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Research article

Galerkin-based solution for the time-fractional diffusion-wave equation

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Abstract: In this work, a new spectral Galerkin approach to solving the time-fractional diffusion-wave equation (TFDWE) with non-homogeneous initial and boundary conditions is presented. A suitable transformation is used to convert the TFDWE governed by non-homogeneous conditions into a modified one governed by homogeneous conditions. New basis functions in terms of specific shifted Horadam polynomials are used. Some new definite integral formulas that are crucial to the numerical implementation are developed, and the Galerkin scheme is analyzed in detail to obtain the approximate solutions. A thorough convergence and error analysis of the proposed expansion is established. A number of numerical experiments are carried out to show the scheme's applicability and accuracy when compared to other methods.

Keywords: Horadam polynomials; diffusion equations; spectral methods; homogeneous conditions; convergence analysis

1. Introduction

The main difference in constructing fractional differential equations (FDEs) compared to classical differential equations is that FDEs have memory effects and non-local interactions, which are particularly useful for modeling real processes. FDEs are more descriptive for systems exhibiting hereditary effects, anomalous diffusion, and complex dynamical behavior. Hence, their role is increasingly important in many fields such as physics, engineering, biology, and finance [1, 2]. In physics and engineering, FDEs find applications in viscoelasticity, heat conduction, electromagnetism, and signal processing, which are all areas that can benefit significantly from modern computer technology; see, for example, [3, 4]. Analytical solutions for FDEs are frequently unavailable; consequently, employing various numerical methods is essential. The following methods have been proposed to solve different FDEs: the variational iteration method [5], the Adomian decomposition method [6], the splines method [7], a predictor-corrector difference method [8], the iterative method [9], the Legendre collocation method [10], the homotopy perturbation method [11], the modified homotopy perturbation method [12], the sixth-order methods [13], the Laplace homotopy analysis method [14], the Hahn wavelets collocation method [15], the Chebyshev wavelet method [16], the meshless finite difference method [17], the wavelet neural networks approach [18], the local meshless radial basis method [19], operational matrix methods [20, 21], and the finite element method [22]. Some other numerical methods can be found in [23, 24].

The time-fractional diffusion-wave equation (TFDWE) is a generalization of the classical diffusion and wave equations in that fractional derivatives are used in time to describe anomalous diffusion phenomena and wave propagation in complex systems. These equations have applications in various fields such as physics, engineering, and finance. More specifically, in electrodynamics, they model phenomena in fractional electrodynamics, where memory effects or non-local behaviors in time are essential [25]. The equations also appear in signal transmission and processing contexts—like electrical networking—where traditional diffusion or wave equations fall short [26]. In addition, applications of fractional diffusion wave models include memory-dependent dynamics in economic modeling and the analysis of systems affected by colored noise [27]. Many contributions have been devoted to solving these equations. In [28], the authors proposed a numerical approach using Bernoulli polynomials to solve certain fractional damped diffusion-wave equations. In [29], the authors utilized a difference scheme to solve the nonlinear TFDWE. In [30], the authors used a generalized finite difference method for solving the TFDWE. An implicit finite difference method was used in [31] for the treatment of the TFDWE. Cubic splines were used in [32] for the TFDWE. Some other methods to treat these equations can be found in [33, 34].

Special functions play an essential role in mathematics, science, and engineering. One can consult [35, 36] for some applications of these polynomials in applied sciences, such as quantum mechanics and physics. In numerical analysis, these functions are widely employed to handle various kinds of DEs. For instance, in [37], the authors introduced certain Chebyshev polynomials and used them to solve the FitzHugh–Nagumo equation. Pell-Lucas polynomials were employed in [38] to treat the nonlinear fractional Duffing equations. In [39], the authors used the shifted Lucas polynomials to handle the time-fractional FitzHugh-Nagumo differential equation. In [40], the authors used Chebyshev polynomials for treating the time-fractional heat equation. In [41], Chelyshkov polynomials were utilized for the numerical solution of the Burgers equation. Certain generalized Chebyshev polynomials were introduced and used in [42] for the numerical solution of the fractional delay pantograph equation.

The Horadam sequences of polynomials and their particular types have been the subject of extensive research from both theoretical and practical perspectives. Horadam sequences extend several famous polynomial sequences, including the Fibonacci, Lucas, Pell, and Pell-Lucas polynomials; they were initially introduced in the 1960s by the mathematician Alwyn Horadam. The Horadam sequences of polynomials and their particular cases have been the subject of extensive research from both theoretical and practical perspectives. Some theoretical results can be found in [43–45]. Numerically, some classes of Horadam polynomials have been used. For example, the shifted Vieta-Fibonacci polynomials were utilized in [46] to solve the nonlinear variable-order time-fractional Burgers-Huxley equations. The shifted Fibonacci polynomials were used in [47] to solve the fractional Burgers equation. In [48], certain shifted Horadam polynomials were utilized to solve certain KdV-type equations.

Spectral methods are important numerical approaches employed to solve all types of differential equations. The philosophy behind applying these methods is based on the assumption that the numerical solutions are represented in terms of suitable special functions that are often special polynomials. Solutions converge exponentially, an essential characteristic of these methods, allowing them to tackle problems with exceptional accuracy. These methods have applications in fluid dynamics, climate modeling, quantum mechanics, and structural analysis, among other scientific and engineering fields; see, for instance, [49, 50]. Prominent spectral technique variations include collocation approaches, which assume collocation points inside the domain; this approach is advantageous since it applies to all kinds of DEs. Several contributions use the collocation approach to handle DEs; see, for instance, [51–54]. Spectral methods also include the Galerkin approach. The philosophy of applying the Galerkin method is based on selecting suitable functions that satisfy the underlying conditions governed by the differential equation, and after that, enforcing the residual of the equation to be orthogonal to the basis functions. One can refer to [55–57] for some contributions regarding the Galerkin method. In contrast to the Galerkin technique, the tau method allows for greater freedom when selecting basis functions; for example, see [58, 59].

The following points are the main contributions of this work:

- Presenting two sets of basis functions in terms of specific shifted Horadam polynomials.
- Utilizing the Galerkin method to treat the TFDWE.
- Investigating the convergence analysis of the proposed Horadam expansion.
- Investigating our numerical algorithm by presenting some numerical examples.

The novelty of the paper can be listed in the following points:

- Employing new basis functions constructed from the shifted Horadam polynomials, used for designing the Galerkin framework for treating the TFDWE.
- Derivation of some new integral identities crucial for the efficient computation of Galerkin matrices.
- Offering a thorough examination of convergence and error, tailored to the suggested basis functions.

The contents of the paper are organized as follows: Section 2 presents some fundamental formulas and provides an overview of fractional calculus. Section 3 describes how to convert the time-fractional diffusion-wave equation, governed by the non-homogeneous conditions, into a homogeneous one with the aid of an appropriate substitution. Section 4 develops in detail a numerical scheme to treat the TFDWE in one dimension. In addition, an account of some extensions of the one-dimensional model is also given in this section. The investigation of the error is discussed in Section 5. Section 6 displays some illustrative examples. Section 7 concludes with a few findings.

2. Fundamentals and basic formulas

This section presents some fundamentals required to design our proposed numerical algorithm. We give an account of the Horadam polynomials and some of their particular classes. This section also provides some essential characteristics of fractional calculus.

2.1. An account on Horadam polynomials

In his essential paper [60], Horadam introduced certain generalized polynomials that are constructed with the aid of the following recursive formula:

$$H_m(z) = r(z)\, H_{m-1}(z) + s(z)\, H_{m-2}(z), \quad H_0(z) = 0,\ H_1(z) = 1. \tag{2.1}$$

The Binet's form for $H_m(z)$ can be written in the following form:

$$H_m(z) = \frac{\left[r(z) + \sqrt{r^2(z) + 4\, s(z)}\right]^m - \left[r(z) - \sqrt{r^2(z) + 4\, s(z)}\right]^m}{2^m \sqrt{r^2(z) + 4\, s(z)}}, \quad m \geq 0. \tag{2.2}$$

The sequence $\{H_m(z)\}_{m\geq 0}$ generalizes some well-known sequences, such as Fibonacci, Lucas, Pell, Pell-Lucas, Fermat, and Fermat-Lucas sequences of polynomials.

Remark 1. *The generalized Fibonacci sequence of polynomials $F_m^{c,d}(z)$ that are generated using the recurrence relation*

$$F_m^{c,d}(z) = a\, z\, F_{m-1}^{c,d}(z) + b\, F_{m-2}^{c,d}(z), \quad F_0^{c,d}(z) = 1,\ F_1^{c,d}(z) = a\, z, \quad m \geq 2, \tag{2.3}$$

is a particular sequence of the Horadam sequence of polynomials $H_{m+1}(z)$.

Remark 2. *It is worth noting here that for every m, $F_m^{c,d}(z)$ is of degree m. Fibonacci, Pell, Fermat, and second-kind Chebyshev polynomials are well-known particular sequences. We have*

$$F_{m+1}(z) = F_m^{(1,1)}(z), \qquad P_{m+1}(z) = F_m^{(2,1)}(z), \tag{2.4}$$

$$\mathcal{F}_{m+1}(z) = F_m^{(3,-2)}(z), \qquad U_m(z) = F_m^{(2,-1)}(z). \tag{2.5}$$

Recently, the authors of [47] used the shifted Fibonacci polynomials to solve the fractional Burgers equation. In addition, the authors of [48] introduced other shifted Horadam polynomials defined as

$$\phi_m(z) = F_m^{-2,-1}(2z - 1), \tag{2.6}$$

and employed them to solve certain KdV-type equations.
The recurrence relation that constructs these polynomials is given by

$$\phi_m(z) = -2\, z\, \phi_{m-1}(z) - \phi_{m-2}(z), \quad \phi_0(z) = 1, \quad \phi_1(z) = -2(2z - 1), \quad m \geq 2. \tag{2.7}$$

In [48], the authors stated and proved this theorem.

