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Research article

Upper bounds of blowup time for nonlinear extensible beam equations

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Abstract: This paper aims to investigate the initial boundary value problem for a class of Kirchhoff-type wave equations modified by Woinowsky-Krieger models, which can be used to describe the nonlinear extensible beam. Using the concave property of the function and a differential inequality, we estimate the upper bounds of the blowup time of the blowup solution at subcritical, critical, and high initial energy levels.

Keywords: extensible beam equation; Kirchhoff-type wave equation; upper bounds of the blowup time

1. Introduction

In this paper, we are concerned with the initial boundary value problem for nonlinear Kirchhoff-type wave equation with strong damping, nonlinear weak damping, and nonlinear source terms

$$\begin{cases} u_{tt} + \Delta^2 u - M(\|\nabla u\|^2)\Delta u - \Delta u_t + |u_t|^{r-1}u_t = |u|^{p-1}u, & \text{in } \Omega\times(0,T), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{1.1}$$

where $r \geq 1$, $p \geq 1$ are real numbers, Δ^2 is the biharmonic operator, Δ is the Laplace operator, $\Omega \subset \mathbb{R}^n$ $(n = 1, 2, \ldots)$ is a bounded domain with smooth boundary $\partial\Omega$, ν is the unit outward normal on $\partial\Omega$. The Kirchhoff term is given by

$$M(s) := 1 + \beta s^{\gamma},$$

with $\beta \geq 0$ and $\gamma \geq 0$. And the exponents involved in (1.1) satisfy

$$(\widetilde{\text{A}}) \qquad \begin{cases} 1 < 2\gamma + 1 < p < \infty, 1 < r < p < \infty, & n \leq 2; \\ 1 < 2\gamma + 1 < p \leq \frac{n}{n-2}, 1 < r < p \leq \frac{n}{n-2}, & n \geq 3. \end{cases}$$

This paper is a continuous work of [1]; hence, we shall give a quick start to draw out our concern. The model equation (1.1) is derived from the Woinowsky-Krieger stretchable beam equation, which is used to study the dynamic buckling of hinged stretchable beams [2], the nonlinear vibrations of an elastic string [3], and large deflections of plates [4]. It plays a vital role in industrial fields, such as vibration of railway track structures [5], micromechanical beams [6, 7], and microbridges [8]. We refer readers to [9, 10] for more detailed physical background. From a mathematical point of view, the Kirchhoff-type equations attracted lots of attentions in the study on the qualitative properties of the solution, including the existence of ground-state solution, global existence, long-time behavior, blowup, blowup time estimation, etc.; see [11–17]. Recently, the problem (1.1) was considered by Yang et al. in [1], who obtained the local solution theorem by the Galerkin method and the Banach fixed-point theorem. They also got the global existence and finite time blowup results at subcritical and critical initial energy levels, i.e., $E(0) \le d$, and pointed out that the global solution decays exponentially, where d is the potential well depth proposed by Payne and Sattinger [18,19]. In addition, the finite time blowup of the solution to problem (1.1) with linear weak damping ($r = 1$) was derived for the arbitrary positive initial energy, i.e., $E(0) > 0$. Then Liu et al. in [20] extended the blowup result at high initial energy in [1] to the nonlinear weak damping case ($r > 1$), and estimated the upper bound of blowup time for this case. In addition, they also gave the lower bounds of the blowup time of the solution at arbitrary initial energy levels. For the sake of clarity, we present a summary of the established results in [1, 20], the results given in this paper, and unresolved issues in the following table.

Table 1. The essence of the present paper and some unsolved problems.

Initial data		Blowup in finite time	Upper bound of blowup time	Lower bound of blowup time
$E(0) < d$	$r = 1$	[1]	**Present paper (Th. 3.4)**	[20]
	$r > 1$	[1]	Unsolved	[20]
$E(0) = d$	$r = 1$	[1]	**Present paper (Th. 3.7)**	[20]
	$r > 1$	[1]	Unsolved	[20]
$E(0) > 0$	$r = 1$	[1]	**Present paper (Th. 3.8)**	[20]
	$r > 1$	[20]	[20]	[20]

From Table 1, we can see that the reference [1] mainly obtained the finite time blowup results of the solution to problem (1.1) at subcritical, critical, and high initial energy levels. The reference [20] mainly focused on the lower bound of blowup time of the solution at the above three different initial energy levels. There are still several critical issues that need to be studied in [1, 20], including the upper bounds of the blowup time for the linear weak damping case, i.e., $r = 1$, and the nonlinear weak damping case, i.e., $r > 1$. In this paper, we focus on the linear weak damping case, i.e., $r = 1$, and use a differential inequality to estimate the upper bounds of blowup time of the blowup solution for subcritical, critical, and high initial energy levels. For the case with nonlinear weak damping, i.e., $r > 1$, the upper bounds of blowup time of the blowup solution for subcritical and critical initial energy levels are still open problems.

This paper is organized as follows. In Section 2, some preliminaries are given to introduce the necessary notations and some lemmas. The main conclusions follow in Section 3, which are about the upper bound of blowup time at subcritical initial energy, critical initial energy, and high initial energy levels.

2. Preliminaries

In this section, we shall give some notions and necessary lemmas that will be used throughout this paper. We denote $H := H_0^2(\Omega)$, $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_p$ for $1 \le p \le \infty$, $\|\cdot\|_{L^2(\Omega)} = \|\cdot\|$ and $\|\cdot\|_{H_0^1}^2 = \|\nabla\cdot\|^2 + \|\cdot\|^2$. We set

$$\|\cdot\|_H^2 := \|\Delta\cdot\|^2 + \|\nabla\cdot\|^2. \tag{2.1}$$

The duality pairing between H and H^* is denoted by $\langle\cdot,\cdot\rangle$, where $H^* = H^{-2}(\Omega)$.

