Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 33(9): 5829–5844.
DOI: 10.3934/era.2025259
Received: 09 April 2025
Revised: 13 August 2025
Accepted: 15 August 2025
Published: 26 September 2025

Research article

On a boundary value problem for a nonlinear differential equation with a Riemann-Liouville fractional derivative of variable order and nonlocal boundary conditions

Alexander A. Potapov[1,2,*], **Vetlugin D. Beybalaev**[3,4,5] **and Abutrab A. Aliverdiev**[3,4,6]

[1] Kotel'nikov Institute of Radioengineering and Electronics, Russian Academy of Sciences, Moscow 125009, Russia

[2] JNU-IRE RAS Joint Laboratory of Information Technology and Fractal Processing of Signals, Jinan University, Guangzhou 510632, China

[3] Dagestan State University, Gadzhieva Str., 43A, Makhachkala 367000, Russia

[4] Institute for Geothermal Researches and Renewable Energy — A Branch of the Joint Institute for High Temperatures, Russian Academy of Sciences Pr. I. Shamilya, 39A, Makhachkala 367030, Russia

[5] Dagestan State University of National Economy, J. Atayev Str., 5, Makhachkala 367000, Russia

[6] Institute of Physics DFRC, Russian Academy of Sciences, M. Yaragskogo Str., 94, Makhachkala 367003, Russia

* **Correspondence:** Email: potapov@cplire.ru.

Abstract: The paper studies the existence of at least two positive solutions for a nonlinear differential equation with a fractional Riemann-Liouville derivative of variable order and nonlocal boundary conditions. To transform the original equation into an integral equation, the Green function is constructed and boundedness conditions for the Green function are obtained. A theorem on the existence of at least two positive solutions to the problem is proven. An example is given that shows the existence of two positive solutions, and approximation graphs are constructed for clarity.

Keywords: fractional derivative; boundary conditions; Green function; non-stationary processes

1. Introduction

Currently, differential equations with fractional derivatives are used to describe various non-local processes in physics, chemistry, and biology [1–4], which has led to the development of the theory of fractional differential equations. Most of the works in this area are related to the study of boundary value problems for linear differential equations with fractional Riemann-Liouville and Gerasimov-Caputo derivatives [5–8]. Recently, many works focused on the existence and uniqueness of a positive solution to boundary value problems for non-linear fractional differential equations.

In [9], the existence and uniqueness of a positive solution to a boundary value problem for a non-linear differential equation with a fractional Riemann-Liouville derivative of the form were studied:

$$D_{0t}^{\alpha}u(t) + f(t, u(t)) = 0, 0 < t < 1, u(0) = u(1) = 0,$$

where $1 < \alpha \leq 2$, and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is a continuous function.

In [10–12], a boundary value problem with a fractional Riemann-Liouville derivative of the following forms was investigated:

$$D_{0t}^{\alpha}u(t) + a(t)f(u) = 0, u(0) = 0, u^{/}(1) = 0, 1 < \alpha \leq 2,$$

$$D_{0t}^{\alpha}u(t) + a(t)f(t, u, u^{/}) = 0, 0 < t < 1, u(0) = u^{/}(0) = u^{//}(0) = u^{//}(1) = 0, 3 < \alpha \leq 4,$$

respectively, where $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is a continuous function, $D_{0t}^{\alpha}u(t) + q(t)f(t, u(t)) = 0$, $0 < t < 1$, $u^{j}(0) = 0$, $0 \leq j \leq n - 2$, $u(1) = \mu \int\limits_0^1 u(s)ds$, and $\alpha \in (n - 1, n]$.

In [13], the authors applied mixed monotone-operator methods to determine the existence and uniqueness of positive solutions to boundary value problems for differential equations with a fractional Riemann-Liouville derivative.

Additionally, [14–18] are devoted to the study of the existence and uniqueness of a positive solution to boundary value problems for a nonlinear differential equation with a fractional derivative.

Moreover, at the same time, the authors of the mentioned works considered boundary value problems for a nonlinear differential equation with a fractional Riemann-Liouville derivative with local boundary conditions, which we think is an erroneous approach. Local boundary conditions on the boundary do not uniquely determine the values of constant coefficients in the solution when constructing the Green function.

Many dynamic systems exhibit fractional behavior that may vary in time or space. This means that fractional differential equations with variable order are a realistic option to describe such processes. Fractional derivatives of variable order are currently used for mathematical modeling in various fields. For example, Coimbra [19] used variable orders to study the physical processes of viscoelasticity. Several definitions of fractional derivatives of variable order have been used [20–23]. As is known, the use of differential equations with fractional derivatives, where the order of the fractional derivative is some constant, is not the final tool to model phenomena in nature. Currently, differential equations with fractional derivatives of variable order are used to describe various processes in complex physical systems. As indicated in the works [24–28], differential equations with a fractional derivative of variable order provide a better description of non-local phenomena with changing dynamics, that is, processes in which the interaction of objects or systems is not limited to local boundaries, but extends over large distances, while the dynamics can change over time. Such phenomena are studied in various fields, including physics, biology and sociology, and require the use of special methods and approaches. Many works are devoted to numerical methods [29–32]. Boundary value problems for differential equations with a variable-order fractional derivative arise in various fields, such as those that describe diffusion processes in an inhomogeneous or heterogeneous medium or processes in which changes in the environment change the dynamics of a particle [24]. In [29], a boundary value problem for the diffusion equation with a variable-order fractional derivative of Caputo was investigated. Using the energy inequalities method, a priori estimates were obtained in the differential and difference forms. In [30], the Chebyshev collocation method of the sixth kind was used to solve the initial value problem for a nonlinear integro-differential equation with a variable-order fractional derivative of Caputo. In [31], the approximate solutions of variable-order fractional differential equations with the Mittag-Leffler kernel in the Liouville-Caputo sense were obtained using an artificial neural network. In [32], an anomalous subdiffusion equation with a variable-order fractional derivative was investigated. In this paper, a difference scheme with first-order temporal accuracy and fourth-order spatial accuracy is proposed, and theorems on the stability and convergence of the difference scheme are proven.

This paper investigates the existence of at least two positive solutions to a boundary value problem for a nonlinear equation with a Riemann-Liouville fractional derivative of variable order and nonlocal boundary conditions.

2. Fractional derivatives of variable order

Let us consider fractional derivatives, where the order of the derivative is some function $\alpha(t)$.