Theorem 1. *For any positive integer m, $\phi_m(z)$ has the following representation:*

$$\phi_m(z) = \frac{-1}{2\sqrt{\pi}} \sum_{j=0}^{m} \frac{(1 + 2m - j)!\, \Gamma\left(-\frac{1}{2} - m + j\right)}{(m - j)!\, j!} z^{m-j}. \tag{2.8}$$

2.2. Caputo's fractional derivative

Definition 1. *[61] In Caputo's sense, the fractional-order derivative of $\xi \in C^r[0, s]$, where $r = \lceil\gamma\rceil$ is defined as*

$$D_s^\gamma \xi(s) = \frac{1}{\Gamma(r-\gamma)} \int_0^s (s-t)^{r-\gamma-1} \xi^{(r)}(t)dt, \quad \gamma > 0, \quad s > 0, \ r-1 < \gamma \le r, \quad r \in \mathbb{N}. \tag{2.9}$$

For D_s^γ with $r-1 < \gamma \le r, \quad r \in \mathbb{N}$, we have

$$D_s^\gamma C = 0, \quad C \text{ is a constant}, \tag{2.10}$$

$$D_s^\gamma s^r = \begin{cases} 0, & \text{if } r \in \mathbb{N}_0 \quad \textit{and} \quad r < \lceil\gamma\rceil, \\ \frac{r!}{\Gamma(r-\gamma+1)} s^{r-\gamma}, & \text{if } r \in \mathbb{N}_0 \quad \textit{and} \quad r \ge \lceil\gamma\rceil, \end{cases} \tag{2.11}$$

where $\mathbb{N} = \{1, 2, ...\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, and $\lceil\gamma\rceil$ is the ceiling function.

Remark 3. *The Caputo fractional derivative is a nonlocal rate of change where a system's current state is determined by its whole historical background, with contributions decaying over time as a power law. Due to this memory effect, it can be used to describe processes that exhibit hereditary behavior in the real world, such as viscoelastic materials and anomalous diffusion, while maintaining physically interpretable initial conditions in terms of common integer-order quantities, like velocity and displacement; see [61, 62].*

Remark 4. *There are several fractional derivatives, such as the local fractional derivative, He's fractional derivative, Atangana-Baleanu fractional derivative, and conformable fractional derivative; see [63, 64]. In this paper, we use Caputo's fractional derivative due to its advantages, such as its well-established physical interpretation and wide applicability in applied science and engineering [65].*

3. The homogeneous form of the time-fractional diffusion wave equation

Consider the following TFDWE [66, 67]:

$$\frac{\partial^\gamma \zeta(z,t)}{\partial t^\gamma} + a \frac{\partial \zeta(z,t)}{\partial t} + b\,\zeta(z,t) - \frac{\partial^2 \zeta(z,t)}{\partial z^2} = f(z,t), \quad 1 < \gamma < 2, \tag{3.1}$$

governed by the following non-homogeneous conditions:

$$\zeta(z,0) = \zeta_0(z), \quad \zeta_t(z,0) = \zeta_1(z), \quad 0 \le z \le 1, \tag{3.2}$$

$$\zeta(0,t) = \zeta_3(t), \quad \zeta(1,t) = \zeta_4(t), \quad 0 \le t \le 1, \tag{3.3}$$

where a and b are the damping and reaction term coefficients, respectively, and $f(z,t)$ is the source term.

We will employ a suitable transformation to convert the non-homogeneous conditions (3.2) and (3.3) into homogeneous ones to derive our Galerkin scheme. More definitely, we make use of the substitution

$$T(z,t) := \zeta(z,t) + \hat{\zeta}(z,t), \tag{3.4}$$

where

$$\hat{\zeta}(z,t) = -\,t\,\left((z-1)\,\zeta_1(0) - z\,\zeta_1(1) + \zeta_1(z)\right) + (z-1)\,\zeta(0,t) \\ -\,z\,\zeta(1,t) - (z-1)\,\zeta(0,0) + z\,\zeta(1,0) - \zeta(z,0). \tag{3.5}$$

Then, Eq (3.4) converts the TFDWE (3.1) governed by (3.2) and (3.3) into the modified equation that follows:

$$\frac{\partial^{\gamma} T(z,t)}{\partial t^{\gamma}} + a\,\frac{\partial T(z,t)}{\partial t} + b\,T(z,t) - \frac{\partial^2 T(z,t)}{\partial z^2} = h(z,t), \quad 1 < \gamma < 2, \tag{3.6}$$

subject to the following homogeneous conditions:

$$T(z,0) = T_t(z,0) = 0, \quad 0 \le z \le 1, \\ T(0,t) = T(1,t) = 0, \quad 0 \le t \le 1, \tag{3.7}$$

where

$$h(z,t) = f(z,t) + \frac{\partial^{\gamma}\hat{\zeta}(z,t)}{\partial t^{\gamma}} + a\,\frac{\partial\hat{\zeta}(z,t)}{\partial t} + b\,\hat{\zeta}(z,t) - \frac{\partial^2\hat{\zeta}(z,t)}{\partial z^2}. \tag{3.8}$$

Thus, we can solve the modified equation (3.6) governed by (3.7) instead of solving (3.1) governed by (3.2) and (3.3).

4. Numerical spectral treatment for time-fractional diffusion wave equation

This section is devoted to proposing a numerical solution of TFDWE (3.1), governed by the homogeneous boundary conditions (3.7). In addition, we will give an account of two generalized models of the TFDWE model.

4.1. Galerkin procedure for (3.6) *governed by* (3.7)

In order to apply the Galerkin method to (3.6), governed by (3.7), we will choose the following two kinds of basis functions:

$$\mathcal{A}_i(z) = z\,(1-z)\,\phi_i(z), \\ \mathcal{B}_j(t) = t^2\,\phi_j(t), \tag{4.1}$$

and consider the following two spaces:

$$\psi_N(\Omega) = \operatorname{span}\{\mathcal{A}_i(z)\,\mathcal{B}_j(t) : i, j = 0, 1, \ldots, N\}, \\ \mathbb{L}_N(\Omega) = \{T(z,t) \in \psi_N(\Omega) : T(z,0) = T_t(z,0) = T(0,t) = T(1,t) = 0\}, \tag{4.2}$$

where $\Omega = [0,1]^2$.
Then, for any $\hat{T}(z,t) \in \mathbb{L}_N(\Omega)$ may be expressed as

$$\hat{T}(z,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\,\mathcal{A}_i(z)\,\mathcal{B}_j(t) = \boldsymbol{\mathcal{A}}\,\boldsymbol{C}\,\boldsymbol{\mathcal{B}}^T, \tag{4.3}$$

where, $\boldsymbol{\mathcal{A}} = [\mathcal{A}_0(z), \mathcal{A}_1(z), \ldots, \mathcal{A}_N(z)]$, $\boldsymbol{\mathcal{B}}^T = [\mathcal{B}_0(t), \mathcal{B}_1(t), \ldots, \mathcal{B}_N(t)]^T$, and $\boldsymbol{C} = (c_{ij})_{0\le i,\, j\le N}$ represents the unknown matrix of order $(N+1)^2$.

Remark 5. *The basis functions in* (4.1) *are constructed to ensure that the approximate solution satisfies the homogeneous conditions* $\hat{T}(0,t) = \hat{T}(1,t) = 0$, *and the initial conditions:* $\hat{T}(z,0) = \hat{T}_t(z,0) = 0$.

Using Eq (3.6), the residual $\mathrm{R}(z,t)$ can now be computed to obtain

$$\mathrm{R}(z,t) = \frac{\partial^{\gamma}\hat{T}(z,t)}{\partial t^{\gamma}} + a\,\frac{\partial\hat{T}(z,t)}{\partial t} + b\,\hat{T}(z,t) - \frac{\partial^{2}\hat{T}(z,t)}{\partial z^{2}} - h(z,t). \tag{4.4}$$

After the Galerkin method is applied, we get

$$\int_0^1\int_0^1 \mathrm{R}(z,t)\,\mathcal{A}_r(z)\,\mathcal{B}_s(t)\,dz\,dt = 0, \quad 0 \le r,\ s \le N. \tag{4.5}$$

Let us define

$$\boldsymbol{H} = (h_{r,s})_{(N+1)\times(N+1)}, \quad h_{r,s} = \int_0^1\int_0^1 h(z,t)\,\mathcal{A}_r(z)\,\mathcal{B}_s(t)\,dz\,dt, \tag{4.6}$$

$$\boldsymbol{\mathcal{Z}} = (z_{i,r})_{(N+1)\times(N+1)}, \quad z_{i,r} = \int_0^1 \mathcal{A}_i(z)\,\mathcal{A}_r(z)\,dz, \tag{4.7}$$

$$\boldsymbol{\mathcal{U}} = (u_{i,r})_{(N+1)\times(N+1)}, \quad u_{i,r} = \int_0^1 \frac{d^2\,\mathcal{A}_i(z)}{d\,z^2}\,\mathcal{A}_r(z)\,dz, \tag{4.8}$$

$$\boldsymbol{\mathcal{Y}} = (y_{j,s})_{(N+1)\times(N+1)}, \quad y_{j,s} = \int_0^1 \mathcal{B}_j(t)\,\mathcal{B}_s(t)\,dt, \tag{4.9}$$

$$\boldsymbol{\mathcal{V}} = (v_{j,s})_{(N+1)\times(N+1)}, \quad v_{j,s} = \int_0^1 \frac{d\,\mathcal{B}_j(t)}{d\,t}\,\mathcal{B}_s(t)\,dt, \tag{4.10}$$

$$\boldsymbol{\mathcal{M}} = (m_{j,s})_{(N+1)\times(N+1)}, \quad m_{j,s} = \int_0^1 D_t^{\gamma}\,\mathcal{B}_j(t)\,\mathcal{B}_s(t)\,dt. \tag{4.11}$$

