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Research article

Finite groups with at most six subgroups not in the Chermak-Delgado lattice

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Abstract: The Chermak-Delgado lattice of a finite group G is a self-dual sublattice of the subgroup lattice of G. In this paper, we determine finite groups with at most six subgroups not in the Chermak-Delgado lattice.

Keywords: Chermak-Delgado lattice; subgroup lattice; finite groups; finite solvable groups; Sylow subgroups

1. Introduction

Suppose that G is a finite group, and H is a subgroup of G. The Chermak-Delgado measure of H (in G) is denoted by $m_G(H)$ and defined as $m_G(H) = |H| \cdot |C_G(H)|$. The maximal Chermak-Delgado measure of G is denoted by $m^*(G)$, and defined as

$$m^*(G) = \max\{m_G(H) \mid H \leq G\}.$$

Let

$$\mathcal{CD}(G) = \{H \mid m_G(H) = m^*(G)\}.$$

Then the set $\mathcal{CD}(G)$ forms a sublattice of $\mathcal{L}(G)$, the subgroup lattice of G, and is called the Chermak-Delgado lattice of G. It was first introduced by Chermak and Delgado [1] and revisited by Isaacs [2]. In the last years, there has been a growing interest in understanding this lattice and related ones both in the finite and infinite cases. For example, some group theorists determine whether a given self-dual lattice can be realized as the Chermak-Delgado lattice of some group, such as a chain (see [3, 4]), a quasi-antichain (see [5, 6]), or some special self-dual lattices (see [7–9]); some group theorists study the Chermak-Delgado lattice of certain groups and some properties of the Chermak-Delgado lattice (see [10–14]); some group theorists study the Chermak-Delgado measure of subgroups (see [15–19]); and for more research results about lattices, see [20–24] and so on.

According to [14], we know that there is always a subgroup of group G that is not in $\mathcal{CD}(G)$. It is natural to ask the question:

Can the number of subgroups not in the Chermak-Delgado lattice determine the group structure?

Some cases of this question have been proposed and solved. Fasolă and Tărnăuceanu [25] classified groups with at most two subgroups not in the Chermak-Delgado lattice. Burrell et al. [26] classified groups with at most four subgroups not in the Chermak-Delgado lattice and showed that the only non-nilpotent group with at most five subgroups not in the Chermak-Delgado lattice is S_3. By [26, Lemma 5.1], a non-abelian nilpotent group with at most six subgroups not in the Chermak-Delgado lattice is a p-group. Liu et al. [27] classified finite p-groups with at most $p^2 + p$ subgroups not in the Chermak-Delgado lattice. In this paper, we focus on finite groups with at most six subgroups not in the Chermak-Delgado lattice. It is easy to check the case of abelian groups. Hence, we only need to consider the case of non-nilpotent groups that have exactly six subgroups not in the Chermak-Delgado lattice.

Following [26], we use $\delta_{\mathcal{CD}}(G)$ to denote the number of subgroups of G not in $\mathcal{CD}(G)$. That is,

$$\delta_{\mathcal{CD}}(G) = |\mathcal{L}(G)| - |\mathcal{CD}(G)|.$$

Our main result is stated as follows.

Main Theorem. Let G be a finite group. If $\delta_{\mathcal{CD}}(G) = 6$, then G is nilpotent.

Combining the results from [25–27] and a simple verification for abelian groups, the finite groups with at most six subgroups not in the Chermak-Delgado lattice are completely classified. For the reader's convenience, we present the results as follows:

Theorem. Let G be a finite group. Then $\delta_{\mathcal{CD}}(G) \leqslant 6$ if and only if G is one of the following groups:

1) C_p or Q_8; ($\delta_{\mathcal{CD}}(G) = 1$)
2) C_{p^2}; ($\delta_{\mathcal{CD}}(G) = 2$)
3) C_{p^3} or $C_p \times C_q$; ($\delta_{\mathcal{CD}}(G) = 3$)
4) C_{p^4}, $C_2 \times C_2$, or $\langle a, b \mid a^9 = b^3 = 1, [a, b] = a^3\rangle$; ($\delta_{\mathcal{CD}}(G) = 4$)
5) C_{p^5}, $C_{p^2} \times C_q$, $C_3 \times C_3$, D_8, or S_3; ($\delta_{\mathcal{CD}}(G) = 5$)
6) C_{p^6}, $\langle a, b \mid a^8 = b^2 = 1, [a, b] = a^4\rangle$ or $\langle a, b \mid a^{25} = b^5 = 1, [a, b] = a^5\rangle$. ($\delta_{\mathcal{CD}}(G) = 6$)

2. Preliminaries

Let G be a finite group. We use n_p to denote the number of Sylow p-subgroups of G, where p is a prime factor of $|G|$. The following properties in Lemma 2.1 are basic and often used in this paper. We will not point out when we use them.

