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Research article

The classical linear duality method in some semilinear and noncoercive Dirichlet problems

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Abstract: We study the following semilinear system:

$$(*)\begin{cases} u \in W_0^{1,2}(\Omega): \ -\operatorname{div}(M(x)\,Du) + u = \theta\,\psi|\psi|^{p'-2} + f(x); \\ \psi \in W_0^{1,2}(\Omega): \ -\operatorname{div}(M(x)\,D\psi) + \psi = u|u|^{p-2} \end{cases}$$

and we prove the existence of bounded weak solutions in $W_0^{1,2}(\Omega)$. Even if the system is nonlinear, we use a duality method.

We dedicate this paper to Patrizia Pucci certain that she will help us for the study of (*) with nonlinear principal part.

Keywords: Dirichlet problems; duality methods; semilinear equations; bounded solutions; elliptic systems

1. Introduction

Let Ω be a bounded, open subset of $\mathbb{R}^N$, with $N \geq 3$, $M(x)$ be measurable, symmetric, uniformly elliptic and bounded; i.e., $\alpha > 0$ and $\beta > 0$ exist such that

$$\alpha|\xi|^2 \leq M(x)\,\xi\cdot\xi, \qquad |M(x)| \leq \beta, \tag{1.1}$$

for every ξ in $\mathbb{R}^N$, and for almost every x in Ω. Furthermore, we assume that

$$f(x) \in L^m(\Omega),\ m > \frac{N}{2}, \tag{1.2}$$

$$0 < \theta < 1, \quad 1 < p \leq \frac{N}{2}. \tag{1.3}$$

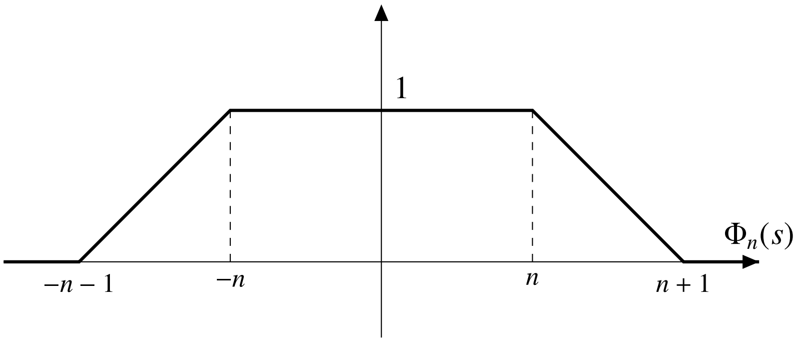
Assumptions (1.1)–(1.3) are used throughout the paper.

We want to find a weak solution (u, ψ) of the semilinear elliptic system

$$\begin{cases} u \in W_0^{1,2}(\Omega) : \ -\operatorname{div}(M(x)\,Du) + u = \theta\,\psi|\psi|^{p'-2} + f(x); \\ \psi \in W_0^{1,2}(\Omega) : \ -\operatorname{div}(M(x)\,D\psi) + \psi = u|u|^{p-2}. \end{cases} \tag{1.4}$$

In some instances, we set $q = p'$ as the Hölder conjugate of p.

We study the existence of solutions of the system (1.4) by approximation; the starting point is the definition of the real function $\Phi_n(s)$, whose graph is shown in the following figure.



Then, for every fixed n, we consider the system

$$\begin{cases} u_n \in W_0^{1,2}(\Omega) : \ -\operatorname{div}(M(x)\,Du_n) + u_n = \theta\,\psi_n|\psi_n|^{p'-2}[\Phi_n(\psi_n)]^{p'-1} + f(x), \\ \psi_n \in W_0^{1,2}(\Omega) : \ -\operatorname{div}(M(x)\,D\psi_n) + \psi_n = u_n|u_n|^{p-2}[\Phi_n(u_n)]^{p-1}. \end{cases} \tag{1.5}$$

The existence of a weak solution (u_n, ψ_n) is a simple consequence of the Schauder Theorem, since the function $\Phi_n(s)$ has compact support. The same property and (1.2) give the boundedness of every u_n and every ψ_n. The key point of the existence is the proof that the sequence $\{u_n\}$ is bounded in $L^\infty(\Omega)$ and the sequence $\{\psi_n\}$ is bounded in $L^\infty(\Omega)$.