Consequently, we can rewrite Eq (4.5) as

$$\sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\,z_{i,r}\,m_{j,s} + a\sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\,z_{i,r}\,v_{j,s} + b\sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\,z_{i,r}\,y_{j,s} - \sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\,u_{i,r}\,y_{j,s} = h_{r,s}. \tag{4.12}$$

which can be represented equivalently as

$$\boldsymbol{\mathcal{Z}}^T\,\boldsymbol{C}\,\boldsymbol{\mathcal{M}} + a\,\boldsymbol{\mathcal{Z}}^T\,\boldsymbol{C}\,\boldsymbol{\mathcal{V}} + b\,\boldsymbol{\mathcal{Z}}^T\,\boldsymbol{C}\,\boldsymbol{\mathcal{Y}} - \boldsymbol{\mathcal{U}}^T\,\boldsymbol{C}\,\boldsymbol{\mathcal{Y}} = \boldsymbol{H}. \tag{4.13}$$

Any suitable numerical algorithm, such as the Gaussian elimination procedure, can be utilized for solving algebraic system equation (4.13) with order $(N+1)^2$. The elements of the matrices $\boldsymbol{\mathcal{Z}}, \boldsymbol{\mathcal{M}}, \boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Y}}$ are explicitly given in the following theorem.

Theorem 2. *The following five useful integral formulas hold:*

$$\begin{aligned}
&(a) \quad \int_0^1 \mathcal{A}_i(z)\,\mathcal{A}_r(z)\,dz = z_{i,r},\\
&(b) \quad \int_0^1 \frac{d^2\,\mathcal{A}_i(z)}{d\,z^2}\,\mathcal{A}_r(z)\,dz = u_{i,r},\\
&(c) \quad \int_0^1 \mathcal{B}_j(t)\,\mathcal{B}_s(t)\,dt = y_{j,s},\\
&(d) \quad \int_0^1 \frac{d\,\mathcal{B}_j(t)}{d\,t}\,\mathcal{B}_s(t)\,dt = v_{j,s},\\
&(e) \quad \int_0^1 [D_t^{\gamma}\,\mathcal{B}_j(t)]\,\mathcal{B}_s(t)\,dt = m_{j,s},
\end{aligned} \tag{4.14}$$

where

$$z_{i,r} = \sum_{s=0}^{i} -\frac{(r+1)!\,\Gamma\left(-s-\frac{1}{2}\right)(i+s+1)!\binom{i}{i-s}}{\sqrt{\pi}(s+3)(s+4)(s+5)i!\,r!}\ {}_3F_2\left(\begin{matrix}-r, r+2, s+3\\ \frac{3}{2}, s+6\end{matrix}\,\middle|\,1\right), \tag{4.15}$$

$$\begin{aligned}
u_{i,r} = &\sum_{s=0}^{i} -\frac{(r+1)!\,\Gamma\left(-s-\frac{1}{2}\right)(i+s+1)!\binom{i}{i-s}}{3\sqrt{\pi}(s+2)(s+3)(s+4)i!\,r!}\\
&\times\left(2r(r+2)(s+1)\ {}_3F_2\left(\begin{matrix}1-r, r+3, s+2\\ \frac{5}{2}, s+5\end{matrix}\,\middle|\,1\right)\right.\\
&\left.-3(s+4)\ {}_3F_2\left(\begin{matrix}-r, r+2, s+1\\ \frac{3}{2}, s+4\end{matrix}\,\middle|\,1\right)\right),
\end{aligned} \tag{4.16}$$

$$y_{j,s} = \sum_{r=0}^{j} -\frac{(s+1)!\,\Gamma\left(-r-\frac{1}{2}\right)(j+r+1)!\binom{j}{j-r}}{2\sqrt{\pi}(r+5)j!\,s!}\ {}_3F_2\left(\begin{matrix}r+5, -s, s+2\\ \frac{3}{2}, r+6\end{matrix}\,\middle|\,1\right), \tag{4.17}$$

$$v_{j,s} = \sum_{r=0}^{j} -\frac{(r+2)(s+1)!\,\Gamma\left(-r-\frac{1}{2}\right)(j+r+1)!\binom{j}{j-r}}{2\sqrt{\pi}(r+4)j!\,s!}\ {}_3F_2\left(\begin{matrix}r+4, -s, s+2\\ \frac{3}{2}, r+5\end{matrix}\,\middle|\,1\right), \tag{4.18}$$

$$\begin{aligned}
m_{j,s} = &\sum_{r=0}^{j} -\frac{(r+1)(r+2)(s+1)(-\gamma+r+3)(-\gamma+r+4)\Gamma\left(-r-\frac{1}{2}\right)(j+r+1)!}{4\,(j-r)!}\\
&\times\ {}_3F_2\left(\begin{matrix}-s, s+2, -\gamma+r+5\\ \frac{3}{2}, -\gamma+r+6\end{matrix}\,\middle|\,1\right).
\end{aligned} \tag{4.19}$$

Proof. Using the power formula (2.8) along with basis functions (4.1), we get

$$\mathcal{A}_i(z) = \frac{-1}{2\sqrt{\pi}\,i!}\sum_{r=0}^{i}\Gamma\left(-r-\frac{1}{2}\right)(i+r+1)!\binom{i}{r}\left(z^{r+1}-z^{r+2}\right), \tag{4.20}$$

$$\frac{d^2 \mathcal{A}_i(z)}{d z^2} = \frac{-1}{2 \sqrt{\pi}\, i!} \sum_{r=0}^{i} \Gamma\left(-r - \frac{1}{2}\right) (i + r + 1)! \binom{i}{r} \times \left(r\,(r + 1)\, z^{r-1} - (r + 1)\,(r + 2)\, z^r\right), \tag{4.21}$$

$$\mathcal{B}_j(t) = \frac{-1}{2 \sqrt{\pi}\, j!} \sum_{r=0}^{j} \Gamma\left(-r - \frac{1}{2}\right) (j + r + 1)! \binom{j}{r} t^{r+2}, \tag{4.22}$$

$$\frac{d\,\mathcal{B}_j(t)}{d t} = \frac{-1}{2 \sqrt{\pi}\, j!} \sum_{r=0}^{j} \Gamma\left(-r - \frac{1}{2}\right) (j + r + 1)! \binom{j}{r} (r + 2)\, t^{r+1}. \tag{4.23}$$

Also, the application of Caputo definition (2.11) enables us to write $D_t^{\gamma}\, \mathcal{B}_j(t)$ as:

$$D_t^{\gamma}\, \mathcal{B}_j(t) = \frac{-1}{2 \sqrt{\pi}\, j!} \sum_{r=0}^{j} \Gamma\left(-r - \frac{1}{2}\right) (j + r + 1)! \binom{j}{r} \frac{(r + 2)!}{\Gamma(-\gamma + r + 3)}\, t^{r+2-\gamma}. \tag{4.24}$$

Now, to prove part (a),

$$\begin{aligned} z_{i,r} &= \int_0^1 \mathcal{A}_i(z)\, \mathcal{A}_r(z)\, dz \\ &= \sum_{s=0}^{i} \sum_{p=0}^{r} \frac{\Gamma\left(-p - \frac{1}{2}\right) \Gamma\left(-s - \frac{1}{2}\right) \binom{i}{i-s} \binom{r}{r-p} (i + s + 1)!\, (p + r + 1)!}{4\,\pi\, i!\, r!} \\ &\times \int_0^1 \left(z^{p+1} - z^{p+2}\right) \left(z^{s+1} - z^{s+2}\right) dz, \end{aligned} \tag{4.25}$$

which is equivalent to

$$\begin{aligned} z_{i,r} &= \int_0^1 \mathcal{A}_i(z)\, \mathcal{A}_r(z)\, dz \\ &= \sum_{s=0}^{i} \sum_{p=0}^{r} \frac{\Gamma\left(-p - \frac{1}{2}\right) \Gamma\left(-s - \frac{1}{2}\right) \binom{i}{s} \binom{r}{p} (i + s + 1)!\, (p + r + 1)!}{4\,\pi\, i!\, r!} \\ &\times \frac{2}{(p + s + 3)(p + s + 4)(p + s + 5)}. \end{aligned} \tag{4.26}$$

Now, the summation $\displaystyle\sum_{p=0}^{r} \frac{\Gamma\left(-p - \frac{1}{2}\right) \binom{r}{p} \Gamma(p + r + 2)}{(p + s + 3)(p + s + 4)(p + s + 5)}$ can be expressed as [68]

$$\sum_{p=0}^{r} \frac{\Gamma\left(-p - \frac{1}{2}\right) \binom{r}{p} \Gamma(p + r + 2)}{(p + s + 3)(p + s + 4)(p + s + 5)} = -\pi\, \Gamma(r + 2)\, \Gamma(s + 3)\ {}_3F_2 \left(\begin{matrix} -r, r + 2, s + 3 \\ \frac{3}{2}, s + 6 \end{matrix} \middle| 1 \right), \tag{4.27}$$

and therefore, inserting Eq (4.27) into Eq (4.26), we get the desired result of $z_{i,r}$.
If we insert the series of relations (4.21)–(4.24) into integrations (b), (c), (d), and (e), following similar steps as part (a) and simplifying the right-hand side of these integrations, we get the desired results.