Lemma 2.1. [9, Theorem 2.1] *Suppose that G is a finite group and $H, K \in \mathcal{CD}(G)$.*

1) *$\langle H, K\rangle = HK$. Hence, a Chermak-Delgado lattice is modular.*

2) *$C_G(H \cap K) = C_G(H)C_G(K)$.*

3) *$C_G(H) \in \mathcal{CD}(G)$ and $C_G(C_G(H)) = H$. Hence a Chermak-Delgado lattice is self-dual, and $Z(G) \leq H$.*

4) *Let M be the maximal member of $\mathcal{CD}(G)$. Then M is characteristic in G and $\mathcal{CD}(M) = \mathcal{CD}(G)$.*

5) *The minimal member of $\mathcal{CD}(G)$ is characteristic, abelian, and contains $Z(G)$.*

Lemma 2.2. [11, Theorem A] *Let G be a finite group. If $H \in \mathcal{CD}(G)$, then $H \lhd \lhd G$.*

Lemma 2.3. [28, Theorem 5.16] *Let a p-group G be neither cyclic nor a 2-group of maximal class, $k \in \mathbb{N}^*$, and $p^k < p^m = |G|$. Then*

$$s_k(G) \equiv 1 + p \pmod{p^2},$$

where $s_k(G)$ is the number of subgroups of order p^k of G.

Lemma 2.4. [26, Theorem B] *Let G be a finite group that is not nilpotent. If $\delta_{\mathcal{CD}}(G) = 5$, then $G \cong S_3$.*

Lemma 2.5. [26, Lemma 3.2] *Let G be a finite group. Then at most one Sylow subgroup of G is in $\mathcal{CD}(G)$. Furthermore, if a Sylow subgroup of G is in $\mathcal{CD}(G)$, then it is normal.*

3. Proof of main theorem

Lemma 3.1. *Let G be a finite group. If $\delta_{\mathcal{CD}}(G) < 13$, then G is solvable.*

Proof. Suppose that G is non-solvable. By the "Burnside $p^a q^b$ Theorem", G has at least three Sylow subgroups P_1, P_2, and P_3 that are non-normal, where $P_i \in \mathrm{Syl}_{p_i}(G), i = 1, 2, 3$, and p_1, p_2, and p_3 are distinct prime factors of $|G|$. Hence G has at least $p_1 + 1 + p_2 + 1 + p_3 + 1$ Sylow subgroups. By Lemma 2.5, $\delta_{\mathcal{CD}}(G) \geqslant 2 + 1 + 3 + 1 + 5 + 1 = 13$. This contradicts the hypothesis. □

Lemma 3.2. *Let G be a finite group and $1 \in \mathcal{CD}(G)$. Let $P \in \mathrm{Syl}_p(G)$, where p is a prime factor of $|G|$. Suppose that $1 < H \leq P$. If $Z(P) \cap H \neq 1$, then $H \notin \mathcal{CD}(G)$. In particular, $P, Z(P) \notin \mathcal{CD}(G)$.*

Proof. Notice that $1 \in \mathcal{CD}(G)$. We have $m^*(G) = |G|$. Since $Z(P)$ is non-trivial, $p \mid |C_G(P)|$. Let $|P| = p^n$. If $Z(P) \cap H \neq 1$, then $H \notin \mathcal{CD}(G)$. If not, then $m_G(H) = |H| \cdot |C_G(H)| = |G|$. Hence $\frac{|G|}{p^n} \mid |C_G(H)|$. Therefore $G = \langle P, C_G(H) \rangle$. Thus $Z(P) \cap H \leq Z(G)$. Hence $m_G(G) = |G| \cdot |Z(G)| > |G| = m^*(G)$, which is a contradiction. In particular, we have $P, Z(P) \notin \mathcal{CD}(G)$. □

Corollary 3.3. *Let G be a finite group and $1 \in \mathcal{CD}(G)$. Suppose that $P \in \mathrm{Syl}_p(G)$, where p is a prime factor of $|G|$.*

1) *If P is abelian, then there are at least $n_p + \tau(|P|) - 2$ subgroups of P not in $\mathcal{CD}(G)$, where $\tau(|P|)$ is the number of all positive divisors of $|P|$;*

2) *If P is non-abelian, then there are at least $n_p + p + \tau(|Z(P)|)$ subgroups of P not in $\mathcal{CD}(G)$, where $\tau(|Z(P)|)$ is the number of all positive divisors of $|Z(P)|$.*

Proof. 1) According to Lemma 3.2, we know that all Sylow p-subgroups are not in $\mathcal{CD}(G)$. Moreover, since P is abelian, all non-trivial proper subgroups of P are also not in $\mathcal{CD}(G)$. Thus $\delta_{\mathcal{CD}}(G) \geqslant n_p + \tau(|P|) - 2$.

