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Research article

Generic properties of Darboux-type curves

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Abstract: In this paper, we present a method for constructing Darboux-type curves by using spherical curves and smooth functions. We show that all Darboux-type curves can be generated using this method. We discuss the relationship between Darboux-type curves and rectifying curves. Moreover, we establish a connection between Darboux-type curves and geodesics on a rectifying developable surface from the viewpoint of curves on ruled surfaces.

Keywords: Darboux-type curves; spherical curves; rectifying curves; rectifying developable surface; geodesics

1. Introduction

As one of the important objects of study in classical differential geometry, curves always attract extensive attention from mathematicians and physicists. The Darboux vector is the angular velocity vector of the Frenet frame of a space curve, which was proposed by Gaston Darboux [1]. When an object moves along a regular curve, using the Frenet frame for space curves, the motion of the object can be described by two vectors: the translation vector and the rotation vector, which is also known as the Darboux vector.

Darboux vectors, a concept familiar to both mathematicians and physicists, have been studied widely in a variety of fields. For example, in [2], Düldül defined new vector fields in Euclidean 4-space and demonstrated that the newly defined planes act as Darboux vectors. In [3], Wang and Pei defined the Darboux vector of the null curve and explained the direction of rotation of the Cartan frame in 3-dimensional Minkowski space. Ekici and Dede [4] studied the Darboux vector of ruled surfaces in pseudo-Galilean space. They observed that in a pseudo-Galilean space the Darboux vector can be interpreted kinematically as a shear along the absolute line. In theoretical kinematics, Bukcu and Karacan [5] defined the Bishop motion of a space curve in Euclidean 3-space and showed that the

Bishop Darboux rotation for the space curve is decomposed into two simultaneous rotations. They proved that one of the axes of these simultaneous rotations is a parallel of the Bishop Darboux vector of the curve. Moreover, Darboux vectors can describe the behavior of kinematics. The centrode is a curve with the Darboux vector as its position vector [6], and it plays an important role in joint kinematics [7–9]. It is well known that Bertrand curves and Mannheim curves are special curves defined using principal normal vectors and binormal vectors. Inspired by these definitions, Yu et al. [10] defined a new type of special curve based on Darboux vectors as follows: A curve $\boldsymbol{\gamma}$ is called a Darboux-type curve if there exists another curve $\boldsymbol{\gamma}_D$ such that the Darboux lines of $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}_D$ are parallel to each other at corresponding points. Subsequently, Qian et al. [11] used Darboux vectors to define the null Darboux mate curves of a null curve in the Minkowski 3-space. Compared with Bertrand curves and Mannheim curves, whose geometric properties are already widely understood by mathematicians, the properties of Darboux-type curves need to be further studied.

In generic differential geometry, the sphere is one kind of model submanifolds with standard properties in 3-dimensional Euclidean space $\mathbb{R}^3$. Therefore, the spherical curve can be used as a standard curve for studying special curves. Mathematicians have given methods for constructing special curves using spherical curves in $\mathbb{R}^3$. These methods can be used to create cylindrical helices [12], Bertrand curves [13], Mannheim curves [14], and rectifying curves [6, 15]. On the other hand, we can study curves from the perspective of curves on ruled surfaces. In [16], Izumiya and Takeuchi investigated Bertrand curves in $\mathbb{R}^3$. They stated the relationship between Bertrand curves and minimal asymptotic curves on ruled surfaces.

This paper is inspired by the above ideas to study Darboux-type curves in 3-dimensional Euclidean space. We offer an approach for creating Darboux-type curves from spherical curves. Furthermore, any Darboux-type curve can be generated by using this way. We prove that rectifying curves are Darboux-typed curves. The spherical Darboux image of a Darboux-type curve is also shown to be equivalent to the spherical evolute of the associated spherical curve. We study Darboux-type curves and rectifying developable surfaces from the viewpoint of curves on ruled surfaces and obtain that two disjoint geodesics on a developable surface are Darboux curve pairs.

In Section 2, we recall fundamental concepts and properties of space curves. We describe the notion of Darboux-type curves and study geometric properties of these curves in Section 3. A method for building Darboux-type curves using spherical curves was provided in Section 4. In Section 5, we investigate Darboux-type curves from the perspective of curves on ruled surfaces. In Section 6, we give an example of Darboux-type curves.

All manifolds and maps considered here are of class C^∞ unless otherwise stated.

2. Preliminaries

We now recall the basic concepts of curves in Euclidean space, which are discussed in detail in [17]. Let $\mathbb{R}^3$ be Euclidean 3-space with standard inner product $\boldsymbol{a} \cdot \boldsymbol{b}$, for any vector $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$ in $\mathbb{R}^3$. We define the norm of $\boldsymbol{a}$ by $\|\boldsymbol{a}\|$. Let I be an open interval. Suppose that $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ is a regular curve (*i.e.*, $\dot{\boldsymbol{\gamma}} = d\boldsymbol{\gamma}/dt \neq 0$). The arc-length of $\boldsymbol{\gamma}$, measured from $\boldsymbol{\gamma}(t_0)$ $(t_0 \in I)$, is $s(t) = \int_{t_0}^{t} \|\dot{\boldsymbol{\gamma}}\| dt$. So we call s the *arc-length parameter* of $\boldsymbol{\gamma}$, if it satisfies the condition $\|\boldsymbol{\gamma}'(s)\| = 1$, where $\boldsymbol{\gamma}'(s) = d\boldsymbol{\gamma}/ds$. Throughout this paper, the arc-length of $\boldsymbol{\gamma}$ is denoted by s. We define the *unit tangent vector* of $\boldsymbol{\gamma}$ at s by $\boldsymbol{T}(s) = \boldsymbol{\gamma}'(s)$. We denote the *curvature* of $\boldsymbol{\gamma}$ by $\kappa(s) = \|\boldsymbol{\gamma}''(s)\|$. If $\kappa(s) \neq 0$, we call $\boldsymbol{N}(s) = \boldsymbol{\gamma}''(s)/\|\boldsymbol{\gamma}''(s)\|$ a *unit principal normal vector* of $\boldsymbol{\gamma}$ at s. We set $\boldsymbol{B}(s) = \boldsymbol{T}(s) \times \boldsymbol{N}(s)$ and call $\boldsymbol{B}(s)$ a *unit binormal vector* of $\boldsymbol{\gamma}$ at s. Then the following Frenet-Serret formulae hold:

