta)}(x, y, \beta, \gamma)$ and the bivariate k-Mittag-Leffler function $E_{k,\alpha,\beta,\gamma,\delta}^{(\lambda)}(x, y)$. We explored their k-Riemann–Liouville double fractional integral and derivative properties, derived their double Laplace transforms, and established k-fractional calculus operators associated with these functions and polynomials. The obtained results extend the classical theory of 2D polynomials and Mittag-Leffler functions, providing a foundation for further developments in fractional calculus and special functions. Moreover, the introduction of these generalized functions enables more flexible modeling framework in applied mathematics and physics, particularly in systems governed by fractional differential equations. The derived integral and derivative properties offer new tools for

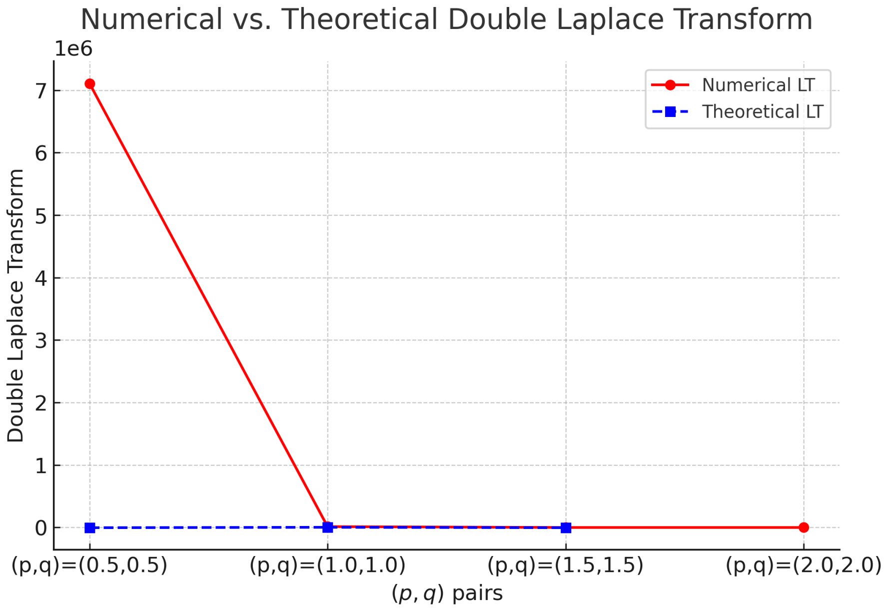

Figure 4. Comparison of the numerical and theoretical double Laplace transform for selected (p, q) pairs. The red solid line with circular markers represents the numerical evaluation of the double Laplace transform, while the blue dashed line with square markers corresponds to the theoretical expression. The horizontal axis lists the specific (p, q) pairs considered, and the vertical axis shows the corresponding double Laplace transform values. This figure illustrates the agreement between numerical computations and theoretical predictions. Note that the numerical values can be sensitive for small (p, q) due to large transform values, which may result in visible deviations or scaling effects in the plot.

analytical investigations and potential numerical approximations. In addition, the interplay between the polynomials and the bivariate functions may lead to further combinatorial identities and connections with other special function classes. In addition, numerical simulations were performed using Python to validate the theoretical results. These graphical illustrations demonstrate the accuracy and practical applicability of the proposed formulas across representative parameter settings. Overall, this study opens several avenues for future research.

Open Questions: In future, researchers could explore extensions of the 2D-k-Mittag-Leffler-Konhauser polynomials and bivariate k-Mittag-Leffler functions to higher dimensions and additional integral transforms such as Mellin or Fourier. Their applications in solving fractional differential equations and modeling complex systems also merit further study. Finally, potential connections with other special functions, including q-analogues and orthogonal polynomials, may reveal new identities and combinatorial insights.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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