2) Since $P/Z(P)$ is non-cyclic, the number of subgroups of P that properly contain $Z(P)$ is greater than $p + 2$. Hence there are at least $n_p + p + \tau(|Z(P)|)$ subgroups of P not in $\mathcal{CD}(G)$. □

Lemma 3.4. *Let G be a finite solvable group. Suppose that $P \in \mathrm{Syl}_p(G)$, where p is a prime factor of $|G|$.*

1) *If $P \in \mathcal{CD}(G)$, then all p'-subgroups of G are not in $\mathcal{CD}(G)$;*

2) *If $P \ntrianglelefteq G$, then $m_G(H) \neq m_G(G)$, where H is a Hall p'-subgroup of G.*

Proof. 1) Since $Z(P)$ is non-trivial, $p \mid |C_G(P)|$. Let $|P| = p^n$. If $P \in \mathcal{CD}(G)$, then $p^{n+1} \mid m^*(G) = |P| \cdot |C_G(P)|$. Let K be a p'-subgroup of G. Then $p \nmid |K|$ and $|C_G(K)| \mid |G|$. It follows that $p^{n+1} \nmid m_G(K)$. Hence $K \notin \mathcal{CD}(G)$.

2) If $P \ntrianglelefteq G$, then $|P| \nmid |C_G(H)|$. Hence $|P| \nmid |H| \cdot |C_G(H)| = m_G(H)$. Since $|P| \mid m_G(G)$, $m_G(H) \neq m_G(G)$. □

Lemma 3.5. *Let G be a solvable non-nilpotent group. If $\delta_{\mathcal{CD}}(G) = 6$, then $|G|$ has exactly two prime factors.*

Proof. Assume the contrary. Then $|G|$ has at least three prime factors. Let $P \in \mathrm{Syl}_p(G)$, where p is a prime factor of $|G|$. Since G is non-nilpotent, there exists at least one non-normal Sylow subgroup of G. Without loss of generality, we may assume $P \ntrianglelefteq G$. Then $P \notin \mathcal{CD}(G)$. Hence $N_G(P) \notin \mathcal{CD}(G)$. Otherwise, $N_G(P) \lhd \lhd G$ by Lemma 2.2. Since $P \trianglelefteq N_G(P)$, $P \lhd \lhd G$. Thus $P \trianglelefteq G$; this is a contradiction. Except for the Sylow p-subgroups, there exists at least one other Sylow subgroup that is not in $\mathcal{CD}(G)$, by Lemma 2.5. We claim that $P = N_G(P)$. Otherwise, $\delta_{\mathcal{CD}}(G) \geqslant 2n_p + 1 \geqslant 2(1 + 2) + 1 = 7$, which is a contradiction. Next, we claim that $n_p \leqslant 5$. If not, then $n_p \geqslant 6$. Hence $\delta_{\mathcal{CD}}(G) \geqslant n_p + 1 \geqslant 7$; this is a contradiction. On the other hand, since $|G|$ has at least three prime factors, $n_p = |G : N_G(P)| = |G : P| \geqslant 2 \times 3 = 6$; this is a contradiction. □

Proof of Main Theorem. By Lemma 3.1, G is solvable. If G is not nilpotent, then there exists at least one Sylow subgroup not normal in G, and $|G|$ exactly has two prime factors, p and q, by Lemma 3.5. Let $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$, and $P \ntrianglelefteq G$. By the proof of the Lemma 3.5, we have $P = N_G(P)$ and $n_p \leqslant 5$. Hence, we have $p = 2$ or 3.