Remark 6. *Regarding the structure of the matrices in the system* (4.13)*, the two matrices* $\mathcal{Z}$ *and* $\mathcal{U}$ *are sparse, while the matrices* $\mathcal{Y}$*,* $\mathcal{V}$*, and* $\mathcal{M}$ *are dense.*

Remark 7. *The following algorithm describes step by step how to obtain the proposed numerical technique.*

Algorithm 1 Coding algorithm for the suggested method

Input γ, a, b, $\zeta_0(z)$, $\zeta_1(z)$, $\zeta_3(t)$, $\zeta_4(t)$ and $f(z,t)$.
Step 1. Using transformation (3.4) to convert the TFDWE (3.1) governed by (3.2) and (3.3), into the modified equation (3.6) governed by (3.7).
Step 2. Consider a numerical solution $\hat{T}(z,t)$ as in (4.3).
Step 3. Apply the Galerkin method to obtain the system in (4.13).
Step 4. Use Theorem 2 to get the elements $z_{i,r}$, $u_{i,r}$, $y_{j,s}$, $v_{j,s}$ and $m_{j,s}$
Step 5. Use *NSolve* command to solve the system in (4.13) to get c_{ij}.
Output $\hat{T}(z,t)$.

4.2. Extension to the algorithm in a more general interval

Although the method described in Subsection 4.1 is designed for solving the TFDWE on the domain $[0,1]\times[0,1]$, it can be extended to solve the same model but on $[0,\ell]\times[0,\ell]$. In this case, the basis functions and the elements of matrices must be redefined over the new interval as follows:

$$\begin{aligned}\bar{\mathcal{A}}_i(z) &= z\,(\ell - z)\,\phi_i\left(\frac{z}{\ell}\right),\\ \bar{\mathcal{B}}_j(t) &= t^2\,\phi_j\left(\frac{t}{\ell}\right),\end{aligned}\tag{4.28}$$

and

$$\begin{aligned}&(a)\quad \int_0^{\ell} \bar{\mathcal{A}}_i(z)\,\bar{\mathcal{A}}_r(z)\,dz = \bar{z}_{i,r},\\ &(b)\quad \int_0^{\ell} \frac{d^2\,\bar{\mathcal{A}}_i(z)}{d\,z^2}\,\bar{\mathcal{A}}_r(z)\,dz = \bar{u}_{i,r},\\ &(c)\quad \int_0^{\ell} \bar{\mathcal{B}}_j(t)\,\bar{\mathcal{B}}_s(t)\,dt = \bar{y}_{j,s},\\ &(d)\quad \int_0^{\ell} \frac{d\,\bar{\mathcal{B}}_j(t)}{d\,t}\,\bar{\mathcal{B}}_s(t)\,dt = \bar{v}_{j,s},\\ &(e)\quad \int_0^{\ell} [D_t^{\gamma}\,\bar{\mathcal{B}}_j(t)]\,\bar{\mathcal{B}}_s(t)\,dt = \bar{m}_{j,s},\end{aligned}\tag{4.29}$$

and the right-hand sides of (4.29) have slight changes from those in (4.14). Therefore, we follow similar steps as in Subsection 4.1 to obtain the approximate solution.

4.3. An account on a generalized TFDWE *in two-dimensions*

The algorithm described in Subsection 4.1 can be extended to solve the following two-dimensional TFDWE:

$$\frac{\partial^\gamma T(z,y,t)}{\partial t^\gamma} + a\,\frac{\partial T(z,y,t)}{\partial t} + b\,\zeta(z,y,t) - \frac{\partial^2 T(z,y,t)}{\partial z^2} - \frac{\partial^2 T(z,y,t)}{\partial y^2} = f(z,y,t), \quad 1 < \gamma < 2, \tag{4.30}$$

governed by the following non-homogeneous conditions:

$$T(z,y,0) = T_t(z,y,0) = 0, \quad 0 \le z, y \le 1, \tag{4.31}$$

$$T(0,y,t) = T(1,y,t) = 0, \quad 0 \le y, t \le 1, \tag{4.32}$$

$$T(z,0,t) = T(z,1,t) = 0, \quad 0 \le z, t \le 1, \tag{4.33}$$

where $f(z,y,t)$ is the source term.
In this case, we consider the following approximate solution:

$$\hat{T}(z,t) = \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N} c_{ijk}\,\mathcal{A}_i(z)\,\mathcal{A}_j(z)\,\mathcal{B}_k(t),$$

and follow similar steps as in Subsection 4.1 to get a linear system of algebraic equations of dimension $(N+1)^3$ in the unknown expansion coefficients c_{ijk} , which can be treated using any suitable numerical algorithm.

5. Convergence and error analysis

This part delves further into the error analysis of the proposed method. Here are four key estimates that we will present.

- Lemma 3 gives an upper bound for $|\mathcal{A}_i(z)|$.
- Lemma 4 gives an upper limit for $|\mathcal{B}_j(t)|$.
- Theorem 4 gives a maximum estimate for $|c_{ij}|$.
- Theorem 5 gives an upper bound on the value of $|T(z,t) - \hat{T}(z,t)|$.

Lemma 1. *Let $g(z)$ be an infinitely differentiable function at the origin. Then, the following inequality holds [48]:*

$$g(z) = \sum_{n=0}^{\infty}\sum_{s=n}^{\infty} \frac{4\,(-1)^n\, g^{(s)}(0)\,(n+1)\,\Gamma\left(s+\frac{3}{2}\right)}{\sqrt{\pi}\,(s-n)!\,(n+s+2)!}\,\phi_n(z). \tag{5.1}$$

Lemma 2. *The following inequality is valid for $\phi_i(z)$ [48]:*

$$|\phi_i(z)| \le i+1, \qquad i \ge 0,\ z \in (0,1). \tag{5.2}$$

Lemma 3. *The following inequality holds:*

$$|\mathcal{A}_i(z)| \le \frac{i+1}{4}. \tag{5.3}$$

Proof. Eq (4.1) enables one to write

$$\begin{aligned} |\mathcal{A}_i(z)| &= |z\,(1-z)\,\phi_i(z)| \\ &\leq \frac{1}{4}\,|\phi_i(z)|, \end{aligned} \tag{5.4}$$

by making use of Lemma 2, the desired inequality (5.3) can be obtained.

Lemma 4. *The following inequality holds:*

$$|\mathcal{B}_j(t)| \leq j+1. \tag{5.5}$$

Proof. The proof of this lemma can be easily obtained after using Eq (4.1) and Lemma 2 as follows:

$$\begin{aligned} |\mathcal{B}_j(t)| &= |t^2\,\phi_j(t)| \\ &\leq |\phi_j(t)| \leq j+1. \end{aligned} \tag{5.6}$$

Theorem 3. *[48] Consider* $u(z,t) = g_1(z)\,g_2(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} v_{ij}\,\phi_i(z)\,\phi_j(t)$*, with* $|g_1^{(i)}(0)| \leq \ell_1^i$*, and* $|g_2^{(i)}(0)| \leq \ell_2^i$*, where* ℓ_1 *and* ℓ_2 *are positive constants. One has*

$$|v_{ij}| \leq \frac{\left(e^{\ell_1}+1\right)\left(e^{\ell_2}+1\right)2^{-2\,(i+j+1)}\,\ell_1^i\,\ell_2^j}{i!\,j!}. \tag{5.7}$$

Moreover, the series is absolutely convergent.

Theorem 4. *If a function* $T(z,t) = z\,(1-z)\,t^2\,g_1(z)\,g_2(t) \in \mathbb{L}_N(\Omega)$*, with* $|g_n^{(i)}(0)| \leq Q_n^i$*,* $n = 1,2$*,* $i \geq 0$*,* Q_n *is a positive constant, and if* $T(z,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}\,\mathcal{A}_i(z)\,\mathcal{B}_j(t)$*, then the following estimation is obtained:*

$$|c_{ij}| \leq \frac{\left(e^{Q_1}+1\right)\left(e^{Q_2}+1\right)Q_1^i\,Q_2^j}{i!\,j!\,2^{2\,(i+j+1)}}. \tag{5.8}$$

Moreover, the series is absolutely convergent.

Proof. By virtue of Lemma 1, $T(z,t)$ is represented as

$$T(z,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}\,\mathcal{A}_i(z)\,\mathcal{B}_j(t), \tag{5.9}$$

where

$$c_{ij} = \sum_{m=i}^{\infty}\sum_{n=j}^{\infty} \frac{16\,(-1)^{i+j}\,g_1^{(m)}(0)\;g_2^{(n)}(0)\,(i+1)\,(j+1)\,\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(m+\frac{3}{2}\right)}{\pi\,(m-i)!\,(i+m+2)!\,(n-j)!\,(j+n+2)!}. \tag{5.10}$$

Taking |.| for both sides of the last equation and using the assumption $|g_{1,2}^{(i)}(0)| \leq Q_{1,2}^{i}$, it allows the double series to be separated as follows:

$$|c_{ij}| \leq \sum_{m=i}^{\infty} \frac{4\,Q_1^{(m)}\,(i+1)\,\Gamma\left(m+\frac{3}{2}\right)}{\sqrt{\pi}\,(m-i)!\,(i+m+2)!} \times \sum_{n=j}^{\infty} \frac{4\,Q_2^{(n)}\,(j+1)\,\Gamma\left(n+\frac{3}{2}\right)}{\sqrt{\pi}\,(n-j)!\,(j+n+2)!}. \tag{5.11}$$

Now, by similarity with the procedures in Theorem 3, we obtain

$$|c_{ij}| \leq \frac{\left(e^{Q_1}+1\right)\left(e^{Q_2}+1\right)\,Q_1^i\,Q_2^j}{i!\,j!\,2^{2\,(i+j+1)}}. \tag{5.12}$$

This ends the proof.

