Electronic Research Archive

https://www.aimspress.com/journal/era

ERA, 33(8): 4658–4678.
DOI: 10.3934/era.2025209
Received: 06 June 2025
Revised: 30 July 2025
Accepted: 05 August 2025
Published: 12 August 2025

Research article

Unique positive solution for a nonlinear p-Laplacian Hadamard fractional differential boundary value problem

Zichun Wang and Kemei Zhang*

School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China

* **Correspondence:** Email: zhkm90@126.com.

Abstract: The purpose of this article is to investigate a class of nonlinear p-Laplacian Hadamard fractional differential equations. By employing two fixed–point theorems of the sum operator, we get the uniqueness of a positive solution for such p-Laplacian equations. In addition, an iterative sequence can be constructed to approximate the unique positive solution. The validity of the results is demonstrated through two illustrative numerical examples in the final of this paper.

Keywords: fixed point theorem; uniqueness; fractional differential equations; p-Laplacian operator

1. Introduction

Recent research advances have demonstrated that differential equations play a pivotal role in modeling complex phenomena across multiple scientific disciplines. Among various fractional-order operators, the Caputo-type derivative has emerged as a powerful tool for characterizing non-Brownian particle motion in heterogeneous media. In such cases, infinite-point conditions naturally arise to represent distributed memory effects in multiscale systems or global constraints imposed by discrete spatial observations [1–3]. Similarly, the Hadamard-type fractional operator provides unique capabilities in material science, offering an accurate description of time-dependent creep behavior in viscoelastic polymers. In such applications, the infinite-point conditions physically correspond to the coupled relaxation processes occurring across multiple time scales within the material's microstructure [4–8]. The p-Laplacian operator can be employed in modeling nonlinear circuit elements (such as memristors), where the fractional-order terms characterize frequency-dependent dissipation effects, as well as in describing the nonlinear constitutive relations of non-Newtonian fluids [9–11]. The convergence analysis of iterative sequences provides an effective computational tool for material parameter identification, while the rigorous proof of solution uniqueness coupled with high-efficiency iterative algorithms is effectively translating theoretical advances into practical applications [12–15]. The numerical solution of high-dimensional differential equations has become a research hotspot due to their widespread applications in physics, engineering, biology, and other fields. To address the complexity of high-dimensional problems, scholars have proposed various efficient numerical methods, among which the Alternating Direction Implicit method [16–19], the Finite Volume method [20], compact difference schemes, and extrapolation techniques [21–23] have garnered significant attention for their advantages in computational efficiency and accuracy. These mathematical tools continue to reveal new insights into complex systems where conventional integer-order models prove inadequate.

In [6], the authors investigated the existence of a positive solution and stability analysis of the following equation:

$$\begin{cases} {}^{H}D^{\beta}_{1^+}\left(\varphi_p\left({}^{H}D^{\alpha}_{1^+}\hbar\right)\right)(r) = P\left(r, \hbar(r)\right), 1 < r < e, \\ \hbar(1) = \hbar'(1) = \hbar'(e) = 0, \\ \varphi_p\left({}^{H}D^{\alpha}_{1^+}\hbar(e)\right) = \rho \int_1^e \varphi_p\left({}^{H}D^{\alpha}_{1^+}\hbar(r)\right)\frac{dr}{r}, \end{cases} \tag{1.1}$$

where $1 < \beta \leq 2$, $2 < \alpha \leq 3$, and $0 < \mu \leq \beta$, $\varphi_p(r) = |r|^{p-2} r$ is the p-Laplacian for $p > 1$, $r \in \mathbb{R}$ and ${}^{H}D^{\beta}_{1^+}$, ${}^{H}D^{\alpha}_{1^+}$ are the Hadamard fractional derivatives. By employing a fixed method, they obtained the positive solutions.

In [24], by employing the fixed point index theory and special properties of Green's function, they obtained the multiple positive solutions of the following BVP:

$$\begin{cases} {}^{H}D^{\alpha}_{1^+}\left(\varphi_p\left({}^{H}D^{\gamma}_{1^+}u\right)\right)(r) + h\left(t, u(r)\right) = 0, 1 < r < e, \\ v^{(m)}(1) = 0, m = 0, 1, \ldots, n-2; {}^{H}D^{r_1}_{1^+}v(e) = \sum_{m=1}^{\infty} \eta_m\, {}^{H}D^{r_2}_{1^+}v(\xi_m), \\ {}^{H}D^{\gamma}_{1^+}v(1) = 0; \varphi_p\left({}^{H}D^{\gamma}_{1^+}v(e)\right) = \sum_{m=1}^{\infty} \zeta_m \varphi_p\left({}^{H}D^{\gamma}_{1^+}v(\xi_m)\right), \end{cases} \tag{1.2}$$

where $\alpha \in (1, 2]$, $\gamma \in (n-1, n]$, $2 \leq r_2 \leq r_1 \leq n-2\,(n \geq 3)$.

In [25], the author investigated the following equation:

$$\begin{cases} -D^{\beta}_{t}\left(\varphi_p\left(-D^{\gamma}_{t}z(\xi) - u\left(\xi, z(\xi), D^{\alpha}_{t}z(\xi)\right)\right)\right) = v\left(\xi, z(\xi), D^{\alpha}_{t}z(\xi)\right), 0 < \xi < 1, \\ D^{\gamma}_{t}z(0) = D^{\gamma+1}_{t}z(0) = D^{\alpha}_{t}z(0) = 0, \\ D^{\gamma}_{t}z(1) = 0, D^{\alpha}_{t}z(1) = \int_0^1 D^{\alpha}_{t}z(s)\,dB(s), \end{cases} \tag{1.3}$$

where D^{α}_{t}, D^{β}_{t}, D^{γ}_{t} denotes the Riemann–Liouville fractional derivatives, $\gamma \in (1, 2]$, $\beta \in (2, 3]$, $\alpha \in (0, 1]$, $\gamma > \alpha + 1$, $\int_0^1 D^{\alpha}_{t}z(r)\,dB(r)$ is a Riemann–Stieltjes integral and B denotes a function of bounded variation. The author obtain the unique positive solution by applying two fixed–point theorems.

Inspired by these outstanding achievements, in this article, we investigate the uniqueness of positive solutions for the following fractional differential equations with p-Laplacian operators.

$$\begin{cases} {}^{H}D_{1^+}^{\alpha}\left(\varphi_p\left(-{}^{H}D_{1^+}^{\beta}y(t)-Q(t,y(t))\right)\right)+P(t,y(t))=0, 1<t<e, \\ y^{(m)}(1)=0, m=0,1,2,\ldots,n-2; {}^{H}D_{1^+}^{r_1}y(e)=\sum_{m=1}^{\infty}\delta_m {}^{H}D_{1^+}^{r_2}y(\varepsilon_m), \\ {}^{H}D_{1^+}^{\beta}y(1)+Q(1,y(1))=0, \\ \varphi_p\left(-{}^{H}D_{1^+}^{\beta}y(e)-Q(e,y(e))\right)=\sum_{m=1}^{\infty}\theta_m\varphi_p\left(-{}^{H}D_{1^+}^{\beta}y(\varepsilon_m)-Q(\varepsilon_m,y(\varepsilon_m))\right), \end{cases} \tag{1.4}$$

where α, $\beta \in \mathbb{R}_+$, with $\alpha \in (1,2]$, $\beta \in (n-1,n]$, $2 \le r_2 \le r_1 \le n-2\,(n \ge 3)$, $0 < \delta_m, \theta_m < 1$, $1 < \varepsilon_m < e\,(m = 1,2,\ldots,\infty)$ and $P, Q \in C([1,e]\times\mathbb{R}_+,\mathbb{R}_+)$.

This paper establishes a uniqueness theorem for positive solutions to a class of singular p-Laplacian Hadamard fractional differential equations. The boundary conditions in [6] are of Dirichlet type and linear integral type, which fail to capture nonlocal effects. In contrast, the present work adopts infinite series summation boundary conditions, enabling the characterization of long-range interactions or discrete sampling effects. While [24] involves only a single Hadamard fractional derivative and does not account for nonlinear perturbation terms, thereby limiting the model's ability to describe complex dynamical behaviors. This study further generalizes the complexity of boundary conditions and incorporates mixed-type nonlinear terms, demonstrating an advanced application of the theoretical tools from [24]. Unlike reference [25], which primarily uses Riemann–Liouville or Caputo fractional derivatives, this study adopts Hadamard-type fractional derivatives. The introduction of the Hadamard-type fractional derivative necessitates the incorporation of logarithmic weight factors in integral transforms and requires the construction of weighted function spaces adapted to the Hadamard operator. Meanwhile, the growth conditions of nonlinear terms P and Q are influenced by the logarithmic scaling.

We arrange this paper as follows: We first present relevant definitions and lemmas, then systematically investigate the key properties of Green's functions in Section 2. Section 3 establishes the uniqueness of a positive solution for the p-Laplacian equations, along with an iterative sequence to approximate the unique positive solution. The validity of the results is demonstrated through two illustrative examples at the end of this paper.

2. Preliminaries and lemmas

This section will systematically introduce some background knowledge to construct the theoretical framework for subsequent theorem proofs.