Case 1. $p = 3$

Now, we have $n_p = 4$. Thus $q = 2$ and $|G| = |N_G(P)| \cdot n_p = |P| \cdot 4 = 3^i 4$, where $i \in \mathbb{Z}^+$. We assert that $Q \trianglelefteq G$. If not, then $n_q \geqslant 3$. Hence $\delta_{\mathcal{CD}}(G) \geqslant 7$; this is a contradiction. If $1 \in \mathcal{CD}(G)$, then $m^*(G) = |G|$ and $i = 1$ by Corollary 3.3. That is $|G| = 12$. Thus, there exists a subgroup of order 4, and its measure is greater than 12, a contradiction. Hence, we have $1 \notin \mathcal{CD}(G)$. Next, we will prove that all Sylow subgroups of G are not in $\mathcal{CD}(G)$. Otherwise, we have $Q \in \mathcal{CD}(G)$. Then $G \notin \mathcal{CD}(G)$ by Lemma 3.42). Hence, there are at least six subgroups not in $\mathcal{CD}(G)$. Since $Q \in \mathcal{CD}(G)$ and $|Q| = 4$, $4^2 \mid m^*(G)$, and there exists a subgroup $K \leq Q$ such that $|K| = 2$. Hence $4^2 \nmid m_G(K)$. Therefore $K \notin \mathcal{CD}(G)$. Consequently, $\delta_{\mathcal{CD}}(G) \geqslant 7$, a contradiction.

Notice that all Sylow subgroups of G are not in $\mathcal{CD}(G)$ and $1 \notin \mathcal{CD}(G)$. Hence, there are at least six subgroups not in $\mathcal{CD}(G)$. In the following, we will prove that $Z(G) > 1$ and $G \in \mathcal{CD}(G)$. If $Z(G) = 1$, then $G \notin \mathcal{CD}(G)$. Hence $\delta_{\mathcal{CD}}(G) \geqslant 7$, a contradiction. Thus $Z(G) > 1$. Moreover, if $G \notin \mathcal{CD}(G)$, then $\delta_{\mathcal{CD}}(G) \geqslant 7$, a contradiction. Consequently, $G \in \mathcal{CD}(G)$. Next, if $6 \mid |Z(G)|$, then the subgroups of $Z(G)$ of order 2 and 3 are not in $\mathcal{CD}(G)$. Hence $\delta_{\mathcal{CD}}(G) \geqslant 8$. This is a contradiction. Therefore $6 \nmid |Z(G)|$. Notice that $|G| = 3^i 4$. Hence, we may assume that $|Z(G)| = 2$ or 3^{i_1}. If $|Z(G)| = 2$, then $PZ(G) \notin \mathcal{CD}(G)$. Since $n_p = 4$, $\delta_{\mathcal{CD}}(G) \geqslant 10$. This is a contradiction. If $|Z(G)| = 3^{i_1}$, then $i_1 < i$ and $QZ(G) \notin \mathcal{CD}(G)$. Moreover, we have $\delta_{\mathcal{CD}}(G) \geqslant 7$. This is a contradiction.

Case 2. $p = 2$

Obviously, $n_p = 5$ or $n_p = 3$.

Subcase 2.1. $n_p = 5$

It is easy to see that $q = 5$ and $|G| = |N_G(P)| \cdot n_p = |P| \cdot 5 = 2^i 5$, where $i \in \mathbb{Z}^+$. If $1 \in \mathcal{CD}(G)$, then $m^*(G) = |G|$ and $i = 1$ by Corollary 3.3. That is $|G| = 10$. Hence there exists a subgroup of order 5, and its measure is greater than 10. This is a contradiction. Therefore $1 \notin \mathcal{CD}(G)$. If $Q \in \mathcal{CD}(G)$, then $G \notin \mathcal{CD}(G)$ by Lemma 3.4(2). Now, we have $\delta_{\mathcal{CD}}(G) \geqslant 7$. This is a contradiction. Hence, all Sylow subgroups of G are not in $\mathcal{CD}(G)$. Consequently, there are at least seven subgroups not in $\mathcal{CD}(G)$. This is a contradiction.

Subcase 2.2. $n_p = 3$

Now, we have $q = 3$ and $|G| = |N_G(P)| \cdot n_p = |P| \cdot 3 = 2^i3$, where $i \in \mathbb{Z}^+$. We assert that $Q \trianglelefteq G$. If not, then $n_q \geqslant 4$. Now, we have $\delta_{\mathcal{CD}}(G) \geqslant 7$. This is a contradiction. By Lemma 2.4 and $\delta_{\mathcal{CD}}(G) = 6$, we have $|G| > 6$. If $1 \in \mathcal{CD}(G)$, then $m^*(G) = |G|$ and $i \leqslant 3$ by Corollary 3.3. It follows that $|G| = 12$ or 24. If $|G| = 12$, then there exists a subgroup of order 4, and its measure is greater than 12. This is a contradiction. Thus $|G| = 24$. Moreover, we have the subgroups of order 8 that are non-abelian. It follows that P has three subgroups of order 4. If all subgroups of P of order 4 are not in $\mathcal{CD}(G)$, then $\delta_{\mathcal{CD}}(G) \geqslant 7$. This is a contradiction. Hence there exists a subgroup $H \leq P$ of order 4 such that $H \in \mathcal{CD}(G)$. Therefore $m_G(H) = |H| \cdot |C_G(H)| = |G| = m^*(G)$. Thus $3 \mid |C_G(H)|$. It follows that $Q \leq C_G(H)$. Hence, HQ is abelian and $|HQ| = 12$. Therefore $m_G(HQ) > m_G(G)$; this is a contradiction. Thus $1 \notin \mathcal{CD}(G)$.