Theorem 5. *This upper bound for the truncation error may be obtained if $T(z,t)$ meets the hypothesis of Theorem 4.*

$$|T(z,t)-\hat{T}(z,t)| < \frac{\mathrm{L}\,Q_2^{N+1}+\mathcal{P}\,Q_1^{N+1}}{2^{2N+5}\,N!}, \tag{5.13}$$

where

$$\begin{aligned} \mathrm{L} &= e^{Q_1/4}\left(e^{Q_1}+1\right)(Q_1+4)\left(e^{Q_2}+1\right)\left(e^{Q_2/4}(Q_2+4)+4\right), \\ \mathcal{P} &= e^{Q_2/4}\left(e^{Q_2}+1\right)(Q_2+4)\left(e^{Q_1}+1\right)\left(e^{Q_1/4}(Q_1+4)+4\right). \end{aligned} \tag{5.14}$$

Proof. From definitions of $T(z,t)$ and $\hat{T}(z,t)$, we obtain

$$\begin{aligned} |T(z,t)-\hat{T}(z,t)| &= \left|\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}\,\mathcal{A}_i(z)\,\mathcal{B}_j(t) - \sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\,\mathcal{A}_i(z)\,\mathcal{B}_j(t)\right| \\ &\leq \left|\sum_{i=0}^{N}\sum_{j=N+1}^{\infty} c_{ij}\,\mathcal{A}_i(z)\,\mathcal{B}_j(t)\right| + \left|\sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty} c_{ij}\,\mathcal{A}_i(z)\,\mathcal{B}_j(t)\right|. \end{aligned} \tag{5.15}$$

The application of Theorem 4 along with Lemmas 3 and 4 yields the following inequalities:

$$\sum_{i=0}^{N} \frac{2^{-2i-1}(i+1)\left(e^{Q_1}+1\right)Q_1^i}{2\,i!} < e^{Q_1/4}\left(e^{Q_1}+1\right)(Q_1+4), \tag{5.16}$$

$$\sum_{i=N+1}^{\infty} \frac{2^{-2i-1}(i+1)\left(e^{Q_1}+1\right)Q_1^i}{2\,i!} < \frac{\left(e^{Q_1}+1\right)\left(e^{Q_1/4}(Q_1+4)+4\right)2^{-2N-5}Q_1^{N+1}}{N!}, \tag{5.17}$$

$$\sum_{j=N+1}^{\infty} \frac{2^{-2j-1}(j+1)\left(e^{Q_2}+1\right)Q_2^j}{j!} < \frac{\left(e^{Q_2}+1\right)\left(e^{Q_2/4}(Q_2+4)+4\right)2^{-2N-5}Q_2^{N+1}}{N!}, \tag{5.18}$$

$$\sum_{j=0}^{\infty} \frac{2^{-2j-1}(j+1)\left(e^{Q_2}+1\right)Q_2^j}{j!} < e^{Q_2/4}\left(e^{Q_2}+1\right)(Q_2+4), \tag{5.19}$$

consequently, we obtain

$$|T(z,t)-\hat{T}(z,t)| < \frac{\mathrm{L}\,Q_2^{N+1}+\mathcal{P}\,Q_1^{N+1}}{2^{2N+5}\,N!}, \tag{5.20}$$

where L and $\mathcal{P}$ are defined in Eq (5.14). This ends the proof.

6. Some numerical examples

In this section, we present some numerical examples to demonstrate the accuracy and efficiency of the suggested numerical algorithm by using the following absolute error (AEs), maximum absolute error (MAEs), and $L_\infty - error$:

$$AEs = \left|u(z,t) - u^N(z,t)\right|, \tag{6.1}$$

$$MAEs = \max_{z_i} \left|u(z_i, z_j) - u^N(z_i, z_j)\right|, \quad z_i \in \left\{\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \dots, 1\right\}, \tag{6.2}$$

$$L_\infty - error = \max_{(z,t)\in\Omega} \left|u(z,t) - u^N(z,t)\right|. \tag{6.3}$$

In addition, some comparisons with some methods are given.

Test Problem 1. *[66, 67] Consider the following* TFDWE*:*

$$\frac{\partial^\gamma u(z,t)}{\partial t^\gamma} + \frac{\partial u(z,t)}{\partial t} - \frac{\partial^2 u(z,t)}{\partial z^2} = f(z,t), \tag{6.4}$$

with the following conditions:

$$u(z,0) = u_t(z,0) = 0, \quad 0 \le z \le 1, \tag{6.5}$$

$$u(0,t) = u(1,t) = 0, \quad 0 \le t \le 1, \tag{6.6}$$

and $f(z,t)$ is selected such that the exact solution is $u(z,t) = t^2 z(1-z)$.

A comparison of the AEs between our technique at $N = 1$ and the methods in [66, 67] is shown in Tables 1 and 2. Furthermore, the CPU time (in seconds) for our method is shown in Tables 1 and 2. These tables demonstrate that our strategy yields accurate results for small N choices. In addition, our strategy outperformed the methods in [66, 67] as shown by these comparisons. Figure 1 displays the AEs acquired at different γ values for $N = 1$ using the proposed method, showing that it successfully approaches the precise solution.

Table 1. Comparison of the AEs at $t = 0.1$ of Test Problem 1.

	Method in [66]		Our method			
z	$\gamma = 1.5$	$\gamma = 1.9$	$\gamma = 1.5$	CPU time (seconds)	$\gamma = 1.9$	CPU time (seconds)
0.1	1.46277×10^{-7}	4.23821×10^{-8}	1.19262×10^{-17}		5.96311×10^{-18}	
0.2	2.73755×10^{-7}	7.78422×10^{-8}	1.4962×10^{-17}		7.15573×10^{-18}	
0.3	3.70496×10^{-7}	1.03582×10^{-7}	1.21431×10^{-17}		4.77049×10^{-18}	
0.4	4.30339×10^{-7}	1.19048×10^{-7}	5.20417×10^{-18}		0	
0.5	4.50539×10^{-7}	1.24200×10^{-7}	4.33681×10^{-18}	6.265	5.20417×10^{-18}	6.577
0.6	4.30339×10^{-7}	1.19048×10^{-7}	1.25767×10^{-17}		1.04083×10^{-17}	
0.7	3.70496×10^{-7}	1.03582×10^{-7}	1.9082×10^{-17}		1.34441×10^{-17}	
0.8	2.73755×10^{-7}	7.78422×10^{-8}	2.05998×10^{-17}		1.36609×10^{-17}	
0.9	1.46277×10^{-7}	4.23821×10^{-8}	1.48536×10^{-17}		9.6494×10^{-18}	

Table 2. Comparison of AEs at $\gamma = 1.9$ of Test Problem 1.

(z, t)	Method in [67]	Our method	Our CPU time (seconds)
(0.1, 0.1)	8.02310×10^{-18}	5.96311×10^{-18}	
(0.3, 0.3)	1.04083×10^{-17}	2.77556×10^{-17}	
(0.5, 0.5)	6.93889×10^{-18}	8.32667×10^{-17}	6.432
(0.7, 0.7)	4.77396×10^{-15}	1.38778×10^{-16}	
(0.9, 0.9)	1.11577×10^{-14}	5.55112×10^{-17}	

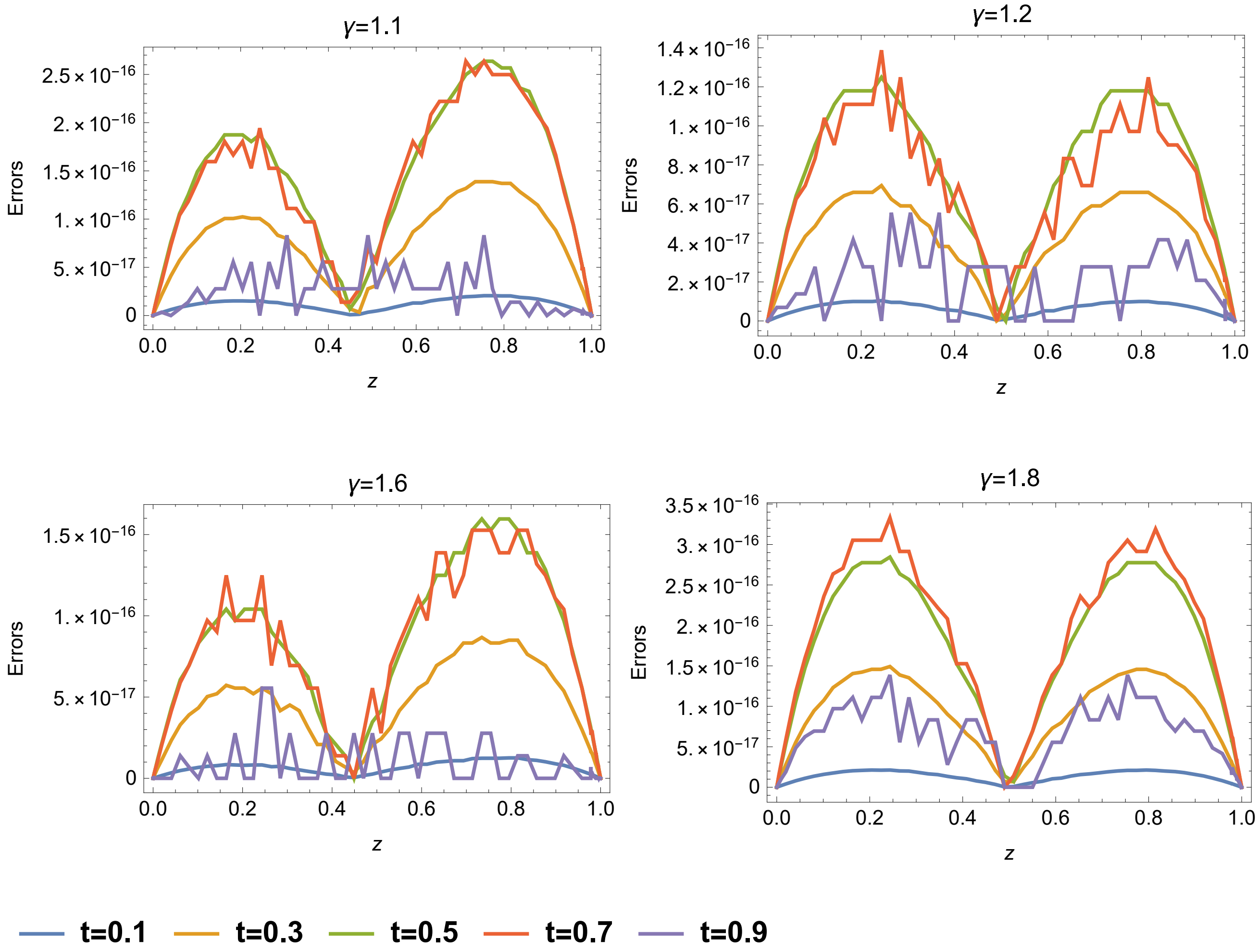

Figure 1. The AEs of Test Problem 1.