Let the functions $u : [a, b] \to R$ and $\alpha : [a, b] \to (0, 1)$ are defined on the segment $u(t), \alpha(t) \in C[a, b]$. Then, the Riemann-Liouville and Caputo fractional derivatives of variable order be defined as follows [27]:

1) Riemann-Liouville fractional derivative of order $\alpha(t)$:

$$D_{at}^{\alpha(t)}u(t) = \frac{1}{\Gamma(1-\alpha(t))}\frac{d}{dt}\int_a^t (t-s)^{-\alpha(t)}u(s)ds.$$

2) Caputo fractional derivative of order $\alpha(t)$:

$$D_{at}^{\alpha(t)}u(t) = \frac{1}{\Gamma(1-\alpha(t))}\int_a^t (t-s)^{-\alpha(t)}u^{/}(s)ds.$$

The Riemann-Liouville fractional integral of variable order is defined as follows [30]:

$$J_{at}^{\alpha(t)}u(t) = \frac{1}{\Gamma(\alpha(t))}\int_a^t (t-s)^{\alpha(t)-1}u(s)ds,$$

where $0 < \alpha(t) < 1$.

The following equality holds [27]:

$$D_{at}^{\alpha(t)}u(t) = \frac{d}{dt}J_{at}^{1-\alpha(t)}u(t).$$

Let $1 < \eta(t) < 2$, and let $\eta(t) = 1 + \beta(t)$, where $\eta(t), \beta(t) \in C[a, b]$, $0 < \beta(t) < 1$. Then,

$$D_{at}^{\eta(t)}u(t) = \frac{d}{dt}D_{at}^{\beta(t)}u(t) = \left(\frac{d}{dt}\right)^2 J_{at}^{1-\beta(t)}u(t).$$

Thus, the Riemann-Liouville fractional derivative of variable order for $1 < \eta(t) < 2$ is defined as follows:

$$D_{at}^{\eta(t)}u(t) = \frac{1}{\Gamma(2-\eta(t))}\frac{d^2}{dt^2}\int_a^t (t-s)^{1-\eta(t)}u(s)ds. \quad (1)$$

3. Statement of the problem

Let us consider a two-point boundary value problem for a nonlinear differential equation with a Riemann-Liouville fractional derivative of variable order of the following form:

$$D_{0t}^{\alpha(t)}u(t) + q(t)f(t,u(t)) = 0, 0 < t < 1, \quad (2)$$

$$\lim_{t\to 0} D_{0t}^{\alpha(t)-2}u(t) = 0, D_{0t}^{\beta(t)}u(1) = \frac{\delta\cdot\Gamma(\alpha(t))}{\Gamma(\nu(t))}\int_0^1 u(s)ds, \quad (3)$$

where $D_{0t}^{\alpha(t)}u(t)$ is the fractional derivative of Riemann-Liouville of variable order (1), $\alpha : [0,1] \to (1,2)$ is a continuous function, $0 \le \delta \le 1$, and $f : [0,1]\times[0,\infty) \to [0,\infty)$ is a continuous function.

Let's consider the case when $\beta(t) = \frac{\alpha(t)-1}{2}$, where $\beta : [0,1] \to (0,1/2)$. Then, $\nu(t) = \alpha(t) - \beta(t)$, $\nu : [0,1] \to (1,3/2)$.

Since the functions $\alpha(t)$ and $\beta(t)$ are continuous and bounded, we have the following:

$$\|\alpha(t)\|_{C[0,1]} = \max_{0<t\le 1}\alpha(t), \|\beta(t)\|_{C[0,1]} = \max_{0<t\le 1}\beta(t), \|\nu(t)\|_{C[0,1]} = \max_{0<t\le 1}\nu(t). \quad (4)$$

4. Construction of the Green's function

First, let us construct the Green's function to transform the original problems (2) and (3) into an integral equation.

Let $1 < \alpha^* < 2$ and $0 < \beta^* < 1/2$ be arbitrary fixed values of the functions $\alpha(t)$ and $\beta(t)$, which satisfy the condition $1 < \alpha^* - \beta^* < 1.5$, and $u \in C(0,1) \cap L_1(0,1)$ be simultaneously continuous and summable over the interval $(0,1)$. Then, the following equality holds [9]:

$$I_{0t}^{\alpha}D_{0t}^{\alpha^*}u(t) = u(t) + C_1t^{\alpha^*-1} + C_2t^{\alpha^*-2}, \quad (5)$$

where $C_i \in R, i = 1,2$.

Let's consider the following boundary value problem:

$$D_{0t}^{\alpha(t)}u(t) + p(t) = 0, 0 < t < 1, \quad (6)$$

$$\lim_{t\to 0} D_{0t}^{\alpha(t)-2}u(t) = 0, D_{0t}^{\beta(t)}u(1) = \frac{\delta\cdot\Gamma(\alpha(t))}{\Gamma(\nu(t))}\int_0^1 u(s)ds. \quad (7)$$

Using (5) and taking the arbitrariness of the choice of α^*, β^* into account, the solution of problem (6), (7) for $1 < \alpha(t) < 2$, $0 < \beta(t) < \frac{1}{2}$, $t \in [0,1]$, can be represented in the following form:

$$u(t) = C_1t^{\alpha(t)-1} + C_2t^{\alpha(t)-2} - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha(t)-1}p(s)ds.$$

Lemma 1. Let $p(t) \in C[0,1]$, $1 < \alpha(t) < 2$, and $0 < \beta(t) < \frac{1}{2}$. Then, the only solution:

$$D_{0t}^{\alpha(t)} u(t) + p(t) = 0, 0 < t < 1, \tag{8}$$

$$\lim_{t\to 0} D_{0t}^{\alpha(t)-2} u(t) = 0, D_{0t}^{\beta(t)} u(1) = \frac{\delta \cdot \Gamma(\alpha(t))}{\Gamma(\nu(t))} \int_0^1 u(s)ds, \tag{9}$$

can be represented as follows:

$$u(t) = \int_0^t G(t,s)p(s)ds. \tag{10}$$

In equality (10), the Green's function has the follows form:

$$G(t,s) = \frac{1}{\lambda(t)\Gamma(\alpha(t))} \begin{cases} t^{\alpha(t)-1}(1-s)^{\nu(t)-1}(\alpha(t) - \delta(1-s)^{\beta(t)}) - (\alpha(t)-\delta)(t-s)^{\alpha(t)-1}, 0 \le s \le t \le 1, \\ t^{\alpha(t)-1}(1-s)^{\nu(t)-1}(\alpha(t) - \delta(1-s)^{\beta(t)}), 0 \le t \le s \le 1, \end{cases} \tag{11}$$

where $\lambda(t) = \frac{1}{(\alpha(t)-\delta)}$.