Of course the systems (1.4) and (1.5) above are understood in the weak formulation: $\forall\, v \in W_0^{1,2}(\Omega)$. It holds true that

$$\begin{cases} u \in W_0^{1,2}(\Omega) : \displaystyle\int_\Omega M(x)\,Du\,Dv + \int_\Omega u\,v = \theta \int_\Omega \psi|\psi|^{p'-2}\,v + \int_\Omega f(x)\,v(x); \\ \psi \in W_0^{1,2}(\Omega) : \displaystyle\int_\Omega M(x)\,D\psi\,Dv + \int_\Omega \psi\,v = \int_\Omega u|u|^{p-2}\,v. \end{cases} \tag{1.6}$$

and

$$\begin{cases} u_n \in W_0^{1,2}(\Omega) : \displaystyle\int_\Omega M(x)\,Du_n\,Dv + \int_\Omega u_n\,v \\ \qquad\qquad = \theta \displaystyle\int_\Omega \psi_n|\psi_n|^{p'-2}[\Phi_n(\psi_n)]^{p'-1}\,v + \int_\Omega f(x)\,v(x); \\ \psi_n \in W_0^{1,2}(\Omega) : \displaystyle\int_\Omega M(x)\,D\psi_n\,Dv + \int_\Omega \psi_n\,v = \int_\Omega u_n|u_n|^{p-2}[\Phi_n(u_n)]^{p-1}\,v. \end{cases} \tag{1.7}$$

The main point is the following Lemma 2.1, where the a priori estimates are proved, despite the noncoercivity of the problem.

Remark 1.1. *Let* $\mathbf{p} = \mathbf{2}$*, so that our system is*

$$\begin{cases} u \in W_0^{1,2}(\Omega) : \int_\Omega M(x)\, Du\, Dv + \int_\Omega u\, v = \theta \int_\Omega \psi\, v + \int_\Omega f(x)\, v(x); \\ \psi \in W_0^{1,2}(\Omega) : \int_\Omega M(x)\, D\psi\, Dw + \int_\Omega \psi\, w = \int_\Omega u\, w. \end{cases}$$

It is possible to prove the existence with the use of the Lax–Milgram theorem if $0 < \theta < 1 + 2\lambda_1$*, with* λ_1 *the first eigenvalue of the differential operator* $-\operatorname{div}(M(x)\, Dv)$*, because the bilinear form*

$$\mathcal{A}((u, \psi), (v, \varphi)) = \int_\Omega [MDuDv + MD\psi D\varphi + uv + \psi\varphi - \theta\psi v - u\varphi]$$

is coercive.

Remark 1.2. *We emphasize that in the problem* (1.4)*, the two principal part are linear, but*

$$\begin{cases} \text{if } 1 < p < 2, \text{ then the right-hand side of the first equation has a growth of} > 1;, \\ \text{if } 2 < p, \text{ then the right-hand side of the second equation has a growth of} > 1, \end{cases}$$

so the proof of the a priori estimates cannot be straight. Beacause of this, we try a **duality approach even if the problem is nonlinear**.

2. Proof of the main result

In this section, we prove the existence result (Theorem 2.2) and we start with some a priori estimates on the sequences $\{u_n\}$, $\{\psi_n\}$.

Lemma 2.1.

$$\text{The sequences } \{u_n\},\ \{\psi_n\} \text{ are bounded in } L^\infty(\Omega). \tag{2.1}$$

Proof. Step 1. Following a duality approach, we use ψ_n as test function of the first equation of (1.7) and u_n as test function of the second equation of (1.7), and we deduce that

$$\theta \int_\Omega |\psi_n|^{p'} [\Phi_n(\psi_n)]^{p'-1} + \int_\Omega f(x)\, \psi_n = \int_\Omega |u_n|^p [\Phi_n(u_n)]^{p-1},$$

so that (when we use $0 \le \Phi_n(\psi_n) \le 1$ and a Brezis-Strauss [1] estimate in the second equation

$$\|\psi_n\|_{p'} \le \|\text{right-hand side}\|_{p'} \tag{2.2}$$

we have

$$\left[\begin{array}{l} \int_\Omega |u_n|^p [\Phi_n(u_n)]^{p-1} = \theta \int_\Omega |\psi_n|^{p'} [\Phi_n(\psi_n)]^{p'-1} + \int_\Omega f(x)\, \psi_n \\ \le \theta \int_\Omega |\psi_n|^{p'} + \|f\|_p \|\psi_n\|_{p'} \\ \le \theta \int_\Omega (|u_n|^{p-1} [\Phi_n(u_n)]^{p-1})^{p'} + \|f\|_p \|\psi_n\|_{p'} \end{array} \right.$$

and

$$(1-\theta)\int_\Omega |u_n|^p[\Phi_n(u_n)]^{p-1} \le \|f\|_p \, \|\psi_n\|_{p'} \le \|f\|_p \Big[\int_\Omega [|u_n|\,\Phi_n(u_n)]^{(p-1)p'}\Big]^{\frac{1}{p'}},$$