Similar to [21], we define the following total energy functional

$$E(t) := \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|u\|_H^2 + \frac{\beta}{2\gamma+2}\|\nabla u\|^{2\gamma+2} - \frac{1}{p+1}\|u\|_{p+1}^{p+1},$$

the potential functional

$$J(u) := \frac{1}{2}\|u\|_H^2 + \frac{\beta}{2\gamma+2}\|\nabla u\|^{2\gamma+2} - \frac{1}{p+1}\|u\|_{p+1}^{p+1},$$

the unstable manifold

$$V := \{u \in H \mid I(u) < 0\},$$

and the Nehari functional

$$I(u) := \|u\|_H^2 + \beta\|\nabla u\|^{2\gamma+2} - \|u\|_{p+1}^{p+1}.$$

The potential well depth can be characterized as

$$d := \inf_{u\in\mathcal{N}} J(u),$$

where

$$\mathcal{N} := \{u \in H\backslash\{0\} \mid I(u) = 0\}.$$

Next, we give the definition of a weak solution to problem (1.1) as follows.

Theorem 2.1 (Weak solution [1, 22]). *Function $u(x,t)$ is called a weak solution to problem (1.1) on $\Omega\times[0,T]$, if*

$$\begin{aligned} u \in C([0,T]; H_0^2(\Omega)) \cap C^1([0,T]; L^2(\Omega)) \cap C^2([0,T]; H^{-2}(\Omega)),\\ u_t \in L^2([0,T]; H_0^1(\Omega)) \cap L^\infty([0,T]; L^{r+1}(\Omega)), \end{aligned} \tag{2.2}$$

satisfies

$$\begin{aligned} &(u_{tt},\eta) + (\Delta u, \Delta\eta) + (\nabla u, \nabla\eta) + \beta\|\nabla u\|^{2\gamma}(\nabla u, \nabla\eta)\\ &+ (\nabla u_t, \nabla\eta) + (|u_t|^{r-1}u_t, \eta) = (|u|^{p-1}u, \eta), \end{aligned} \tag{2.3}$$

where $\eta \in H$, $u(x,0) = u_0(x) \in H$ and $u_t(x,0) = u_1(x) \in H_0^1(\Omega)$.

We recall the non-increasing property of the energy function $E(t)$ with respect to time t established in [1], which will be used in the later proof.

Lemma 2.1 (Non-increasing energy functional $E(t)$ [1]). *Let $u(x,t)$ be a weak solution to problem (1.1) satisfying (2.2), then we obtain*

$$E(t) + \int_0^t \|\nabla u_\tau\|^2 \mathrm{d}\tau + \int_0^t \|u_\tau\|_{r+1}^{r+1} \mathrm{d}\tau = E(0), \tag{2.4}$$

namely,

$$E'(t) = -\|\nabla u_t\|^2 - \|u_t\|_{r+1}^{r+1} \leq 0.$$

Proof. We define the approximation function for u_t

$$D_h u := \frac{\hat{u}(t+h) - \hat{u}(t-h)}{2h},$$

where

$$\hat{u} := \begin{cases} u(x,0), & t \leq 0, \\ u(x,t), & 0 < t < T, \\ u(x,T), & t \geq T, \end{cases}$$

and T is any time in $(0, T_{max})$. Due to $\hat{u} \in H$ for any $t \in [0, T_{max})$, we know $D_h u \in H$. Then, by taking $\eta = D_h u$ and integrating both sides over $[0,T]$ in (2.3), we have

$$\begin{aligned} &\int_0^T (u_{tt}, D_h u)\mathrm{d}t + \int_0^T (\Delta u, \Delta(D_h u))\mathrm{d}t \\ &+ \int_0^T M(\|\nabla u\|_2^2)(\nabla u, \nabla(D_h u))\mathrm{d}t + \int_0^T (\nabla u_t, \nabla(D_h u))\mathrm{d}t \\ &+ \int_0^T (|u_t|^{r-1} u_t, D_h u)\mathrm{d}t = \int_0^T (|u|^{p-1} u, D_h u)\mathrm{d}t. \end{aligned} \tag{2.5}$$

According to $u \in C^1([0,T]; L^2(\Omega))$, i.e., $u_t \in C([0,T]; L^2(\Omega))$, given by the similar arguments to the proof of Theorem 1.1 in [22], $u \in C([0,T]; H)$ shown in (2.2), and Proposition 4.1 in [23], we have

$$\begin{aligned} \int_0^T (u_{tt}, D_h u)\mathrm{d}t &= \int_\Omega \int_0^T u_{tt} D_h u \mathrm{d}t\mathrm{d}x \\ &= \int_\Omega (D_h u(T) u_t(T) - D_h u(0) u_t(0))\, \mathrm{d}x - \int_\Omega \int_0^T u_t (D_h u)_t \mathrm{d}t\mathrm{d}x \\ &= (u_t(T), D_h u(T)) - (u_t(0), D_h u(0)) - \int_0^T (u_t, (D_h u)_t)\, \mathrm{d}t \\ &= (u_t(T), D_h u(T)) - (u_t(0), D_h u(0)), \end{aligned}$$

which means that (2.5) becomes

$$\begin{aligned}&(u_t(T), D_hu(T)) - (u_t(0), D_hu(0)) + \int_0^T (\Delta u, \Delta\,(D_hu))\mathrm{d}t \\ &+ \int_0^T M(\|\nabla u\|_2^2)(\nabla u, \nabla\,(D_hu))\mathrm{d}t + \int_0^T (\nabla u_t, \nabla\,(D_hu))\mathrm{d}t \\ &+ \int_0^T (|u_t|^{r-1}u_t, D_hu)\mathrm{d}t \\ =& \int_0^T (|u|^{p-1}u, D_hu)\mathrm{d}t.\end{aligned} \tag{2.6}$$

