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Research article

On λ-biharmonic hypersurfaces in $L^m \times \mathbb{R}$

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Abstract: This paper investigated λ-biharmonic hypersurfaces in $L^m \times \mathbb{R}$, where L^m represents an Einstein space, and $\mathbb{R}$ denotes the real line. We demonstrated that such hypersurfaces with some curvature conditions are either of two types: minimal or vertical cylinders over λ-biharmonic hypersurfaces in L^m. Particularly, when the Einstein space L^m has constant sectional curvature, we classify λ-biharmonic hypersurfaces as totally umbilical or semi-parallel.

Keywords: λ-biharmonic hypersurfaces; product manifold; minimal; vertical cylinders

1. Introduction

Let (M^m, g) and (N^n, g^N) be Riemannian manifolds with dimensions m and n, and consider a smooth map $\varphi \in C^\infty(M^m, N^n)$; the energy functional $E(\varphi)$ and bienergy functional $E_2(\varphi)$ are given by

$$E(\varphi) := \frac{1}{2}\int_M |\mathrm{d}\varphi|^2 \mathrm{d}\, v_g \ \text{ and } \ E_2(\varphi) := \frac{1}{2}\int_M |\tau(\varphi)|^2 \mathrm{d}\, v_g,$$

where

$$\tau(\varphi) := \mathrm{tr}(\nabla \mathrm{d}\varphi)$$

is the tension field of φ. The critical points of $\mathrm{E}(\varphi)$ and $\mathrm{E}_2(\varphi)$ are harmonic and biharmonic maps, respectively, whose Euler-Lagrange equations are given by $\tau(\varphi) = 0$ and

$$\tau_2(\varphi) := \mathrm{tr}\left(\nabla^\varphi \nabla^\varphi \tau(\varphi) - \nabla^\varphi_\nabla \tau(\varphi)\right) - \mathrm{tr}\widetilde{\mathrm{R}}\big(\mathrm{d}\varphi, \tau(\varphi)\big)\mathrm{d}\varphi = 0,$$

where ∇^{φ} and ∇ are induced connections of the vector bundle $\varphi^{-1}\mathrm{TN}^{\mathrm{n}}$ and Levi-Civita connection on M^m, and $\tilde{R}$ is the curvature operator on N^n, which is given by

$$\tilde{R}(K,M)N = \nabla^{\varphi}{}_K\nabla^{\varphi}{}_M N - \nabla^{\varphi}{}_M\nabla^{\varphi}{}_K N - \nabla^{\varphi}{}_{[K,M]}N,$$

where K, M, N are any tangent vector fields of N^n. Let λ be a real constant, and the λ-biharmonic map is a critical point of

$$\mathrm{E}_{2,\lambda}(\varphi) \coloneqq \mathrm{E}_2(\varphi) + \lambda\mathrm{E}(\varphi).$$

Then the Euler-Lagrange equation of $E_{2,\lambda}(\varphi)$ is given by

$$\tau_2(\varphi) - \lambda\tau(\varphi) = 0.$$

We say that the submanifold M^m is λ-biharmonic if the λ-biharmonic map φ is an isometric immersion. The research of λ-biharmonic submanifolds contributes to the comprehension of finite-type submanifold theory, which is instrumental in analyzing the geometric properties of Riemannian manifolds with spectral behaviors. One can refer to [1] for more details.

Currently, research on λ-biharmonic submanifolds is mainly focused on the classification of hypersurfaces. The pioneering work on λ-biharmonic hypersurfaces was established by Chen [2] in 1988, showing that all such hypersurfaces in $\mathbb{R}^3$ are minimal or locally isometric to a circular cylinder. Then, Ferrández and Lucas studied the case in higher dimensions and demonstrated in [3] that λ-biharmonic hypersurfaces in $\mathbb{R}^{m+1}$ with no more than two different principal curvatures are minimal or locally isometric to $\mathbb{R}^k \times S^{m-k}(a)$. Subsequently, Chen and Garay [4] indicated that $\delta(2)$ ideal λ-biharmonic hypersurfaces in $\mathbb{R}^{m+1}$ are minimal or locally congruent to $\mathbb{R} \times S^{n-1}\left(\frac{\lambda}{n-1}\right)$. In non-flat space forms, L. Du considered λ-biharmonic hypersurfaces with no more than two different principal curvatures and provided a classification result [5]. Additionally, several papers [6–10] obtained that λ-biharmonic hypersurfaces have constant mean curvatures under some curvature conditions.

This paper focuses on the study of λ-biharmonic hypersurfaces in $L^m \times \mathbb{R}$, where L^m represents an Einstein space, and $\mathbb{R}$ denotes the real line. In Section 3, we discuss the component maps and equivalent equations of λ-biharmonic isometric immersions, which will be useful for our main results in Sections 4 and 5. In Section 4, we show that the λ-biharmonic hypersurfaces with constant mean curvature, constant angle function, or nonnegative Ricci curvature in $L^m \times \mathbb{R}$ are minimal or vertical cylinders in L^m. Furthermore, in Section 5, for the case that L^m has constant sectional curvature c, we provide classifications of these hypersurfaces as totally umbilical or semi-parallel.

2. Preliminaries

Let φ: $M^m \to (N^m \times \mathbb{R}, g^N + \mathrm{d}t^2)$ be an isometric immersion from the Riemannian hypersurface M^m to the product manifold $(N^m \times \mathbb{R},\ g^N + \mathrm{d}t^2)$ formed by the Riemannian manifold $(N^m,\ g^N)$ and the real line $(\mathbb{R}, \mathrm{d}t^2)$. The projection $\pi_2 \circ \varphi$ of φ onto the factor $\mathbb{R}$ is the height function of the hypersurface M^m, denoted by h. Let ξ, T, and α be defined as the unit normal vector of M^m, the tangent component of ∂_t, and the angle between ∂_t and ξ, respectively. Then, ∂_t can be expressed as

$$\partial_t = T + \cos(\alpha)\,\xi,$$

and the following equation holds (cf. [11])

$$\nabla h = T.$$

Let ∇ and A denote the Levi-Civita connection and the shape operator of M^m. Note that ∂_t is a parallel vector field in $N^m \times \mathbb{R}$; then, for any tangent vector field X of M^m, we have

$$\begin{cases} \nabla_X T = \cos(\alpha)\, AX, \\ X(\cos(\alpha)) = -\langle AX, T\rangle. \end{cases} \tag{1}$$

