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Research article

Dynamic analysis and optimal control of a fractional order predator-prey model with economic threshold

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Abstract: In this paper, a fractional order differential algebraic predator-prey system with Holling type functional response was presented. The dynamic behavior of populations under the influence of economic benefits was thoroughly investigated. Specifically, with economic benefit as the bifurcation parameter, the existence of singularity-induced bifurcation was analyzed based on the theory of differential algebraic system. After that, a state feedback controller was designed to eliminate the singularity induced bifurcation and stabilize the system near the corresponding interior equilibrium. Furthermore, an optimal control problem was formulated, and the necessary conditions were derived. In addition, the validity of the theoretical results was confirmed through extensive numerical simulations.

Keywords: predator-prey model; fractional order; differential algebraic equations; singularity-induced bifurcation; stability; optimal control

1. Introduction

In the fields of fishery, forestry, and other resource development sectors, accurately predicting population dynamics is crucial for formulating scientifically sound management strategies. Over the past few years, there has been a growing interest in population dynamics modeling within these domains [1]. Mathematical models, particularly predator-prey models, serve as powerful tools for analyzing the interactions between resource populations and predator populations [2–6].

The predator-prey model has been widely studied in various forms due to its universality and significance. Numerous ecologists and mathematicians have explored the qualitative properties of predator-prey models within population dynamics [7–12]. Lotka in 1925 and Volterra in 1926 introduced the first predator-prey model. Following this, many more intricate but realistic predator-prey models, with various forms of “functional responses”, have been proposed by economists. Among these, one of the most interesting predator-prey models was introduced by Abhijit [13] in 2021, which has the Holling type-I and Holling type-II functional response. Khajanchi et al. [14, 15] introduced stage structure into the predator-prey model with ratio-dependent and Holling type-IV functional response and studied its stability. Rihan et al. [16] investigated a delayed predator-prey model incorporating Allee effects, with their research demonstrating that during the early stages of low population density, the model exhibits high sensitivity to minute variations in Allee parameters, though this sensitivity gradually diminishes over time. However, the majority of the models studied in the literature are based on integer-order differential equations.

With the advancement of mathematical modeling, researchers have increasingly turned to fractional differential equations for investigating predator-prey dynamics [17, 18]. Unlike traditional integer-order models, fractional-order systems inherently possess memory properties that better capture nonlocal temporal effects and hereditary traits in ecological interactions, such as predator maturation delays, gestation periods, and prey refuge adaptation mechanisms [19–22]. These characteristics enable more precise descriptions of memory dependent biological processes in predator prey systems. Notable contributions in this field include Ramesh's work on a fractional-order delayed predator-prey model incorporating Holling type-II functional response and prey refuge, which established quantitative relationships between refuge availability and dynamical system stability [23]. Almatrafi and Berkal [24] applied bifurcation theory to analyze a fractional-order predator-prey dynamical system with Allee effects, investigating the system's dynamic characteristics through piecewise constant approximation and conformable derivative discretization methods.

From an economic perspective, biological resources in predator-prey ecosystems are frequently harvested for commercial gain through fishing and sales. These economic activities profoundly impact the dynamic evolution of prey and predator populations. To better analyze such systems, economic factors influencing harvesting efforts can be integrated into the model, typically represented by algebraic equations [25–29]. For instance, Gao et al. [30] introduced an algebraic equation to quantify the economic benefits of commercial fishing in protected fisheries. Building on this work, Liu et al. [31] further advanced the approach by incorporating time delays into algebraic equations, thereby proposing a novel model to examine the conditions for singularity induced bifurcation and Hopf bifurcation. In a related study, Komeil et al. [32] integrated fractional-order dynamics into a singular system with Holling type-II functional response, investigating how changes in economic benefits alter the systems dynamic behavior.

Optimal control theory serves as a powerful framework for determining the most efficient fishing harvest strategies. This methodological approach has been extensively applied across interdisciplinary domains, including biological sciences, economics, business management, and engineering systems [33–36]. Pontryagin's maximum principle remains the predominant analytical tool in such studies. In their research, Gao et al. [37] laid a theoretical foundation for fishery economic development through optimal control analysis of net economic returns. Meanwhile, Liu and Huang [38] addressed harvesting challenges involving functional responses, demonstrating how bang-bang control and singular control strategies can be systematically applied to formulate optimal harvesting policies.