Definition 1. [26, 27] The Hadamard fractional integral of $\beta\,(\beta > 0)$ order of a function $g : (0,\infty) \longrightarrow \mathbb{R}_+$ is given by

$${}^{H}I_{1^+}^{\beta}g(u)=\frac{1}{\Gamma(\beta)}\int_1^u\left(\ln\frac{u}{v}\right)^{\beta-1}\frac{g(v)}{v}dv.$$

Definition 2. [26, 27] The Hadamard fractional derivative of $\beta\,(\beta > 0)$ order of a function $g : (0,\infty) \longrightarrow \mathbb{R}_+$ is given by

$$ {}^{H}D_{1^+}^{\beta} g(u) = \frac{1}{\Gamma(n-\beta)} \left(u \frac{d}{du}\right)^n \int_1^u \left(\ln \frac{u}{v}\right)^{n-\beta-1} \frac{g(v)}{v} dv. $$

where $n = [\beta] + 1$.

Lemma 1. *[26, 27] If β, $\gamma > 0$, then*

$$ {}^{H}I_{1^+}^{\beta} (\ln u)^{\gamma-1} = \frac{\Gamma(\gamma)}{\Gamma(\gamma+\beta)} (\ln u)^{\gamma+\beta-1}, $$

$$ {}^{H}D_{1^+}^{\beta} (\ln u)^{\gamma-1} = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\beta)} (\ln u)^{\gamma-\beta-1}. $$

Lemma 2. *[27] Suppose that $\beta > 0$ and $g \in C[0,\infty) \cap L^1[0,\infty)$; then the solution of the equation ${}^{H}D_{1^+}^{\beta} g(u) = 0$ is*

$$ g(u) = c_1 (\ln u)^{\alpha-1} + c_2 (\ln u)^{\alpha-2} + \cdots + c_n (\ln u)^{\alpha-n}, \ c_i \in \mathbb{R}\,(I = 0, 1, \ldots, N), \ n = [\alpha] + 1. $$

Lemma 3. *[25] Let P be continuous. Then the solution of the BVP*

$$ \begin{cases} {}^{H}D_{1^+}^{\beta} y(t) + P(t) = 0, t \in (1, e), \\ y^{(m)}(1) = 0, m = 0, 1, \ldots, n-2, \\ {}^{H}D_{1^+}^{r_1} y(e) = \sum\limits_{m=1}^{\infty} \delta_m \, {}^{H}D_{1^+}^{r_2} y(\varepsilon_m), \end{cases} \tag{2.1} $$

can be expressed by

$$ y(t) = \int_1^e \Theta(t, r) P(r) \frac{dr}{r}, \tag{2.2} $$

where

$$ \Theta(t, r) = \frac{1}{\sigma\Gamma(\beta)} \begin{cases} \Gamma(\beta) (\ln t)^{\beta-1} L(r) (1 - \ln r)^{\beta-r_1-1} - \sigma (\ln t - \ln r)^{\beta-1}, 1 \le r \le t \le e, \\ \Gamma(\beta) (\ln t)^{\beta-1} L(r) (1 - \ln r)^{\beta-r_1-1}, 1 \le t \le r \le e, \end{cases} \tag{2.3} $$

in which

$$ L(r) = \frac{1}{\Gamma(\beta-r_1)} - \frac{1}{\Gamma(\beta-r_2)} \sum_{r \le \varepsilon_m} \delta_m \left(\frac{\ln \varepsilon_m - \ln r}{1 - \ln r}\right)^{\beta-r_2-1} (1 - \ln r)^{r_1-r_2}, \tag{2.4} $$

$$ \sigma = \frac{\Gamma(\beta)}{\Gamma(\beta-r_1)} - \frac{\Gamma(\beta)}{\Gamma(\beta-r_2)} \sum_{m=1}^{\infty} \delta_m (\ln \varepsilon_m)^{\beta-r_2-1} \neq 0. \tag{2.5} $$

Lemma 4. *The Green functions* (2.3) *show these properties:*

(1) $\Theta(t, r) > 0$, $\forall (t, r) \in (1, e) \times (1, e)$;
(2) $\Theta(t, r) \ge (\ln t)^{\beta-1} \Theta(e, r)$, $\forall (t, r) \in [1, e] \times [1, e]$;
(3) $\frac{\Theta(t,r)}{1+(\ln t)^{\beta-1}} \le \frac{\Theta(t,r)}{(\ln t)^{\beta-1}} \le \frac{L(e)}{\sigma}$, $\forall (t, r) \in [1, e] \times [1, e]$.

Proof. (1) (2) The demonstration parallels the proof of Lemma 2.5 in [25]; hence, we will not include it here.

(3) By simple calculation, we have $L'(s) > 0$. In other words, $L(s) \leq L(e)$ for $s \in [1, e]$. Then, in view of Lemma 3, $\forall (t, r) \in [1, e] \times [1, e]$, we can obtain

$$\begin{aligned}\frac{\Theta(t,r)}{(\ln t)^{\beta-1}} &\leq \frac{\Gamma(\beta)(\ln t)^{\beta-1} L(r)(1-\ln r)^{\beta-r_1-1}}{\sigma\Gamma(\beta)(\ln t)^{\beta-1}} \\ &\leq \frac{L(r)}{\sigma} \\ &\leq \frac{L(e)}{\sigma}.\end{aligned}$$

On the other hand, we can easily prove that $\frac{\Theta(t,r)}{1+(\ln t)^{\beta-1}} \leq \frac{\Theta(t,r)}{(\ln t)^{\beta-1}}$, when $(t, r) \in [1, e] \times [1, e]$. □

Lemma 5. *The equation of the BVPs* (1.4) *can be represented by*

$$y(t) = \int_1^e \Theta(t,r)\varphi_q\left(\int_1^e \Lambda(r,\omega) P(\omega, y(\omega)) \frac{\mathrm{d}\omega}{\omega}\right)\frac{\mathrm{d}r}{r} + \int_1^e \Theta(t,r) Q(r, y(r)) \frac{\mathrm{d}r}{r}, \tag{2.6}$$

where

$$\Lambda(t,r) = \frac{1}{\Pi\Gamma(\alpha)}\begin{cases}\Gamma(\alpha)(\ln t)^{\alpha-1} K(r)(1-\ln r)^{\alpha-1} - \Pi(\ln t - \ln r)^{\alpha-1}, \ 1 \leq r \leq t \leq e, \\ \Gamma(\alpha)(\ln t)^{\alpha-1} K(r)(1-\ln r)^{\alpha-1}, \ 1 \leq t \leq r \leq e,\end{cases} \tag{2.7}$$

in which

$$T(r) = \frac{1}{\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha)}\sum_{r\leq\varepsilon_m}\theta_m\left(\frac{\ln\varepsilon_m - \ln r}{1-\ln r}\right)^{\alpha-1}, \tag{2.8}$$

$$\Pi = 1 - \sum_{m=1}^{\infty}\theta_m(\ln\varepsilon_m)^{\alpha-1} \neq 0. \tag{2.9}$$

Proof. Let

$$x(t) = \varphi_p\left(-{}^H D_{1^+}^{\beta} y(t) - Q(t, y(t))\right), \ v(t) = P(t, y(t))$$

then the solution of the BVP

$$\begin{cases}{}^H D_{1^+}^{\beta} y(t) + \varphi_q(x(t)) + Q(t, y(t)) = 0, \\ y^{(m)}(1) = 0, m = 0, 1, \ldots, n-2, \\ {}^H D_{1^+}^{r_1} y(e) = \sum_{m=1}^{\infty}\delta_m \, {}^H D_{1^+}^{r_2} y(\varepsilon_m),\end{cases} \tag{2.10}$$

can be expressed as

$$y(t) = \int_1^e \Theta(t,r)\varphi_q(x(r))\frac{\mathrm{d}r}{r} + \int_1^e \Theta(t,r) Q(r, y(r))\frac{\mathrm{d}r}{r}. \tag{2.11}$$

Since

$$x(t) = \varphi_p\left(-{}^H D_{1^+}^{\beta} y(t) - Q(t, y(t))\right), \ v(t) = P(t, y(t)),$$

then, we obtain

$$\begin{cases} {}^{H}D_{1^+}^{\alpha}x(t)+v(t)=0,\ 1<t<e,\\ x(1)=0,\\ x(e)=\sum\limits_{m=1}^{\infty}\theta_m x(\varepsilon_m). \end{cases} \tag{2.12}$$