Let $\tilde{R}$, $\widetilde{\mathrm{Ric}}$, and $\tilde{S}$ be the Riemannian curvature, Ricci curvature, and scalar curvature of $N^m \times \mathbb{R}$, R, Ric, and S be the corresponding ones of M^m, and H be the mean curvature. Then, for any tangent vector fields X, Y of M^m,

$$\widetilde{\mathrm{Ric}}(X,Y) = \mathrm{Ric}(X,Y) + \langle AX, AY\rangle - mH\langle AX, Y\rangle + \tilde{R}(X,\xi,Y,\xi),$$

and

$$\tilde{S} = S + |A|^2 - m^2H^2 + 2\widetilde{\mathrm{Ric}}(\xi,\xi). \tag{2}$$

The Laplacian of $\theta \coloneqq \cos(\alpha)$ can be expressed as (cf. [11])

$$\Delta\theta = -m\langle \nabla H, \partial_t\rangle - \theta\left(|A|^2 + \widetilde{\mathrm{Ric}}(\xi,\xi)\right). \tag{3}$$

Let B represent the second fundamental form of M^m, then M^m is totally umbilical if

$$\langle B(X,Y), \xi\rangle = \langle X, Y\rangle H,$$

which is equivalent to $A(X) = HX$. The hypersurface M^m is termed semi-parallel if

$$B(R(X,Y)U,V) + B(U,R(X,Y)V) = 0,$$

where X, Y, U, V are any tangent vector fields of M^m.

Next, we define $L^m(c)$ as an m dimensional manifold with constant sectional curvature c. For any tangent vector fields $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ of $L^m(c) \times \mathbb{R}$, Riemannian curvature tensor $\tilde{R}$ of $L^m(c) \times \mathbb{R}$ can be expressed as (cf. [9])

$$\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z} = c\{\langle \tilde{Y},\tilde{Z}\rangle\tilde{X} - \langle \tilde{X},\tilde{Z}\rangle\tilde{Y} - \langle \tilde{Y},\partial_t\rangle\langle \tilde{Z},\partial_t\rangle\tilde{X} + \langle \tilde{X},\partial_t\rangle\langle \tilde{Z},\partial_t\rangle\tilde{Y}$$
$$+\langle \tilde{X},\tilde{Z}\rangle\langle \tilde{Y},\partial_t\rangle\partial_t - \langle \tilde{Y},\tilde{Z}\rangle\langle \tilde{X},\partial_t\rangle\partial_t\}. \tag{4}$$

For the case of $c = 1$, referencing [12], a rotational hypersurface M^m in $S^m \times \mathbb{R}$, parameterizing the profile curve as

$$\gamma(s) = \big(\cos(s), 0, \cdots, 0, \sin(s), q(s)\big),$$

where q is a smooth function, and M^m can be parameterized as

$$f(s, v_1, \cdots, v_{m-1}) = \big(\cos(s), \varphi_1(v_1,\cdots,v_{m-1})\sin(s), \cdots, \varphi_m(v_1,\cdots,v_{m-1})\sin(s), q(s)\big), \tag{5}$$

where $\varphi = (\varphi_1, \cdots \varphi_m)$ is an orthogonal parameterization of S^{m-1} in $\mathbb{R}^m$. Thus, M^m have two distinct principal curvatures

$$\kappa = -\frac{q''(s)}{(1+q'(s)^2)^{\frac{3}{2}}} \quad \text{and} \quad \nu = -\frac{q'(s)\cot(s)}{(1+q'(s)^2)^{\frac{1}{2}}},$$

with multiplicities 1 and $m-1$, respectively. For the rotational hypersurface M^m in the product space $H^m \times \mathbb{R}$, the profile curve can be parameterized as

$$\gamma(s) = (\cosh(s), 0, \cdots, 0, \sinh(s), q(s)),$$

M^m can be expressed as

$$f = (\varphi_1(v_1, \cdots, v_{m-1}) \cosh(s), \cdots, \varphi_m(v_1, \cdots, v_{m-1}) \cosh(s), \sinh(s), q(s)),$$

where $\varphi = (\varphi_1, \cdots \varphi_m)$ is an orthogonal parameterization of H^{m-1} in $\mathbb{R}^m$. The two distinct principal curvatures of M^m are

$$\kappa = -\frac{q''(s)}{(1+q'(s)^2)^{\frac{3}{2}}} \quad \text{and} \quad \nu = -\frac{q'(s)\tanh(s)}{(1+q'(s)^2)^{\frac{1}{2}}},$$

with multiplicities 1 and $m-1$, respectively.

3. Some lemmas

In this section, we study the isometric immersion $\varphi: (M^m, g) \to (N^m \times \mathbb{R}, g^N + \mathrm{d}t^2)$ and show that φ is λ-biharmonic precisely when its component maps $\pi_1 \circ \varphi$ and $\pi_2 \circ \varphi$ are λ-biharmonic (cf. Lemma 3.1). Furthermore, we explore the λ-biharmonic height function $h = \pi_2 \circ \varphi$ and derive an important equation (cf. Lemma 3.3) for the proof of our main theorems. Additionally, we obtain the equivalent equations (cf. Lemmas 3.4) for hypersurfaces to be λ-biharmonic.