$$\begin{cases} \boldsymbol{T}'(s) = \kappa(s)\boldsymbol{N}(s), \\ \boldsymbol{N}'(s) = -\kappa(s)\boldsymbol{T}(s) + \tau(s)\boldsymbol{B}(s), \\ \boldsymbol{B}'(s) = -\tau(s)\boldsymbol{N}(s), \end{cases}$$

where $\tau(s)$ is the torsion of $\boldsymbol{\gamma}$ at s. For a regular curve $\boldsymbol{\gamma} : I \to \mathbb{R}^3$, we call $\boldsymbol{D}(s) = \tau(s)\boldsymbol{T}(s) + \kappa(s)\boldsymbol{B}(s)$ the *Darboux vector* of $\boldsymbol{\gamma}$. One defines the normalization of the Darboux vector as

$$\boldsymbol{d}(s) = \frac{\tau(s)\boldsymbol{T}(s) + \kappa(s)\boldsymbol{B}(s)}{\sqrt{\tau^2(s) + \kappa^2(s)}},$$

which is called the *spherical Darboux image* of $\boldsymbol{\gamma}$. If $\kappa(s) \neq 0$, the vector $\widetilde{\boldsymbol{D}}(s) = \boldsymbol{D}(s)/\kappa(s)$ is called the *modified Darboux vector* of $\boldsymbol{\gamma}$. Let t be a general parameter of the regular curve $\boldsymbol{\gamma}$; then the curvature and torsion of $\boldsymbol{\gamma}$ can be determined as follows:

$$\kappa(t) = \frac{\|\dot{\boldsymbol{\gamma}}(t) \times \ddot{\boldsymbol{\gamma}}(t)\|}{(\dot{\boldsymbol{\gamma}}(t) \cdot \dot{\boldsymbol{\gamma}}(t))^{\frac{3}{2}}}, \qquad \tau(t) = \frac{det(\dot{\boldsymbol{\gamma}}(t), \ddot{\boldsymbol{\gamma}}(t), \dddot{\boldsymbol{\gamma}}(t))}{\|\dot{\boldsymbol{\gamma}}(t) \times \ddot{\boldsymbol{\gamma}}(t)\|^2}.$$

A regular curve $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ with $\kappa(s) \neq 0$ is called a *cylindrical helix* if the tangent lines of $\boldsymbol{\gamma}$ make a constant angle with a fixed direction. The curve $\boldsymbol{\gamma}$ is known to be a cylindrical helix if and only if $(\tau/\kappa)(s) = \text{constant}$. A regular curve $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ with $\kappa(s) \neq 0$ is said to be a *rectifying curve* if the position vector of $\boldsymbol{\gamma}$ always lies in the rectifying plane of $\boldsymbol{\gamma}$. A regular curve $\boldsymbol{\gamma}(s)$ is regarded as a rectifying curve if and only if there exist constants $a \neq 0$ and b such that $(\tau/\kappa)(s) = as + b$ for any $s \in I$.

3. Darboux-type curves

By using Darboux vectors, Yu et al. defined a new special curve in $\mathbb{R}^3$ as follows:

Defination 3.1. [10] Let $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}_D : I \to \mathbb{R}^3$ be different regular curves with Darboux vectors $\boldsymbol{D}(t)$ and $\boldsymbol{D}_D(t)$, respectively. We call $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}_D$ *Darboux curve pairs* if there is a smooth function $\omega : I \to \mathbb{R}$ satisfying $\boldsymbol{\gamma}_D(t) = \boldsymbol{\gamma}(t) + \omega(t)\boldsymbol{D}(t)$ and $\boldsymbol{D}(t) = \pm\boldsymbol{D}_D(t)$ for any $t \in I$.

The regular curve $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ is called a *Darboux-type curve* if there is another regular curve $\boldsymbol{\gamma}_D : I \to \mathbb{R}^3$ such that $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}_D$ are Darboux curve pairs; we also call $\boldsymbol{\gamma}_D$ the *Darboux mate curve* of $\boldsymbol{\gamma}$. In [10], the authors gave some geometrical properties of Darboux curve pairs as follows:

Theorem 3.2. *The Darboux lines of two regular curves are parallel if and only if their principal normal lines are parallel to each other.*

Theorem 3.3. *If the principal normal lines of two regular curves are parallel to each other, then the tangent lines of these two curves make a constant angle at the corresponding points.*

By Theorems 3.2 and 3.3, we know that if the Darboux lines of two curves are parallel to each other, then their tangent lines make a constant angle at the corresponding points.

For two regular curves $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}_D$, suppose that at corresponding points their principal normal lines are parallel and their tangent lines form an angle ϕ. By Theorem 3.3, ϕ is a constant. Next, we present the following formulae:

$$\begin{cases} \boldsymbol{T}_D = \cos\phi \boldsymbol{T} + \sin\phi \boldsymbol{B}, \\ \boldsymbol{N}_D = \varepsilon \boldsymbol{N}, \\ \boldsymbol{B}_D = -\varepsilon \sin\phi \boldsymbol{T} + \varepsilon \cos\phi \boldsymbol{B}, \end{cases} \tag{3.1}$$

where $\varepsilon = \pm 1$. Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ be a Darboux-type curve and

$$\boldsymbol{\gamma}_D(s) = \boldsymbol{\gamma}(s) + \omega(s)\boldsymbol{D}(s) \tag{3.2}$$

be a Darboux mate curve of $\boldsymbol{\gamma}$. Assume that the arc-length parameters of $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}_D$ are s and $\bar{s}$, respectively. Differentiating (3.2) with respect to s, we obtain

$$\frac{d\bar{s}}{ds}\boldsymbol{T}_D(s) = (1 + (\omega\tau)'(s))\boldsymbol{T}(s) + (\omega\kappa)'(s)\boldsymbol{B}(s).$$

From (3.1), we obtain

$$\begin{cases} \dfrac{d\bar{s}}{ds}\cos\phi = 1 + (\omega\tau)'(s), \\ \\ \dfrac{d\bar{s}}{ds}\sin\phi = (\omega\kappa)'(s), \\ \\ \dfrac{1 + (\omega\tau)'(s)}{(\omega\kappa)'(s)} = \cot\phi, \end{cases}$$

where $\phi \in (0, \pi)$ is a constant. Thus, Yu et al. obtained the following conclusion in [10].