On the basis of above equation, we obtain

$$x(t)=-{}^{H}I_{1^+}^{\alpha}v(t)\sum_{n=1}^{2}c_n(\ln t)^{\alpha-n}.$$

It follows from $y(1)=0$ that $c_2=0$. Consequently, we notice

$$x(t)=-{}^{H}I_{1^+}^{\alpha}v(t)+c_1(\ln t)^{\alpha-1}.$$

Hence, we obtain

$$x(e)=-{}^{H}I_{1^+}^{\alpha}v(e)+c_1. \tag{2.13}$$

On the other hand, $x(e)=\sum_{m=1}^{\infty}\theta_m x(\varepsilon_m)$, together with (2.13), yields that

$$\begin{aligned} c_1&=\int_1^e\frac{(1-\ln r)^{\alpha-1}}{\Pi\Gamma(\alpha)}v(r)\frac{\mathrm{d}r}{r}-\sum_{m=1}^{\infty}\theta_m\int_1^{\varepsilon_m}\frac{(\ln\varepsilon_m-\ln r)^{\alpha-1}}{\Pi\Gamma(\alpha)}v(r)\frac{\mathrm{d}r}{r}\\ &=\int_1^e\frac{(1-\ln r)^{\alpha-1}}{\Pi\Gamma(\alpha)}v(r)\frac{\mathrm{d}r}{r}-\int_1^{\varepsilon_m}\frac{(1-\ln r)^{\alpha-1}}{\pi}\frac{\sum_{m=1}^{\infty}\theta_m(\ln\varepsilon_m-\ln r)^{\alpha-1}}{\Gamma(\alpha)(1-\ln r)}v(r)\frac{\mathrm{d}r}{r}\\ &=\int_1^e\frac{(1-\ln r)^{\alpha-1}T(r)}{\Pi}v(r)\frac{\mathrm{d}r}{r}, \end{aligned}$$

where $T(r)$ is similar to (2.8), and Π is similar to (5). Thus,

$$\begin{aligned} x(t)&=-{}^{H}I_{1^+}^{\alpha}v(t)+\int_1^e\frac{(1-\ln r)^{\alpha-1}T(r)(\ln t)^{\alpha-1}}{\Pi}v(r)\frac{\mathrm{d}r}{r}\\ &=-\frac{1}{\Gamma(\alpha)}\int_1^t(\ln t-\ln r)^{\alpha-1}v(r)\frac{\mathrm{d}r}{r}+\int_1^e\frac{(1-\ln r)^{\alpha-1}T(r)(\ln t)^{\alpha-1}}{\Pi}v(r)\frac{\mathrm{d}r}{r}\\ &=\int_1^e\Lambda(t,r)v(r)\frac{\mathrm{d}r}{r}. \end{aligned}$$

Then combining with (2.11), we can conclude that (2.6) hold. □

Lemma 6. *Here are the properties of Green function (2.7):*

(1) $\Lambda(t,r)>0$, $t,r\in(1,e)$;

(2) $(\ln t)^{\alpha-1}\Lambda(e,r)\le\Lambda(t,r)\le\frac{(\ln t)^{\alpha-1}T(e)}{\Pi}$, $t,r\in[1,e]$.

Proof. The demonstration parallels the proof of Lemma 4; hence, we will not include it here. □

Through the discussion of Lemmas 4–6, we have obtained the Green's function for Eq (1.4) and its properties, which lays the foundation for the subsequent study of the uniqueness of positive solutions.

In the subsequent sections, we will present two fixed–point theorems for the sum operator, which are employed in this paper. Let $(\aleph, \|\cdot\|)$ be a real Banach space. $\aleph$ is partially ordered by a cone $K \subset \aleph$, i.e., $u \leq v$ if and only if $v - u \in K$. Moreover, a cone K is normal if there exists a positive constant N when $\theta \leq u \leq v$, $\|u\| \leq N \|v\|$ holds for all $u, v \in \aleph$.

When $u, v \in \aleph$, the expression $u \sim v$ denotes that there exist two positive numbers λ_1 and λ_2 such that $\lambda_1 u \leq v \leq \lambda_2 u$. It's evident that "$\sim$" serves as an equivalence relation. If $h > \theta$ and we define a set $P_h = \{u \in \aleph : u \sim h\}$, obviously $P_h \subset K$.

Definition 3. [28] Let $0 < \delta < 1$. An operator $A : P \to P$ is said to be δ-concave if $A(tu) \geq t^{\delta} A(u)$ for $t \in (0, 1)$, $u \in P$.

Definition 4. [28] An operator $A : P \to P$ is said to be sub-homogeneous if $A(tu) \geq tA(u)$ for $t > 0$, $u \in P$.

Lemma 7. *[29] Let $\aleph$ be a real Banach space. The cone K is a normal cone in $\aleph$, $A : K \to K$ is an increasing δ-concave operator, and $B : K \to K$ is an increasing sub-homogeneous operator. Assume that*
(1) *There exists $k > \theta$ such that $Ak \in P_k$ and $Bk \in P_k$;*
(2) *There is a positive constant δ_0 such that $Au \geq \delta_0 Bu$, $u \in K$.*
Then the operator equation

$$Au + Bu = u$$

has a unique solution u^* in P_k. Moreover, making the sequence $v_n = Av_{n-1} + Bv_{n-1}$, $n = 1, 2, \ldots,$ for any initial value $v_0 \in P_k$, one has $v_n \to u^*$ as $n \to \infty$.

Lemma 8. *[30] Let $\aleph$ be a real Banach space. The cone K is a normal cone in $\aleph$, $A : K \to K$ is an increasing operator, and $B : K \to K$ is a decreasing operator. In addition,*
(1) *For $u \in K$ and $s \in (0, 1)$, there exist $\hbar_i(s) \in (s, 1)$, $i = 1, 2$ such that*

$$A(su) \geq \hbar_1(s) Au, \ B(su) \leq \frac{1}{\hbar_2(s)} Bu;$$

(2) *There is $k_0 \in P_k$ such that $Ak_0 + Bk_0 \in P_k$.*
Then the operator equation

$$Au + Bu = u$$

has a unique solution u^* in P_k. Further, for any initial value u_0, $v_0 \in P_k$, making the sequences

$$u_n = Au_{n-1} + Bv_{n-1}, \ v_n = Av_{n-1} + Bu_{n-1}, \ n = 1, 2, \ldots,$$

one has $u_n \to u^*$, $v_n \to v^*$ as $n \to \infty$.

Remark 1. The conclusions of Lemmas 7 and 8 continue to hold even when B is the zero operator.

3. Main results

Now, resetting

$$\aleph = \left\{ y \in C([1, e], \mathbb{R}) \,|\, sup_{t \in [1,e]} \frac{|y(t)|}{1 + (\ln t)^{\beta - 1}} < +\infty \right\}$$

with the norm

$$\|y\|_{\aleph} = sup_{t\in[1,e]} \frac{|y(t)|}{1+(\ln t)^{\beta-1}},$$

then $(\aleph, \|\cdot\|_{\aleph})$ is a Banach space. The cone $P \subset \aleph$ is defined as $P = \{y \in \aleph | y(t) \geq 0\}$. $\aleph$ has a partial order

$$x_1 \leq x_2 \Longleftrightarrow x_1(t) \leq x_2(t),\ t \in [1, e].$$

It follows from $0 \leq x_1(t) \leq x_2(t)$, that

$$sup_{t\in[1,e]} \frac{|x_1(t)|}{1+(\ln t)^{\beta-1}} \leq sup_{t\in[1,e]} \frac{|x_2(t)|}{1+(\ln t)^{\beta-1}}.$$

Hence, $\|x_1\| \leq \|x_2\|$. In other words, the cone P is a normal.

To obtain our main results, the following assumptions must be made:
(H_1) $P, Q : [1, e] \times [0, +\infty] \to [0, +\infty]$ are increasing in the second argument. Moreover, $P(t, 0) \not\equiv 0$, $Q(t, 0) \not\equiv 0$, $t \in [1, e]$;
(H_2) there is a bounded set Ω when $y \in \Omega$, $P\left(t, \left(1 + (\ln t)^{\beta-1}\right) y\right)$ and $Q\left(t, \left(1 + (\ln t)^{\beta-1}\right) y\right)$ are bounded for $t \in [1, e]$;
(H_3) $P(t, \lambda y) \geq \varphi_p(\lambda) P(t, y)$ for $t \in [1, e]$, $\lambda \in (0, 1)$, $y \in \mathbb{R}_+$, and $Q(t, \lambda y) \geq \lambda^\delta Q(t, y)$ for all $\lambda \in (0, 1)$, $t \in [1, e]$, $y \in \mathbb{R}_+$, where δ is a positive constant;
(H_4) there exists a positive constant δ_0 such that $P(t, y) \leq \delta_0 \leq Q(t, 0)$ for all $y \in \mathbb{R}_+$ and $t \in [1, e]$;
(H_5) $P : [1, e] \times [0, +\infty] \to [0, +\infty]$ is decreasing in the second argument, $Q : [1, e] \times [0, +\infty] \to [0, +\infty]$ is increasing in the second argument. In addition, $P(t, 1) \not\equiv 0$, $Q(t, 0) \not\equiv 0$, $t \in [1, e]$;
(H_6) for $\lambda \in (0, 1)$, there exist $\phi_1(\lambda)$, $\phi_2(\lambda) \in (\lambda, 1)$ such that $Q(t, \lambda y) \geq \phi_1(\lambda) Q(t, y)$ and $P(t, \lambda y) \leq \varphi_p\left(\frac{1}{\phi_2(\lambda)}\right) P(t, y)$, $t \in [1, e]$.