Lemma 3.1. *Let $\varphi: (M^m, g) \to (N^m \times \mathbb{R}, g^N + dt^2)$ be an isometric immersion. Then, φ is λ-biharmonic precisely when $\pi_1 \circ \varphi: (M, g) \to (N, g^N)$ and $\pi_2 \circ \varphi: (M, g) \to (\mathbb{R}, dt^2)$ are both λ-biharmonic. Moreover, the height function $h = \pi_2 \circ \varphi$ is λ-biharmonic.*

Proof. Let ∇^{φ} be the induced connection of the vector bundle $\varphi^{-1}T(N^m \times \mathbb{R})$. Since

$$\mathrm{d}\varphi(X) = \mathrm{d}(\pi_1 \circ \varphi)(X) + \mathrm{d}(\pi_2 \circ \varphi)(X),$$

we know

$$\nabla_X^{\varphi}(\mathrm{d}\varphi(Y)) = \nabla_X^{\varphi}(\mathrm{d}(\pi_1 \circ \varphi))(Y) + \nabla_X^{\varphi}(\mathrm{d}(\pi_2 \circ \varphi))(Y).$$

Let $\{e_i\}_{i=1}^m$ be a local orthonormal frame, then

$$\tau(\varphi) = \sum_{i=1}^m \{\nabla_{e_i}^{\varphi}(\mathrm{d}\varphi(e_i)) - \mathrm{d}\varphi(\nabla_{e_i} e_i)\}$$

$$= \sum_{i=1}^m \{\nabla_{e_i}^{\pi_1 \circ \varphi}(\mathrm{d}(\pi_1 \circ \varphi)(e_i)) - \mathrm{d}(\pi_1 \circ \varphi)(\nabla_{e_i} e_i)\}$$

$$+\sum_{i=1}^{m}\{\nabla_{e_i}^{\pi_2\circ\varphi}d(\pi_2\circ\varphi)(e_i)-\mathrm{d}(\pi_2\circ\varphi)(\nabla_{e_i}e_i)\}$$
$$=\tau(\pi_1\circ\varphi)+\tau(\pi_2\circ\varphi).$$

Note that

$$\tau_2(\varphi)=-J^{\varphi}(\tau(\varphi)),$$

where

$$J^{\varphi}(X)\coloneqq-\{\mathrm{tr}(\nabla^{\varphi}\nabla^{\varphi}-\nabla_{\nabla}^{\varphi})X-\mathrm{tr}\tilde{R}(\mathrm{d}\varphi,X)\mathrm{d}\varphi\}.$$

We obtain

$$\tau_2(\varphi)=-J^{\varphi}(\tau(\varphi))$$
$$=-J^{\pi_1\circ\varphi}(\tau(\pi_1\circ\varphi))-J^{\pi_2\circ\varphi}(\tau(\pi_2\circ\varphi))$$
$$=\tau_2(\pi_1\circ\varphi)+\tau_2(\pi_2\circ\varphi).$$

Therefore, φ is λ-biharmonic, i.e., $\tau_2(\varphi)-\lambda\tau(\varphi)=0$ if and only if

$$\begin{cases}\tau_2(\pi_1\circ\varphi)-\lambda\tau(\pi_1\circ\varphi)=0,\\ \tau_2(\pi_2\circ\varphi)-\lambda\tau(\pi_2\circ\varphi)=0,\end{cases}$$

which means $\pi_1\circ\varphi$ and $\pi_2\circ\varphi$ are λ-biharmonic.

Lemma 3.2. *Let* $\varphi:(M^m,g)\to(N^m\times\mathbb{R},g^N+dt^2)$ *be an isometric immersion. Assuming that* $h=\pi_2\circ\varphi$ *is* λ*-biharmonic, then we have*

$$\Delta^2h=\lambda\Delta h.$$

Proof. Let $\{e_i\}_{i=1}^m$ be a local orthonormal frame on M^m. Then

$$\tau(h)=\mathrm{tr}(\nabla\mathrm{d}h)$$
$$=\nabla_{e_i}^{h}(\mathrm{d}h(e_i))-\mathrm{d}h(\nabla_{e_i}e_i)$$
$$=e_ie_ih-(\nabla_{e_i}e_i)h$$
$$=\Delta h.$$

Considering that the curvature operator $\bar{R}$ on $(\mathbb{R},\mathrm{d}t^2)$ is zero, we obtain

$$\tau_2(h)=\Delta\tau(h)-\mathrm{tr}\bar{R}(\mathrm{d}h,\tau(h)dh)=\Delta^2h.$$

Because h is λ-biharmonic, that is, $\tau_2(h)-\lambda\tau(h)=0$, then we have $\Delta^2h=\lambda\Delta h$.

Lemma 3.3. *Let* M^m *be a* λ*-biharmonic hypersurface in* $N^m\times\mathbb{R}$. *Then*

$$\Delta(H\theta)=\lambda H\theta. \tag{6}$$

Proof. We know from [11] that $\Delta h = m\theta H$. It follows that

$$\Delta^2 h = \Delta(\Delta h) = m\Delta(H\theta),$$

which, together with $\Delta^2 h = \lambda\Delta h$ (cf. Lemma 3.2), deduced our lemma.

Now, we provide the equivalent equations that characterize the hypersurface M^m as being λ-biharmonic.

Lemma 3.4. *Let $\varphi: M^m \to N^{m+1}$ be an isometric immersion. Then the hypersurface M^m is λ-biharmonic if*

$$\begin{cases} \Delta H - H|A|^2 + H\widetilde{Ric}(\xi,\xi) - \lambda H = 0, \\ 2A(\nabla H) + \frac{m}{2}\nabla H^2 - 2H\left(\widetilde{Ric}(\xi)\right)^{\top} = 0, \end{cases} \tag{7}$$

where $\langle \widetilde{Ric}(\cdot), \cdot \rangle = \widetilde{Ric}(\cdot, \cdot)$.