Test Problem 2. *[66] Consider the following* TFDWE*:*

$$\frac{\partial^{\gamma} u(z,t)}{\partial t^{\gamma}} - \frac{\partial^2 u(z,t)}{\partial z^2} = f(z,t), \tag{6.7}$$

constrained by the conditions:

$$u(z,0) = 0, \quad u_t(z,0) = -\sin(\pi z), \quad 0 \le z \le 1, \tag{6.8}$$
$$u(0,t) = u(1,t) = 0, \quad 0 \le t \le 1, \tag{6.9}$$

and $f(z,t)$ is selected to meet the exact solution $u(z,t) = \left(t^2 - t\right)\sin(\pi z)$.

The AEs of our technique for $N = 6$ and of the methods in [66] are compared in Table 3. This comparison reveals the superior performance of our technique over the method in [66]. The AEs at various values of γ when $N = 6$ are shown in Table 4. Furthermore, Tables 3 and 4 display the CPU time for our technique. We note that the errors that follow show that our technique successfully provides a very close approximation to the exact solution. At $\gamma = 1.5$ and $\gamma = 1.95$, Figures 2 and 3 show the MAEs and L_∞-error at different N values, respectively. The results demonstrate a strong correlation between the approximate and precise solutions and confirm that the proposed method consistently decreases errors across the domain.

Table 3. Comparison of the AEs of Test Problem 2 at $t = 0.2$ and $\gamma = 1.8$.

z	Method in [66]	Our method	Our CPU time (seconds)
0.1	4.673×10^{-4}	1.74667×10^{-9}	
0.2	6.451×10^{-4}	2.48633×10^{-9}	
0.3	6.474×10^{-4}	1.65384×10^{-9}	
0.4	5.484×10^{-4}	9.38278×10^{-10}	
0.5	3.978×10^{-4}	2.44196×10^{-9}	13.546
0.6	2.229×10^{-4}	9.89533×10^{-10}	
0.7	5.111×10^{-5}	1.69246×10^{-9}	
0.8	1.619×10^{-4}	2.49459×10^{-9}	
0.9	3.037×10^{-4}	1.75614×10^{-9}	

Table 4. The AEs at various values of γ of Test Problem 2 when $N = 6$.

(z,t)	$\gamma = 1.2$	CPU time (seconds)	$\gamma = 1.4$	CPU time (seconds)	$\gamma = 1.6$	CPU time (seconds)
(0.1, 0.1)	3.92408×10^{-11}		3.89197×10^{-10}		9.53868×10^{-10}	
(0.2, 0.2)	2.38662×10^{-9}		2.39724×10^{-9}		2.19488×10^{-9}	
(0.3, 0.3)	3.85614×10^{-9}		4.17539×10^{-9}		2.81513×10^{-9}	
(0.4, 0.4)	3.60788×10^{-9}		3.58954×10^{-9}		3.66384×10^{-9}	
(0.5, 0.5)	1.74089×10^{-8}	11.656	1.71391×10^{-8}	12.359	1.81422×10^{-8}	13.142
(0.6, 0.6)	8.09673×10^{-9}		8.08484×10^{-9}		8.01134×10^{-9}	
(0.7, 0.7)	2.31155×10^{-8}		2.32862×10^{-8}		2.24449×10^{-8}	
(0.8, 0.8)	3.85016×10^{-8}		3.85138×10^{-8}		3.86699×10^{-8}	
(0.9, 0.9)	3.40007×10^{-8}		3.40222×10^{-8}		3.36636×10^{-8}	

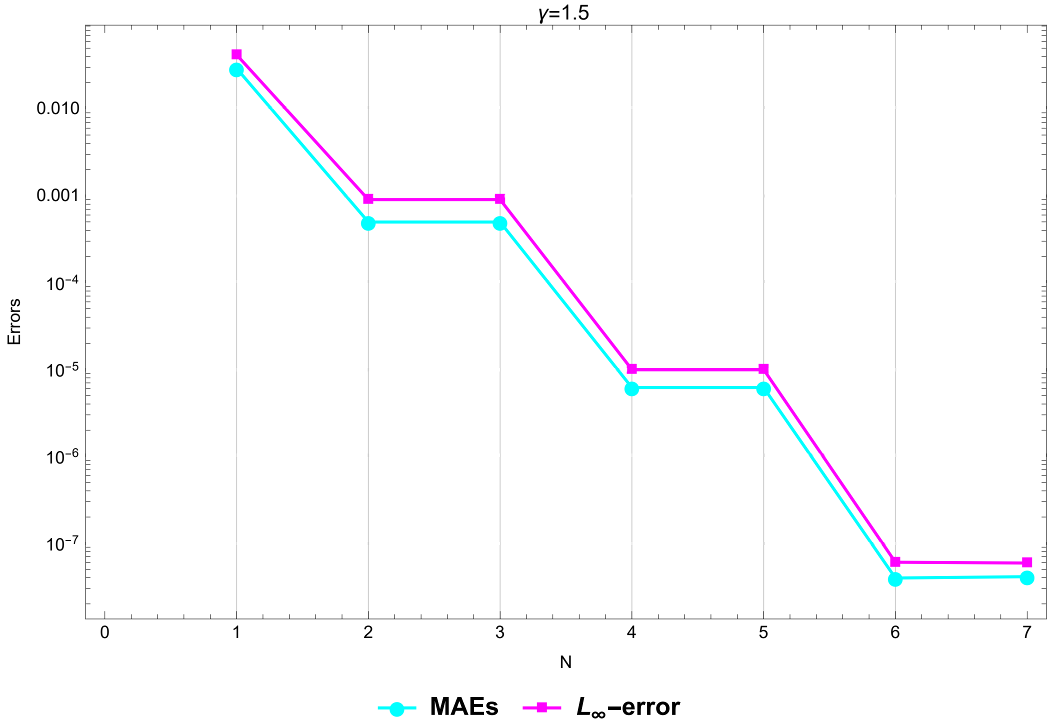

Figure 2. The errors of Test Problem 2 at various values of N when $\gamma = 1.5$.

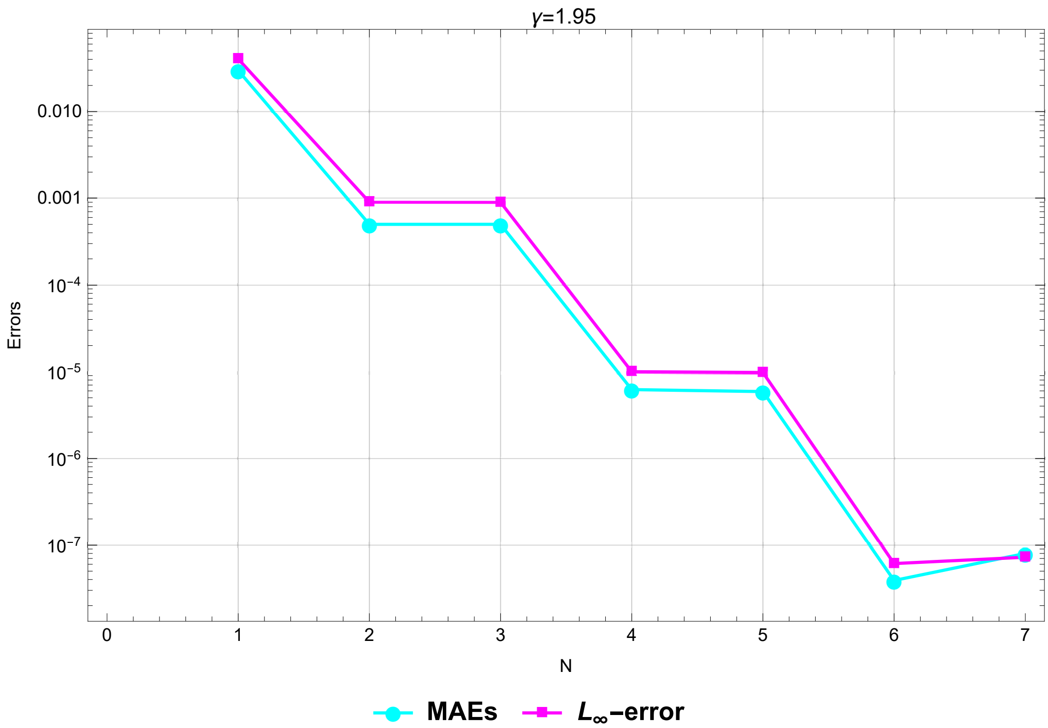

Figure 3. The errors of Test Problem 2 at various values of N when $\gamma = 1.95$.

Test Problem 3. *[67, 69] Consider the following* TFDWE*:*

$$\frac{\partial^{\gamma} u(z,t)}{\partial t^{\gamma}} + \frac{\partial u(z,t)}{\partial t} - \frac{\partial^2 u(z,t)}{\partial z^2} = f(z,t), \tag{6.10}$$

governed by the following constraints:

$$u(z,0) = u_t(z,0) = 0, \quad 0 \le z \le 1, \tag{6.11}$$

$$u(0,t) = t^3, \quad u(1,t) = t^3\, e, \quad 0 \le t \le 1, \tag{6.12}$$

and $f(z,t)$ is selected such that the exact solution is $u(z,t) = t^3\, e^z$.