Theorem 3.4. *Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ be a regular space curve with arc-length parameter s. The curve $\boldsymbol{\gamma}$ is a Darboux-type curve if and only if there exists a smooth function $\omega(s)$ and a constant $\phi \in (0, \pi)$ such that*

$$\frac{1 + (\omega\tau)'(s)}{(\omega\kappa)'(s)} = \cot\phi$$

for any $s \in I$.

We get the following corollary.

Corollary 3.5. *A regular space curve $\boldsymbol{\gamma}$ is a Darboux-type curve if and only if there exists a smooth function $\omega(s)$, two constant numbers C and $\phi \in (0, \pi)$ such that* $\cot\phi\omega(\mathrm{s})\kappa(\mathrm{s}) - \omega(\mathrm{s})\tau(\mathrm{s}) = \mathrm{s}+C$ *for any $s \in I$.*

Proof. By Theorem 3.4, the sufficient and necessary condition for $\boldsymbol{\gamma}$ to be a Darboux-type curve is that there exists a smooth function $\omega(s)$ and a constant $\phi \in (0, \pi)$ such that $[1 + (\omega\tau)'(s)]/(\omega\kappa)'(s) = \cot\phi$. The above equation is equivalent to the condition that $(\cot\phi\omega\kappa - \omega\tau)'(\mathrm{s}) = 1$. By integrating both sides of the final equation, we get $\cot\phi\omega(\mathrm{s})\kappa(\mathrm{s}) - \omega(\mathrm{s})\tau(\mathrm{s}) = \mathrm{s}+C$.

□

In [10], the authors have proved that cylindrical helices are Darboux-type curves and the Darboux mate curve of a cylindrical helix is also a cylindrical helix. Using the geometric properties of rectifying curves, we derive the following theorem.

Theorem 3.6. *Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ be a rectifying curve with curvature $\kappa \neq 0$. Then $\boldsymbol{\gamma}$ is a Darboux-type curve, and the Darboux mate curve of $\boldsymbol{\gamma}$ is also a rectifying curve.*

Proof. We define a curve

$$\boldsymbol{\gamma}_D(s) = \boldsymbol{\gamma}(s) + \omega(s)\widetilde{\boldsymbol{D}}(s), \tag{3.3}$$

where $\omega(s)$ is a nonzero constant and $\widetilde{\boldsymbol{D}}(s)$ is the modified Darboux vector. Suppose that $\bar{s}$ is the arc-length parameter of $\boldsymbol{\gamma}_D$. A straightforward calculation, we obtain

$$\frac{d\boldsymbol{\gamma}_D}{ds} = \frac{d\bar{s}}{ds}\boldsymbol{T}_D(\bar{s}(s)) = (1 + \omega(\frac{\tau}{\kappa})'(s))\boldsymbol{T}(s). \tag{3.4}$$

We now assume that $d\bar{s}/ds = 1 + \omega(\tau/\kappa)'(s)$. Given that $\boldsymbol{\gamma}(s)$ is a rectifying curve, we deduce $d\bar{s}/ds$ as a constant. Differentiating (3.4) with respect to s, we get

$$\frac{d^2\boldsymbol{\gamma}_D}{ds^2} = (\frac{d\bar{s}}{ds})^2\kappa_D(\bar{s}(s))\boldsymbol{N}_D(\bar{s}(s)) = (1 + \omega(\frac{\tau}{\kappa})'(s))\kappa(s)\boldsymbol{N}(s). \tag{3.5}$$

This means that $\boldsymbol{N}_D$ and $\boldsymbol{N}$ are parallel to each other. By Theorem 3.2, we know that $\boldsymbol{\gamma}$ is a Darboux-type curve and $\boldsymbol{\gamma}_D$ is the Darboux mate curve of $\boldsymbol{\gamma}$. Continuing the calculations, we have

$$\frac{d^3\boldsymbol{\gamma}_D}{ds^3} = (1 + \omega(\frac{\tau}{\kappa})'(s))(-\kappa^2(s)\boldsymbol{T}(s) + \kappa'(s)\boldsymbol{N}(s) + \kappa(s)\tau(s)\boldsymbol{B}(s)).$$

Thus, according to the formulae for curvature and torsion under general parameters, we compute as follows:

$$\kappa_D(s) = \frac{\kappa(s)}{1 + \omega(s)(\frac{\tau}{\kappa})'(s)}, \qquad \tau_D(s) = \frac{\tau(s)}{1 + \omega(s)(\frac{\tau}{\kappa})'(s)}.$$

These formulae show that $\tau_D/\kappa_D = \tau/\kappa$, which implies that $\boldsymbol{\gamma}_D$ is a rectifying curve. □

For a unit speed curve $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ with $\kappa > 0$ and $\tau \neq 0$, we call the curve denoted by $\widetilde{\boldsymbol{D}}(s) = (\tau/\kappa)(s)\boldsymbol{T}(s) + \boldsymbol{B}(s)$ the *dilated centrode* of $\boldsymbol{\gamma}$. Chen et al. have shown that if $\boldsymbol{\gamma}(s)$ is neither a helix nor a planar curve, then $\widetilde{\boldsymbol{D}}(s)$ is a rectifying curve in [18]. From this, we can conclude the following corollary.

Corollary 3.7. *Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ be a curve with $\kappa(s) > 0$ and $\tau(s) \neq 0$. Assume that $\boldsymbol{\gamma}(s)$ is not a helix. Then the dilated centrode $\boldsymbol{\beta}(s)$ is a Darboux-type curve.*

By straightforward calculations, it can be deduced that the position vector of dilated centrode $\boldsymbol{\beta}(s)$ and the modified Darboux vector of $\boldsymbol{\beta}(s)$ are parallel to each other. Hence, the Darboux mate curve $\overline{\boldsymbol{\beta}}(s)$ of dilated centrode $\boldsymbol{\beta}(s)$ satisfies $\overline{\boldsymbol{\beta}}(s) = f(s)\boldsymbol{\beta}(s)$, where $f(s)$ is a differential function.