Proof. Let $\{e_i\}_{i=1}^m$ be a local orthonormal frame on M^m such that $\{d\varphi(e_1), \cdots, d\varphi(e_m), \xi\}$ forms an orthonormal basis of N^{m+1}. Combining with $\tau(\varphi) = mH\xi$, we have

$$\tau_2(\varphi) = \sum_{i=1}^{m} \{\nabla^{\varphi}_{e_i} \nabla^{\varphi}_{e_i}(mH\xi) - \nabla^{\varphi}_{\nabla_{e_i}e_i}(mH\xi) - \tilde{R}(\mathrm{d}\varphi(e_i), mH\xi)\mathrm{d}\varphi(e_i)\}$$

$$= m(\Delta H)\xi - 2mA(\nabla H) - mH\Delta^{\varphi}\xi - mH\sum_{i=1}^m \tilde{R}(\mathrm{d}\varphi(e_i), \xi)\mathrm{d}\varphi(e_i).$$

Since

$$\sum_{i=1}^m \langle \tilde{R}(\mathrm{d}\varphi(e_i), \tau(\varphi))\mathrm{d}\varphi(e_i), \xi \rangle = -mH\widetilde{\mathrm{Ric}}(\xi,\xi),$$

and

$$\langle \Delta^{\varphi}\xi, \xi \rangle = \sum_{i=1}^m \langle \nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\xi - \nabla^{\varphi}_{\nabla_{e_i}e_i}\xi, \xi \rangle$$

$$= \sum_{i=1}^m \langle \nabla^{\varphi}_{e_i}\xi, \nabla^{\varphi}_{e_i}\xi \rangle = |A|^2,$$

the normal part of $\tau_2(\varphi) - \lambda\tau(\varphi)$ is

$$\left(\tau_2(\varphi) - \lambda\tau(\varphi)\right)^{\perp} = \langle \tau_2(\varphi) - \lambda\tau(\varphi), \xi \rangle \xi$$

$$= m\left(\Delta H - H|A|^2 + H\widetilde{\mathrm{Ric}}(\xi,\xi) - \lambda H\right)\xi.$$

Considering

$$\sum_{k=1}^m \langle \Delta^{\varphi}\xi, e_k \rangle e_k = \sum_{i,k=1}^m \left\langle \nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\xi - \nabla^{\varphi}_{\nabla_{e_i}e_i}\xi, e_k \right\rangle e_k$$

$$= m(\nabla H) - \left(\widetilde{\mathrm{Ric}}(\xi)\right)^{\top},$$

and

$$\sum_{i,k=1}^{m} \langle \tilde{R}(\mathrm{d}\varphi(e_i), \xi)\mathrm{d}\varphi(e_i), e_k\rangle e_k = -\sum_{k=1}^{m} \left[\widetilde{\mathrm{Ric}}(\xi, e_k)\right] e_k = -\left(\widetilde{\mathrm{Ric}}(\xi)\right)^{\top},$$

then the tangent part of $\tau_2(\varphi) - \lambda\tau(\varphi)$ is given by

$$\left(\tau_2(\varphi) - \lambda\tau(\varphi)\right)^{\top} = \langle \tau_2(\varphi) - \lambda\tau(\varphi), e_k\rangle e_k$$

$$= m\left(-2A(\nabla H) - \frac{m}{2}(\nabla H^2) + 2H\left(\widetilde{\mathrm{Ric}}(\xi)\right)^{\top}\right).$$

Therefore, M^m is λ-biharmonic precisely when the tangent and normal parts of $\tau_2(\varphi) - \lambda\tau(\varphi)$ are both zero, i.e., Eq (7) holds.

Remark 3.5. *When ambient space is $L^m(c) \times \mathbb{R}$, we can rephrase λ-biharmonic Eq (7) into the following form with angle function α by combining Eq (4).*

$$\begin{cases} \Delta H - H(|A|^2 - c(m-1)\,sin^2(\alpha) + \lambda) = 0, \\ A(\nabla H) + \frac{m}{2} H \nabla H + c(m-1)\,cos(\alpha)\,HT = 0. \end{cases}$$

Lemma 3.6. *([13]) (Yau's maximum principle)*

(i) Let M^m be a Riemannian manifold whose Ricci curvature is nonnegative and u be a smooth function. If $u > 0$ and $\Delta u = 0$, then u is a constant;

(ii) Let M^m be a Riemannian manifold and u be a smooth function. If $u \geq 0$ and $\Delta u \geq 0$, then $\int_M u^p = +\infty$ for $p > 1$, unless u is a constant.

Lemma 3.7. *([14]) Let $u \in (0, C](C > 0)$ be a solution to the differential inequality $\Delta u \leq 0$ on a complete noncompact manifold M^m. If*

$$\int_M \left(log^{(k)} \frac{Ce^{(k)}}{u}\right)^p dv_g < +\infty, for\ some\ p > 0, k \in N,$$

where $log^{(k)} = log\left(log^{(k-1)}\right)$ and $e^{(k)} = e^{e^{(k-1)}}$, $log^{(1)} = log$ and $e^{(1)} = e$. Then u is a constant.

4. λ-biharmonic hypersurface in $L^m \times \mathbb{R}$

This section discusses λ-biharmonic hypersurfaces in $L^m \times \mathbb{R}$ and proves that such hypersurfaces are either minimal or vertical cylinders under some curvature conditions. First, we consider hypersurfaces with constant mean curvature.

Theorem 4.1. *Let L^m be an Einstein space with $Ric^L = \mu g^L$. Then, every λ-biharmonic hypersurface $(\lambda \geq 0)$ M^m in $L^m \times \mathbb{R}$, whose mean curvature is a constant, is minimal or a vertical cylinder in L^m.*

Proof. Assume $H \neq 0$, Eq (6) can be reduced to

$$\Delta\theta = \lambda\theta. \tag{8}$$

As demonstrated in [11], we have

$$\widetilde{\mathrm{Ric}}(\xi, \xi) = \mu(1 - \theta^2). \tag{9}$$

Substituting Eq (9) into (3), we find

$$\Delta\theta = -\theta\big(|A|^2 + \mu(1 - \theta^2)\big).$$

By incorporating Eq (9) into the first equation of Eq (7), we obtain

$$|A|^2 = \mu(1 - \theta^2) - \lambda. \tag{10}$$

Considering that H is a constant, combining the two above equations leads to

$$\Delta\theta = -\theta(2|A|^2 + \lambda). \tag{11}$$

Combining Eqs (8) and (11), we arrive at $\theta(|A|^2 + \lambda) = 0$. Since $|A|^2 > 0$ and $\lambda \geq 0$, we obtain $\theta \equiv 0$. Combining with $\theta = \cos(\alpha) = \langle \partial_t, \xi \rangle$, we deduce that ∂_t is perpendicular to ξ, which means that ∂_t is tangent to M^m. Thus, M^m is a vertical cylinder $P^{m-1} \times \mathbb{R}$. Let φ be the isometric immersion of the hypersurface M^m. According to Lemma 3.1, we infer that the component map $\pi_1 \circ \varphi$ of φ is λ-biharmonic, indicating that P^{m-1} is a λ-biharmonic hypersurface in L^m.