Next, we claim that $Q \notin \mathcal{CD}(G)$. If not, then $|Q|\cdot|C_G(Q)| = m_G(Q) > m_G(1) = |G|$. Thus $|C_G(Q)| > \frac{|G|}{3}$. Next, since G is non-nilpotent, we have $C_G(Q) < G$. Notice that $|G| = 2^i3$. We have $|C_G(Q)| = \frac{|G|}{2}$ and $m^*(G) = \frac{3|G|}{2}$. Let $S \in \mathrm{Syl}_p(C_G(Q))$. Thus, there exists $P_1 \in \mathrm{Syl}_p(G)$ such that $S = P_1 \cap C_G(Q)$. Then $|S| = \frac{|P_1|}{2}$ and $Z(P_1) \cap S \leq Z(G)$. If $Z(P_1) \cap S \neq 1$, then $m_G(Z(P_1) \cap S) = |Z(P_1) \cap S| \cdot |G| \geqslant 2|G|$, a contradiction. Hence $Z(P_1) \cap S = 1$. Since $Z(P_1)Z(S)Q \leq C_G(S)$, $|C_G(S)| \geqslant 4|Q|$. Therefore $m_G(S) = |S| \cdot |C_G(S)| \geqslant \frac{|P_1|}{2} \cdot 4|Q| = 2|G| > m_G(Q)$, which contradicts $Q \in \mathcal{CD}(G)$. Hence, all Sylow subgroups of G are not in $\mathcal{CD}(G)$. Therefore, there are at least five subgroups not in $\mathcal{CD}(G)$.

In the following, we claim that $Z(G) > 1$. If not, then $G \notin \mathcal{CD}(G)$. Let M be the maximal member in $\mathcal{CD}(G)$. If $|Z(M)| = 2$, then $m_G(M) = |M| \cdot |Z(M)| \leqslant \frac{|G|}{2} \cdot 2 = |G|$, which is a contradiction. Thus $|Z(M)| \neq 2$. Moreover, we have $2 \mid |Z(M)|$. Hence the subgroup of $Z(M)$ of order 2 is not in $\mathcal{CD}(G)$. Therefore $\delta_{\mathcal{CD}}(G) \geqslant 7$, a contradiction.

We claim that $G \in \mathcal{CD}(G)$. If not, then $Z(G) \notin \mathcal{CD}(G)$. Thus $\delta_{\mathcal{CD}}(G) \geqslant 7$; this is a contradiction. Notice that $|G| = 2^i3$ and G is non-nilpotent. Hence $|Z(G)| = 2^s$, where $1 \leqslant s < i$. It follows that $QZ(G) \notin \mathcal{CD}(G)$. Now, there are at least six subgroups not in $\mathcal{CD}(G)$. If $i = 2$, then $|G| = 12$ and $|Z(G)| = 2$. Thus there exists an abelian subgroup of order 6, and its measure is greater than $m_G(G)$. It is a contradiction to $G \in \mathcal{CD}(G)$. Therefore $i > 2$. Hence there exists a subgroup $T \neq Z(G)$ of order 4 such that $T \in \mathcal{CD}(G)$. If not, then $\delta_{\mathcal{CD}}(G) \geqslant 7$, a contradiction. Thus $T \in \mathcal{CD}(G)$. Hence $m_G(T) = |T| \cdot |C_G(T)| = |G| \cdot |Z(G)| = m_G(G)$. Therefore $3 \mid |C_G(T)|$. Thus $Q \leq C_G(T)$. Hence TQ is abelian. It follows that $9 \mid m_G(TQ)$. Hence $m_G(TQ) \neq m_G(G)$ by $9 \nmid m_G(G)$. Thus $TQ \notin \mathcal{CD}(G)$. Now, we have $\delta_{\mathcal{CD}}(G) \geqslant 7$. This is a contradiction. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors report there are no competing interests to declare.

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