Here, we use (2.2) again, we note that $0 \le \Phi_n \le 1$ implies $[\Phi_n(u_n)]^p \le [\Phi_n(u_n)]^{p-1}$, so that

$$(1-\theta)\int_\Omega |u_n|^p[\Phi_n(u_n)]^{p-1} \le \|f\|_p \, \|\psi_n\|_{p'} \le \|f\|_p \Big[\int_\Omega [|u_n|^p\,\Phi_n(u_n)]^{p-1}\Big]^{\frac{1}{p'}}.$$

Thus we deduce

$$(1-\theta)\Big[\int_\Omega |u_n|^p[\Phi_n(u_n)]^{p-1}\Big]^{\frac{1}{p}} \le \|f\|_p. \tag{2.3}$$

Recall that

$$(1-\theta)\Big[\int_\Omega |u_n|^p[\Phi_n(u_n)]^{p}\Big]^{\frac{1}{p}} \le (1-\theta)\Big[\int_\Omega |u_n|^p[\Phi_n(u_n)]^{p-1}\Big]^{\frac{1}{p}},$$

and we have thus proved, where the assumption $0 < \theta < 1$ is fundamental, that

$$(1-\theta)\Big[\int_\Omega |u_n|^p[\Phi_n(u_n)]^{p}\Big]^{\frac{1}{p}} \le \|f\|_p,$$

that is

$$\text{the sequence } \{[u_n\,\Phi_n(u_n)]^{p-1}\} \text{ is bounded in } L^{p'}(\Omega); \tag{2.4}$$

i.e., the right-hand side of the second equation is bounded in $L^{p'}(\Omega)$. Once more, the use of a Brezis-Strauss estimate in the second equation says that

$$\text{the sequence } \{\psi_n\} \text{ is bounded in } L^{p'}(\Omega) = L^q(\Omega). \tag{2.5}$$

Step 2. We now work on the first equation. Then, since $0 \le \Phi(t) \le 1$, we have

$$\int_\Omega (|\psi_n\,\Phi_n(\psi_n)|^{(p'-1)})^p = \int_\Omega |\psi_n\,\Phi_n(\psi_n)|^{p'} \le \int_\Omega |\psi_n|^{p'},$$

so that

$$\text{the sequence } [\psi_n\,\Phi_n(\psi_n)]^{(p'-1)} \text{ is bounded in } L^p(\Omega). \tag{2.6}$$

Here, we point out that

- (2.4) and $p' > \frac{N}{2}$ imply that the sequence $\{\psi_n\}$ is bounded in $L^\infty(\Omega)$, which implies in the first equation, that the sequence $\{u_n\}$ is bounded in $L^\infty(\Omega)$,
- (2.6) and $p > \frac{N}{2}$ imply that the sequence $\{u_n\}$ is bounded in $L^\infty(\Omega)$, which implies in the second equation that also the sequence $\{\psi_n\}$ is also bounded in $L^\infty(\Omega)$,

Summing up, $p < \frac{N}{N-2}$ or $p > \frac{N}{2}$ give the boundedness of the two sequences $\{u_n\}$, $\{\psi_n\}$.

From now on, we work with

$$\frac{N}{N-2} \leq p \leq \frac{N}{2}.$$

We now use a Brezis-Strauss estimate in the first equation and we have

$$\left[\begin{array}{l} \|u_n\|_p \leq \|\theta\, \psi_n|\psi_n|^{p'-2}[\Phi_n(\psi_n)]^{p'-1} + f(x)\|_p \\ \leq \theta\|\psi_n|\psi_n|^{p'-2}[\Phi_n(\psi_n)]^{p'-1}\|_p + \|f(x)\|_p \leq \theta\|\psi_n\|_{p'}^{p-1} + \|f(x)\|_p. \end{array}\right.$$

Thus

$$\text{the sequence } \{u_n\} \text{ is bounded in } L^p(\Omega). \tag{2.7}$$

Now, the right-hand side of the second equation is bounded in $L^{p'}(\Omega)$; in fact, we have

$$\|\text{right-hand side}\|_m \leq \|u_n\|_{(p-1)m}, \tag{2.8}$$

with $m = p'$ thanks to (2.7).