When the scalar curvature of the Einstein manifold L^m is nonzero, we can remove the condition $\lambda \geq 0$ in Theorem 4.1 and obtain the following theorem:

Theorem 4.2. *Let L^m be an Einstein space with $Ric^L = \mu g^L$ and $\mu \neq 0$. Then, every λ-biharmonic hypersurface M^m in $L^m \times \mathbb{R}$, whose mean curvature is a constant, is minimal or a vertical cylinder in L^m.*

Proof. When $\lambda \geq 0$, Theorem 4.1 has given the proof. When $\lambda < 0$ and $H \neq 0$, we also have Eqs (8) and (10) and $\theta(|A|^2 + \lambda) = 0$. Suppose $\theta \neq 0$ on some open subset U, then $|A|^2 + \lambda \equiv 0$ on U. Considering that $\mu \neq 0$, then from Eq (10) we have $\theta^2 \equiv 1$ on U and $\Delta\theta = 0$ at some point. Compared with Eq (8), we conclude that $\theta = 0$ at some point, a contradiction. So, $\theta \equiv 0$, which implies that ∂_t is tangent to the hypersurface M^m. The conclusion has been proved.

For non-minimal λ-biharmonic hypersurfaces in $L^m \times \mathbb{R}$ having constant mean curvature, there is a relation between λ and μ as follows:

Proposition 4.3. *Let L^m be an Einstein space with $Ric^L = \mu g^L$ and $\mu \neq 0$. Then, every non-minimal λ-biharmonic hypersurface M^m in $L^m \times \mathbb{R}$ having constant mean curvature satisfies $\lambda = \mu - |A|^2$. Moreover, we have*

$$\lambda < \mu.$$

Proof. From Theorem 4.2, M^m is the vertical cylinder $P^{m-1} \times \mathbb{R}$. Denote the mean curvatures of $P^{m-1} \times \mathbb{R}$ and P^{m-1} by H and H_0, respectively; then, we have $H_0 = \frac{m}{m-1} H$ from [15]. Since H is a nonzero constant, from Lemma 3.4 we have the equation

$$\Delta H_0 - H_0 |A_0|^2 = (\lambda - \mu) H_0,$$

where A_0 is the shape operator of M^{m-1}, which can be reduced to

$$\lambda = \mu - |A_0|^2.$$

By a straightforward computation, we have $|A|^2 = |A_0|^2$, then $\lambda = \mu - |A|^2$.

Especially when the ambient space is the product space $L^m(c) \times \mathbb{R}$, we can provide more specific classification results for $m = 3$. Combining Theorem 4.2 with the classification results [5] for non-minimal λ-biharmonic surfaces of $L^3(c)$, $c = 1$ or -1, we have the following theorem:

Theorem 4.4. *Let* (M^3, g) *be a non-minimal* λ*-biharmonic hypersurface of* $L^3(c) \times \mathbb{R}$ *with constant mean curvature. If* $c = 1$*, then* M^3 *is a vertical cylinder* $P^2 \times \mathbb{R}$*, where* P^2 *is locally isometric to*

(i) $\mathbb{S}^2\left(\frac{4-\lambda}{2}\right)$ *with* $\lambda < 2$*,*

(ii) or $\mathbb{S}^1\left(\frac{4-\lambda+\sqrt{\lambda^2-4\lambda}}{2}\right) \times \mathbb{S}^1\left(\frac{4-\lambda-\sqrt{\lambda^2-4\lambda}}{2}\right)$ *with* $\lambda < 0$.

If $c = -1$*, then* M^3 *is a vertical cylinder* $P^2 \times \mathbb{R}$*, where* P^2 *is locally isometric to*

(iii) $\mathbb{R}^2$ *with* $\lambda = -4$*,*

(iv) or $\mathbb{S}^2\left(\frac{-4-\lambda}{2}\right)$ *with* $\lambda < -4$*,*

(v) or $\mathbb{H}^2\left(\frac{-4-\lambda}{2}\right)$*with* $-4 < \lambda < -2$*,*

(vi) or $\mathbb{S}^1\left(\frac{-4-\lambda+\sqrt{\lambda^2+4\lambda}}{2}\right) \times \mathbb{H}^1\left(\frac{-4-\lambda-\sqrt{\lambda^2+4\lambda}}{2}\right)$ *with* $\lambda < -4$.

Remark 4.5. *From Theorem 4.4, it can be observed that the inequality* $\lambda < \mu$ *in Proposition 4.3 is optimal. When* $\lambda = 0$*, the result of Theorem 4.4 is coincident with the result in [15]. When* $c = 1$*, the result of Theorem 4.4 is coincident with the result in [16].*

We now explore the λ-biharmonic hypersurfaces characterized by a constant angle function.

Theorem 4.6. *Let* L^m *be an Einstein space and* M^m *a* λ*-biharmonic hypersurface* $(\lambda \geq 0)$ *in* $L^m \times \mathbb{R}$*. If* M^m *has constant angle function* α *and* $H \geq 0$*,* $H \in L^p(M)$ *for some* $p > 1$*, then* M^m *is minimal or a vertical cylinder in* L^m.

Proof. We know that θ is a constant because of $\theta = \cos(\alpha)$. If $\theta = 0$, M^m is a vertical cylinder in L^m. If $\theta \neq 0$, we derive$\Delta H = \lambda H$ by (6), and from $\lambda \geq 0$ and $H \geq 0$, we have$\Delta H \geq 0$. Additionally, with $H \in L^p(M)$ for some $p > 1$, Lemma 3.6 leads us to know that H is a constant, and the result follows from Theorem 4.1.