4. Darboux-type curves and Spherical curves

In this section, we show the method of generating Darboux-type curves using spherical curves. Let $\widetilde{\boldsymbol{\alpha}} : I \to S^2$ be a unit speed spherical curve parameterized by arc-length u. We call $\widetilde{\boldsymbol{t}}(u) = d\widetilde{\boldsymbol{\alpha}}/du$ a unit tangent vector of $\widetilde{\boldsymbol{\alpha}}$ at u. Then we set a vector $\widetilde{\boldsymbol{n}}(u) = \widetilde{\boldsymbol{\alpha}}(u) \times \widetilde{\boldsymbol{t}}(u)$. The orthonormal frame

$\{\widetilde{\boldsymbol{\alpha}}(u), \widetilde{\boldsymbol{t}}(u), \widetilde{\boldsymbol{n}}(u)\}$ along $\widetilde{\boldsymbol{\alpha}}$ is called the *Sabban frame* of $\widetilde{\boldsymbol{\alpha}}$ (see [19]). Then the following spherical Frenet-Serret formulae hold:

$$\begin{cases} \widetilde{\boldsymbol{\alpha}}'(u) = \widetilde{\boldsymbol{t}}(u), \\ \widetilde{\boldsymbol{t}}'(u) = -\widetilde{\boldsymbol{\alpha}}(u) + k_g(u)\widetilde{\boldsymbol{n}}(u), \\ \widetilde{\boldsymbol{n}}'(u) = -k_g(u)\widetilde{\boldsymbol{t}}(u), \end{cases}$$

where $k_g(u)$ is the *geodesic curvature* of the curve $\widetilde{\boldsymbol{\alpha}}(u)$ on S^2.

We now define a curve

$$\boldsymbol{\gamma}(u) = \cot\phi \int_{u_0}^{u} \frac{\omega(\sigma)}{\omega_1(\sigma)}\widetilde{\boldsymbol{\alpha}}(\sigma)d\sigma - \int_{u_0}^{u} \frac{\omega(\sigma)}{\omega_1(\sigma)}\widetilde{\boldsymbol{n}}(\sigma)d\sigma + \mathbf{C},$$

where ω is a smooth function, $\mathbf{C}$ is a constant vector , $\omega_1(\sigma) = \sigma + C$, C and $\phi \in (0, \pi)$ are constant numbers. From this defined curve, we obtain the following important theorem.

Theorem 4.1. *According to the above notation, $\boldsymbol{\gamma}$ is a Darboux-type curve. In addition, all Darboux-type curves can be created in this way.*

Proof. By direct calculations, we obtain that

$$\begin{aligned} \dot{\boldsymbol{\gamma}}(u) &= \frac{\omega(u)}{\omega_1(u)}(\cot\phi\widetilde{\boldsymbol{\alpha}}(u) - \widetilde{\boldsymbol{n}}(u)), \\ \ddot{\boldsymbol{\gamma}}(u) &= \frac{d}{du}(\frac{\omega}{\omega_1})(u)(\cot\phi\widetilde{\boldsymbol{\alpha}}(u) - \widetilde{\boldsymbol{n}}(u)) + \frac{\omega(u)}{\omega_1(u)}(\cot\phi + k_g(u))\widetilde{\boldsymbol{t}}(u), \\ \dddot{\boldsymbol{\gamma}}(u) &= [\frac{d^2}{du^2}(\frac{\omega}{\omega_1})(u)\cot\phi - \frac{\omega(u)}{\omega_1(u)}(\cot\phi + k_g(u))]\widetilde{\boldsymbol{\alpha}}(u) + [2\frac{d}{du}(\frac{\omega}{\omega_1})(u)(\cot\phi + k_g(u)) \\ &\quad + \frac{\omega(u)}{\omega_1(u)}k_g'(u)]\widetilde{\boldsymbol{t}}(u) + [-\frac{d^2}{du^2}(\frac{\omega}{\omega_1})(u) + \frac{\omega(u)}{\omega_1(u)}(\cot\phi + k_g(u))k_g(u)]\widetilde{\boldsymbol{n}}(u). \end{aligned}$$

Thus, according to the formulae for curvature and torsion under general parameters, we can compute as follows:

$$\kappa(u) = \sin^2\phi\frac{\omega_1(u)}{\omega(u)}|\cot\phi + k_g(u)|, \qquad \tau(u) = \sin^2\phi\frac{\omega_1(u)}{\omega(u)}(\cot\phi k_g(u) - 1).$$

Case (1): $\cot\phi + k_g(u) > 0$. In this case, we obtain

$$\cot\phi\omega(u)\kappa(u) - \omega(u)\tau(u) = \omega_1(u).$$

By Corollary 3.5, it implies that $\boldsymbol{\gamma}$ is a Darboux-type curve.

Case (2): $\cot\phi + k_g(u) < 0$. In this case we obtain

$$\cot\phi\omega(u)\kappa(u) + \omega(u)\tau(u) = -\omega_1(u).$$

Let $\phi_1 = -\phi$, the above equation can be written as

$$\cot\phi_1\omega(u)\kappa(u) - \omega(u)\tau(u) = \omega_1(u).$$

Since $\cot(\pi+\phi_1)=\cot\phi$ and $\pi+\phi_1\in(0,\pi)$, we get

$$\cot(\pi+\phi_1)\omega(u)\kappa(u)-\omega(u)\tau(u)=\omega_1(u).$$

In case (2), it also implies that $\boldsymbol{\gamma}$ is a Darboux-type curve.

Conversely, let $\boldsymbol{\gamma}$ be a Darboux-type curve with arc-length parameter s. We assume that $\{\boldsymbol{T},\boldsymbol{N},\boldsymbol{B}\}$ is the Frenet frame of $\boldsymbol{\gamma}$. There is a smooth function ω and a constant number ϕ such that

$$\frac{1+(\omega\tau)'}{(\omega\kappa)'}=\cot\phi.$$

Using the Frenet frame of $\boldsymbol{\gamma}$, we define a spherical curve $\widetilde{\boldsymbol{\alpha}}$ with arc-length parameter u written as

$$\widetilde{\boldsymbol{\alpha}}(s)=\cos\phi\boldsymbol{T}(s)+\sin\phi\boldsymbol{B}(s).$$

Then we obtain

$$\frac{d\widetilde{\boldsymbol{\alpha}}}{ds}=\sin\phi(\cot\phi\kappa(s)-\tau(s))\boldsymbol{N}(s)=\frac{du}{ds}\boldsymbol{N}(s).$$

Assuming that $\omega(s)$ is a nonzero smooth function, we get

$$\frac{du}{ds}=\frac{\sin\phi}{\omega(s)}(\cot\phi\omega(s)\kappa(s)-\omega(s)\tau(s))=\frac{\omega_1(s)}{\omega(s)}\sin\phi,$$

where $\omega_1(s)=s+C$ and C is a constant number.