Proof. Let $1 < \alpha^* < 2$, and let $0 < \beta^* < \frac{1}{2}$, and $1 < \nu^* < \frac{3}{2}$ be arbitrary fixed values of the functions $\beta(t)$ and $\nu(t)$, respectively. Then, from the left boundary condition, we obtain $C_2 = 0$.

We have the following:

$$D_{0t}^{\beta^*} u(t) = C_1 \frac{\Gamma(\alpha^*) \cdot t^{\nu^*-1}}{\Gamma(\alpha^* - \beta^*)} - \frac{1}{\Gamma(\alpha^* - \beta^*)} \int_0^t (t-s)^{\nu^*-1} p(s)ds.$$

Then, from the right boundary condition, we obtain the following:

$$C_1 \frac{\Gamma(\alpha^*)}{\Gamma(\nu^*)} - \frac{1}{\Gamma(\nu^*)} \int_0^1 (1-s)^{\nu^*-1} p(s)ds = \frac{\delta \cdot \Gamma(\alpha^*)}{\Gamma(\nu^*)} \int_0^1 u(s)ds,$$

that is, $C_1 = \frac{1}{\Gamma(\alpha^*)} \int_0^1 (1-s)^{\nu^*-1} p(s)ds + \delta \int_0^1 u(s)ds$.

Consequently, equality (7) will take the following form:

$$\begin{aligned} u(t) &= \frac{t^{\alpha^*-1}}{\Gamma(\alpha^*)} \int_0^1 (1-s)^{\nu^*-1} p(s)ds + \delta \cdot t^{\alpha^*-1} \int_0^1 u(s)ds - \frac{1}{\Gamma(\alpha^*)} \int_0^t (t-s)^{\alpha^*-1} p(s)ds \\ &= \frac{1}{\Gamma(\alpha^*)} \int_0^t \left[t^{\alpha^*-1}(1-s)^{\nu^*-1} - (t-s)^{\alpha^*-1} \right] p(s)ds + \frac{1}{\Gamma(\alpha^*)} \int_t^1 t^{\alpha^*-1}(1-s)^{\nu^*-1} p(s)ds + \delta \cdot t^{\alpha^*-1} \int_0^1 u(s)ds \\ &= \int_0^t G(t,s)p(s)ds. \end{aligned} \tag{12}$$

By integrating both parts of equality (12) over the segment, we obtain the following:

$$\begin{aligned}\int_0^1 u(s)ds &= \int_0^1\left[\int_0^1 G_1^*(\tau,s)p(s)d\tau + \delta\cdot s^{\alpha^*-1}\int_0^1 u(\tau)d\tau\right]ds\\ &= \int_0^1\int_0^1 G_1^*(\tau,s)p(s)d\tau ds + \delta\int_0^1 u(s)ds\cdot\int_0^1 z^{\alpha^*-1}dz,\end{aligned}$$

where $G_1^*(\tau,s) = \frac{1}{\Gamma(\alpha^*)}\left[\tau^{\alpha^*-1}(1-s)^{\nu^*-1} - (\tau-s)^{\alpha^*-1}\right]$, or

$$\int_0^1 u(s)ds = \frac{\alpha^*}{(\alpha^*-\delta)}\int_0^1\int_0^1 G_1^*(\tau,s)p(s)d\tau ds = \frac{1}{(\alpha^*-\delta)\Gamma(\alpha^*)}\int_0^1 (1-s)^{\nu^*-1}(1-(1-s)^{\beta^*})p(s)ds. \quad (13)$$

Consequently, taking the arbitrariness of the choice of fixed values α^*, β^*, ν^* and equality (12) and (13) into account, the Green's function will take the form (11).

Lemma 2. The Green's function defined by equality (11) satisfies the following conditions:

$$0 < G(t,s) \le \frac{t^{\alpha(t)-1}(\alpha(t)-\delta(1-s)^{\beta(t)})}{(\alpha(t)-\delta)\Gamma(\alpha(t))}(1-s)^{\nu(t)-1}, \quad t,s\in[0,1], \quad (14)$$

$$r(t)\cdot g(s) \le G(t,s) \le g(s), \quad t,s\in[0,1], \quad (15)$$

where

$$g(s) = \frac{\|\alpha(t)\|(\|\alpha(t)\|-\delta)+\delta}{(\|\alpha(t)\|-\delta)\Gamma(\|\alpha(t)\|)}s^{\|\beta(t)\|}(1-s)^{\|\nu(t)\|-1},\ r(t) = \frac{t^{\|\alpha(t)\|-1}(1-t)\|\beta(t)\|(1+\delta)}{\|\alpha(t)\|(\|\alpha(t)\|-\delta)+\delta}.$$

Proof. Let us represent the Green's function as follows:

$$G(t,s) = G_1(t,s) + \frac{t^{\alpha(t)-1}\delta(1-(1-s)^{\beta(t)})}{(\alpha(t)-\delta)\Gamma(\alpha(t))}(1-s)^{\nu(t)-1} \ge 0,$$

where

$$G_1(t,s) = \frac{1}{\Gamma(\alpha(t))}\left[t^{\alpha(t)-1}(1-s)^{\nu(t)-1} - (t-s)^{\alpha(t)-1}\right] \ge 0,$$

$$G_2(t,s) = \frac{t^{\alpha(t)-1}\delta(1-(1-s)^{\beta(t)})}{(\alpha(t)-\delta)\Gamma(\alpha(t))}(1-s)^{\nu(t)-1} \ge \frac{t^{\alpha(t)-1}\delta\cdot\beta(t)\cdot s^{\beta(t)}}{2(\alpha(t)+1)(\alpha(t)-\delta)\Gamma(\alpha(t))}(1-s)^{\nu(t)-1} \ge 0,$$

$0 \le s \le t \le 1$.