Theorem 4.7. *Let* L^m *be an Einstein space. If a* λ*-biharmonic hypersurface* M^m *in* $L^m \times \mathbb{R}$ *is totally umbilical and* α *is a constant, then* M^m *is minimal or a vertical cylinder in* L^m.

Proof. For $\lambda = 0$, the result has been proved in [17]. For $\lambda \neq 0$, set $\{e_i\}_{i=1}^m$ to be a local orthonormal frame on M^m, then $A(e_i) = He_i$, $|A|^2 = mH^2$. Substitute Eq (9) into the first equation of Eq (7), and we obtain

$$\Delta H - mH^3 + H\mu(1 - \theta^2) - \lambda H = 0. \tag{12}$$

Since α is a constant, we know that $\theta = \cos(\alpha)$ is a constant. When $\theta = 0$, it means that ∂_t is tangent to the hypersurface M^m, then M^m is a vertical cylinder in L^m.

When $\theta \neq 0$, Eq (6) tells us that $\Delta H = \lambda H$. So, Eq (12) can be reduced to

$$mH^3 = H\mu(1 - \theta^2).$$

Combined with $\Delta H = \lambda H$, we have $H = 0$, i.e., M^m is minimal.

Next, we discuss the case of λ-biharmonic hypersurfaces with nonnegative Ricci curvature in $L^m \times \mathbb{R}$.

Theorem 4.8. *Let M^m be a λ-biharmonic $(\lambda \geq 0)$ hypersurface in $L^m \times \mathbb{R}$ with $Ric \geq 0$. If H is harmonic and satisfies $H \geq -C$ for some positive C, then M^m is minimal or a vertical cylinder in L^m.*

Proof. Define $u := H + C + 1 > 0$, from $\Delta H = 0, \Delta u = 0$. By Lemma 3.6, $H = u - C - 1$ is a constant. The result follows from Theorem 4.2.

Theorem 4.9. *Let M^m be a λ-biharmonic hypersurface in $L^m \times \mathbb{R}$. If M^m has nonnegative Ricci curvature and*

$$\int_M H^{2p}\, dv_g < +\infty,$$

$$\int_M \left(log^{(k)} \frac{e^{(k)}}{\theta^2 + \varepsilon}\right)^q dv_g < +\infty, \tag{13}$$

for some $p > 1$, $q > 0$, $k \in N$, and $\varepsilon > 0$. Then M^m is minimal or a vertical cylinder in L^m.

Proof. If $\lambda = 0$, one can refer to [17]. If $\lambda \neq 0$, we have from Lemma 3.3 that

$$\Delta(H\theta)^2 = 2|\nabla(H\theta)|^2 + 2\lambda(H\theta)^2 \geq 0.$$

Since $-1 \leq \theta = \cos(\alpha) \leq 1$, it follows that

$$\int_M (H\theta)^{2p}\, dv_g \leq \int_M H^{2p}\, dv_g < +\infty, \text{ for some } p > 1.$$

By Lemma 3.6, it leads to the conclusion that $(H\theta)^2$ is a constant, that is, $H\theta$ is a constant.

Given that $\Delta h = mH\theta$ (cf. [11]) and $H\theta$ is a constant, the Ricci identity

$$\Delta\nabla_i h = \nabla_i \Delta h + (\mathrm{Ric})_{ij} \nabla_j h$$

can be simplified as

$$\Delta\nabla_i h = (\mathrm{Ric})_{ij} \nabla_j h.$$

Because $\nabla h = T$ and due to the nonnegativity of Ricci curvature, we deduce from the above equation that

$$\langle \Delta T, T\rangle = \mathrm{Ric}(T,T) \geq 0.$$

Hence,

$$\frac{1}{2}\Delta|T|^2 = |\nabla T|^2 + \langle \Delta T, T\rangle = |\nabla T|^2 + \mathrm{Ric}(T,T) \geq 0. \tag{14}$$

With $\partial_t = T + \theta$, we obtain $\langle T, T\rangle = 1 - \theta^2$. Thus,

$$\frac{1}{2}\Delta|T|^2 = \frac{1}{2}\Delta(1-\theta^2) = -\frac{1}{2}\Delta\left(\frac{1}{2} + \theta^2\right). \tag{15}$$

By combining Eqs (14) and (15), we can deduce

$$\Delta\left(\theta^2 + \frac{1}{2}\right) \leq 0.$$

Hence, $\theta^2 + \frac{1}{2}$ is a superharmonic function. Notably, $\theta^2 + \frac{1}{2}$ also satisfies Eq (13). By Lemma 3.7, we conclude that $\theta^2 + \frac{1}{2}$ is a constant, that is, θ is a constant, which implies that H is a constant. It follows from Eq (6) that $H = 0$ or $\theta = 0$. So, M^m is either minimal or a vertical cylinder $P^{m-1} \times \mathbb{R}$. From Lemma 3.1, we infer that P^{m-1} is a λ-biharmonic hypersurface.

Theorem 4.10. *Let M^m be a λ-biharmonic hypersurface in $L^m \times \mathbb{R}$ with $Ric \geq 0$. If S is a constant and θ is harmonic, then M^m is minimal or a vertical cylinder in L^m.*

Proof. The case of $\lambda = 0$ follows by [17]. For $\lambda \neq 0$, set $u = \theta + 2$, then $\Delta u = \Delta\theta = 0$ and $u > 0$ by $-1 \leq \theta \leq 1$. By Lemma 3.6, u and θ are constants. If $\theta = 0$, M^m is a vertical cylinder.

If $\theta \neq 0$, we have $\Delta H = \lambda H$. Substituting this into the first equation of Eq (7), combining with $\widetilde{\mathrm{Ric}}(\xi,\xi) = \mu(1-\theta^2)$, we obtain

$$H\left(|A|^2 - \mu(1-\theta^2)\right) = 0.$$

If $H \neq 0$ at some point $p \in M$, then

$$|A|^2 = \mu(1-\theta^2).$$

Consequently, Eq (2) becomes $S = \tilde{S} - 3\mu(1-\theta^2) + m^2H^2$, where $\tilde{S} = m\mu$. This yields $\nabla S = 2m^2H\nabla H$. Because S is a constant, we have $\nabla S = 0$, which together with $2m^2H\nabla H = \nabla S$ leads to H being a constant. A contradiction arises from $\Delta H = \lambda H$, that is, $H \equiv 0$.