Due to $u \in C([0,T]; H(\Omega))$, $\|\Delta u\|_2$ is a norm of H, and $u_t \in L^2([0,T]; H_0^1(\Omega)) \subseteq L^1([0,T]; H_0^1(\Omega))$, we can use Proposition 4.1 in [23] to give

$$\lim_{h\to 0}\int_0^T (\Delta u, \Delta(D_hu))\mathrm{d}t = \frac{1}{2}(\|\Delta u(T)\|_2^2 - \|\Delta u(0)\|_2^2), \tag{2.7}$$

$$D_hu \in L^2([0,T]; H_0^1(\Omega)),$$

$$\|D_hu\|_{L^2([0,T];H_0^1(\Omega))} \le \|u_t\|_{L^2([0,T];H_0^1(\Omega))},$$

and

$$D_hu \longrightarrow u_t \quad \text{weakly in} \quad L^2([0,T]; H_0^1(\Omega)) \text{ as } h \to 0.$$

Then, by the definition of weak convergence, we know that for any bounded linear functional F on $L^2([0,T]; H_0^1(\Omega))$, we have

$$F(D_hu) \longrightarrow F(u_t). \tag{2.8}$$

By (2.8), we have

$$\int_0^T M(\|\nabla u\|_2^2)(\nabla u, \nabla(D_hu))\mathrm{d}t \longrightarrow \int_0^T M(\|\nabla u\|_2^2)(\nabla u, \nabla u_t)\mathrm{d}t, \tag{2.9}$$

where

$$\begin{aligned}&\int_0^T M(\|\nabla u\|_2^2)(\nabla u, \nabla u_t)\mathrm{d}t \\ =& \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla u\|_2^2 + \frac{\beta}{\gamma+1}\|\nabla u\|^{2\gamma+2}\right)\mathrm{d}t,\end{aligned}$$

$$\begin{aligned}\int_0^T (|u|^{p-1}u, D_hu)\mathrm{d}t \longrightarrow& \int_0^T (|u|^{p-1}u, u_t)\mathrm{d}t \\ =& \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{p+1}\|u\|_{p+1}^{p+1}\right)\mathrm{d}t.\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}\int_0^T (\nabla u_t, \nabla(D_h u))\mathrm{d}t &\longrightarrow \int_0^T (\nabla u_t, \nabla u_t)\mathrm{d}t \\ &= \int_0^T \|\nabla u_t\|_2^2 \mathrm{d}t.\end{aligned} \tag{2.11}$$

According to Proposition 4.2 in [23], $u \in C([0,T];H) \subseteq C([0,T];H_0^1(\Omega)) \subseteq C([0,T];L^{r+1}(\Omega))$ and $u_t \in L^{r+1}([0,T];L^{r+1}(\Omega)) \subseteq L^1([0,T];L^{r+1}(\Omega))$, we have

$$D_h u \in L^{r+1}([0,T];L^{r+1}(\Omega)),$$

$$\|D_h u\|_{L^{r+1}([0,T];L^{r+1}(\Omega))} \leq \|u_t\|_{L^{r+1}([0,T];L^{r+1}(\Omega))},$$

and

$$D_h u \longrightarrow u_t \quad \text{in} \quad L^{r+1}([0,T];L^{r+1}(\Omega)) \text{ as } h \to 0. \tag{2.12}$$

According to (2.12), we have

$$D_h u \rightharpoonup u_t \quad \text{in} \quad L^{r+1}([0,T];L^{r+1}(\Omega)) \text{ as } h \to 0,$$

where r satisfies $\widetilde{\mathrm{A}}$. We define

$$G(D_h u) := \int_0^T (|u_t|^{r-1}u_t, D_h u)\mathrm{d}t.$$

Since $G(\cdot)$ is bounded on $L^{r+1}([0,T];L^{r+1}(\Omega))$, we have

$$G(D_h u) \longrightarrow G(u_t),$$

i.e.,

$$\begin{aligned}\int_0^T (|u_t|^{r-1}u_t, D_h u)\mathrm{d}t &\longrightarrow \int_0^T (|u_t|^{r-1}u_t, u_t)\mathrm{d}t \\ &= \int_0^T \|u_t\|_{r+1}^{r+1}\mathrm{d}t.\end{aligned} \tag{2.13}$$

And from $u_t \in C([0,T];L^2(\Omega))$ and Proposition 4.2 in [23], we have

$$D_h u(0) \longrightarrow \frac{1}{2}u_t(0) \quad \text{weakly in} \ \ L^2(\Omega)$$

and

$$D_h u(T) \longrightarrow \frac{1}{2}u_t(T) \quad \text{weakly in} \ \ L^2(\Omega).$$

Similarly, we have

$$(u_t(T), D_h u(T)) \longrightarrow (u_t(T), \frac{1}{2}u_t(T)) = \frac{1}{2}\|u_t(T)\|_2^2 \tag{2.14}$$

and

$$(u_t(0), D_h u(0)) \longrightarrow (u_t(0), \frac{1}{2}u_t(0)) = \frac{1}{2}\|u_t(0)\|_2^2, \tag{2.15}$$

respectively. According to (2.7), (2.9)–(2.11), and (2.13)–(2.15), by letting $h \to 0$ in (2.6), we have

$$\begin{aligned}
&\frac{1}{2}\|u_t(T)\|_2^2 - \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|\Delta u(T)\|_2^2 - \frac{1}{2}\|\Delta u_0\|_2^2 + \frac{1}{2}\|\nabla u(T)\|_2^2\\
&+ \frac{\beta}{\gamma+1}\|\nabla u(T)\|_2^{2\gamma+2} - \frac{1}{2}\|\nabla u_0\|_2^2 - \frac{\beta}{\gamma+1}\|\nabla u_0\|_2^{2\gamma+2} - \frac{1}{p+1}\|u(T)\|_{p+1}^{p+1}\\
&+ \frac{1}{p+1}\|u_0\|_{p+1}^{p+1}\\
=& -\int_0^T \|\nabla u_t\|_2^2 \mathrm{d}t - \int_0^T \|u_t\|_{r+1}^{r+1}\mathrm{d}t,
\end{aligned}$$

i.e.,

$$E(T) - E(0) = -\int_0^T \|\nabla u_t\|_2^2 \mathrm{d}t - \int_0^T \|u_t\|_{r+1}^{r+1}\mathrm{d}t. \tag{2.16}$$

Since T is any time in $(0, T_{max})$, to facilitate discussion, we replace T with $t \in [0, T_{max})$, which means that (2.16) becomes

$$E(t) - E(0) = -\int_0^t \|\nabla u_t\|_2^2 \mathrm{d}\tau - \int_0^t \|u_t\|_{r+1}^{r+1}\mathrm{d}\tau. \tag{2.17}$$

By differentiating both sides of (2.17) with respect to t, we finish the proof of the energy equality for weak solutions. □

In the following lemma, we introduce a differential inequality, which is used to obtain the upper bound of the blowup time of the solution to problem (1.1).