The MAEs of our technique at $N = 4$ and of the methods in [67,69] are compared in Table 5. Table 6 presents the MAEs and L_∞-error at various values of N when $\gamma = 1.6$. The AEs at various values of γ when $N = 4$ are shown in Table 7. The table results show that the present method is efficient and accurate. Moreover, the CPU time for our method is presented in Table 7. Figure 4 illustrates the AEs at various values of N. Figure 5 shows the MAEs and L_∞-error at various values of N when $\gamma = 1.3$. The figure confirms that the proposed method efficiently decreases errors over the whole domain and demonstrates a strong correlation between the approximate and precise solutions.

Table 5. Comparison of MAEs at $\gamma = 1.85$ of Test Problem 3.

Method	Error
Method in [67] ($\mathcal{M} = 5$, $\bar{\mathcal{M}} = 2$)	1.25×10^{-7}
Method in [69] ($N = M = 8$)	2.64×10^{-5}
Method in [69] ($N = M = 16$)	1.25×10^{-6}
Our method at $N = 4$	5.13275×10^{-8}

Table 6. The errors of Test Problem 3 at various values of N when $\gamma = 1.6$.

N	1	2	3	4
MAEs	5.61411×10^{-4}	2.13193×10^{-5}	7.46835×10^{-7}	3.38471×10^{-8}
L_∞-error	7.39044×10^{-4}	2.99301×10^{-4}	1.56769×10^{-6}	5.02555×10^{-8}

Table 7. The AEs at various values of γ of Test Problem 3 when $N = 4$.

(z,t)	$\gamma = 1.1$	CPU time (seconds)	$\gamma = 1.5$	CPU time (seconds)	$\gamma = 1.9$	CPU time (seconds)
(0.1, 0.1)	2.59191×10^{-9}		1.78048×10^{-10}		3.66682×10^{-9}	
(0.2, 0.2)	1.46909×10^{-8}		5.0335×10^{-9}		2.45845×10^{-8}	
(0.3, 0.3)	1.74919×10^{-9}		7.77898×10^{-10}		3.79485×10^{-9}	
(0.4, 0.4)	2.64584×10^{-10}		5.44976×10^{-9}		5.35764×10^{-9}	
(0.5, 0.5)	1.07529×10^{-8}	4.265	5.10597×10^{-9}	4.484	2.06719×10^{-8}	4.016
(0.6, 0.6)	1.24458×10^{-8}		1.17496×10^{-8}		4.3198×10^{-9}	
(0.7, 0.7)	1.63205×10^{-8}		6.09226×10^{-9}		7.31176×10^{-9}	
(0.8, 0.8)	1.85922×10^{-8}		2.69138×10^{-8}		3.63235×10^{-8}	
(0.9, 0.9)	6.89892×10^{-9}		1.49644×10^{-8}		2.83055×10^{-8}	

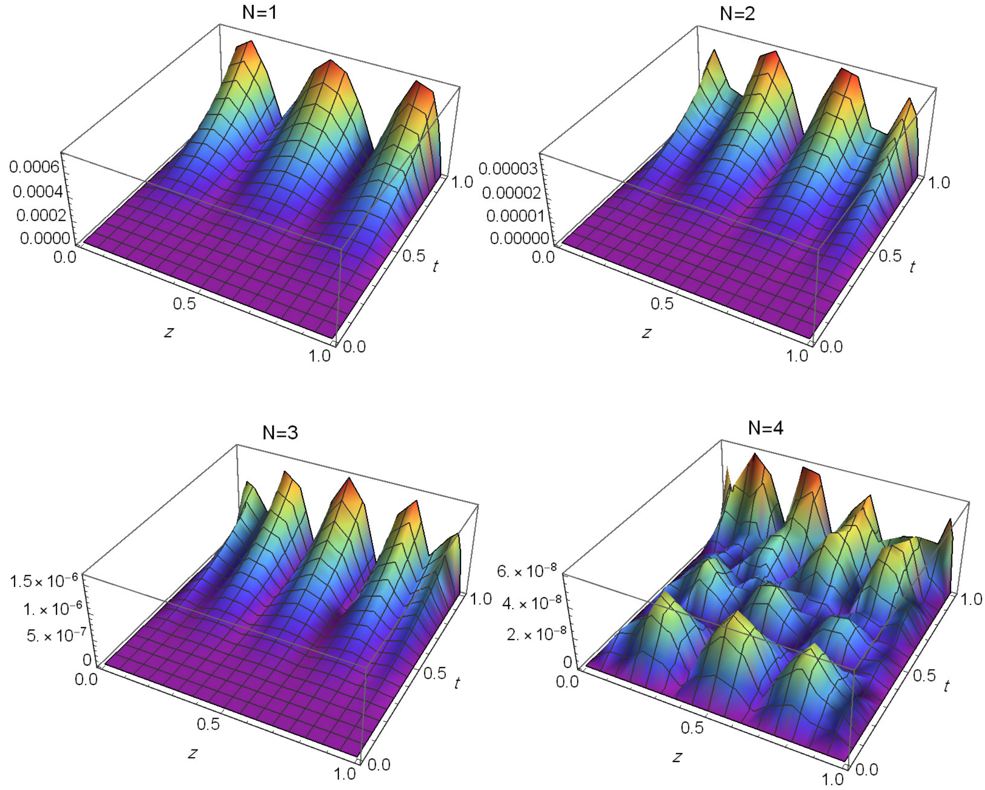

Figure 4. The AEs of Test Problem 3 at various values of N when $\gamma = 1.7$.

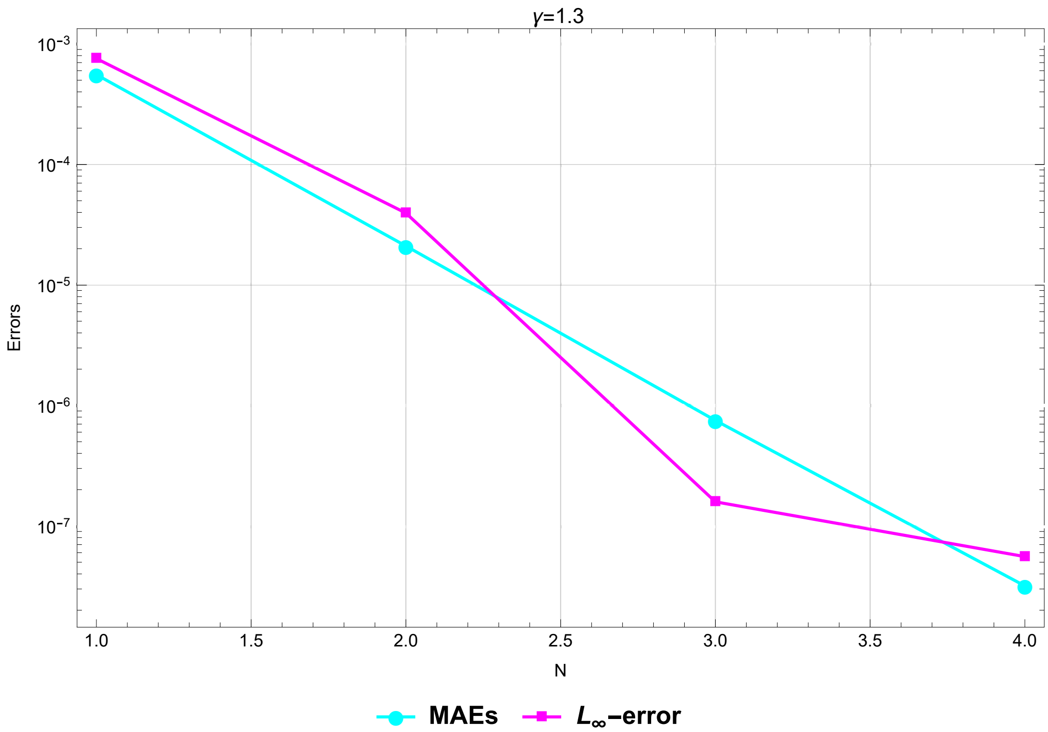

Figure 5. The errors of Test Problem 3 at various values of N when $\gamma = 1.3$.

Remark 8. *The sharp drop in L_∞ error at $N = 3$ in Figure 5, compared to MAEs, is due to faster reduction in the maximum pointwise error, indicating improved accuracy at specific points.*

Test Problem 4. *Consider the following* TFDWE*:*

$$\frac{\partial^\gamma u(z,t)}{\partial t^\gamma} + \frac{\partial u(z,t)}{\partial t} - \frac{\partial^2 u(z,t)}{\partial z^2} = f(z,t), \tag{6.13}$$

with the following conditions:

$$u(z,0) = u_t(z,0) = 0, \quad 0 \le z \le 1, \tag{6.14}$$

$$u(0,t) = u(1,t) = 0, \quad 0 \le t \le 1, \tag{6.15}$$

and $f(z,t)$ is selected such that the exact solution is $u(z,t) = t^\gamma z(1-z)$.

The AEs at various values of t when $\gamma = 1.8$ and $N = 8$ are shown in Table 8. Also, the AEs at various values of t when $\gamma = 1.2$ and $N = 9$ are shown in Table 9. The table results show that the present method is efficient and accurate. Figure 6 illustrates the MAEs at various values of N when $\gamma = 1.5$. This figure verifies that the suggested approach reduces errors consistently throughout the domain and shows a good agreement with the approximate solution.