Therefore, condition (14) is satisfied. Let us prove that condition (15) is satisfied.

$$\begin{aligned}G(t,s) &= G_1(t,s) + \frac{t^{\alpha(t)-1}\delta(1-(1-s)^{\beta(t)})}{(\alpha(t)-\delta)\Gamma(\alpha(t))}(1-s)^{\nu(t)-1}\\ &\le \frac{\|\alpha(t)\|\cdot s^{\|\beta(t)\|}(1-s)^{\|\nu(t)\|-1}}{\Gamma(\|\alpha(t)\|)} + \frac{\delta}{(\|\alpha(t)\|-\delta)\Gamma(\|\alpha(t)\|)}s^{\|\beta(t)\|}(1-s)^{\|\nu(t)\|-1}\\ &= s^{\|\beta(t)\|}(1-s)^{\|\nu(t)\|-1}\frac{\|\alpha(t)\|(\|\alpha(t)\|-\delta)+\delta}{(\|\alpha(t)\|-\delta)\Gamma(\|\alpha(t)\|)}\\ &= g(s),\end{aligned}$$

$$
\begin{aligned}
G(t,s) &= G_1(t,s) + \frac{t^{\|\alpha(t)\|-1}\delta(1-(1-s)^{\|\beta(t)\|})}{(\|\alpha(t)\|-\delta)\Gamma(\|\alpha(t)\|)}(1-s)^{\nu(t)-1} \\
&\geq t^{\|\alpha(t)\|-1}(1-t)\frac{\|\beta(t)\|\, s^{\|\beta(t)\|}(1-s)^{\|\nu(t)\|-1}}{(\|\alpha(t)\|-\delta)\Gamma(\|\alpha(t)\|)} + \frac{\|\beta(t)\|\,\delta \cdot t^{\|\alpha(t)\|-1}(1-t)}{(\|\alpha(t)\|-\delta)\Gamma(\|\alpha(t)\|)} s^{\|\beta(t)\|}(1-s)^{\|\nu(t)\|-1} \\
&= t^{\|\alpha(t)\|-1}(1-t)\frac{\|\beta(t)\|\,(1+\delta)}{(\|\alpha(t)\|-\delta)\Gamma(\|\alpha(t)\|)} s^{\|\beta(t)\|}(1-s)^{\|\nu(t)\|-1} \\
&= r(t)\cdot g(s).
\end{aligned}
$$

5. Study of existence of two positive solutions of nonlinear problem

Now we will study problems (2) and (3). To prove the existence of at least two positive solutions, we will use the algorithm proposed in [12] and generalize it to the case of a fractional derivative of variable order.

The function will be a solution to problems (2) and (3) if and only if the function will be a solution to the nonlinear integral equation:

$$
u(t) = \int_0^1 G(t,s)q(s)f(t,u(t))dt,
$$

where $t \in [0,1]$, $u \in C[0,1]$, and $G(t,s)$ is the Green's function defined by equality (11). We introduce the following notation:

$$
Y(t) = \left(\frac{1}{(\alpha(t)-\delta)\Gamma(\alpha(t))} \int_0^1 (1-s)^{\nu(t)-1}(\alpha(t)-\delta(1-s)^{\beta(t)})q(s)ds \right)^{-1}. \tag{16}
$$

Based on condition (15), we can assert that $Y(t) > 0$, where $t \in [0,1]$. It follows from condition (15) that the function $G(t,s)$ is continuous and there exist numbers m, and $M > 0$ such that the following inequality holds:

$$
m \leq G(t,s) \leq M, t, s \in [0,1]. \tag{17}
$$

Suppose that the functions $f(t,u(t))$, and $q(t)$ satisfy the following conditions:

1) $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous non-decreasing function, $f(t,0) \neq 0$,

$$
f(t,u) \leq f(t,z) \leq Y(t)\cdot\gamma, 0 \leq u \leq z, \gamma = const, t \in [0,1]. \tag{18}
$$

2) $q : (0,1) \to [0,\infty)$ is a continuous function and

$$
0 < \int_0^1 (1-s)^{\nu(t)-1}q(s)ds < \infty. \tag{19}
$$

Consider a real Banach space $C[0,1]$ with the norm $\|u\| = \max_{0\le t\le 1} |u(t)|$. In this space, we define a cone K in the following form:

$$K = \{u \in C[0,1] : u(t) \ge r(t)\,\|u\|\,, t \in [0,1]\}. \tag{20}$$

We define the operator $T : C[0,1] \to C[0,1]$ as follows:

$$(Tu)(t) = \int_0^1 G(t,s)q(s)f(s,u(s))ds, t \in [0,1], u \in C[0,1]. \tag{21}$$

The operator T is a completely continuous operator and leaves the cone invariant K, (i.e., $T(K) \subseteq K$, $\forall u \in K$).

The operator T is equicontinuous and uniformly bounded. Then, based on the Arzela-Ascoli theorem, it is not difficult to prove that the operator is a completely continuous operator. We will prove that $T(K) \subseteq K$, $\forall u \in K$.

Indeed, for any $u \in K$, using conditions (15), (17), and (18), we obtain:

$$(Tu)(t) = \int_0^1 G(t,s)q(s)f(s,u(s))ds \le \int_0^1 g(s)q(s)f(s,u(s))ds, t \in [0,1]. \tag{22}$$

It follows from inequality (22) that

$$\|Tu\| \le \int_0^1 g(s)q(s)f(s,u(s)ds, \tag{23}$$

$$(Tu)(t) = \int_0^1 G(t,s)q(s)f(s,u(s))ds \ge r(t)\int_0^1 g(s)q(s)f(s,u(s))ds, t \in [0,1]. \tag{24}$$

Using inequality (23), we obtain the following:

$$(Tu)(t) \ge r(t) \cdot \|Tu\|\,. \tag{25}$$

Thus, based on inequality (23), we can assert that $Tu \in K$.

Let us consider the the following cone:

$$K_\gamma = \{u \in C[0,1] : \|u(t)\|_C \le \gamma(\forall t \in [0,1])\}. \tag{26}$$

Let K_γ be the cone defined by equality (26). Then, $T(K_\gamma) \subseteq K_\gamma$. In fact, if $u \in K_\gamma$, $0 \le u(s) \le \|u\| \le$

γ, and $s \in [0,1]$, then based on (14) and (18), we obtain the following:

$$\begin{aligned} 0 \le (Tu)(t) &= \int_0^1 G(t,s)q(s)f(s,u(s))ds \\ &\le \frac{t^{\alpha(t)-1}}{(\alpha(t)-\delta)\Gamma(\alpha)} \int_0^1 (1-s)^{\nu(t)-1}(\alpha(t)-\delta(1-s)^{\beta(t)})q(s)f(s,\gamma) \\ &\le \frac{Y(t)\gamma}{(\alpha(t)-\delta)\Gamma(\alpha)} \int_0^1 (1-s)^{\nu(t)-1}(\alpha(t)-\delta(1-s)^{\beta(t)})q(s)ds \\ &= \gamma, \quad t \in [0,1]. \end{aligned} \tag{27}$$

It follows from (23) that $\|Tu\| \le \gamma$; therefore, $T(K_\gamma) \subseteq K_\gamma$.