To sum up, existing studies still have shortcomings in aspects such as fractional-order modeling of multi-species ecosystems, coupling analysis of economic factors and ecological dynamics, control of singularity-induced bifurcations, and optimal management strategies. Therefore, this paper constructs a fractional-order singular system of two-prey and two-predator with economic thresholds, aiming to explore the ability of fractional-order derivatives to depict ecological complexity, analyze the system dynamics and singularity-induced bifurcation phenomena under different economic benefit scenarios, design controllers, and optimal control strategies, and, finally, verify the theoretical results and the effectiveness of the controllers through numerical simulations, so as to provide theoretical support for the sustainable management of ecosystems and resource development.

The remainder of this paper is structured as follows. In Section 2, the fundamental model is established, and the solvability conditions for the system are presented. Section 3 examines the combined effects of economic interest on population dynamics. The existence of singularity-induced bifurcation is investigated, and state feedback controller is designed to eliminate such bifurcation. Section 4 formulates the optimal control problem. Numerical simulations are carried out in Section 5 to validate the theoretical results. Finally, this paper is concluded with a summary.

2. Model formulation

In this section, inspired by [13, 32], we establish a predator-prey model with fractional order derivatives as follows:

$$
\begin{cases}
\mathrm{D}^{\alpha} x(t) &= rx\left(1 - \dfrac{x}{K}\right) - a_1 xy - \omega_1 xw - \omega_2 xv, \\
\mathrm{D}^{\alpha} y(t) &= sy\left(1 - \dfrac{y}{L}\right) - a_2 xy - \dfrac{\omega_3 wy}{m + y} - \omega_4 vy, \\
\mathrm{D}^{\alpha} w(t) &= n_1 \omega_1 xw + \dfrac{n_2 \omega_3 yw}{m + y} - b_1 wv - d_1 w, \\
\mathrm{D}^{\alpha} v(t) &= n_3 \omega_2 xv + n_4 \omega_4 yv - b_2 wv - d_2 v,
\end{cases}
\tag{2.1}
$$

with initial condition

$$x(0) \geq 0,\ y(0) \geq 0,\ w(0) \geq 0,\ v(0) \geq 0.$$

Here, D^{α} is the Caputo fractional order derivative, $\alpha \in (0, 1]$. x and y represent the populations of two types of prey, while w and v denote the populations of two types of predators. Here, species y and v are respectively larger in size compared to species x and w.

This fractional-order definition enhances the flexibility of the predator-prey model and aligns it more closely with the reality of prey-predator interactions, as it enables precise descriptions of nonlinear phenomena. However, it is clear that model (2.1) exhibits a dimensional inconsistency along the time axis, as the dimensions on both sides of the equation do not match. To ensure dimensional consistency, we employ the correction method proposed in [39]. As a result, system (2.1) should be rewritten as:

$$\begin{cases} \frac{1}{\sigma^{1-\alpha}}\mathrm{D}^{\alpha}x(t) &= rx\left(1-\frac{x}{K}\right)-a_1xy-\omega_1xw-\omega_2xv, \\ \frac{1}{\sigma^{1-\alpha}}\mathrm{D}^{\alpha}y(t) &= sy\left(1-\frac{y}{L}\right)-a_2xy-\frac{\omega_3wy}{m+y}-\omega_4vy, \\ \frac{1}{\sigma^{1-\alpha}}\mathrm{D}^{\alpha}w(t) &= n_1\omega_1xw+\frac{n_2\omega_3yw}{m+y}-b_1wv-d_1w, \\ \frac{1}{\sigma^{1-\alpha}}\mathrm{D}^{\alpha}v(t) &= n_3\omega_2xv+n_4\omega_4yv-b_2wv-d_2v. \end{cases} \tag{2.2}$$

Harvesting is widely recognized to significantly impact the dynamic evolution of populations. Over recent years, increasing attention has been paid to predator-prey models with the Holling-type functional response. For instance, in fishery model [30], biological resources are harvested and sold for economic gain.