Step 3. Here, we rewrite (2.5) and (2.7) as follows:

$$\begin{cases} \text{the sequence } \{\psi_n\} \text{ is bounded in } L^q(\Omega) \\ \text{the sequence } \{u_n\} \text{ is bounded in } L^p(\Omega), \end{cases} \tag{2.9}$$

which imply that

$$\begin{cases} \text{the right-hand side of the first equation is bounded in } L^p(\Omega) \\ \text{the right-hand side of the second equation is bounded in } L^q(\Omega). \end{cases}$$

Observe that if $p > \frac{N}{2}$, the Stampacchia boundedness theorem says that the sequence $\{u_n\}$ is bounded in $L^\infty(\Omega)$ (and then also $\{\psi_n\}$ is bounded in $L^\infty(\Omega)$); if $p' > \frac{N}{2}$ (i.e., $p < \frac{N}{N-2}$), the sequence $\{\psi_n\}$ is bounded in $L^\infty(\Omega)$ (and then also $\{u_n\}$ is bounded in $L^\infty(\Omega)$) [2]. In the case where $\frac{N}{N-2} \leq p \leq \frac{N}{2}$, we can continue.

With the use of the Stampacchia summability theory, we can deduce that

$$\begin{cases} \text{the sequence } \{u_n\} \text{ is bounded in } L^{p^{**}}(\Omega), \\ \text{the sequence } \{\psi_n\} \text{ is bounded in } L^{q^{**}}(\Omega). \end{cases} \tag{2.10}$$

We now point out the improvement of the summability stated in (2.10) compared with that in (2.9) and we quantify this improvement

$$\begin{cases} \dfrac{1}{p^{**}} = \dfrac{1}{p} - \dfrac{2}{N}, \\ \dfrac{1}{q^{**}} = \dfrac{1}{q} - \dfrac{2}{N}. \end{cases} \tag{2.11}$$

Thus now we are done if $p^{**} > \frac{N}{2}$ or if $q^{**} > \frac{N}{2}$. If not, we continue and we prove (as above) that

$$\begin{cases} \text{the sequence } \{u_n\} \text{ is bounded in } L^{(p^{**})^{**}}(\Omega), \\ \text{the sequence } \{\psi_n\} \text{ is bounded in } L^{(q^{**})^{**}}(\Omega). \end{cases} \tag{2.12}$$

- Observe that $(p^{**})^{**} = p^{****}$.
- Define $p_1 = p^{**}$, $p_2 = p_1^{**}$, $\cdots$ $p_{k+1} = p_k^{**}$, such that

$$\frac{1}{p_k} = \frac{1}{p} - k\frac{2}{N},$$

- Notice that with $\sigma > 1$, we have

$$\frac{1}{p_k} < \frac{1}{\sigma} \iff \frac{1}{p} - k\frac{2}{N} < \frac{1}{\sigma} \iff \frac{1}{p} \cdot \sigma < (k+1).$$

- The same comment applies about q.

Thus, after a finite number of steps, the sequence $\{u_n\}$ (or $\{\psi_n\}$) is bounded in $L^\sigma(\Omega)$, $\sigma > 1$.

Then $u_n|u_n|^{p-2}[\Phi_n(u_n)]^{p-1}$ is bounded in $L^{\frac{\sigma}{p-1}}(\Omega)$ and $\frac{\sigma}{p-1} > \frac{N}{2}$ if we choose $\sigma > (p-1)\frac{N}{2}$, which implies (in the second equation) that the sequence $\{\psi_n\}$ is bounded in $L^\infty(\Omega)$, thanks to the Stampacchia boundedness theorem; the same discussion holds for $\{u_n\}$.

Theorem 2.2. *Under the assumptions* (1.1)–(1.3)*, there is a pair* (u, ψ) *of weak solutions of the system* (1.4)*; that is a solution of the Dirichlet problem* (1.6).

Proof. Estimate (2.1) says that the sequences $\{u_n\}$, $\{\psi_n\}$ are bounded in $L^\infty(\Omega)$; then

$$\text{the two right-hand sides of (1.7) are bounded in } L^\infty(\Omega), \tag{2.13}$$

which implies that the solutions $\{u_n\}$, $\{\psi_n\}$ are compact in $W_0^{1,2}(\Omega)$, so that u, $\psi \in W_0^{1,2}(\Omega)$ exist with (up to subsequences)

$$\begin{cases} \{u_n\} \text{ converges strongly in } W_0^{1,2}(\Omega) \text{ and a.e. to } u; \\ \{\psi_n\} \text{ converges strongly in } W_0^{1,2}(\Omega) \text{ and a.e. to } \psi. \end{cases} \tag{2.14}$$

Now with (2.14), it is easy to pass to the limit in (1.7) and our statement is proved.

Remark 2.3. *Observe that u and ψ are bounded functions.*

Open Problems 2.4.

- *In Theorem 2.2, allow $\theta = 1$.*
- *Nonlinear principal part.*

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there are no conflicts of interest.

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