Let $h(t) = (\ln t)^{\beta-1}$, $t \in [1, e]$. As $sup_{t\in[1,e]} \frac{|h(t)|}{1+(\ln t)^{\beta-1}} = \frac{1}{2}$, then $h(t) \in P$, Next, we will need a set $P_h = \{y \in \aleph | y \sim h\}$. According to Lemma 5, the problem (1.4) can be expressed through the integral formulation (2.6). We now give the definitions of the two operators $A, B : P \to \aleph$:

$$Ay(t) = \int_1^e \Theta(t, r) Q(r, y(r)) \frac{\mathrm{d}r}{r}, \tag{3.1}$$

$$By(t) = \int_1^e \Theta(t, r) \varphi_q\left(\int_1^e \Lambda(r, \omega) P(\omega, y(\omega)) \frac{\mathrm{d}\omega}{\omega}\right) \frac{\mathrm{d}r}{r}. \tag{3.2}$$

Then we can see that y is the solution of problem (1.4) if and only if Y is the solution of $Ay + By = y$.

Lemma 9. *Suppose that (H_1) and (H_2) are satisfied. Then $A, B : P \to P$.*

Proof. Let $y \in P$, then $sup_{t\in[1,e]} \frac{|y(t)|}{1+(\ln t)^{\beta-1}} < +\infty$. According to (H_2), there are $M_y > 0$ and $M_y^{'} > 0$ such that

$$P\left(r, \frac{\left[1 + (\ln r)^{\beta-1}\right] y(r)}{1 + (\ln r)^{\beta-1}}\right) \leq M_y,\ Q\left(r, \frac{\left[1 + (\ln r)^{\beta-1}\right] y(r)}{1 + (\ln r)^{\beta-1}}\right) \leq M_y^{'}.$$

And, by (H_1) and Lemma 4 ,

$$\begin{aligned}\frac{Ay(t)}{1+(\ln t)^{\beta-1}} &\leq \int_1^e \frac{L(e)}{\sigma} Q\left(r, \frac{\left[1+(\ln r)^{\beta-1}\right]y(r)}{1+(\ln r)^{\nu-1}}\right)\frac{\mathrm{d}r}{r} \\ &= \frac{L(e)}{\sigma} M_y'.\end{aligned}$$

Then we can conclude that $Ay \in \aleph$, and clearly $Ay(t)$ is nonnegative; in other words, $A : P \to P$.
Similarly, the proof holds for operator $B : P \to P$. □

Lemma 10. *Suppose that* (H_1) *and* (H_3) *hold. Then the operator A is increasing and* δ*-concave, and B is increasing and sub-homogeneous.*

Proof. First of all, we claim that A is an increasing operator. When $a \geq c$ for any $a, c \in P$, there is $a(t) \geq c(t)$, for $t \in [1, e]$. Moreover, according to (H_1) and Lemma 4, we get

$$\begin{aligned}Aa(t) &= \int_1^e \Theta(t,r) Q(r, a(r)) \frac{\mathrm{d}r}{r} \\ &\geq \int_1^e \Theta(t,r) Q(r, c(r)) \frac{\mathrm{d}r}{r} \\ &= Ac(t).\end{aligned}$$

So, A is increasing operators. Similarly, B is also increasing.
Next, we prove that A is a δ-concave operator. Based on (H_1), (H_3), and Lemma 4, we can easily derive that for $y \in P$, $\lambda \in (0, 1)$,

$$\begin{aligned}A(\lambda y)(t) &= \int_1^e \Theta(t,r) Q(r, \lambda y(r)) \frac{\mathrm{d}r}{r} \\ &\geq \lambda^\delta \int_1^e \Theta(t,r) Q(r, x(r)) \frac{\mathrm{d}r}{r} \\ &= \lambda^\delta Ax(t).\end{aligned}$$

Therefore, A is a δ-concave operator.
Finally, we are committed to proving that B is sub-homogeneous. Since (H_1), (H_3), and Lemma 4 that for $y \in P$, $\lambda \in (0, 1)$,

$$\begin{aligned}B(\lambda y)(t) &= \int_1^e \Theta(t,r)\varphi_q\left(\int_1^e \Lambda(r,\omega) P(\omega, \lambda y(\omega)) \frac{\mathrm{d}\omega}{\omega}\right)\frac{\mathrm{d}r}{r} \\ &\geq \int_1^e \Theta(t,r)\varphi_q\left(\int_1^e \Lambda(r,\omega)\varphi_p(\lambda) P(\omega, y(\omega)) \frac{\mathrm{d}\omega}{\omega}\right)\frac{\mathrm{d}r}{r} \\ &= \lambda \int_1^e \Theta(t,r)\varphi_q\left(\int_1^e \Lambda(r,\omega) P(\omega, y(\omega)) \frac{\mathrm{d}\omega}{\omega}\right)\frac{\mathrm{d}r}{r} \\ &= \lambda By(t).\end{aligned}$$

Then, B is a sub-homogeneous operator. □

Lemma 11. *Suppose that* (H_1) *and* (H_2) *are satisfied. Then* $Ah \in P_h$ *and* $Bh \in P_h$.

Proof. Since $h \in P$, then $sup_{t\in[1,e]} \frac{|h(t)|}{1+(\ln t)^{\gamma-1}} < +\infty$ for all $t \in [1, e]$. In view of (H_2), there exists $M_h > 0$ and $M_h^{'} > 0$ such that

$$P\left(r, \frac{\left[1+(\ln r)^{\beta-1}\right]h(r)}{1+(\ln r)^{\beta-1}}\right) \le M_h,\ Q\left(r, \frac{\left[1+(\ln r)^{\beta-1}\right]h(r)}{1+(\ln r)^{\beta-1}}\right) \le M_h^{'}.$$

Let

$$l_1 = \int_1^e \Theta(e,r)\,Q(r,0)\,\frac{\mathrm{d}r}{r},$$
$$l_2 = \int_1^e \frac{L(e)}{\sigma} M_h \frac{\mathrm{d}r}{r}.$$

We begin by proving $l_1 h(t) \le Ah(t) \le l_2 h(t)$. Based on (H_1),

$$\begin{aligned} Ah(t) &= \int_1^e \Theta(t,r)\,Q(r,h(r))\,\frac{\mathrm{d}r}{r} \\ &\ge \int_1^e \Theta(t,r)\,Q(r,0)\,\frac{\mathrm{d}r}{r} \\ &\ge (\ln t)^{\beta-1} \int_1^e \Theta(e,r)\,Q(r,0)\,\frac{\mathrm{d}r}{r} \\ &= l_1 h(t). \end{aligned}$$

Similarly,

$$\begin{aligned} Ah(t) &= \int_1^e \Theta(t,r)\,Q(r,h(r))\,\frac{\mathrm{d}r}{r} \\ &\le \int_1^e \Theta(t,r)\,M_h \frac{\mathrm{d}r}{r} \\ &\le (\ln t)^{\beta-1} \int_1^e \frac{L(e)}{\sigma} M_h \frac{\mathrm{d}r}{r} \\ &= l_2 h(t). \end{aligned}$$

Next, we prove that $Ah \in P_h$. Note that $g(s,0) \not\equiv 0$ and $\Psi(e,s) > 0$, so we easily know $g(s,0) \times \Psi(e,s) \not\equiv 0$, therefore $l_1 > 0$.
On the other hand,

$$\begin{aligned} \Theta(e,r) &\le \frac{\Gamma(\beta)\,L(r)\,(1-\ln r)^{\beta-r_1-1}}{\sigma\Gamma(\beta)} \\ &= \frac{L(r)\,(1-\ln r)^{\beta-r_1-1}}{\sigma} \\ &\le \frac{L(r)}{\sigma} \\ &\le \frac{L(e)}{\sigma}. \end{aligned}$$

From (H_1) and (H_2) we see that $Q(r,0) \leq Q(r,h(r)) \leq M_h$. Then $0 < l_1 \leq l_2$ holds. We can conclude that $Ah \in P_h$.

Finally, we are committed to proving that $Bh \in P_h$. Also, set

$$m_1 = \int_1^e \Theta(e,r)\,\varphi_q\left(\int_1^e \Lambda(r,\omega)\,P(\omega,0)\,\frac{d\omega}{\omega}\right)\frac{dr}{r},$$

$$m_2 = \int_1^e \frac{L(e)}{\sigma}\varphi_q\left(\int_1^e \Lambda(r,\omega)\,M_h'\frac{d\omega}{\omega}\right)\frac{dr}{r}.$$

By (H_1) and Lemma 4

$$\begin{aligned} Bh(t) &= \int_1^e \Theta(t,r)\,\varphi_q\left(\int_1^e \Lambda(r,\omega)\,P(\omega,h(\omega))\,\frac{d\omega}{\omega}\right)\frac{dr}{r} \\ &\geq \int_1^e \Theta(t,r)\,\varphi_q\left(\int_1^e \Lambda(r,\omega)\,P(\omega,0)\,\frac{d\omega}{\omega}\right)\frac{dr}{r} \\ &\geq (\ln t)^{\beta-1}\int_1^e \Theta(e,r)\,\varphi_q\left(\int_1^e \Lambda(r,\omega)\,P(\omega,0)\,\frac{d\omega}{\omega}\right)\frac{dr}{r} \\ &= m_1 h(t). \end{aligned}$$

$$\begin{aligned} Bh(t) &= \int_1^e \Theta(t,r)\,\varphi_q\left(\int_1^e \Lambda(r,\omega)\,P(\omega,h(\omega))\,\frac{d\omega}{\omega}\right)\frac{dr}{r} \\ &\leq \int_1^e \Theta(t,r)\,\varphi_q\left(\int_1^e \Lambda(r,\omega)\,M_h'\frac{d\omega}{\omega}\right)\frac{dr}{r} \\ &\leq (\ln t)^{\gamma-1}\int_1^e \frac{L(e)}{\sigma}\varphi_q\left(\int_1^e \Lambda(r,\omega)\,M_h'\frac{d\omega}{\omega}\right)\frac{dr}{r} \\ &= m_2 h(t). \end{aligned}$$