Lemma 2.2 ([24, 25]). *Let $\phi(t)$ be a positive twice-differentiable function that satisfies the inequality*

$$\phi(t)\phi''(t) - (1+\alpha)(\phi'(t))^2 \geq 0, \qquad \alpha > 0.$$

If $\phi(0) > 0$ and $\phi'(0) > 0$, then there exists $0 < t_0 \leq \frac{\phi(0)}{\alpha\phi'(0)}$ such that $\lim_{t\to t_0}\phi(t) = \infty$.

3. Upper bound of the blowup time

The study of the blowup time for solutions to nonlinear evolution equations has attracted considerable attention [26–28]. In this section, we give the main results of the present paper, i.e., estimating the upper bound of blowup time of the blowup solution to problem (1.1) for subcritical initial energy (Subsection 3.1), critical initial energy (Subsection 3.2), and high initial energy (Subsection 3.3).

3.1. Upper bound of the blowup time with subcritical initial energy $E(0) < d$ and $r = 1$

In this subsection, we aim to estimate the upper bound of the blowup time of the solution to problem (1.1) with $E(0) < d$. Before stating this result, we first introduce the invariance of the unstable set established in [1], which is necessary for the later proof.

Lemma 3.1 ([1]). *Suppose $u_0 \in H$, $u_1(x) \in H_0^1(\Omega)$, and also $(\widetilde{\mathrm{A}})$ holds. Then we obtain $u(t) \in V$ for all $t \in (0, T_{\max})$ as long as $u_0 \in V$ and $E(0) < d$, where $T_{\max}$ is the maximum existence time of the solution.*

The relationship between $\|u\|_H$ and the potential well depth d is given in the following lemma.

Lemma 3.2 ([1]). *Suppose $u_0 \in H$, $u_1(x) \in H_0^1(\Omega)$, and also $(\widetilde{\mathrm{A}})$ holds. If $u(t) \in V$, we obtain*

$$\|u\|_H^2 > \frac{2(p+1)}{p-1} d, \tag{3.1}$$

where the depth of the potential well

$$d := \frac{p-1}{2(p+1)} \left(\frac{1}{C^{p+1}} \right)^{\frac{2}{p-1}},$$

and C is the optimal embedding constant of $H \hookrightarrow L^{p+1}(\Omega)$.

With the help of the above lemmas, the blowup result to problem (1.1) with $E(0) < d$ is established.

Proposition 3.3 (Blowup in finite time for subcritical initial energy $E(0) < d$ [1]). *Suppose $u_0 \in H$, $u_1(x) \in H_0^1(\Omega)$, and also $(\widetilde{\mathrm{A}})$ holds. Then the solution to problem (1.1) blows up in finite time provided that $E(0) < d$ and $u_0 \in V$.*

Next, we show the upper bound of the blowup time of the solution to problem (1.1) when the initial energy satisfies $E(0) < d$.

Theorem 3.4 (Upper bound of the blowup time with subcritical initial energy $E(0) < d$ and $r = 1$). *Let the conditions in Proposition 3.3 hold. Then, the maximum existence time of the solution can be estimated by*

$$T_{\max} \leq \frac{2\|u_0\|^2 + \rho_1^2 (d - E(0))^{-1}}{(p-1)\left((u_0, u_1) + \rho_1\right) - 2\|u_0\|_{H_0^1}^2}, \tag{3.2}$$

where

$$\rho_1 := \frac{\sqrt{\sigma_1^2 + 2(p-1)^2 (d - E(0)) \|u_0\|^2} + \sigma_1}{p-1}, \tag{3.3}$$

and

$$\sigma_1 := 2\|u_0\|_{H_0^1}^2 - (p-1)(u_0, u_1). \tag{3.4}$$

Proof. By Proposition 3.3, we know that the solution to problem (1.1) blows up in finite time when the initial data satisfy $E(0) < d$ and $u_0 \in V$. Then it is easy to get

$$\lim_{t\to T_{\max}} \|u\|_H = \infty. \tag{3.5}$$

Next, we shall estimate the upper bound of the blowup time. We introduce an auxiliary function

$$K(t) := \|u\|^2 + \int_0^t \|u\|_{H_0^1}^2 \mathrm{d}\tau + (T_{\max} - t)\|u_0\|_{H_0^1}^2 + \zeta(t+\mu)^2 > 0, \tag{3.6}$$

where ζ and μ are positive constants. By direct calculation, it follows that

$$K'(t) = 2(u, u_t) + 2\int_0^t (u, u_\tau)\mathrm{d}\tau + 2\int_0^t (\nabla u, \nabla u_\tau)\mathrm{d}\tau + 2\zeta\,(t+\mu)\,. \tag{3.7}$$