Table 8. The AEs of Test Problem 4 at $\gamma = 1.8$.

z	$t = 0.1$	$t = 0.5$	$t = 0.9$
0.1	1.50471×10^{-6}	3.55139×10^{-5}	6.19824×10^{-6}
0.2	8.64845×10^{-6}	6.63774×10^{-5}	9.63653×10^{-6}
0.3	2.20929×10^{-5}	8.84147×10^{-5}	1.06229×10^{-5}
0.4	4.07402×10^{-5}	9.6838×10^{-5}	9.20425×10^{-6}
0.5	6.12705×10^{-5}	9.15213×10^{-5}	6.46939×10^{-6}
0.6	8.04591×10^{-5}	7.69198×10^{-5}	3.89078×10^{-6}
0.7	9.38894×10^{-5}	5.79232×10^{-5}	1.75942×10^{-6}
0.8	9.04752×10^{-5}	3.79341×10^{-5}	2.64986×10^{-7}
0.9	7.33972×10^{-5}	1.88164×10^{-5}	7.33614×10^{-8}

Table 9. The AEs of Test Problem 4 at $\gamma = 1.2$.

z	$t = 0.3$	$t = 0.6$	$t = 0.9$
0.1	6.112901×10^{-5}	1.23592×10^{-5}	1.59359×10^{-7}
0.2	1.158385×10^{-4}	2.28075×10^{-5}	3.25241×10^{-7}
0.3	1.57899×10^{-4}	2.96216×10^{-5}	9.13092×10^{-7}
0.4	1.86902×10^{-4}	3.44439×10^{-5}	1.17112×10^{-6}
0.5	1.94317×10^{-4}	3.31497×10^{-5}	1.23643×10^{-6}
0.6	1.82986×10^{-4}	2.81069×10^{-5}	2.10049×10^{-6}
0.7	1.55974×10^{-4}	2.12061×10^{-5}	4.59616×10^{-6}
0.8	1.09507×10^{-4}	9.68978×10^{-6}	2.47897×10^{-6}
0.9	5.44606×10^{-5}	4.67141×10^{-7}	1.92896×10^{-7}

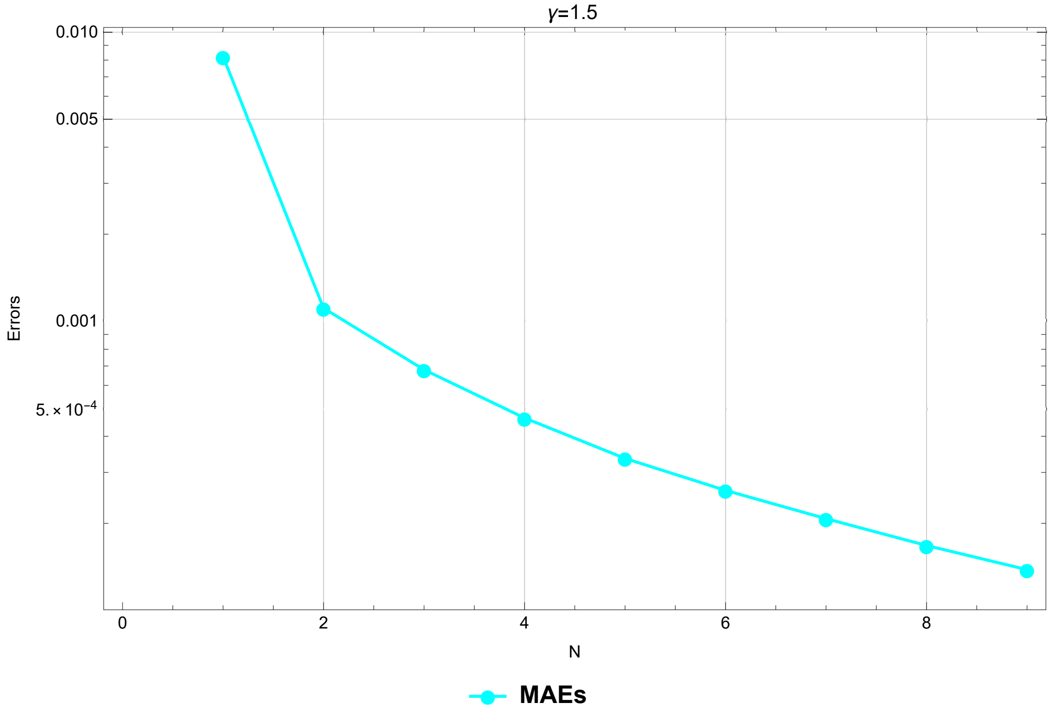

Figure 6. The MAEs of Test Problem 4 at various values of N when $\gamma = 1.5$.

Test Problem 5. *Consider the following* TFDWE*:*

$$\frac{\partial^{\gamma} u(z,t)}{\partial t^{\gamma}} + \frac{\partial u(z,t)}{\partial t} - \frac{\partial^2 u(z,t)}{\partial z^2} = f(z,t), \tag{6.16}$$

with the following conditions:

$$u(z,0) = u_t(z,0) = 0, \quad 0 \le z \le 10, \tag{6.17}$$

$$u(0,t) = u(10,t) = 0, \quad 0 \le t \le 10, \tag{6.18}$$

and $f(z,t)$ is selected such that the exact solution is $u(z,t) = t^2 z (10 - z)$.

Table 10. The AEs of Test Problem 5 at $\gamma = 1.5$.

z	$t = 3$	$t = 6$	$t = 9$
1	4.54747×10^{-13}	1.13687×10^{-12}	1.25056×10^{-12}
2	7.95808×10^{-13}	2.04636×10^{-12}	2.27374×10^{-12}
3	1.02318×10^{-12}	2.50111×10^{-12}	2.72848×10^{-12}
4	1.19371×10^{-12}	2.95586×10^{-12}	3.18323×10^{-12}
5	1.25056×10^{-12}	3.41061×10^{-12}	3.18323×10^{-12}
6	1.13687×10^{-12}	3.06954×10^{-12}	2.72848×10^{-12}
7	9.9476×10^{-13}	2.72848×10^{-12}	2.95586×10^{-12}
8	7.67386×10^{-13}	2.04636×10^{-12}	2.27374×10^{-12}
9	4.26326×10^{-13}	1.13687×10^{-12}	1.25056×10^{-12}

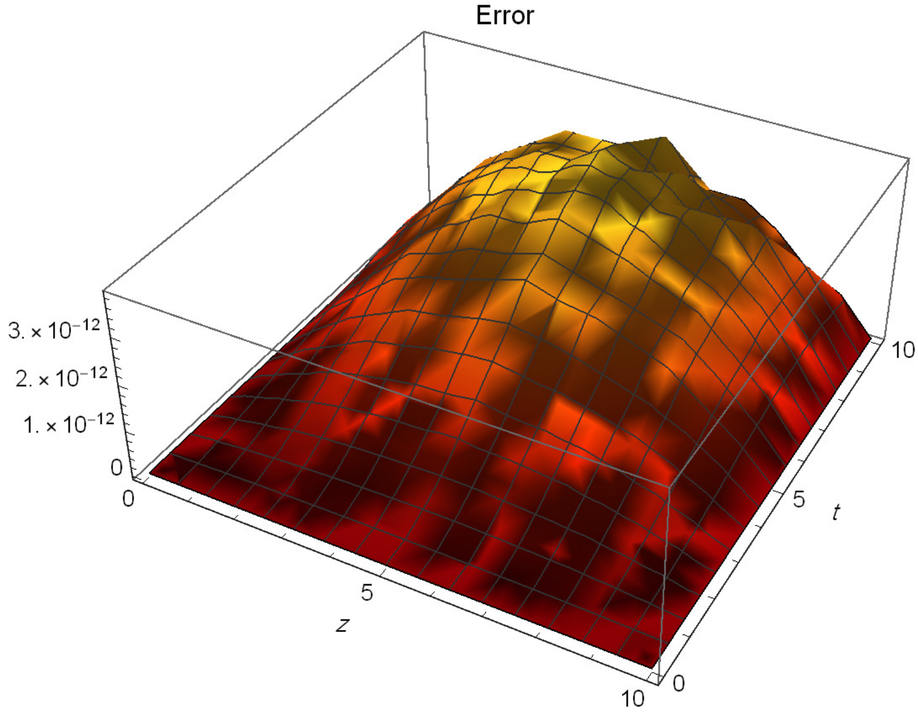

Figure 7. The AEs of Test Problem 5 at $N = 1$ when $\gamma = 1.5$.

The AEs at $\gamma = 1.5$ and $N = 1$ are shown in Figure 7. Also, the AEs at various values of t when $\gamma = 1.5$ and $N = 1$ are shown in Table 10. These results verify that the suggested approach reduces errors consistently throughout the domain and shows a good agreement with the approximate solution.

7. Concluding remarks

We have successfully proposed a spectral Galerkin method for the solution of the TFDWE. First, we proposed a suitable transformation that converts the TFDWE with its governing non-homogeneous conditions into a modified one governed by homogeneous conditions. A particular sequence of Horadam polynomials was used as a basis functions; based on which, the approximate solutions are expressed. The numerical experiments verified that the presented spectral scheme is accurate. As an expected future work, we think that the Galerkin approach presented in this paper can be extended to treat two-dimensional fractional wave models. Furthermore, some other generalized Horadam polynomial sequences will soon be a target to use as basis functions in forthcoming papers. We aim, in the near future, to use the collocation method along with the Horadam polynomial to handle more complicated and nonlinear problems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Author contributions

Waleed Mohamed Abd-Elhameed: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Writing—original draft preparation, Writing—review and editing, Supervision; Mohamed A. Abdelkawy: Methodology, Validation, Investigation, Funding acquisition; Naher Mohammed A. Alsafri: Methodology, Validation, Investigation; Ahmed Gamal Atta: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Writing—original draft preparation, Writing—review and editing. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare that they have no competing interests.

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