On the basis of $P(\omega,0) \not\equiv 0$, $\Theta(e,r)$, $\Lambda(r,\omega) > 0$, and $\Theta(e,r) \leq \frac{L(e)}{\sigma}$, $P(\omega,0) \leq P(\omega,h(\omega)) \leq M_h'$ for $\omega \in [1,e]$, we conclude $Bh \in P_h$. □

Theorem 1. *Suppose that (H_1)–(H_4) are fulfilled. Then the problem (1.4) has a unique positive solution y^* in P_h. For arbitrary initial point x_0 within the space P_h, making a sequence by*

$$y_{n+1}(t) = \int_1^e \Theta(t,r)\,\varphi_q\left(\int_1^e \Lambda(r,\omega)\,P(\omega,y_n(\omega))\,\frac{d\omega}{\omega}\right)\frac{dr}{r} + \int_1^e \Theta(t,r)\,Q(r,y_n(r))\,\frac{dr}{r},$$

$n = 0,1,\ldots,$ we get $y_n(t) \to y^*(t)$ as $n \to \infty$.

Proof. On the basis of Lemmas 9–11, we just need to demonstrate the second condition of Lemma 7. In view of Lemma 6 and (H_4), for $x \in P$,

$$\begin{aligned} By(t) &= \int_1^e \Theta(t,r)\,\varphi_q\left(\int_1^e \Lambda(r,\omega)\,P(\omega,y(\omega))\,\frac{d\omega}{\omega}\right)\frac{dr}{r} \\ &\leq \int_1^e \Theta(t,r)\,\varphi_q\left(\int_1^e \frac{(\ln t)^{\alpha-1}\,T(e)}{\Pi}P(\omega,y(\omega))\,\frac{d\omega}{\omega}\right)\frac{dr}{r} \end{aligned}$$

$$
\begin{aligned}
&\le \int_1^e \Theta(t,r)\,\varphi_q\left(\int_1^e \frac{T(e)}{\Pi} P(\omega, y(\omega))\,\frac{\mathrm{d}\omega}{\omega}\right)\frac{\mathrm{d}r}{r}\\
&= \left(\frac{T(e)}{\Pi}\right)^{q-1}\int_1^e \Theta(t,r)\,\varphi_q\left(\int_1^e P(\omega, y(\omega))\,\frac{\mathrm{d}\omega}{\omega}\right)\frac{\mathrm{d}r}{r}\\
&\le \left(\frac{T(e)}{\Pi}\right)^{q-1}\int_1^e \Theta(t,r)\,\varphi_q\left(\int_1^e \delta_0\frac{\mathrm{d}\omega}{\omega}\right)\frac{\mathrm{d}r}{r}\\
&= \left(\frac{T(e)}{\Pi}\right)^{q-1}\int_1^e \Theta(t,r)\,\varphi_q(\delta_0)\,\frac{\mathrm{d}r}{r}\\
&= \left(\frac{T(e)}{\Pi}\right)^{q-1}(\delta_0)^{q-2}\int_1^e \Theta(t,r)\,\delta_0\frac{\mathrm{d}r}{r}\\
&\le \left(\frac{T(e)}{\Pi}\right)^{q-1}(\delta_0)^{q-2}\int_1^e \Theta(t,r)\,Q(r,0)\,\frac{\mathrm{d}r}{r}\\
&\le \left(\frac{T(e)}{\Pi}\right)^{q-1}(\delta_0)^{q-2}\int_1^e \Theta(t,r)\,Q(s,y(r))\,\frac{\mathrm{d}r}{r}\\
&= \left(\frac{T(e)}{\Pi}\right)^{q-1}(\delta_0)^{q-2}\,Ay(t).
\end{aligned}
$$

Let $\delta_0' = (\delta_0)^{2-q}\left(\frac{T(e)}{\Pi}\right)^{1-q}$, then we can conclude $Ay(t) \ge \delta_0' By(t)$.

By Lemma 7 and the preceding discussion, the sequences

$$
y_{n+1}(t) = \int_1^e \Theta(t,r)\,\varphi_q\left(\int_1^e \Lambda(r,\omega)\,P(\omega, y_n(\omega))\,\frac{\mathrm{d}\omega}{\omega}\right)\frac{\mathrm{d}r}{r} + \int_1^e \Theta(t,r)\,Q(r, y_n(r))\,\frac{\mathrm{d}r}{r},
$$

$n = 0, 1, 2, \ldots$ converges to $y^*(t)$ as $n \to \infty$. □

Corollary 1. Let $\alpha, \beta, r_1, r_1, \delta_m\ \theta_m\ \varepsilon_m\ (m = 1, 2, \ldots, \infty)$ be given in (1.4), and let
$\left(H_1'\right)$ $g : [1, e] \times \mathbb{R}_+ \to \mathbb{R}_+$ is increasing in the second argument. Moreover, $Q(t, 0) \not\equiv 0$, for $t \in [1, e]$;
$\left(H_2'\right)$ when y is in a bounded set, $Q\left(t, \left(1 + (\ln t)^{\beta-1}\right) y\right)$ is bounded for $t \in [1, e]$;
$\left(H_3'\right)$ there is a positive constant δ for which it holds that $Q(t, \lambda y) \ge \lambda^\delta Q(t, y)$, $\lambda \in (0, 1)$, $t \in [1, e]$, $y \in \mathbb{R}_+$.
Then letting $h(t) = (\ln t)^{\gamma-1}$, $t \in [1, e]$, the following problem

$$
\begin{cases}
{}^H D_{1^+}^{\alpha}\left(\varphi_p\left(-{}^H D_{1^+}^{\beta} y(t) - Q(t, y(t))\right)\right) = 0, 1 < t < e,\\
y^{(m)}(1) = 0, m = 0, 1, 2, \ldots, n-2; {}^H D_{1^+}^{r_1} y(e) = \sum\limits_{m=1}^{\infty} \delta_m {}^H D_{1^+}^{r_2} y(\varepsilon_m),\\
{}^H D_{1^+}^{\beta} y(1) + Q(1, y(1)) = 0,\\
\varphi_p\left(-{}^H D_{1^+}^{\beta} y(e) - Q(e, y(e))\right) = \sum\limits_{m=1}^{\infty} \theta_m \varphi_p\left(-{}^H D_{1^+}^{\beta} y(\varepsilon_m) - Q(\varepsilon_m, y(\varepsilon_m))\right),
\end{cases}
$$

admits a unique solution $y^* \in P_h$. Furthermore, for an arbitrary initial point y_0 within the space P_h, the sequence defined by

$$
y_{n+1}(t) = \int_1^e \Theta(t,r)\,Q(r, y_n(r))\,\frac{\mathrm{d}r}{r}, n = 0, 1, 2, \ldots
$$

converges to $y^*(t)$ as $n \to \infty$.

Proof. First of all, by simple calculation, we obtain the solution of the above equation:

$$y(t) = \int_1^e \Theta(t,r) Q(r, y(r)) \frac{\mathrm{d}r}{r}.$$

We now give the definitions of the two operators $A, B : P \to \aleph$:

$$Ay(t) = \int_1^e \Theta(t,r) Q(r, y(r)) \frac{\mathrm{d}r}{r},$$

$$By(t) = 0.$$

Then, based on $\left(H_1'\right)$ and $\left(H_2'\right)$, we can prove that the operator: $A : P \to P$ and $Ah \in P_h$. Furthermore, from conditions $\left(H_1'\right)$ and $\left(H_3'\right)$, we know that the operator A is increasing and δ-concave. Finally, according to Lemma 4, we conclude that $Ax \geq 0$. By Lemma 7 and the preceding discussion, the result is established. This proof follows a similar approach to Theorem 1, and thus the detailed arguments are omitted here for brevity.