Theorem 1. Let conditions (14) and (15) be satisfied for the function $G(t,s)$, and let $\gamma > 0$ be such that condition (18) holds, where $Y(t)$ is defined by equality (16). Then, problems (2) and (3) have two positive solutions, $\overline{u_1}(t)$ and $\overline{u_2}(t)$, which satisfy conditions $0 < \|\overline{u_1}(t)\| \le \|\overline{u_2}(t)\| \le \gamma$. In this case, successive approximations $u_1^{k+1} = Tu_1^k$ and $u_2^{k+1} = Tu_2^k$, when $k = 0,1,2,\ldots$, converge in the norm in space C to positive solutions $u_1(t)$ and $u_2(t)$, respectively, where $u_1^0(t) = 0$ and $u_2^0(t) = \gamma \cdot t^{\nu(t)-1}$, $t \in [0,1]$, and also, the following inequalities hold:

$$u_1^0(t) \le u_1^2(t) \le \ldots \le u_1^k(t) \le \ldots \le \overline{u_1}(t) \le \overline{u_2}(t) \le \ldots \le u_2^k(t) \le \ldots \le u_2^1(t) \le u_2^0(t), t \in [0,1].$$

Proof. Let us show that successive approximations $\{u_1^k\}$ form a monotonically increasing sequence that tends in the limit to $\overline{u}_1 \in K_\gamma$, that is, $\lim_{k\to\infty} \|v_k - v^*\| = 0$, and $\overline{u_1}(t) \in K_\gamma$ is a positive solution to problems (2) and (3). We have $Tu_1^0 = T0 = 0$, $u_1^0 \in K_\gamma$. It follows from the complete continuity of the operator T that the set of successive approximations $\{u_1^k\}_{k=0}^\infty$ forms a sequentially compact set [33], that is, from any sequence of its elements, one can select a subsequence converging to $\overline{u_1}(t)$. Since $u_1^1 = Tu_1^0 = T0 \in K_\gamma$, then

$$\gamma \ge u_1^1(t) = (Tu_1^0)(t) = (T0)(t) = u_1^0(t), t \in [0,1].$$

It follows from (2) that the operator T is non-decreasing; therefore,

$$u_1^2(t) = (Tu_1^1)(t) \ge (Tu_1^0)(t) = u_1^1(t), t \in [0,1].$$

Let the inequality $u_1^k(t) \ge u_1^{k-1}(t)$, where $t \in [0,1]$ hold for an arbitrary approximation $u_1^k \in K_\gamma$. Then,

$$u_1^{k+1}(t) = (Tu_1^k)(t) \ge (Tu_1^{k-1})(t) = u_1^k(t), t \in [0,1], k = 0,1,2,\ldots$$

From the above reasoning, it follows that there is a solution $\overline{u_1}(t) \in K_\gamma$ such that $\lim_{k\to\infty} \left\|u_1^k - \overline{u_1}\right\| = 0$. Due to the continuity of the operator T and the equation $u_1^{k+1} = Tu_1^k$, we obtain $T\overline{u_1} = \overline{u_1}$. Since zero is not a solution to problems (2) and (3), then $\|\overline{u_1}\| > 0$.

From the definition of a cone, we have that $\overline{u_1}(t) \ge r(t)\,\|\overline{u_1}(t)\| > 0$, where $t \in (0,1)$. Thus, $\overline{u_1}(t)$ is a positive solution to problems (2) and (3).

The sequence of approximations $\{u_2^k\}$ forms a monotonically decreasing sequence and there exists such $\overline{u_2}(t) \in K_\gamma$ that $\lim_{k\to\infty} \left\| u_2^k - \overline{u_2} \right\| = 0$, and where $\overline{u_2}(t) \in K_\gamma$ is a positive solution to problems (2) and (3). It is easy to see that $u_2^0 \in K_\gamma$. Since $T : K_\gamma \to K_\gamma$, then $u_2^k \in T(K_\gamma) \subseteq K_\gamma$, $k = 1, 2, ...$ From the complete continuity of the operator T, it follows that the set of successive approximations $\{u_2^k\}_{k=0}^{\infty}$ forms a sequentially compact set. We have $u_2^1 = Tu_2^0 \in K_\gamma$. Then, by virtue of (12), we obtain the following:

$$
\begin{aligned}
(Tu_2^0)(t) &= \int_0^1 G(t,s)q(s)f(s,u_2^0(s))ds \\
&\le \frac{t^{\alpha(t)-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\nu(t)-1}(\alpha(t) - \delta(1-s)^{\beta(t)})q(s)f(s,u_2^0(s)) \\
&\le \frac{t^{\nu(t)-1} Y\gamma}{\Gamma(\alpha)} \int_0^1 (1-s)^{\nu(t)-1}(\alpha(t) - \delta(1-s)^{\beta(t)})q(s)ds \\
&= \gamma \cdot t^{\nu(t)-1} \\
&= u_2^0(t), \quad t \in [0,1].
\end{aligned}
$$

Consequently, $u_2^1(t) \le u_2^0(t)$. Taking inequality $u_2^1(t) \le u_2^0(t)$ into account and using inequality (20), we obtain the following:

$$
u_2^2(t) = (Tu_2^1)(t) = \int_0^1 G(t,s)q(s)f(s,u_2^1(s))ds \le \int_0^1 G(t,s)q(s)f(s,u_2^0(s))ds = (Tu_2^0)(t) = u_2^1(t),\ t \in [0,1].
$$

Continuing this process, we obtain $u_2^{k+1}(t) \le u_2^k(t)$, $t \in [0,1]$, $k = 1, 2, ...$ From the above reasoning, it follows that there is a solution $\overline{u_2}(t) \in K_\gamma$, such that $\lim_{k\to\infty} \left\| u_2^k - \overline{u_2} \right\| = 0$. Due to the continuity of the operator T, it can be stated that $\overline{u_2}(t)$ is a positive solution to problems (2) and (3). Since zero is not a solution to problems (2) and (3), then $\|\overline{u_2}(t)\| > 0$.

Using the inequality $u_1^0(t) \le u_2^0(t)$, $t \in [0,1]$, we obtain:

$$
u_1^1(t) = (Tu_1^0)(t) = \int_0^1 G(t,s)q(s)f(s,u_1^0(s))ds \le \int_0^1 G(t,s)q(s)f(s,u_2^0(s))ds = (Tu_2^0)(t) = u_2^1(t),\ t \in [0,1].
$$

Then, based on the induction method, we can state that $\overline{u_1}(t) \le \overline{u_2}(t)$, $t \in [0,1]$, when $k = 1, 2, ...$ Thus, the theorem is proven.