Taking $r = 1$ in (1.1), multiplying (1.1) by u and integrating over Ω, we have

$$\langle u_{tt}, u\rangle + (\Delta u, \Delta u) + M(\|\nabla u\|^2)(\nabla u, \nabla u) + (\nabla u_t, \nabla u) + (u_t, u) = (|u|^{p-1}u, u),$$

where $u_{tt} \in H^*$. This fact together with (3.7) and (2.1), gives

$$\begin{aligned} K''(t) &= 2\|u_t\|^2 + 2\langle u_{tt}, u\rangle + 2(u, u_t) + 2(\nabla u, \nabla u_t) + 2\zeta \\ &= 2\|u_t\|^2 - 2\|u\|_H^2 - 2\beta\|\nabla u\|^{2\gamma+2} + 2\|u\|_{p+1}^{p+1} + 2\zeta. \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), it is easy to get

$$\begin{aligned} &K(t)K''(t) - \frac{p+3}{4}(K'(t))^2 \\ =&2K(t)\left(\|u_t\|^2 - \|u\|_H^2 - \beta\|\nabla u\|^{2\gamma+2} + \|u\|_{p+1}^{p+1} + \zeta\right) \\ &- (p+3)\left((u, u_t) + \int_0^t (u, u_\tau)\mathrm{d}\tau + \int_0^t (\nabla u, \nabla u_\tau)\mathrm{d}\tau + \zeta\,(t+\mu)\right)^2 \\ =&2K(t)\left(\|u_t\|^2 - \|u\|_H^2 - \beta\|\nabla u\|^{2\gamma+2} + \|u\|_{p+1}^{p+1} + \zeta\right) \\ &+ (p+3)\left(S(t) - \left(K(t) - (T_{\max} - t)\|u_0\|_{H_0^1}^2\right)\left(\|u_t\|^2 + \int_0^t \|u_t\|_{H_0^1}^2\, d\tau + \zeta\right)\right), \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} S(t) :=& \left(\|u\|^2 + \int_0^t \|u\|_{H_0^1}^2 \mathrm{d}\tau + \zeta\,(t+\mu)^2\right)\left(\|u_t\|^2 + \int_0^t \|u_t\|_{H_0^1}^2\, d\tau + \zeta\right) \\ &- \left((u, u_t) + \int_0^t (u, u_\tau)\mathrm{d}\tau + \int_0^t (\nabla u, \nabla u_\tau)\mathrm{d}\tau + \zeta\,(t+\mu)\right)^2. \end{aligned}$$

Applying the Hölder inequality and Young's inequality, it is straightforward to verify that $S(t) \geq 0$. Then, (3.9) turns into

$$K(t)K''(t) - \frac{p+3}{4}(K'(t))^2 \geq K(t)\chi_1(t), \tag{3.10}$$

where

$$\begin{aligned}\chi_1(t) :=& -(p+1)\|u_t\|^2 - 2\|u\|_H^2 - 2\beta\|\nabla u\|^{2\gamma+2} + 2\|u\|_{p+1}^{p+1}\\ &-(p+3)\int_0^t \|u_\tau(\tau)\|_{H_0^1}^2 \mathrm{d}\tau - (p+1)\zeta.\end{aligned} \tag{3.11}$$

Using the definition of $E(t)$ and (2.4), it is clear that

$$\begin{aligned}2\|u\|_{p+1}^{p+1} =& -2(p+1)E(0) + (p+1)\|u_t\|^2 + (p+1)\|u\|_H^2\\ &+ \frac{\beta(p+1)}{\gamma+1}\|\nabla u\|^{2\gamma+2} + 2(p+1)\int_0^t \|u_\tau(\tau)\|_{H_0^1}^2 \mathrm{d}\tau.\end{aligned} \tag{3.12}$$

Substituting (3.12) into (3.11), and combining Lemmas 3.1 and 3.2, Eq (2.1), and $E(0) < d$ gives

$$\begin{aligned}\chi_1(t) =& -2(p+1)E(0) + (p-1)\|u\|_H^2 + (p-1)\int_0^t \|u_\tau(\tau)\|_{H_0^1}^2 \mathrm{d}\tau\\ &+ \frac{\beta(p-2\gamma-1)}{\gamma+1}\|\nabla u\|^{2\gamma+2} - (p+1)\zeta\\ >& -2(p+1)E(0) + 2(p+1)d + (p-1)\int_0^t \|u_\tau(\tau)\|_{H_0^1}^2 \mathrm{d}\tau\\ &+ \frac{\beta(p-2\gamma-1)}{\gamma+1}\|\nabla u\|^{2\gamma+2} - (p+1)\zeta\\ \geq& -2(p+1)E(0) + 2(p+1)d - (p+1)\zeta \geq 0,\end{aligned} \tag{3.13}$$

where

$$0 < \zeta \leq 2(d - E(0)) := \zeta_1. \tag{3.14}$$

Then, from (3.6), (3.10), and (3.13), it is easy to see that

$$K(t)K''(t) - \frac{p+3}{4}\left(K'(t)\right)^2 \geq 0. \tag{3.15}$$

By (3.6) and (3.7), we have

$$K(0) = \|u_0\|^2 + T_{\max}\|u_0\|_{H_0^1}^2 + \mu^2\zeta,$$

and

$$K'(0) = 2\mu\zeta + 2(u_0, u_1).$$

In order to guarantee $K(0) > 0$ and $K'(0) > 0$, we take

$$\mu > \max\left\{0, -\frac{(u_0, u_1)}{\zeta}\right\}. \tag{3.16}$$

Then, by (3.15) and Lemma 2.2, we obtain

$$0 < T_{\max} \leq \frac{4K(0)}{(p-1)K'(0)} = \frac{2\left(\|u_0\|^2 + T_{\max}\|u_0\|_{H_0^1}^2 + \mu^2\zeta\right)}{(p-1)\left(\mu\zeta + (u_0, u_1)\right)}.$$

This gives

$$
\begin{aligned}
T_{\max} &\le \frac{2\left(\|u_0\|^2+\zeta\mu^2\right)}{(p-1)(\mu\zeta+(u_0,u_1))}\left(1-\frac{2\|u_0\|^2_{H_0^1}}{(p-1)(\mu\zeta+(u_0,u_1))}\right)^{-1}\\
&= \frac{2\left(\|u_0\|^2+\zeta\mu^2\right)}{(p-1)(\mu\zeta+(u_0,u_1))-2\|u_0\|^2_{H_0^1}}\\
&= \frac{2\left(\|u_0\|^2+\rho^2\zeta^{-1}\right)}{(p-1)(\rho+(u_0,u_1))-2\|u_0\|^2_{H_0^1}}\\
&:= Y(\rho,\zeta),
\end{aligned}
\tag{3.17}
$$

where

$$
\rho := \mu\zeta > \begin{cases} -(u_0,u_1), & (u_0,u_1)\le 0,\\ 0, & (u_0,u_1)>0. \end{cases}
\tag{3.18}
$$

The parameters ρ and ζ are introduced in (3.18) and (3.6), which are independent of the parameters involved in problem (1.1). For any μ and ζ that satisfy (3.16) and (3.14), we can obtain a series of effective upper bounds on the blowup time of the solution, which are related to the choice of μ and ζ. Hence, we need to seek μ and ζ that satisfy (3.16) and (3.14) to minimize function $Y(\rho,\zeta)$.