□

Lemma 12. *Suppose that (H_2) and (H_5) are satisfied. Then $Ah + Bh \in P_h$.*

Proof. First of all, we claim that $n_1 h(t) \leq Ah(t) + Bh(t) \leq n_2 h(t)$. Since $h \in P$, then $sup_{t\in[1,e]} \frac{|h(t)|}{1+(\ln t)^{\gamma-1}} < +\infty$. In view of (H_2), there exists $M_h > 0$ and $M_h' > 0$ such that

$$P\left(r, \frac{\left[1 + (\ln r)^{\beta-1}\right] h(r)}{1 + (\ln r)^{\beta-1}}\right) \leq M_h,\ Q\left(r, \frac{\left[1 + (\ln r)^{\beta-1}\right] h(r)}{1 + (\ln r)^{\beta-1}}\right) \leq M_h'.$$

Let

$$n_1 = \int_1^e \Theta(e,r) Q(r,0) \frac{\mathrm{d}r}{r} + \int_1^e \Theta(e,r) \varphi_q\left(\int_1^e \Lambda(r,\omega) P(\omega,1) \frac{\mathrm{d}\omega}{\omega}\right) \frac{\mathrm{d}r}{r},$$

$$n_2 = \int_1^e \frac{L(e)}{\sigma} M_h \frac{\mathrm{d}r}{r} + \int_1^e \frac{L(e)}{\sigma} \varphi_q\left(\int_1^e \Lambda(r,\omega) M_h' \frac{\mathrm{d}\omega}{\omega}\right) \frac{\mathrm{d}r}{r}.$$

In view of lemma 4 and (H_5), that

$$\begin{aligned}
Ah(t) + Bh(t) &= \int_1^e \Theta(t,r) Q(r, h(r)) \frac{\mathrm{d}r}{r} + \int_1^e \Theta(t,r) \varphi_q\left(\int_1^e \Lambda(r,\omega) P(\omega, h(\omega)) \frac{\mathrm{d}\omega}{\omega}\right) \frac{\mathrm{d}r}{r} \\
&\geq (\ln t)^{\beta-1} \int_1^e \Theta(e,r) Q(r, h(r)) \frac{\mathrm{d}r}{r} \\
&\quad + (\ln t)^{\beta-1} \int_1^e \Theta(e,r) \varphi_q\left(\int_1^e \Lambda(r,\omega) P(\omega, h(\omega)) \frac{\mathrm{d}\omega}{\omega}\right) \frac{\mathrm{d}r}{r} \\
&\geq (\ln t)^{\beta-1} \int_1^e \Theta(e,r) Q(r,0) \frac{\mathrm{d}r}{r} \\
&\quad + (\ln t)^{\beta-1} \int_1^e \Theta(e,r) \varphi_q\left(\int_1^e \Lambda(r,\omega) P(\omega,1) \frac{\mathrm{d}\omega}{\omega}\right) \frac{\mathrm{d}r}{r}
\end{aligned}$$

$$= n_1 h(t).$$

Also, by (H_2)

$$\begin{aligned} Ah(t) + Bh(t) &= \int_1^e \Theta(t,r) Q(r,h(r)) \frac{\mathrm{d}r}{r} + \int_1^e \Theta(t,r) \varphi_q \left(\int_1^e \Lambda(r,\omega) P(\omega, h(\omega)) \frac{\mathrm{d}\omega}{\omega} \right) \frac{\mathrm{d}r}{r} \\ &\le (\ln t)^{\beta-1} \int_1^e \frac{L(e)}{\sigma} Q(r,h(r)) \frac{\mathrm{d}r}{r} \\ &\quad + (\ln t)^{\beta-1} \int_1^e \frac{L(e)}{\sigma} \varphi_q \left(\int_1^e \Lambda(r,\omega) P(\omega, h(\omega)) \frac{\mathrm{d}\omega}{\omega} \right) \frac{\mathrm{d}r}{r} \\ &\le (\ln t)^{\beta-1} \int_1^e \frac{L(e)}{\sigma} M_h \frac{\mathrm{d}r}{r} + (\ln t)^{\beta-1} \int_1^e \frac{L(e)}{\sigma} \varphi_q \left(\int_1^e \Lambda(r,\omega) M_h^{'} \frac{\mathrm{d}\omega}{\omega} \right) \frac{\mathrm{d}r}{r} \\ &= n_2 h(t). \end{aligned}$$

In view of $\Theta(t,r) > 0$, $\Lambda(r,\omega) > 0$, $Q(r,0) \not\equiv 0$, $P(\omega,1) \not\equiv 0$, then $n_1 > 0$. Based on (H_2) and (H_5), we get $Q(t,0) \le Q(t,h(t)) \le M_h$ and $P(t,1) \le P(t,h(t)) \le M_h^{'}$ for $t \in [1,e]$. Then, together with lemma 4, yields that $0 < n_1 \le n_2$. Therefore, $Ah + Bh \in P_h$. □

Theorem 2. *We suppose that* (H_2)*,* (H_5)*, and* (H_6)*, are fulfilled. Then the problem (1.4) admits a unique positive solution* y^* *in* P_h*. For an arbitrary initial point* x_0*,*y_0 *within the space* P_h*, the sequences defined by*

$$x_{n+1}(t) = \int_1^e \Theta(t,r) \varphi_q \left(\int_1^e \Lambda(r,\omega) P(\omega, x_n(\omega)) \frac{\mathrm{d}\omega}{\omega} \right) \frac{\mathrm{d}r}{r} + \int_1^e \Theta(t,r) Q(r, y_n(r)) \frac{\mathrm{d}r}{r},$$

$$y_{n+1}(t) = \int_1^e \Theta(t,r) \varphi_q \left(\int_1^e \Lambda(r,\omega) P(\omega, y_n(\omega)) \frac{\mathrm{d}\omega}{\omega} \right) \frac{\mathrm{d}r}{r} + \int_1^e \Theta(t,r) Q(r, x_n(r)) \frac{\mathrm{d}r}{r},$$

$n = 0, 1, \ldots,$ *we obtain* $x_n(t) \to y^*(t)$*,* $y_n(t) \to y^*(t)$ *as* $n \to \infty$*.*

Proof. Based on (H_2), (H_5), and Lemmas 9 and 10, we claim that A is increasing at the same time B is decreasing. According to (H_6), we obtain

$$\begin{aligned} A(\lambda y)(t) &= \int_1^e \Theta(t,r) Q(r, \lambda y(r)) \frac{\mathrm{d}r}{r} \\ &\ge \phi_1(\lambda) \int_1^e \Theta(t,r) Q(r, x(r)) \frac{\mathrm{d}r}{r} \\ &= \phi_1 Ay(t). \end{aligned}$$

$$\begin{aligned} B(\lambda y)(t) &= \int_1^e \Theta(t,r) \varphi_q \left(\int_1^e \Lambda(r,\omega) P(\omega, \lambda y(\omega)) \frac{\mathrm{d}\omega}{\omega} \right) \frac{\mathrm{d}r}{r} \\ &\le \int_1^e \Theta(t,r) \varphi_q \left(\int_1^e \Lambda(r,\omega) \varphi_p \left(\frac{1}{\phi_2(\lambda)} \right) P(\omega, y(\omega)) \frac{\mathrm{d}\omega}{\omega} \right) \frac{\mathrm{d}r}{r} \\ &= \frac{1}{\phi_2(\lambda)} \int_1^e \Theta(t,r) \varphi_q \left(\int_1^e \Lambda(r,\omega) P(\omega, y(\omega)) \frac{\mathrm{d}\omega}{\omega} \right) \frac{\mathrm{d}r}{r} \end{aligned}$$

$$= \frac{1}{\phi_2(\lambda)} By(t).$$

So y^* is the unique solution for problem (1.4). For an arbitrary initial point x_0,y_0 within the space P_h, constructing sequences by

$$x_{n+1}(t) = \int_1^e \Theta(t,r)\varphi_q\left(\int_1^e \Lambda(r,\omega) P(\omega, x_n(\omega)) \frac{d\omega}{\omega}\right)\frac{dr}{r} + \int_1^e \Theta(t,r) Q(r, y_n(r)) \frac{dr}{r},$$

$$y_{n+1}(t) = \int_1^e \Theta(t,r)\varphi_q\left(\int_1^e \Lambda(r,\omega) P(\omega, y_n(\omega)) \frac{d\omega}{\omega}\right)\frac{dr}{r} + \int_1^e \Theta(t,r) Q(r, x_n(r)) \frac{dr}{r},$$

$n = 0, 1, 2, \ldots$, there is $x_n(t) \to y^*(t)$, $y_n(t) \to y^*(t)$ as $n \to \infty$. □

4. Examples

In this section, we give two simple examples to explain the main results.