6. Examples

As an example, consider the following boundary value problem:

$$
D_{0t}^{\alpha(t)} u(t) + \frac{1}{4}u^3 + \sin t + 1 = 0, \qquad 0 < t < 1,
$$

$$\lim_{t\to 0} D_{0t}^{\alpha(t)-2}u(t) = 0, D_{0t}^{\beta(t)}u(t) = \frac{\delta \cdot \Gamma(\alpha(t))}{\Gamma(\alpha(t)-\beta(t))}\int_0^1 u(s)ds.$$

Let us take $\alpha(t) = 1 + \sin(t+1)$, $q(t) \equiv 1$, $f(t,u) = \frac{1}{4}u^3 + \sin t + 1$, $\gamma = 2$, $\delta = 1$, $\beta(t) = \frac{1}{2}\sin(t+1)$, and $\nu(t) - 1 = \frac{1}{2}\sin(t+1)$. Then,

$$Y(t) = \left(\frac{1}{(\alpha(t)-\delta)\Gamma(\alpha(t))}\int_0^1 (1-s)^{\nu(t)-1}(\alpha(t)-\delta(1-s)^{\beta(t)})ds\right)^{-1} \geq 3.555.$$

Figure 1 shows the graph of the dependence $Y(t)\cdot\gamma, t \in [0,1]$. From the figure, $Y^*\cdot\gamma = \gamma\cdot\min_{0\le t\le 1} Y(t) \geq 7.1$. Therefore, the condition $f(t,u) \leq f(t,2) \leq f(1,2) \leq 4 < 7.1 \leq Y(t)\cdot\gamma$ is satisfied.

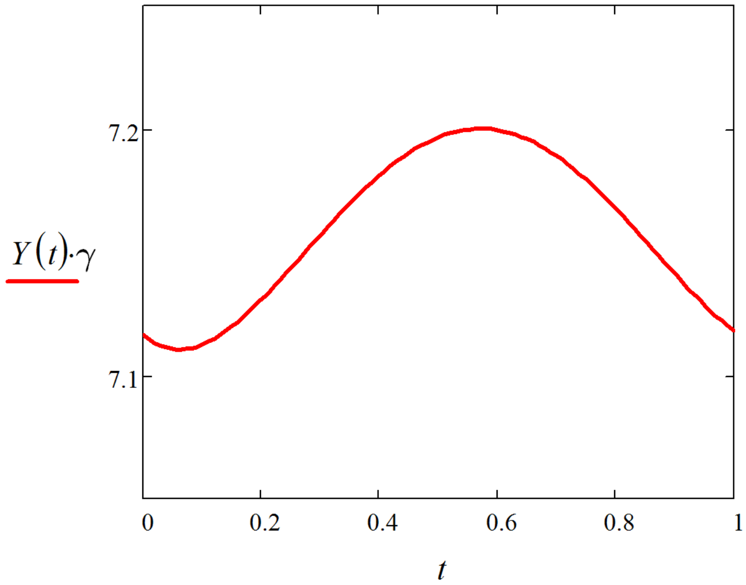

Figure 1. Dependency $Y(t)\cdot\gamma, t \in [0,1]$ graph.

Let us construct a sequence of approximations in the following form:

$$u_1^0(t) = 0,$$

$$u_1^{k+1}(t) = \frac{t^{\alpha(t)-1}}{(\alpha(t)-1)\Gamma(\alpha(t))}\int_0^1 (1-s)^{\nu(t)-1}(\alpha(t)-(1-s)^{\beta(t)})\left(\frac{1}{4}\left(u_1^k(s)\right)^3 + \sin s + 1\right)ds$$
$$-\frac{1}{(\alpha(t)-1)\Gamma(\alpha(t))}\int_0^t (t-s)^{\alpha(t)-1}(\alpha(t)-1)\left(\frac{1}{4}\left(u_1^k(s)\right)^3 + \sin s + 1\right)ds, \quad t\in[0,1], k = 0,1,...,$$

and

$$u_2^0(t) = 2t^{\nu(t)-1},$$

$$u_2^{k+1}(t) = \frac{t^{\alpha(t)-1}}{(\alpha(t)-1)\Gamma(\alpha(t))} \int_0^1 (1-s)^{\gamma(t)-1}(\alpha(t)-(1-s)^{\beta(t)}) \left(\frac{1}{4}\left(u_2^k(s)\right)^3 + \sin s + 1 \right) ds$$
$$- \frac{1}{(\alpha(t)-1)\Gamma(\alpha(t))} \int_0^t (t-s)^{\alpha(t)-1}\alpha(t) \left(\frac{1}{4}\left(u_2^k(s)\right)^3 + \sin s + 1 \right) ds, \quad k = 0, 1, ...$$

We construct the approximation graphs using the Mathcad package. Figure 2 shows the first approximations $u_1^1(t)$, $u_2^1(t)$, and $u_2^0(t)$.

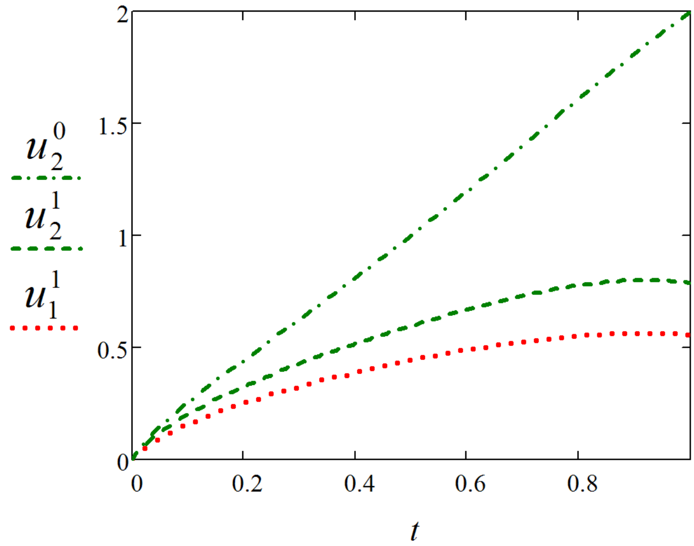

Figure 2. Graphs of approximations $u_1^1(t)$, $u_2^1(t)$, and $u_2^0(t)$.