Remark 4.11. *The condition $\lambda \geq 0$ in Theorem 4.6 can be replaced by $\mu \neq 0$ or reduced to $|A|^2 + \lambda \neq 0$. When $\lambda = 0$, the results of Theorems 4.6–4.10 degenerate into some results in [17].*

5. λ −biharmonic hypersurface in $L^m(c) \times \mathbb{R}$

This section classifies λ-biharmonic hypersurfaces as either totally umbilical or semi-parallel when the ambient space is $L^m(c) \times \mathbb{R}$. It is noteworthy that a totally umbilical hypersurface in $L^m(c) \times \mathbb{R}$ may not be semi-parallel (cf. [17,18]).

Theorem 5.1. *If the λ-biharmonic hypersurface M^m in $L^m(c)\times\mathbb{R}$ is totally umbilical, then M^m must be minimal.*

Proof. When $\lambda=0$, the result follows from [17]. When $\lambda\neq 0$, since M^m is totally umbilical, from Remark 3.5 we have

$$\begin{cases}\Delta H - H(mH^2 - c(m-1)\sin^2(\alpha)+\lambda)=0,\\ \frac{m+2}{2}H\nabla H + c(m-1)\cos(\alpha)\,HT=0.\end{cases} \tag{16}$$

Assuming that $H\neq 0$. If $\sin(\alpha)\equiv 0$, $\theta=\cos(\alpha)=\pm 1$. Taking $X=\nabla H$ in the second equation of Eq (1), combining with $A(\nabla H)=H\nabla H$ leads to $|\nabla H|=0$, that is, H is a constant. Then, Eq (6) tells us that $\lambda\theta=\Delta\theta=0$, a contradiction. If $\sin(\alpha)\neq 0$, at some point $p\in M^m$. We find $|T|^2=\sin^2(\alpha)$ from $T=\partial_t-\cos(\alpha)$. Choose a local orthonormal frame $\{e_i\}_{i=1}^m$ such that

$$T=\sin(\alpha)\,e_1. \tag{17}$$

Inserting $\nabla H=\sum_{i=1}^m e_i(H)e_i$ and Eq (17) into the second equation of Eq (16) yields

$$e_2(H)=e_3(H)=\cdots=e_m(H)=0,$$

and

$$e_1(H)=-\frac{2c(m-1)}{m+2}\sin(\alpha)\cos(\alpha). \tag{18}$$

Applying the first equation of Eq (1), for $2\le j\le m$, we have

$$H\cos(\alpha)=\langle\cos(\alpha)\,Ae_j,e_j\rangle=\langle\nabla e_jT,\,e_j\rangle$$

$$=e_j\langle T,e_j\rangle-\langle T,\nabla_{e_j}e_j\rangle=-\sin(\alpha)\langle e_1,\nabla_{e_j}e_j\rangle,$$

which implies that

$$\left\langle e_1,\nabla_{e_j}e_j\right\rangle=-H\cot(\alpha),$$

and

$$\begin{aligned}\Delta H&=e_1e_1(H)-\sum_i\left(\nabla_{e_i}e_i\right)(H)\\&=e_1e_1(H)-(m-1)\cot(\alpha)\,He_1(H).\end{aligned} \tag{19}$$

Setting $X=e_1$ in the second equation of Eq (1), using $T=\sin(\alpha)\,e_1$, we obtain

$$H=e_1(\alpha). \tag{20}$$

By differentiating Eq (18) along e_1, we get

$$e_1e_1(H)=-\frac{2c(m-1)}{m+2}\cos(2\alpha)\,H. \tag{21}$$

Substituting Eqs (18), (20) and (21) into (19), gives

$$\Delta H = \frac{2c(m-1)}{m+2}\{(m-1)\cos^2(\alpha) - \cos(2\alpha)\} e_1(\alpha).$$

Using Eq (20) and the above equation, from Eq (16) we deduce

$$e_1(\alpha)Q(\alpha) = 0, \tag{22}$$

where

$$Q(\alpha) = \frac{2c(m-1)}{m+2}\{\cos(2\alpha) - (m-1)\cos^2(\alpha)\} - c(m-1)\sin^2(\alpha) + m\big(e_1(\alpha)\big)^2 + \lambda.$$

If $e_1(\alpha) \equiv 0$, then $H \equiv 0$, a contradiction. If $e_1(\alpha) \neq 0$, then Eq (22) becomes $Q(\alpha) = 0$. By differentiating this equation along e_1 and using Sine-Gordon equation (cf. [18])

$$e_1 e_1(2\alpha) + c\sin(2\alpha) = 0,$$

we obtain

$$\frac{4(m-1) - 2(m-1)^2}{m+2}(2m-1)2c\sin(\alpha)\cos(\alpha) = 0,$$

that is,

$$(11m-8)\sin(2\alpha) = 0.$$

This leads to $\sin(2\alpha) \equiv 0$. Thus, α is a constant and $e_1(\alpha) = 0$, a contradiction.

Remark 5.2. *Compared with Theorem 4.7, we delete the condition "constant angle" in Theorem 5.1, but require the ambient space to be* $L^m(c) \times \mathbb{R}$, *and obtain the more correct conclusion.*

In the following, we classify semi-parallel λ-biharmonic hypersurfaces. For semi-parallel hypersurfaces of $S^m \times \mathbb{R}$ and $H^m \times \mathbb{R}$, the following classification results are provided in [18,19].