Example 1. We consider the following BVP:

$$\begin{cases} {}^H D_{1^+}^{\frac{5}{4}}\left(\varphi_p\left(-{}^H D_{1^+}^{\frac{17}{4}} y(t) - \frac{(ty)^{\frac14}}{1+(\ln y)^{\frac{13}{4}}} - 3\right)\right) + \frac{y^{\frac13}}{\left[1+(\ln t)^{\frac{13}{4}}\right] + y^{\frac13}} + \cos^2 t = 0, 1 < t < e, \\ y(1) = y'(1) = y^{(2)}(1) = y^{(3)}(1); {}^H D_{1^+}^{\frac{11}{4}} y(e) = \sum_{m=1}^{\infty} \delta_m^H D_{1^+}^{\frac{9}{4}} y(\varepsilon_m), \\ {}^H D_{1^+}^{\frac{17}{4}} y(1) + \frac{(y)^{\frac14}}{1+(\ln y)^{\frac{13}{4}}} + 3 = 0, \\ \varphi_p\left(-{}^H D_{1^+}^{\frac{17}{4}} y(e) - \frac{(ey)^{\frac14}}{1+(\ln y)^{\frac{13}{4}}} - 3\right) = \sum_{m=1}^{\infty} \theta_m \varphi_p\left(-{}^H D_{1^+}^{\frac{17}{4}} y(\varepsilon_m) - \frac{(\varepsilon_m y)^{\frac14}}{1+(\ln y)^{\frac{13}{4}}} - 3\right), \end{cases} \tag{4.1}$$

where$\delta_m = \frac{1}{2m^2}$, $\varepsilon_m = e^{\frac{1}{m^4}}$, $\theta_m = \frac{1}{2m^2}$, $h(t) = (\ln t)^{\frac{13}{4}}$. Moreover,

$$P(t,y) = \frac{y^{\frac13}}{\left[1+(\ln t)^{\frac{13}{4}}\right] + y^{\frac13}} + \cos^2 t, \ Q(t,y) = \frac{(ty)^{\frac14}}{1+(\ln y)^{\frac{13}{4}}} + 3.$$

Clearly, $P, Q : [1,e] \times \mathbb{R}_+ \to \mathbb{R}_+$ and $P(t,0)$, $Q(t,0) \not\equiv 0$, for $t \in [1,e]$. Further, $P(t,y)$ and $Q(t,y)$ are increasing in y. So the condition (H_1) is satisfied.

In addition, when $0 \le y \le M$,

$$\begin{aligned} P\left(t, \left[1+(\ln t)^{\frac{13}{4}}\right] y\right) &= \frac{y^{\frac13}}{\left[1+(\ln t)^{\frac{13}{4}}\right]^{\frac23} + y^{\frac13}} + \cos^2 t \\ &\le \frac{M^{\frac13}}{1+M^{\frac13}} + \cos^2 t. \end{aligned}$$

$$Q\left(t,\left[1+(\ln t)^{\frac{13}{4}}\right]y\right)=\frac{(ty)^{\frac{1}{4}}}{\left[1+(\ln t)^{\frac{13}{4}}\right]^{\frac{3}{4}}}+3 \\ \leq (tM)^{\frac{1}{4}}+3.$$

Then we can conclude the condition (H_2) is fulfilled.

Suppose $\delta=\frac{1}{2}$, $\lambda\in(0,1)$, $t\in[1,e]$, $y\in\mathbb{R}_+$

$$\begin{aligned} P(t,\lambda y) &= \cos^2 t+\frac{(\lambda y)^{\frac{1}{3}}}{\left[1+(\ln t)^{\frac{13}{4}}\right]+(\lambda y)^{\frac{1}{3}}} \\ &\geq \lambda^2\cos^2 t+\lambda^2\frac{y^{\frac{1}{3}}}{\left[1+(\ln t)^{\frac{13}{4}}\right]+y^{\frac{1}{3}}} \\ &= \lambda^2 P(t,y). \end{aligned}$$

$$\begin{aligned} Q(t,\lambda y) &= \frac{\lambda^{\frac{1}{4}}(ty)^{\frac{1}{4}}}{1+(\ln t)^{\frac{13}{4}}}+3 \\ &\geq \lambda^{\frac{1}{2}}\left[\frac{(ty)^{\frac{1}{4}}}{1+(\ln t)^{\frac{13}{4}}}+3\right] \\ &= \lambda^{\frac{1}{2}}Q(t,y). \end{aligned}$$

Hence, condition (H_3) holds.

Moreover, setting $\delta_0=2$, we have

$$P(t,y)=\frac{y^{\frac{1}{3}}}{\left[1+(\ln t)^{\frac{13}{4}}\right]+y^{\frac{1}{3}}}+\cos^2 t\leq 2\leq 3=Q(t,0).$$

In other words, (H_4) is met. So, on the basis of Theorem 1, we know the problem (4.1) possesses a unique positive solution in P_h.

Example 2. We consider the following boundary value problem

$$\begin{cases} {}^{H}D_{1^+}^{\frac{5}{4}}\left(\varphi_3\left(-{}^{H}D_{1^+}^{\frac{17}{4}}x(t)-\dfrac{(tx)^{\frac{1}{3}}}{1+(\ln t)^{\frac{13}{4}}}-3\right)\right)+\dfrac{1+(1+x)^{-\frac{1}{4}}}{1+(\ln t)^{\frac{13}{4}}}=0,1<t<e, \\ y(1)=y^{'}(1)=y^{(2)}(1)=y^{(3)}(1);{}^{H}D_{1^+}^{\frac{11}{4}}y(e)=\displaystyle\sum_{m=1}^{\infty}\delta_m^{H}D_{1^+}^{\frac{9}{4}}y(\delta_m), \\ {}^{H}D_{1^+}^{\frac{17}{4}}y(1)+y^{\frac{1}{3}}(1)+3=0, \\ \varphi_p\left(-{}^{H}D_{1^+}^{\frac{17}{4}}y(e)-\dfrac{(ey(e))^{\frac{1}{3}}}{2}-3\right)=\displaystyle\sum_{m=1}^{\infty}\theta_m\varphi_p\left(-{}^{H}D_{1^+}^{\frac{17}{4}}y(\delta_m)-\dfrac{(\delta_m y(\delta_m))^{\frac{1}{3}}}{1+(\ln\delta_m)^{\frac{13}{4}}}-3\right), \end{cases} \tag{4.2}$$

where $\delta_m=\frac{1}{2m^2}$, $\varepsilon_m=e^{\frac{1}{m^4}}$, $\theta_m=\frac{1}{2m^2}$, $h(t)=(\ln t)^{\frac{13}{4}}$. Moreover,

$$P(t,y)=\frac{1+(1+y)^{-\frac{1}{4}}}{1+(\ln t)^{\frac{13}{4}}},\ Q(t,y)=\frac{(ty)^{\frac{1}{3}}}{1+(\ln t)^{\frac{13}{4}}}+3.$$

Obviously, P, $Q : [1, e] \times \mathbb{R}_+ \to \mathbb{R}_+$ and $P(t, 1)$, $Q(t, 0) \not\equiv 0$, for $t \in [1, e]$. Further, $P(t, y)$ is decreasing in y, and $Q(t, y)$ is increasing in the second argument. So the condition (H_5) is satisfied.

In addition, when $0 \le y \le M$,

$$\begin{aligned} P\left(t, \left[1 + (\ln t)^{\frac{13}{4}}\right] y\right) &= \frac{1 + \left[1 + \left(1 + (\ln t)^{\frac{13}{4}}\right) y\right]^{-\frac{1}{4}}}{1 + (\ln t)^{\frac{13}{4}}} \\ &\le 2. \end{aligned}$$

$$\begin{aligned} Q\left(t, \left[1 + (\ln t)^{\frac{13}{4}}\right] y\right) &= \frac{(ty)^{\frac{1}{3}}}{\left[1 + (\ln t)^{\frac{13}{4}}\right]^{\frac{2}{3}}} + 3 \\ &\le (tM)^{\frac{1}{4}} + 3. \end{aligned}$$

Thus the condition (H_2) holds.

Take $\phi_1(\lambda) = \lambda^{\frac{1}{3}}$ and $\phi_2(\lambda) = \lambda^{\frac{1}{4}}$; then $\phi_1(\lambda), \phi_2(\lambda) \in (\lambda, 1)$ for $t \in [1, e]$. Therefore,

$$\begin{aligned} P(t, \lambda y) &= \frac{1 + [1 + \lambda y]^{-\frac{1}{4}}}{1 + (\ln t)^{\frac{13}{4}}} \\ &= \lambda^{-\frac{1}{2}} \frac{\lambda^{\frac{1}{2}} + \lambda^{\frac{1}{2}} [1 + \lambda y]^{-\frac{1}{4}}}{\left[1 + (\ln t)^{\frac{13}{4}}\right]} \\ &\le \lambda^{-\frac{1}{2}} \frac{1 + [1 + \lambda y]^{-\frac{1}{4}}}{1 + (\ln t)^{\frac{13}{4}}}. \\ &\le \varphi_3\left(\frac{1}{\phi_2(\lambda)}\right) P(t, y) \end{aligned}$$

$$\begin{aligned} Q(t, \lambda y) &= \frac{\lambda^{\frac{1}{3}} (ty)^{\frac{1}{3}}}{[1 + (\ln t)^{\frac{13}{4}}} + 3 \\ &\ge \phi_1(\lambda) Q(t, y). \end{aligned}$$

So (H_6) is fulfilled. Together with theorem 2, yields that the problem (4.2) has a unique positive solution.

The two numerical examples proposed in this study introduce more complex multi-point and infinite-series boundary conditions compared to reference [6], employing fixed-point theorems on cones. By constructing a specialized cone P_h and verifying the concavity of the operators, the approach is better suited for strongly nonlinear problems. In contrast to reference [24], the work further generalizes the complexity of boundary conditions and incorporates mixed-type nonlinear terms, demonstrating an advanced application of the theoretical tools from [24]. Unlike reference [25], which primarily uses Riemann–Liouville or Caputo fractional derivatives, this study adopts Hadamard-type fractional derivatives, making it more applicable to problems with logarithmic nonlinearities or geometric growth. The complexity and nonlocal characteristics of these two examples make them suitable for multiscale, nonlinear systems with memory/heritability effects, particularly offering advantages in scenarios requiring precise modeling of non-classical diffusion, memory-dependent processes, or hierarchical interactions.