By direct calculations, we obtain

$$\frac{\omega(s)}{\omega_1(s)}\frac{du}{ds}\cot\phi\widetilde{\boldsymbol{\alpha}}(s)=\cos\phi(\cos\phi\boldsymbol{T}(s)+\sin\phi\boldsymbol{B}(s))$$

and

$$\frac{\omega(s)}{\omega_1(s)}\frac{du}{ds}\widetilde{\boldsymbol{\alpha}}(s)\times\frac{d\widetilde{\boldsymbol{\alpha}}}{du}=\sin\phi(\cos\phi\boldsymbol{B}(s)-\sin\phi\boldsymbol{T}(s)).$$

Since $\widetilde{\boldsymbol{\alpha}}\times\dfrac{d\widetilde{\boldsymbol{\alpha}}}{du}=\widetilde{\boldsymbol{n}}$, we have

$$\cot\phi\int_{u_0}^{u}\frac{\omega(\sigma)}{\omega_1(\sigma)}\widetilde{\boldsymbol{\alpha}}(\sigma)d\sigma-\int_{u_0}^{u}\frac{\omega(\sigma)}{\omega_1(\sigma)}\widetilde{\boldsymbol{n}}(\sigma)d\sigma$$

$$=\int_{s_0}^{s}\cos\phi(\cos\phi\boldsymbol{T}(t)+\sin\phi\boldsymbol{B}(t))dt-\int_{s_0}^{s}\sin\phi(\cos\phi\boldsymbol{B}(t)-\sin\phi\boldsymbol{T}(t))dt$$

$$=\int_{s_0}^{s}\boldsymbol{T}(t)dt=\boldsymbol{\gamma}(s)+\mathbf{C}.$$

□

From the above theorem, we get the following corollary.

Corollary 4.2. *A Darboux-type curve $\boldsymbol{\gamma}$ is a cylindrical helix if and only if the corresponding spherical curve $\widetilde{\boldsymbol{\alpha}}$ is a circle.*

Proof. From the proof of Theorem 4.1, the curvature and torsion of $\boldsymbol{\gamma}$ satisfy

$$\frac{d}{du}(\frac{\tau}{\kappa})(u) = \varepsilon k_g'(u)\frac{1+\cot^2\phi}{(\cot\phi + k_g(u))^2}, \tag{4.1}$$

where $\varepsilon = \pm 1$. $\boldsymbol{\gamma}$ is a cylindrical helix if and only if τ/κ =contant. This condition is equivalent to the geodesic curvature of a spherical curve $\widetilde{\boldsymbol{\alpha}}$ being constant. It implies that $\widetilde{\boldsymbol{\alpha}}$ is a circle.

□

Corollary 4.3. *If the spherical curve $\widetilde{\boldsymbol{\alpha}}$ satisfies*

$$k_g(u) = \frac{1}{au+c} - \cot\phi,$$

where a, c, ϕ are constant numbers, then the corresponding Darboux-type curve $\boldsymbol{\gamma}$ is a rectifying curve.

Proof. By straightforward calculations, we get

$$\frac{k_g'(u)}{(\cot\phi + k_g(u))^2} = -a.$$

Based on the above equation and (4.1), it implies that

$$\frac{d}{du}(\frac{\tau}{\kappa})(u) = -\varepsilon a(1+\cot^2\phi).$$

This completes the proof.

□

In [13], Izumiya and Takeuchi introduced the concept of the spherical evolute of a spherical curve as follows: Let $\widetilde{\boldsymbol{\alpha}} : I \to S^2$ be a spherical curve with $k_g(u) \neq 0$. The spherical evolute $\mathbf{e}_{\widetilde{\alpha}}$ of $\widetilde{\boldsymbol{\alpha}}$ is defined by

$$\mathbf{e}_{\widetilde{\alpha}}(u) = \frac{1}{\sqrt{k_g^2(u)+1}}(k_g(u)\widetilde{\boldsymbol{\alpha}}(u) + \widetilde{\boldsymbol{n}}(u)).$$

The spherical evolute $\mathbf{e}_{\widetilde{\alpha}}$ of $\widetilde{\boldsymbol{\alpha}}$, as defined above, is the locus of the center of curvature of $\widetilde{\boldsymbol{\alpha}}$, which is consistent with the geometric definition of evolute.

Then we have the following proposition.

Proposition 4.4. *Let $\widetilde{\boldsymbol{\alpha}} : I \to S^2$ be a spherical curve and $\boldsymbol{\gamma}$ be the Darboux-type curve corresponding to $\widetilde{\boldsymbol{\alpha}}$. Then the spherical Darboux image of $\boldsymbol{\gamma}$ is equal to the spherical evolute of $\widetilde{\boldsymbol{\alpha}}$.*

Proof. Let u and s be the arc-length parameters of $\widetilde{\boldsymbol{\alpha}}$ and $\boldsymbol{\gamma}$, respectively. By direct calculations, we have

$$\boldsymbol{T}(u) = \dot{\boldsymbol{\gamma}}(u)\frac{du}{ds},$$

and

$$\kappa(u)\boldsymbol{N}(u) = \ddot{\boldsymbol{\gamma}}(u)(\frac{du}{ds})^2 + \dot{\boldsymbol{\gamma}}(u)\frac{d^2u}{ds^2}.$$