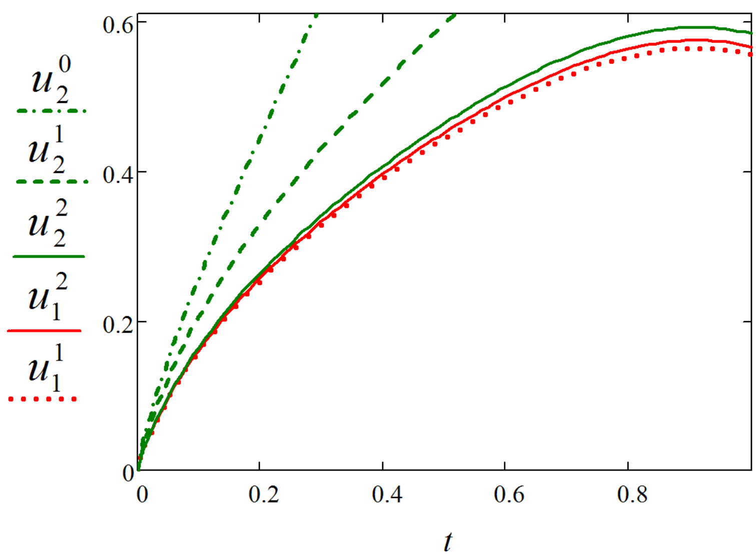

Figure 3. Graphs of approximations $u_1^1(t)$, $u_2^1(t)$, $u_2^2(t)$, $u_1^2(t)$, and $u_2^0(t)$.

From Figure 3, $0 = u_1^0 \le u_1^1 \le u_1^2 \le u_2^2 \le u_2^1(t) \le u_2^0 = 2 \cdot t^{\nu(t)-1}$.

Now, let us consider the case when the orders of fractional derivatives are defined as a logistic function:

$$\alpha(t) = 1 + \frac{1}{1+e^{-bt}}, \beta(t) = \frac{1}{2(1+e^{-bt})}, \nu(t) - 1 = \frac{1}{2(1+e^{-t})}. \tag{28}$$

Let us take $q(t) \equiv 1$, $\gamma = 2$, $\delta = 1$, and $b = 1$. Then, $Y^* \cdot \gamma = \gamma \cdot \min_{0 \le t \le 1} Y(t) \ge 7.2$, $f(t,u) \le f(t,2) \le f(1,2) \le 4 < 7.1 \le Y^*\gamma$.

Figure 4 shows the approximation graphs $u_1^1(t)$, $u_1^2(t)$, $u_1^3(t)$, $u_2^3(t)$, $u_2^2(t)$, $u_2^1(t)$, and $u_2^0(t)$ in the case when the variable orders of fractional derivatives are defined by equalities (28). As we can see from the figure, the approximations satisfy the following conditions: $0 = u_1^0(t) \le u_1^1(t) \le u_1^2(t) \le u_2^2(t) \le u_2^1 \le u_2^0(t) = 2t^{\nu(t)-1}$.

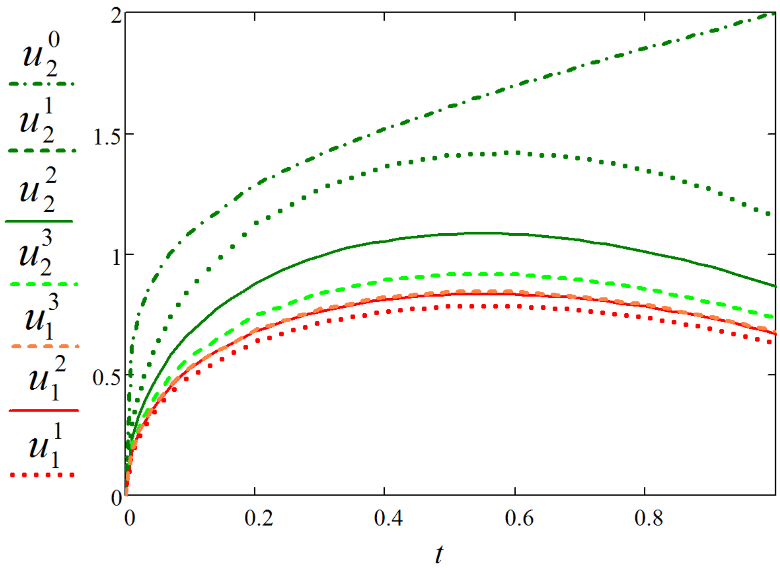

Figure 4. Graphs of the first 3rd approximations $u_2^0(t)$, $u_2^1(t)$, $u_2^2(t)$, $u_2^3(t)$, $u_1^3(t)$, $u_1^2(t)$, and $u_1^1(t)$.

7. Conclusions

The paper studied a boundary value problem for a differential equation with a fractional Riemann-Liouville derivative of variable order and nonlocal boundary conditions. In the case of a fractional derivative of variable order, the solutions have the property of a continuous transition between the derivatives of integer and fractional orders. This may be of interest for media whose properties have an explicit dependence on time. Physically, this may be due to a restructuring external effect (electromagnetic or other) with a possible subsequent degradation of the induced properties. The approach that we considered in the paper consisted of defining an operator that provides a smooth continuous transition of all orders of differentiation between integer orders, which is of great importance from the point of view of physical modeling of non-stationary processes in complex dynamic systems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by ongoing institutional funding. No additional grants to carry out or direct this particular research were obtained.

Conflict of interest

The authors declare there is no conflict of interest.

References

1. V. V. Uchaikin, R. T. Sibatov, Fractional differential kinetics of dispersive transport as the consequence of its self-similarity, *JETP Lett.*, **86** (2007), 512–516. https://doi.org/10.1134/S0021364007200040
2. V. D. Beybalaev, A. A. Aliverdiev, J. Hristov, Transient heat conduction in a semi-infinite domain with a memory effect: Analytical solutions with a robin boundary condition, *Fractal Fract.*, **7** (2023), 770. https://doi.org/10.3390/fractalfract7100770
3. J. Hristov, Constitutive fractional modeling, in *Mathematical Modelling: Principle and Theory* (eds. Hemen Dutta), American Mathematical Society, **786** (2023), 37–140. https://doi.org/10.1090/conm/786/15795
4. V. D. Beybalaev, A. A. Aliverdiev, A. Z. Yakubov, S. A. Ninalalov, A. A. Amirova, Mathematical model of heat conduction for a semi-infinite body, taking into account memory effects and spatial correlations, *Fractal Fract.*, **7** (2023), 265. https://doi.org/10.3390/fractalfract7030265
5. K. S. Miller, Fractional differential equations, *J. Fract. Calc.*, **3** (1993), 49–57.
6. A. M. Nakhushev, The Sturm–Liouville problem for a second order ordinary differential equation with fractional derivatives in the lower terms, *Dokl. Akad. Nauk SSSR*, **234** (1977), 308–311.
7. I. Podlubny, *Fractional Differential Equations, Mathematics in Science and Engineering*, Academic Press, 1999.
8. I. Podlubny, The Laplace transform method for linear differential equations of the fractional order, *Inst. Expe. Phys., Slov. Acad. Sci.*, **1994** (1994).
9. Z. B. Bai, H. S. Lü, Positive solutions for boundary value problem of nonlinear fractional differential equation, *J. Math. Anal. Appl.*, **311** (2005), 495–505. https://doi.org/10.1016/j.jmaa.2005.02.052
10. E. R. Kaufmann, E. Mboumi, Positive solutions of a boundary value problem for a nonlinear fractional differential equation, *Electron. J. Qual. Theory Differ. Equ.*, **3** (2008), 1–11. https://doi.org/10.14232/ejqtde.2008.1.3