5. Conclusions

In this article, we began our analysis by reformulating solutions to the fractional differential equation as fixed points of an operator equation. Employing two fixed-point theorems of the sum operator, we obtained a unique positive solution of the boundary value problems. Finally, to further illustrate the applicability of our findings, we presented two interesting examples. A key contribution of this work lies in the identification of a unique positive solution within a partially ordered space. Furthermore, we showed that this solution could be approximated through iterative sequences, constructed from arbitrary initial points within the space. In addition, this study introduced more sophisticated multi-point and infinite-series boundary conditions. By employing fixed-point theorems on cones and constructing a specialized cone P_h with verified operator concavity, the proposed approach demonstrated enhanced applicability for strongly nonlinear problems.

Looking ahead, this study opens up several promising research directions: extending the framework to variable-order fractional operators for enhanced modeling of multi-scale phenomena, incorporating stochastic elements to construct uncertainty-aware system models, and exploring coupled systems involving p-Laplacian operators to improve modeling capabilities for complex physical processes. These extensions would significantly advance both theoretical foundations and practical applications of fractional differential equations in scientific computing and engineering analysis.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under Grant (62473227, 62073186).

Conflict of interest

The authors declare there are no conflicts of interest.

References

1. L. Guo, C. Li, J. Zhao, Existence of monotone positive solutions for caputo-hadamard nonlinear fractional differential equation with infinite-point boundary value conditions, *Symmetry*, **15** (2023), 970. https://doi.org/10.3390/sym15050970

2. Y. Li, Y. Liu, Multiple solutions for a class of boundary value problems of fractional differential equations with generalized Caputo derivatives, *AIMS Math.*, **6** (2021), 13119–13142. https://doi.org/10.3934/math.2021758
3. X. Zuo, W. Wang, Existence of solutions for fractional differential equation with periodic boundary condition, *AIMS Math.*, **7** (2022), 6619–6633. https://doi.org/10.3934/math.2022369
4. C. Ciftci, F. Deren, Analysis of p-Laplacian Hadamard fractional boundary value problems with the derivative term involved in the nonlinear term, *Math. Method Appl. Sci.*, **46** (2023), 8945–8955. https://doi.org/10.1002/mma.9028
5. L. Guo, H. Liu, C. Li, J. Zhao, J. Xu, Existence of positive solutions for singular p-Laplacian hadamard fractional differential equations with the derivative term contained in the nonlinear term, *Nonlinear Anal. Modell. Control*, **28** (2023), 491–515. https://doi.org/10.15388/namc.2023.28.31728
6. K. Zhang, J. Wang, W. Ma, Solutions for integral boundary value problems of nonlinear Hadamard fractional differential equations, *J. Funct. Spaces*, **2018** (2018), 1–10. https://doi.org/10.1155/2018/2193234
7. A. Tudorache, R. Luca, Positive solutions to a system of coupled Hadamard fractional boundary value problems, *Fractal Fract.*, **8** (2024). https://doi.org/10.3390/fractalfract8090543
8. J. Xu, J. Liu, D. O'Regan, Solvability for a Hadamard-type fractional integral boundary value problem, *Nonlinear Anal. Model. Control*, **28** (2023), 672–696. https://doi.org/10.15388/namc.2023.28.32130
9. H. Lu, Z. Han, S. Sun, Existence on positive solutions for boundary value problems of nonlinear fractional differential equations with p-Laplacian, *Adv. Differ. Equations*, **30** (2013). https://doi.org/10.1186/1687-1847-2013-30
10. L. Guo, Y. Wang, H. Liu, C. Li, W. Wang, Y. Cui, et al., On iterative positive solutions for a class of singular infinite-point p-Laplacian fractional differential equation with singular source terms, *J. Appl. Anal. Comput.*, **13** (2023), 2827–2842. https://doi.org/10.15388/NA.2018.2.3
11. K. Jong, H. Choi, Y. Ri, Existence of positive solutions of a class of multi-point boundary value problems for p-Laplacian fractional differential equations with singular source terms, *Commun. Nonlinear Sci. Numer. Simul.*, **72** (2019), 272–281. https://doi.org/10.1016/j.cnsns.2018.12.021
12. J. Wu, X. Zhang, L. Liu, Y. Wu, Y. Cui, The convergence analysis and error estimation for unique solution of a p-Laplacian fractional differential equation with singular decreasing nonlinearity, *Bound Value Probl.*, **82** (2018). https://doi.org/10.1186/s13661-018-1003-1
13. X. Zhang, C. Mao, L. Liu, Y. Wu, Exact iterative solution for an abstract fractional dynamic system model for bioprocess, *Qual. Theory Dyn. Syst.*, **16** (2017), 205–222. https://doi.org/10.1007/s12346-015-0162-z
14. X. Zhang, L. Liu, Y. Wu, Y. Cui,The existence and nonexistence of entire large solutions for a quasilinear Schrödinger elliptic system by dual approach, *J. Math. Anal. Appl.*, **464** (2018), 1089–1106. https://doi.org/10.1016/j.jmaa.2018.04.040

15. X. Zhang, P. Chen, Y. Wu, B. Wiwatanapataphee, A necessary and sufficient condition for the existence of entire large solutions to a k-Hessian system, *Appl. Math. Lett.*, **145** (2023). https://doi.org/10.1016/j.aml.2023.108745

16. Z. Zhang, X. Yang, S. Wang, The alternating direction implicit difference scheme and extrapolation method for a class of three dimensional hyperbolic equations with constant coefficients, *Electron. Res. Arch.*, **33** (2025), 3348–3377. https://doi.org/10.3934/era.2025148

17. J. Zhang, X. Yang, S. Wang, The ADI difference and extrapolation scheme for high-dimensional variable coefficient evolution equations, *Electron. Res. Arch.*, **33** (2025), 3305–3327. https://doi.org/10.3934/era.2025146

18. T. Liu, H. Zhang, X. Yang, The ADI compact difference scheme for three-dimensional integro-partial differential equation with three weakly singular kernels, *J. Appl. Math. Comput.*, **71** (2025), 1–29. https://doi.org/10.1007/s12190-025-02386-3

19. K. Liu, Z. He, H. Zhang, X. Yang, A Crank-Nicolson ADI compact difference scheme for the three-dimensional nonlocal evolution problem with a weakly singular kernel, *Comput. Appl. Math.*, **44** (2025), 164. https://doi.org/10.1007/s40314-025-03125-x

20. X. Yang, Z. Zhang, Analysis of a new NFV scheme preserving DMP for two-dimensional sub-diffusion equation on distorted meshes, *J. Sci. Comput.*, **99** (2024), 80. https://doi.org/10.1007/s10915-024-02511-7

21. J. Wang, X. Jiang, H. Zhang, X. Yang, A new fourth-order nonlinear difference scheme for the nonlinear fourth-order generalized Burgers-type equation, *J. Appl. Math. Comput.*, (2025), 1–31. https://doi.org/10.1007/s12190-025-02467-3

22. X. Yang, Z. Zhang, Superconvergence analysis of a robust orthogonal Gauss collocation method for 2D fourth-order subdiffusion equations, *J. Sci. Comput.*, **100** (2024), 62. https://doi.org/10.1007/s10915-024-02616-z

23. Y. Shi, X. Yang, The pointwise error estimate of a new energy-preserving nonlinear difference method for supergeneralized viscous Burgers' equation, *Comput. Appl. Math.*, **44** (2025), 257. https://doi.org/10.1007/s40314-025-03222-x

24. L. Guo, W. Wang, C. Li, J. Zhao, D. Min, Existence results for a class of nonlinear singular p-Laplacian Hadamard fractional differential equations, *Electron. Res. Arch.*, **32** (2024), 928–944. http://doi.org/10.3934/era.2024045

25. C. Zhai, Y. Ma, H. Li, Unique positive solution for a p-Laplacian fractional differential boundary value problem involving Riemann-Stieltjes integral, *AIMS Math.*, **5** (2020), 4754–4769. https://doi.org/10.3934/math.2020304

26. B. Ahmad, A. Alsaedi, S. Ntouyas, J. Tariboon, *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*, Springer: Cham, Switzerland, 2017. http://dx.doi.org/10.1007/978-3-319-52141-1

27. A. Kilbas, H. Srivastava, J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier: Amsterdam, 2003. http://doi.org/10.1016/S0304-0208(06)80001-0

28. D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones. Academic Press*, Academic Press, Boston, 1998. https://doi.org/10.1016/c2013-0-10750-7

29. C. Zhai, D. Anderson, A sum operator equation and applications to nonlinear elastic beam equations and Lane-Emden-Fowler equations, *J. Math. Anal. Appl.*, **375** (2011), 388–400. https://doi.org/10.1016/j.jmaa.2010.09.017

30. C. Yang, C. Zhai, M. Hao, Uniqueness of positive solutions for several classes of sum operator equations and applications, *J. Inequal. Appl.*, **2014** (2014), 58. https://doi.org/10.1186/1029-242X-2014-58