By the proof of Theorem 4.1 and $\boldsymbol{B}(u) = \boldsymbol{T}(u) \times \boldsymbol{N}(u)$, we get

$$\boldsymbol{T}(u) = \frac{\omega(u)}{\omega_1(u)}\frac{du}{ds}(\cot\phi\widetilde{\boldsymbol{\alpha}}(u) - \widetilde{\boldsymbol{n}}(u)),$$

and

$$\kappa(u)\boldsymbol{B}(u) = (\frac{\omega(u)}{\omega_1(u)})^2(\frac{du}{ds})^3(\cot\phi + k_g(u))(\cot\phi\widetilde{\boldsymbol{n}}(u) + \widetilde{\boldsymbol{\alpha}}(u)).$$

Since

$$\tau(u) = \sin^2\phi\frac{\omega_1(u)}{\omega(u)}(\cot\phi k_g(u) - 1), \qquad (\frac{\omega(u)}{\omega_1(u)}\frac{du}{ds})^2 = \sin^2\phi,$$

we easily obtain

$$\begin{aligned}\boldsymbol{D}(u) &= \tau(u)\boldsymbol{T}(u) + \kappa(u)\boldsymbol{B}(u)\\ &= \frac{du}{ds}(\cos\phi k_g(u) - \sin\phi)(\cos\phi\widetilde{\boldsymbol{\alpha}}(u) - \sin\phi\widetilde{\boldsymbol{n}}(u))\\ &\quad + \frac{du}{ds}(\cos\phi + k_g(u)\sin\phi)(\cos\phi\widetilde{\boldsymbol{n}}(u) + \sin\phi\widetilde{\boldsymbol{\alpha}}(u))\\ &= \frac{du}{ds}(k_g(u)\widetilde{\boldsymbol{\alpha}}(u) + \widetilde{\boldsymbol{n}}(u)).\end{aligned}$$

Further, we know that

$$\boldsymbol{d}(u) = \frac{\boldsymbol{D}(u)}{||\boldsymbol{D}(u)||} = \frac{k_g(u)\widetilde{\boldsymbol{\alpha}}(u) + \widetilde{\boldsymbol{n}}(u)}{||k_g(u)\widetilde{\boldsymbol{\alpha}}(u) + \widetilde{\boldsymbol{n}}(u)||} = \mathbf{e}_{\widetilde{\alpha}}(u).$$

This completes the proof.

□

5. Curves on rectifying developable

In this section, we study the Darboux-type curves from the viewpoint of curves on the rectifying developable surfaces. Developable surfaces are the ruled surfaces whose Gauss curvature of the regular part of the surface vanishes. Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ and $\boldsymbol{\sigma} : I \to \mathbb{R}^3 \setminus \{0\}$ be smooth mappings. We call the map $G_{(\gamma,\sigma)} : I \times \mathbb{R} \to \mathbb{R}^3$ defined by $G_{(\gamma,\sigma)}(t, z) = \boldsymbol{\gamma}(t) + z\boldsymbol{\sigma}(t)$ a ruled surface in $\mathbb{R}^3$. Curves $\boldsymbol{\gamma}$ and $\boldsymbol{\sigma}$ are called the *base curve* and the *director curve*, respectively.

Let $\boldsymbol{\gamma}$ be a unit speed space curve with $\kappa(s) \neq 0$. We call the ruled surface $G_{(\gamma,\widetilde{D})}(s, z) = \boldsymbol{\gamma}(s) + z\widetilde{\boldsymbol{D}}(s)$ the *rectifying developable* of $\boldsymbol{\gamma}$. In [20], the authors studied the geodesic of the rectifying developable as follows:

Proposition 5.1. *Let $\boldsymbol{\gamma}$ be a regular curve on the ruled surface M with $\kappa \neq 0$. Then the following conditions are equivalent:*
(1) *M is the rectifying developable of $\boldsymbol{\gamma}$.*
(2) *$\boldsymbol{\gamma}$ is a geodesic of M that is transversal to rulings, and M is a developable surface.*

We consider the other geodesic of the rectifying developable $G_{(\gamma,\widetilde{D})}$ except $\boldsymbol{\gamma}$ itself.

Theorem 5.2. *Let $\boldsymbol{\gamma}$ be a Darboux-type curve and M be a rectifying developable of $\boldsymbol{\gamma}$. If $\boldsymbol{\gamma}_D$ is the Darboux mate curve of $\boldsymbol{\gamma}$, then $\boldsymbol{\gamma}_D$ is a geodesic of M that is transversal to rulings.*

Proof. Suppose M is denoted by

$$G_{(\gamma,\widetilde{D})}(s,z) = \boldsymbol{\gamma}(s) + z\widetilde{\boldsymbol{D}}(s).$$

By Proposition 5.1, $\boldsymbol{\gamma}$ is the geodesic of $G_{(\gamma,\widetilde{D})}$. Then the principal normal direction of $\boldsymbol{\gamma}(s)$ at $p = \boldsymbol{\gamma}(s_0)$ is parallel to the unit normal $\boldsymbol{n}$ of $G_{(\gamma,\widetilde{D})}$ at p. We assume that $\boldsymbol{\gamma}_D(s) = \boldsymbol{\gamma}(s) + z(s)\widetilde{\boldsymbol{D}}(s)$ is a Darboux mate curve of $\boldsymbol{\gamma}$ and $z(s)$ satisfies

$$1 + (z\frac{\tau}{\kappa})'(s) = \cot\phi z'(s),$$

where ϕ is a constant. By the Frenet-Serret formulae, we obtain

$$\frac{d\boldsymbol{\gamma}_D}{ds}(s) = [1 + (z\frac{\tau}{\kappa})'(s)]\boldsymbol{T}(s) + z'(s)\boldsymbol{B}(s),$$
$$\frac{d^2\boldsymbol{\gamma}_D}{ds^2}(s) = (z\frac{\tau}{\kappa})''(s)\boldsymbol{T}(s) + z''(s)\boldsymbol{B}(s) + [(1 + (z\frac{\tau}{\kappa})'(s))\kappa(s) - z'(s)\tau(s)]\boldsymbol{N}(s).$$

From the relation

$$\det(\frac{d^2\boldsymbol{\gamma}_D}{ds^2}, \boldsymbol{n}, \frac{d\boldsymbol{\gamma}_D}{ds}) = 0,$$

it follows that $\boldsymbol{\gamma}_D$ is a geodesic of $G_{(\gamma,\widetilde{D})}$ which is transversal to rulings.