By the definition of $Y(\rho,\zeta)$ and (3.14), it is clear that

$$
\inf_{\rho,\zeta} Y(\rho,\zeta) = \inf_{\rho} Y(\rho,\zeta_1) = \inf_{\rho} \frac{2\left(\|u_0\|^2+\rho^2\zeta_1^{-1}\right)}{(p-1)(\rho+(u_0,u_1))-2\|u_0\|^2_{H_0^1}}.
\tag{3.19}
$$

Now, the problem was reduced to finding the minimum of the function $Y(\rho,\zeta_1)$ that depends only on ρ. It is clear that if there exists a point ρ_1 that satisfies

$$
\rho_1 := \mu\zeta_1 > \begin{cases} -(u_0,u_1), & (u_0,u_1)\le 0,\\ 0, & (u_0,u_1)>0, \end{cases}
\tag{3.20}
$$

such that $Y'(\rho_1)=0$, then the function $Y(\rho,\zeta_1)$ reaches the minimum at ρ_1, i.e.,

$$
\inf_{\rho} Y(\rho,\zeta_1) = Y(\rho_1,\zeta_1).
\tag{3.21}
$$

We first show the existence of ρ_1. Taking the derivative of $Y(\rho,\zeta_1)$ with respect to ρ, it follows that

$$
\begin{aligned}
Y'(\rho) &= \frac{4\rho\zeta_1^{-1}\left((p-1)(\rho+(u_0,u_1))-2\|u_0\|^2_{H_0^1}\right)-2(p-1)\left(\|u_0\|^2+\rho^2\zeta_1^{-1}\right)}{\left((p-1)(\rho+(u_0.u_1))-2\|u_0\|^2_{H_0^1}\right)^2}\\
&= \frac{2\left((p-1)\rho^2+2(p-1)(u_0,u_1)\rho-4\|u_0\|^2_{H_0^1}\rho-(p-1)\zeta_1\|u_0\|^2\right)}{\zeta_1\left((p-1)(\rho+(u_0,u_1))-2\|u_0\|^2_{H_0^1}\right)^2}.
\end{aligned}
$$

By a simple computation, we obtain that ρ_1 defined in (3.3) satisfies $Y'(\rho_1) = 0$.

Next, we consider the following two cases, i.e., $(u_0, u_1) \leq 0$ and $(u_0, u_1) > 0$, to verify that ρ_1 satisfies (3.20).

Case I: $(u_0, u_1) \leq 0$. By (3.3), (3.4), $(\widetilde{\mathrm{A}})$, and $E(0) < d$, we have

$$\rho_1 > \frac{\sqrt{\sigma_1^2 + 2\,(p-1)^2\,(d - E(0))\,\|u_0\|^2}}{p-1} + |\,(u_0, u_1)\,|. \tag{3.22}$$

Case II: $(u_0, u_1) > 0$. Similar to Case I, we obtain

$$\rho_1 > \frac{|\sigma_1| + \sigma_1}{p-1} \geq 0. \tag{3.23}$$

These two cases mean that ρ_1 satisfies (3.20). Then combining (3.17), (3.19), and (3.21), the conclusion (3.2) is derived. This completes the proof. □

3.2. Upper bound of the blowup time with critical initial energy $E(0) = d$ and $r = 1$

The purpose of this subsection is to estimate the upper bound of the blowup time of the solution to problem (1.1) with $E(0) = d$. We first introduce the blowup result of problem (1.1) at critical initial energy given in [1].

Proposition 3.5 (Blowup in finite time for critical initial energy $E(0) = d$ [1])**.** *Suppose $u_0 \in H$, $u_1(x) \in H_0^1(\Omega)$, and also $(\widetilde{\mathrm{A}})$ holds. Then the solution to problem (1.1) blows up in finite time provided the initial data satisfy $E(0) = d$ and $u_0 \in V$.*

The following lemma shows that V is invariant under the flow of problem (1.1).

Lemma 3.6 (Invariant unstable set V for critical initial energy $E(0) = d$ [1])**.** *Let the conditions in Proposition 3.5 hold; then we have $u(t) \in V$ for all $t \in (0, T_{\max})$, where $T_{\max}$ is the maximum existence time of the solution.*

Using Lemmas 3.6 and 3.2, we can also obtain the inequality (3.1) under the conditions $E(0) = d$ and $u_0 \in V$. By defining the same auxiliary function $K(t)$ as in (3.6), it is clear that

$$K(t)K''(t) - \frac{p+3}{4}\,(K'(t))^2 \geq K(t)\chi_1(t),$$

where $\chi_1(t)$ is defined in (3.11). However, due to $E(0) = d$, by (3.12), (3.13), and (3.1), we cannot find a positive constant ζ such that $\chi_1(t) \geq 0$, which implies that the inequality (3.15) no longer holds. Next, we shall show the upper bound of the blowup time of the solution to problem (1.1) with linear weak damping when initial energy satisfies $E(0) = d$ by introducing a new auxiliary function $L(t)$.