Given that commercial harvesting is influenced by various factors such as harvesting costs, income, and market demand, it is practical to treat the harvesting effort as a variable. Consequently, the harvesting function $h(t)$ can be expressed as $h(t) = z(t)v(t)$, where $z(t)$ represents the harvesting effort exerted by the static population.

Drawing on economic theory and utilizing the proposed algebraic equations to explore the economic benefits of the harvesting effort [40], the predator-prey fractional-order model (2.2) can be further developed. Thus, the economic benefit equation is expressed as:

$$ph(t) - cz(t) = M, \tag{2.3}$$

where M denotes the net economic profit, $ph(t)$ is total revenue, and $cz(t)$ is total cost. Then, p and c are respectively the price per unit of harvested biomass and the cost per unit of effort. Thus, we obtain the following fractional order singular (FOS) system :

$$\begin{cases} \frac{1}{\sigma^{1-\alpha}}\mathrm{D}^{\alpha}x(t) &= rx\left(1-\frac{x}{K}\right)-a_1xy-\omega_1xw-\omega_2xv, \\ \frac{1}{\sigma^{1-\alpha}}\mathrm{D}^{\alpha}y(t) &= sy\left(1-\frac{y}{L}\right)-a_2xy-\frac{\omega_3wy}{m+y}-\omega_4vy, \\ \frac{1}{\sigma^{1-\alpha}}\mathrm{D}^{\alpha}w(t) &= n_1\omega_1xw+\frac{n_2\omega_3yw}{m+y}-b_1wv-d_1w, \\ \frac{1}{\sigma^{1-\alpha}}\mathrm{D}^{\alpha}v(t) &= n_3\omega_2xv+n_4\omega_4yv-b_2wv-d_2v-zv, \\ 0 &= z(pv-c)-M, \end{cases} \tag{2.4}$$

with $x(0) \geq 0,\ y(0) \geq 0,\ w(0) \geq 0,\ v(0) \geq 0,\ 0 < \alpha \leq 1$. All the parameters are described in Table 1. System (2.4) can be transformed into a matrix form as follows:

$$E\frac{1}{\sigma^{1-\alpha}}\mathrm{D}^{\alpha}Y(t) = F(Y(t)),\ 0 < \alpha \leq 1, \tag{2.5}$$

Table 1. The biological meanings of the variables and parameters of system (2.1).

Variables	**Description**		
$x(t)$	The density of the smaller prey		
$y(t)$	The density of the larger prey		
$w(t)$	The density of the smaller predator		
$v(t)$	The density of the larger predator		
Parameters	**Description**	**Value**	**Reference**
r	The intrinsic growth rate of smaller prey x	0.85	[13]
s	The intrinsic growth rate of larger prey y	0.7	[13]
K	The environmental carrying capacity for smaller prey x	6000	[13]
L	The environmental carrying capacity for larger prey y	2000	[13]
a_1	Inter-species interference coefficient of prey y on prey x	0.000001	[13]
a_2	Inter-species interference coefficient of prey x on prey y	0.0009	[13]
b_1	Inter-species interference coefficients of predator v on predator w	0.002	[13]
b_2	Inter-species interference coefficients of predator w on predator v	0.0005	[13]
ω_1	The searching efficiency of smaller predator w with respect to smaller prey x	0.013	[13]
ω_3	The searching efficiency of smaller predator w with respect to larger prey y	0.02	[13]
ω_2	The searching efficiency of larger predator v with respect to smaller prey x	0.005	[13]
ω_4	The searching efficiency of larger predator v with respect to larger prey y	0.03	[13]
n_1	Conversion factors denoting the number of newly born smaller predator per unit consumption of smaller prey	0.007	[13]
n_2	Conversion factors denoting the number of newly born smaller predator per unit consumption of larger prey	0.0104	[13]
n_3	Conversion factors denoting the number of newly born larger predator per unit consumption of smaller prey	0.001	[13]
n_4	Conversion factors denoting the number of newly born larger predator per unit consumption of larger prey	0.0093	[13]
m	The half saturation constant	1200	[13]
d_1	Death rates of smaller predator	0.04	[13]
d_2	Death rates of larger predator	0.03	[13]
p	The prices of a unit of the harvested biomass	1.5	[32]
c	The cost of a unit of the effort	$1, 7$	–
M	The net economic profit	$0, 0.0001$	–
A	An economic constant	0.125	[30]
β	The instantaneous annual discount rate	0.01	[30]