□

Finally, we will characterize Darboux-type curves from the viewpoint of curves on rectifying developable.

Theorem 5.3. *Let $G_{(\gamma,\delta)}(s,z) = \boldsymbol{\gamma}(s) + z\boldsymbol{\delta}(s)$ be a regular ruled surface with $\|\boldsymbol{\delta}\| = 1$. If $G_{(\gamma,\delta)}$ is a developable surface and two disjoint geodesics on it that are transversal to rulings, then the two geodesics are Darboux curve pairs.*

Proof. Let $\boldsymbol{\gamma}(s)$ and $\boldsymbol{\gamma}_1(s) = \boldsymbol{\gamma}(s) + z(s)\boldsymbol{\delta}(s)$ be two geodesics that are transversal to rulings. Let s and s_1 be the arc-length parameters of $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}_1$, respectively. By Proposition 5.1, $G_{(\gamma,\delta)}$ is the rectifying developable of $\boldsymbol{\gamma}(s)$. It implies that if $\boldsymbol{\delta}$ and $\widetilde{\boldsymbol{D}}$ are parallel, then the curve $\boldsymbol{\gamma}_1(s)$ can be rewritten as $\boldsymbol{\gamma}_1(s) = \boldsymbol{\gamma}(s) + \omega(s)\widetilde{\boldsymbol{D}}(s)$, where $\omega(s) = z(s)\kappa(s)/\sqrt{\tau^2(s) + \kappa^2(s)}$. Moreover, the principal normal direction of $\boldsymbol{\gamma}(s)$ at $p = \boldsymbol{\gamma}(s_0)$ is also parallel to the unit normal $\boldsymbol{n}$ of $G_{(\gamma,\delta)}$ at p. According to the Frenet–Serret formulae, we get

$$\frac{d\boldsymbol{\gamma}_1}{ds}(s) = [1 + (\omega\frac{\tau}{\kappa})'(s)]\boldsymbol{T}(s) + \omega'(s)\boldsymbol{B}(s), \tag{5.1}$$

$$\frac{d^2\boldsymbol{\gamma}_1}{ds^2}(s) = (\omega\frac{\tau}{\kappa})''(s)\boldsymbol{T}(s) + \omega''(s)\boldsymbol{B}(s) + [(1 + (\omega\frac{\tau}{\kappa})'(s))\kappa(s) - \omega'(s)\tau(s)]\boldsymbol{N}(s). \tag{5.2}$$

Since $\boldsymbol{\gamma}_1(s)$ is also the geodesic on $G_{(\gamma,\delta)}$, we have

$$\det(\frac{d^2\boldsymbol{\gamma}_1}{ds^2}, \boldsymbol{n}, \frac{d\boldsymbol{\gamma}_1}{ds}) = \omega'(\omega\frac{\tau}{\kappa})'' - \omega''[1 + (\omega\frac{\tau}{\kappa})'] = 0.$$

If $\omega' \neq 0$, then

$$\left(\frac{1 + (\omega\frac{\tau}{\kappa})'}{\omega'}\right)' = \frac{\omega'(\omega\frac{\tau}{\kappa})'' - \omega''[1 + (\omega\frac{\tau}{\kappa})']}{(\omega')^2}.$$

In this case, we have $1 + (\omega\tau/\kappa)'(s) = C\omega'(s)$, where C is a constant. By Theorem 3.4, $\boldsymbol{\gamma}(s)$ is a Darboux-type curve, and $\boldsymbol{\gamma}_1(s)$ is the Darboux mate curve of $\boldsymbol{\gamma}(s)$.

If $\omega' = 0$, from equations (5.1) and (5.2), we have

$$\frac{ds_1}{ds}\boldsymbol{T}_1(s) = [1 + (\omega\frac{\tau}{\kappa})'(s)]\boldsymbol{T}(s),$$

$$\frac{d^2 s_1}{ds^2}\boldsymbol{T}_1(s) + (\frac{ds_1}{ds})^2\kappa_1(s)\boldsymbol{N}_1(s) = (\omega\frac{\tau}{\kappa})''(s)\boldsymbol{T}(s) + [1 + (\omega\frac{\tau}{\kappa})'(s)]\kappa(s)\boldsymbol{N}(s).$$

From the above equations, we deduce that the principal normals of $\boldsymbol{\gamma}(s)$ and $\boldsymbol{\gamma}_1(s)$ are parallel to each other. By Theorem 3.2, $\boldsymbol{\gamma}(s)$ is a Darboux-type curve, and $\boldsymbol{\gamma}_1(s)$ is the Darboux mate curve of $\boldsymbol{\gamma}(s)$.

□

6. Examples

In this section, we present an example of the Darboux-type curve.

Example 1. We consider a spherical curve $\widetilde{\boldsymbol{\alpha}}(u) = (\sin u, \sin u \cos u, \cos^2 u)$, where $u \in [\pi, 2\pi)$. We can calculate that $(d\widetilde{\boldsymbol{\alpha}}/du)(u) = (\cos u, \cos^2 u - \sin^2 u, -2\cos u \sin u)$, then we obtain

$$\tilde{\boldsymbol{t}}(u) = (\frac{\sqrt{2}\cos u}{\sqrt{3+\cos 2u}}, \frac{\sqrt{2}(\cos^2 u - \sin^2 u)}{\sqrt{3+\cos 2u}}, \frac{-2\sqrt{2}\cos u \sin u}{\sqrt{3+\cos 2u}}),$$

and

$$\tilde{\boldsymbol{n}}(u) = \widetilde{\boldsymbol{\alpha}}(u) \times \tilde{\boldsymbol{t}}(u) = (-\frac{\sqrt{2}\cos^2 u}{\sqrt{3+\cos 2u}}, \frac{\sqrt{2}(\cos^3 u + 2\sin^2 u \cos u)}{\sqrt{3+\cos 2u}}, \frac{-\sqrt{2}\sin^3 u}{\sqrt{3+\cos 2u}}).$$

Through direct calculations, we get $k_g(u) = \dfrac{-\sqrt{2}\sin u(5+\cos 2u)}{(3+\cos 2u)^{\frac{3}{2}}}$. According to Theorem 4.1, we obtain the following Darboux-type curve ($\omega(\sigma) = \omega_1(\sigma)\sin\sigma\cos\sigma, \omega_1(\sigma) = \sigma, \mathbf{C} = \mathbf{0}, \cot\phi = 2$).