Theorem 3.7 (Upper bound of blowup time for critical initial energy $E(0) = d$ and $r = 1$)**.** *Let the conditions in Proposition 3.5 hold, if the initial data satisfy*

$$(u_0, u_1) > \frac{2\|u_0\|_{H_0^1}}{p+1} > 0, \tag{3.24}$$

then the maximum existence time of the solution to problem (1.1) is given by

$$T_{\max} \leq \frac{2\,\|u_0\|^2}{(p+1)(u_0, u_1) - 2\|u_0\|_{H_0^1}^2}. \tag{3.25}$$

Proof. We define

$$L(t) := \|u\|^2 + \int_0^t \|u\|_{H_0^1}^2 \mathrm{d}\tau + (T_{\max} - t)\|u_0\|_{H_0^1}^2 > 0. \tag{3.26}$$

Then, we have

$$L'(t) = 2(u, u_t) + 2\int_0^t (u, u_\tau)\mathrm{d}\tau + 2\int_0^t (\nabla u, \nabla u_\tau)\mathrm{d}\tau. \tag{3.27}$$

By similar arguments of (3.8)–(3.10), it is clear that

$$L(t)L''(t) - \frac{p+3}{4}\left(L'(t)\right)^2 \geq L(t)\theta(t), \tag{3.28}$$

where

$$\theta(t) := -(p+1)\|u_t\|^2 - 2\|u\|_H^2 - 2\beta\|\nabla u\|^{2\gamma+2} + 2\|u\|_{p+1}^{p+1} - (p+3)\int_0^t \|u_\tau(\tau)\|_{H_0^1}^2 \mathrm{d}\tau. \tag{3.29}$$

Substituting (3.12) into (3.29), and combining Lemmas 3.2 and 3.6, $(\widetilde{\mathrm{A}})$ and $E(0) = d$, we have

$$\begin{aligned}
\theta(t) =& -2(p+1)E(0) + (p-1)\|u\|_H^2 + (p-1)\int_0^t \|u_\tau(\tau)\|_{H_0^1}^2 \mathrm{d}\tau \\
&+ \frac{\beta(p-2\gamma-1)}{\gamma+1}\|\nabla u\|^{2\gamma+2} \\
>& -2(p+1)d + 2(p+1)d + (p-1)\int_0^t \|u_\tau(\tau)\|_{H_0^1}^2 \mathrm{d}\tau \\
&+ \frac{\beta(p-2\gamma-1)}{\gamma+1}\|\nabla u\|^{2\gamma+2} \geq 0,
\end{aligned}$$

which, together with (3.26) and (3.28), yields

$$L(t)L''(t) - \frac{p+3}{4}\left(L'(t)\right)^2 \geq 0. \tag{3.30}$$

By (3.26), (3.27), and (3.24), we obtain

$$L(0) = \|u_0\|^2 + T_{\max}\|u_0\|_{H_0^1}^2 > 0,$$

and

$$L'(0) = 2(u_0, u_1) > 0.$$

Using these facts, (3.30) and Lemma 2.2, it follows that

$$0 < T_{\max} \leq \frac{4L(0)}{(p-1)L'(0)} = \frac{2\left(\|u_0\|^2 + T_{\max}\|u_0\|_{H_0^1}^2\right)}{(p-1)(u_0, u_1)},$$

which gives (3.25). This completes the proof. □

3.3. Upper bound of the blowup time with high initial energy $E(0) > 0$ and $r = 1$

In this subsection, we obtain the upper bound of the blowup time of the solution to problem (1.1) when initial energy satisfies $E(0) > 0$. By the proof of Theorem 3.1, we observe that the upper bound of the blowup time at a subcritical initial energy level strongly depends on the relationship between the initial energy and the potential well depth. However, for high initial energy levels, i.e., when the initial energy is not controlled by the potential well depth, we need to seek strategies and techniques to get the upper bound of the blowup time. In the following, we present the relevant results established in [1], and then estimate the upper bound of the blowup time at high initial energy based on these results.

Lemma 3.8 ([1]). *Suppose $u_0 \in H$, $u_1(x) \in H_0^1(\Omega)$, and also $(\widetilde{\mathrm{A}})$ holds. We set*

$$\tilde{c} := \min\{1, \widetilde{C}\} > 0, \quad \|u\|_H^2 \geq \widetilde{C}\|u\|_{H_0^1}^2, \tag{3.31}$$

$$Q_0 := \|\nabla u_0\|^2 + \|u_0\|^2 + 2(u_0, u_1) - \frac{4(p+1)}{(p-1)\tilde{c}}E(0), \tag{3.32}$$

and

$$B(t) := \|\nabla u(t)\|^2 + \|u(t)\|^2 + 2(u, u_t).$$

Assume that the initial data satisfy

$$I(u_0) - \|u_1\|^2 < 0, \quad Q_0 > 0,$$

then the function $B(t)$ is strictly increasing.

Proposition 3.9 (Blowup in finite time with high initial energy $E(0) > 0$ and $r = 1$ [1]). *Let the assumptions of Lemma 3.8 hold. Then the solution blows up in finite time.*

After the above preparations, we estimate the upper bound of the blowup time of the solution to problem (1.1) when the initial energy satisfies $E(0) > 0$.

Theorem 3.10 (Upper bound of blowup time for high initial energy $E(0) > 0$ and $r = 1$). *Let the conditions in Proposition 3.9 hold; then we have*

$$T_{\max} \leq \frac{4\left(\|u_0\|^2 + \rho_2^2\zeta_2^{-1}\right)}{(p-1)(\rho_2 + (u_0, u_1)) - 4\|u_0\|_{H_0^1}^2}, \tag{3.33}$$

where

$$\rho_2 := \frac{\sqrt{\sigma_2^2 + \tilde{c}Q_0\,(p-1)^3(p+3)^{-1}\,\|u_0\|^2} + \sigma_2}{p-1}, \tag{3.34}$$

$$\sigma_2 := 4\,\|u_0\|_{H_0^1}^2 - (p-1)\,(u_0, u_1)\,,$$

and $\tilde{c}$ and Q_0 are defined in (3.31) and (3.32), respectively.