Lemma 5.3. *([18]) Let* M^m *be a semi-parallel hypersurface of* $H^m \times \mathbb{R}$ *(*$m \geq 3$*), then there are three possibilities:*

(i) M^m *is totally umbilical.*

(ii) M^m *is locally congruent to a rotation hypersurface, whose profile curve* γ *is given by one of the following:*

$$\gamma(s) = (\,cosh\,(s), 0, \cdots, 0, sinh\,(s), \int \sqrt{Ccosh^2(s) - 1}\, ds),$$

$$\gamma(s) = (\,cosh\,(s), 0, \cdots, 0, sinh\,(s), \int \sqrt{Csinh^2(s) - 1}\, ds),$$

$$\gamma(s) = \left(s, 0, \cdots, -\frac{1}{2s}, \int \sqrt{C - \frac{1}{2s^2}}\, ds\right).$$

(iii) $M^m \subseteq \widetilde{M}^{m-1} \times \mathbb{R}$, *where* $\widetilde{M}^{m-1}$ *is a semi-parallel hypersurface of* H^m.

Lemma 5.4. *([19]) Let M^m be a semi-parallel hypersurface of $S^m \times \mathbb{R}$ ($m \geq 3$), then there are three possibilities:*

(i) M^m is totally umbilical

(ii) M^m is locally congruent to a rotation hypersurface, whose profile curve is a vertical line, or can be parameterized as

$$\gamma(s) = \left(co\, s(s)\, , 0, \cdots, 0, si\, n(s)\, , \int \sqrt{Ccos^2(s) - 1} ds\right).$$

(iii) $M^m \subset \widetilde{M}^{m-1} \times R$, where $\widetilde{M}^{m-1}$ is a semi-parallel hypersurface of S^m.

Let M^m be a semi-parallel λ-biharmonic hypersurface of $S^m \times \mathbb{R}$ or $H^m \times \mathbb{R}$. Now we discuss the three cases of Lemmas 5.3 and 5.4. For case (i), that is, M^m is totally umbilical, we know M^m is minimal by Theorem 5.1. For case (iii), Lemma 3.1 leads to the conclusion that M^m is a vertical cylinder in S^m or H^m. When M^m is locally congruent to a rotation hypersurface, as in case (ii), a contradiction can be derived by employing the following lemma.

Lemma 5.5. *Let M^m be a λ-biharmonic rotation hypersurface in $L^m(c) \times \mathbb{R}$ ($c = 1$ or -1), then*

$$\left(\frac{m}{2} H - \alpha' \, cos(\alpha)\right) H' - c(m-1) \, sin(\alpha) \, H = 0. \tag{23}$$

Proof. For $c = 1$, according to Eq (5) and [12], there exists a local orthonormal frame $\{e_i\}_{i=1}^m$ such that

$$e_1 = \frac{1}{\sqrt{1+(q')^2}} df(\partial_s), \quad e_i = \frac{1}{\sqrt{\sin^2(s)\left(\sum_{k=1}^m \frac{\partial \varphi_k}{\partial v_i}\right)}} df(\partial_i), \; 2 \leq i \leq m,$$

and

$$\xi = \frac{1}{\sqrt{1+(q')^2}} (-q' \sin(s)\, , \varphi_1 q' \cos(s)\, , \cdots, \varphi_m q' \cos(s)\, , -1),$$

such that

$$\cos(\alpha) = \langle \xi, \partial_t \rangle = -\frac{1}{\sqrt{1+(q')^2}},$$

$$\sin(\alpha) = \frac{q'}{\sqrt{1+(q')^2}}.$$

Then

$$A(e_1) = \kappa e_1, \; A(e_i) = \nu e_i, \; 2 \leq i \leq m, \tag{24}$$

where

$$\kappa = -\frac{q''}{(1+(q')^2)^{\frac{3}{2}}} = -\alpha' \cos(\alpha),$$

$$\nu = -\frac{q' \cot(s)}{(1+(q')^2)^{\frac{1}{2}}} = -\sin(\alpha)\cot(s).$$

Thus,

$$T = \partial_t - \cos(\alpha)\,\xi = \sin(\alpha)\, e_1, \tag{25}$$

and

$$H = -\frac{1}{m}(\alpha' \cos(\alpha) + (m-1)\cot(s)\sin(\alpha)),$$

which leads to

$$\nabla H = e_1(H)e_1 = -\cos(\alpha)\, H' e_1. \tag{26}$$

By substituting Eqs (25) and (26) into the second equation of Eq (3.3), and combining Eq (24),

$$\left(\frac{m}{2}H - \alpha'(s)\cos(\alpha)\right)H' = c(m-1)\sin(\alpha)\, H.$$

The case $c = -1$ is similar.

Theorem 5.6. *If* M^m *is a semi-parallel* λ*-biharmonic hypersurface in* $L^m(c) \times \mathbb{R}$ $(m \geq 3,\ c = 1, -1)$*, then* M^m *must be minimal or a vertical cylinder in* $L^m(c)$*.*

Proof. We deal with the case $c = 1$. The discussion for the case $c = -1$ is similar. Let M^m be a λ-biharmonic semi-parallel hypersurface of $S^m \times \mathbb{R}$, and there are three cases (i)–(iii) as given by Lemma 5.4. For the cases (i) and (iii), we can derive the conclusion by using Theorem 5.1 and Lemma 3.1. Now, let us focus on case (ii).

From [19], we know that $\kappa\nu = -\cos^2(\alpha)$, where κ and ν are two different principal curvatures. Moreover, from the proof of Lemma 5.5, we have $\kappa\nu = \alpha' \sin(\alpha)\cos(\alpha)\cot(s)$. It follows that

$$-\cos^2(\alpha) = \alpha' \sin(\alpha)\cos(\alpha)\cot(s).$$

By setting $u = -\sin(\alpha)$, we can simplify the above equation to

$$uu' \cot(s) = u^2 - 1.$$

Solving this equation, we have $u = \pm\sqrt{1 + C\sec^2(s)}$ for some real constant C. However, this solution does not meet the criteria in Eq (23), leading to a contradiction.

Remark 5.7. *For semi-parallel biharmonic hypersurface (i.e.,* $\lambda = 0$*), it must be minimal when the ambient space is* $H^m \times \mathbb{R}$ *(cf. [17]).*

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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