$$\begin{aligned}\boldsymbol{\gamma}(u) &= 2\int_0^u \sin\sigma\cos\sigma\widetilde{\boldsymbol{\alpha}}(\sigma)d\sigma - \int_0^u \sin\sigma\cos\sigma\tilde{\boldsymbol{n}}(\sigma)d\sigma\\ &= (\int_0^u \sin\sigma\cos\sigma(2\sin\sigma + \frac{\sqrt{2}\cos^2\sigma}{\sqrt{3+\cos 2\sigma}})d\sigma, \int_0^u \sin\sigma\cos\sigma(2\sin\sigma\cos\sigma -\\ &\quad \frac{\sqrt{2}\cos\sigma(1+\sin^2\sigma)}{\sqrt{3+\cos 2\sigma}})d\sigma, \int_0^u \sin\sigma\cos\sigma(2\cos^2\sigma + \frac{\sqrt{2}\sin^3\sigma}{\sqrt{3+\cos 2\sigma}})d\sigma).\end{aligned}$$

Since $\boldsymbol{T}(u) = \dfrac{1}{\sqrt{5}}(2\widetilde{\boldsymbol{\alpha}}(u) - \tilde{\boldsymbol{n}}(u))$, $\boldsymbol{N}(u) = \tilde{\boldsymbol{t}}(u)$ and $\boldsymbol{B}(u) = \dfrac{1}{\sqrt{5}}(\widetilde{\boldsymbol{\alpha}}(u) + 2\tilde{\boldsymbol{n}}(u))$. By the Frenet–Serret formula of $\boldsymbol{\gamma}$, we get that

$$\kappa(u) = \frac{1}{5\sin u\cos u}(2 + k_g(u)), \quad \tau(u) = \frac{1}{5\sin u\cos u}(2k_g(u) - 1).$$

The Darboux mate curve $\boldsymbol{\gamma}_D$ of $\boldsymbol{\gamma}$ is

$$\boldsymbol{\gamma}_D(u) = \boldsymbol{\gamma}(u) + \omega(u)\boldsymbol{D}(u)$$

$$= \boldsymbol{\gamma}(u) + u \sin u \cos u \boldsymbol{D}(u),$$

where $\boldsymbol{D}(u) = \dfrac{1}{\sqrt{5} \sin u \cos u}(k_g(u)\widetilde{\boldsymbol{\alpha}}(u) + \tilde{\boldsymbol{n}}(u))$.

In Figure 1, we draw the images of the spherical curve $\widetilde{\boldsymbol{\alpha}}$, the Darboux-type curve $\boldsymbol{\gamma}$ and the Darboux mate curve $\boldsymbol{\gamma}_D$. The rectifying developable of $\boldsymbol{\gamma}$ is written as

$$G_{(\gamma,\tilde{D})}(u, z) = \boldsymbol{\gamma}(u) + z\tilde{\boldsymbol{D}}(u),$$

where $z \in \mathbb{R}$ and $\tilde{\boldsymbol{D}}(u) = \dfrac{\sqrt{5}}{2 + k_g(u)}(k_g(u)\widetilde{\boldsymbol{\alpha}}(u) + \tilde{\boldsymbol{n}}(u))$. The image of the rectifying developable of $\boldsymbol{\gamma}$ is plotted in Figure 2.

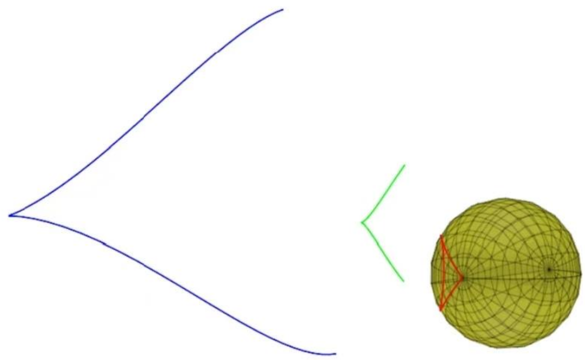

Figure 1. The red curve is $\widetilde{\boldsymbol{\alpha}}$, the green curve is $\boldsymbol{\gamma}$ and the blue curve is $\boldsymbol{\gamma}_D$.

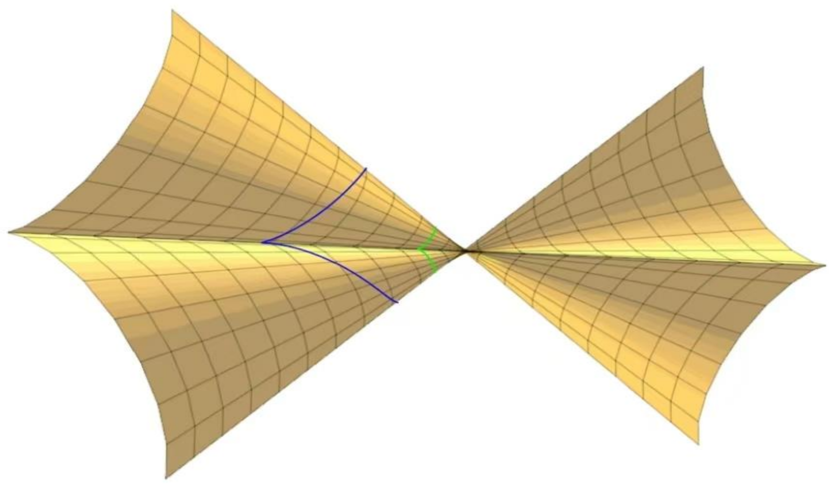

Figure 2. The rectifying developable of $\boldsymbol{\gamma}$.

7. Conclusions

The above research shows that we can construct a Darboux-type curve from a spherical curve and a smooth function. We know that cylindrical spiral curves and rectifying curves are Darboux-type curves, and also that geodesics on developable surfaces are closely related to Darboux-type curves. Based on these studies, we can next extend the definition of Darboux-type curves to higher-dimensional Euclidean spaces or non-Euclidean spaces. This helps to enrich the study of geometric properties of Darboux-type curves in higher-dimensional spaces and in non-Euclidean spaces.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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