Proof. It follows from Proposition 3.9 that the solution blows up in finite time, i.e., (3.5). Next, we are going to estimate the upper bound of the blowup time. We define the same auxiliary function $K(t)$ as (3.6). Then using (3.7)–(3.10), we have

$$K(t)K''(t) - \frac{\eta+3}{4}\left(K'(t)\right)^2 \geq K(t)\chi_2(t), \quad \eta > 0, \tag{3.35}$$

where

$$\begin{aligned}\chi_2(t) :=& -(\eta+1)\|u_t\|^2 - 2\|u\|_H^2 - 2\beta\|\nabla u\|^{2\gamma+2} + 2\|u\|_{p+1}^{p+1}\\ &- (\eta+3)\int_0^t \|u_\tau(\tau)\|_{H_0^1}^2 \mathrm{d}\tau - (\eta+1)\zeta.\end{aligned} \tag{3.36}$$

Taking $\eta = \frac{p+1}{2}$ and substituting (3.12) into (3.36), we derive

$$\begin{aligned}\chi_2(t) =& -2(p+1)E(0) + \frac{p-1}{2}\|u_t\|^2 + (p-1)\|u\|_H^2 + \frac{\beta(p-2\gamma-1)}{\gamma+1}\|\nabla u\|^{2\gamma+2}\\ &+ \frac{3(p-1)}{2}\int_0^t \|u_\tau(\tau)\|_{H_0^1}^2 \mathrm{d}\tau - \frac{p+3}{2}\zeta,\end{aligned}$$

which, together with $(\widetilde{\mathrm{A}})$, (3.31), the Cauchy-Schwarz inequality and Lemma 3.8 yields

$$\begin{aligned}\chi_2(t) &\geq -2(p+1)E(0) + \frac{p-1}{2}\|u_t\|^2 + (p-1)\|u\|_H^2 - \frac{p+3}{2}\zeta\\ &\geq -2(p+1)E(0) + \frac{p-1}{2}(\|u_t\|^2 + 2\widetilde{C}\|u\|_{H_0^1}) - \frac{p+3}{2}\zeta\\ &\geq -2(p+1)E(0) + \frac{(p-1)\tilde{c}}{2}(2(u,u_t) + \|\nabla u\|^2 + \|u\|^2) - \frac{p+3}{2}\zeta\\ &= \frac{(p-1)\tilde{c}}{2}Q_0 - \frac{p+3}{2}\zeta,\end{aligned}$$

where

$$0 < \zeta \leq \zeta_2 := \frac{\tilde{c}(p-1)Q_0}{p+3}. \tag{3.37}$$

Now, we take μ that satisfies (3.16) such that $K(0) > 0$ and $K'(0) > 0$. Then by Lemma 2.2 and (3.35), we obtain

$$0 < T_{\max} \leq \frac{8K(0)}{(p-1)K'(0)} = \frac{4\left(\|u_0\|^2 + T_{\max}\|u_0\|_{H_0^1}^2 + \mu^2\zeta\right)}{(p-1)\left(\mu\zeta + (u_0,u_1)\right)},$$

which gives

$$\begin{aligned}T_{\max} &\leq \frac{4\left(\|u_0\|^2 + \zeta\mu^2\right)}{(p-1)(\mu\zeta + (u_0,u_1))}\left(1 - \frac{4\|u_0\|_{H_0^1}^2}{(p-1)(\mu\zeta + (u_0,u_1))}\right)^{-1}\\ &= \frac{4\left(\|u_0\|^2 + \zeta\mu^2\right)}{(p-1)(\mu\zeta + (u_0,u_1)) - 4\|u_0\|_{H_0^1}^2}\\ &= \frac{4\left(\|u_0\|^2 + \rho^2\zeta^{-1}\right)}{(p-1)(\rho + (u_0,u_1)) - 4\|u_0\|_{H_0^1}^2}\\ &:= Z(\rho,\zeta),\end{aligned} \tag{3.38}$$

where ρ is defined the same as (3.18). Based on the similar discussions under (3.18), we are going to find μ and ζ that satisfy (3.16) and (3.37) to minimize the function $Z(\rho,\zeta)$.

Using (3.37), we minimize the function $Y(\rho,\zeta)$ for ζ yields

$$\inf_{\rho,\zeta} Z(\rho,\zeta) = \inf_{\rho} Z(\rho,\zeta_2) = \inf_{\rho} \frac{4\,(\|u_0\|^2 + \rho^2\zeta_2^{-1})}{(p-1)(\rho + (u_0,u_1)) - 4\|u_0\|^2_{H_0^1}}. \tag{3.39}$$

At this stage, the problem was simplified to minimizing the function $Z(\rho,\zeta_2)$, which depends only on ρ. It is clear that if we can find a point ρ_2 that satisfies

$$\rho_2 := \mu\zeta_2 > \begin{cases} -(u_0,u_1), & (u_0,u_1) \le 0, \\ 0, & (u_0,u_1) > 0, \end{cases} \tag{3.40}$$

such that $Z'(\rho_2) = 0$, then the function $Z(\rho,\zeta_2)$ attains its minimum at ρ_2, i.e.,

$$\inf_{\rho} Z(\rho,\zeta_2) = Z(\rho_2,\zeta_2). \tag{3.41}$$

We first prove the existence of ρ_2. By differentiating $Z(\rho,\zeta_2)$ with respect to ρ, we derive

$$Z'(\rho,\zeta_2) = \frac{4\Big((p-1)\rho^2 + 2(p-1)(u_0,u_1)\rho - 8\|u_0\|^2_{H_0^1}\rho - (p-1)\zeta_2\|u_0\|^2\Big)}{\zeta_2\Big((p-1)(\rho + (u_0,u_1)) - 4\|u_0\|^2_{H_0^1}\Big)^2}.$$

A straightforward calculation shows that ρ_2 defined in (3.34) satisfies $Z'(\rho_2) = 0$.

The remaining proof is to verify that ρ_2 satisfies (3.40). Using similar arguments as (3.22) and (3.23), it is easy to get that ρ_2 satisfies (3.40). Hence, we omit it here. Then we use (3.38), (3.39), and (3.41) to conclude the conclusion (3.33). This completes the proof. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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