11. C. Z. Bai, Triple positive solutions for a boundary value problem of nonlinear fractional differential equation, *Electron. J. Qual. Theory Differ. Equ.*, **24** (2008), 1–10.
12. Y. Sun, Y. Sun, Positive solutions and monotone iterative sequences for a fractional differential equation with integral boundary conditions, *Adv. Differ. Equ.*, **2014** (2014), 29. https://doi.org/10.1186/1687-1847-2014-29
13. C. B. Zhai, M. R. Hao, Mixed monotone operator methods for the existence and uniqueness of positive solutions to Riemann-Liouville fractional differential equation boundary value problems, *Bound. Value Probl.*, **2013** (2013), 85. https://doi.org/10.1186/1687-2770-2013-85
14. S. Q. Zhang, The existence of a positive solution for a nonlinear fractional differential equation, *J. Math. Anal. Appl.*, **252** (2000), 804–812. https://doi.org/10.1006/jmaa.2000.7123
15. X. G. Zhang, Y. F. Han, Existence and uniqueness of positive solutions for higher order nonlocal fractional differential equations, *Appl. Math. Lett.*, **25** (2012), 555–560. https://doi.org/10.1016/j.aml.2011.09.058
16. X. G. Zhang, L. S. Liu, Y. H. Wu, Multiple positive solutions of a singular fractional differential equation with negatively perturbed term, *Math. Comput. Modell.*, **55** (2012), 1263–1274. https://doi.org/10.1016/j.mcm.2011.10.006
17. X. G. Zhang, L. S. Liu, Y. H. Wu, Y. N. Lu, The iterative solutions of nonlinear fractional differential equations, *Appl. Math. Comput.*, **219** (2013), 4680–4691. https://doi.org/10.1016/j.amc.2012.10.082
18. C. F. Li, X. N. Luo, Y. Zhou, Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations, *Comput. Math. Appl.*, **59** (2010), 1363–1375. https://doi.org/10.1016/j.camwa.2009.06.029
19. C. F. M. Coimbra, Mechanics with variable-order differential operators, *Ann. Phys.*, **515** (2003), 692–703. https://doi.org/10.1002/andp.200310032
20. D. Ingman, J. Suzdalnitsky, Control of damping oscillations by fractional differential operator with time-dependent order, *Comput. Methods Appl. Mech. Eng.*, **193** (2004), 5585–5595. https://doi.org/10.1016/j.cma.2004.06.029
21. C. F. Lorenzo, T. T. Hartley, Variable order and distributed order fractional operators, *Nonlinear Dyn.*, **29** (2009), 57–98. https://doi.org/10.1023/A:1016586905654
22. B. P. Moghaddam, J. A. T. Machado, Extended algorithms for approximating variable order fractional derivatives with applications, *J. Sci. Comput.*, **71** (2017), 1351–1374. https://doi.org/10.1007/s10915-016-0343-1
23. L. Ramirez, C. Coimbra, On the selection and meaning of variable order operators for dynamic modeling, *Int. J. Differ. Equ.*, **1** (2010), 1–16. https://doi.org/10.1155/2010/846107
24. H. G. Sun, W. Chen, Y. Q. Chen, Variable order fractional differential operators in anomalous diffusion modeling, *Physica A*, **388** (2009), 4586–4592. https://doi.org/10.1016/j.physa.2009.07.024
25. S. Zhang, L. Hu, Unique existence result of approximate solution to initial value problem for fractional differential equation of variable order involving the derivative arguments on the half-axis, *Mathematics*, **7** (2019), 286. https://doi.org/10.3390/math7030286

26. D. Valerio, J. Sa da Costa, Variable-order fractional derivative and their numerical approximations, *Signal Process.*, **91** (2011), 470–483. https://doi.org/10.1016/j.sigpro.2010.04.006

27. D. Tavares, R. Almeida, D. F. M. Torres, Caputo derivatives of fractional variable order: Numerical approximations, *Commun. Nonlinear Sci. Numer. Simul.*, **35** (2016), 69–87. https://doi.org/10.1016/j.cnsns.2015.10.027

28. A. Razminia, A. F. Dizaji, V. J. Majd, Solution existence for non-autonomous variable-order fractional differential equations, *Math. Comput. Modell.*, **55** (2012), 1106–1117. https://doi.org/10.1016/j.mcm.2011.09.034

29. A. A. Alikhanov, Boundary value problems for the equation of the variable order in differential and difference settings, *Appl. Math. Comput.*, **219** (2012), 3938–3946. https://doi.org/10.1016/j.amc.2012.10.029

30. A. Babaei, H. Jafari, S. Banihashemi, Numerical solution of variable order fractional nonlinear quadratic integro-differential equations based on the sixth-kind Chebyshev collocation method, *J. Comput. Appl. Math.*, **377** (2020), 112908. https://doi.org/10.1016/j.cam.2020.112908

31. C. J. Zuniga-Aguilar, H. M. Romero-Ugalde, J. F. Gomez-Aguilar, R. F. Escobar-Jimenez, M. Valtierra-Rodríguez, Solving fractional differential equations of variable-order involving operator with Mittag-Leffler kernel using artificial neural networks, *Chaos, Solitons Fractals*, **103** (2017), 382–403. https://doi.org/10.1016/j.chaos.2017.06.030

32. C. M. Chen, F. Liu, V. Anh, I. Turner, Numerical schemes with high spatial accuracy for a variableorder anomalous subdiffusion equation, *SIAM J. Sci. Comput.*, **32** (2010), 1740–1760. https://doi.org/10.1137/090771715

33. M. O. Korpusov, A. A. Panin, *Lectures on Linear and Nonlinear Functional Analysis, Part III: Nonlinear Analysis*, Physics Faculty of Moscow State University, 2015.