where $F : R^5 \longrightarrow R^5$, $Y(t) \in R^5$, and the matric $E \in R^{5\times 5}$ have the following forms:

$$Y(t) = \begin{pmatrix} x(t) \\ y(t) \\ w(t) \\ v(t) \\ z(t) \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$F = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix} = \begin{pmatrix} rx\left(1 - \frac{x}{K}\right) - a_1xy - \omega_1xw - \omega_2xv \\ sy\left(1 - \frac{y}{L}\right) - a_2xy - \frac{\omega_3wy}{m+y} - \omega_4vy \\ n_1\omega_1xw + \frac{n_2\omega_3yw}{m+y} - b_1wv - d_1w \\ n_3\omega_2xv + n_4\omega_4yv - b_2wv - d_2v - zv \\ z(pv - c) - M \end{pmatrix}. \tag{2.6}$$

Remark 1. *Dimensionless scaling simplifies the system of equations by removing all parameters that carry units. Nevertheless, when interpreting the system from a biological perspective, it is crucial to restore the parameters to their original values.*

Theorem 1. *System* (2.5) *is solvable if* $v \neq \frac{c}{p}$.

Proof. System (2.5) can be linearized into the following constant-coefficient system

$$E\frac{1}{\sigma^{1-\alpha}}\mathrm{D}^{\alpha}Y(t) = JY(t),\ 0 < \alpha \le 1, \tag{2.7}$$

where J denotes the Jacobian matrix in the neighborhood of equilibrium Y^*, as shown below:

$$J(Y^*) = \begin{pmatrix} A_1 & -a_1x^* & -\omega_1x^* & -\omega_2x^* & 0 \\ -a_2y^* & A_2 & -\frac{\omega_3y^*}{m+y^*} & -\omega_4y^* & 0 \\ n_1\omega_1w^* & \frac{mn_2\omega_3w^*}{(m+y^*)^2} & A_3 & -b_1w^* & 0 \\ n_3\omega_2v^* & n_4\omega_4v^* & -b_2v^* & A_4 & -v^* \\ 0 & 0 & 0 & pz^* & pv^* - c \end{pmatrix}, \tag{2.8}$$

where

$$\begin{aligned} A_1 &= r - \frac{2rx^*}{K} - a_1y^* - \omega_1w^* - \omega_2v^*, \\ A_2 &= s - \frac{2sy^*}{L} - a_2x^* - \frac{m\omega_3w^*}{(m+y^*)^2} - \omega_4v^*, \\ A_3 &= n_1\omega_1x^* + \frac{n_2\omega_3y^*}{m+y^*} - b_1v^* - d_1, \\ A_4 &= n_3\omega_2x^* + n_4\omega_4y^* - b_2w^* - d_2 - z^*. \end{aligned}$$

We need to prove the following two points:

(1) System (2.7) is always regular,

(2) System (2.7) is impulse-free at all points except $v = \frac{c}{p}$.

For (1), we calculate the determinant of the matrix $|E\lambda - J| \neq 0$ for some $\lambda \in C$, which C is the field of complex number. So, the condition of regularity is always satisfied [41].

For (2), when $v \neq \frac{c}{p}$, we have deg(det|Eλ − J|) = rank E, which ensures that system (2.7) satisfies the impulse-freeness condition. However,

$$\mathrm{deg}(\mathrm{det}|\mathrm{E}\lambda - \mathrm{J}|)|_{\mathrm{v}=\frac{\mathrm{c}}{\mathrm{p}}} \neq \mathrm{rank\ E}.$$

In conclusion, in accordance with Definition 3.1 and Remark 3.1 in [42], system (2.5) is of index one. Consequently, it exhibits no impulsive behavior in the vicinity of equilibrium Y^*. Furthermore, by applying the implicit function theorem [43], system (2.5) is solvable. □

3. Qualitative analysis results for system (2.5)

In this section, our primary objective is to investigate the dynamics of internal equilibrium and deduce sufficient condition for the existence of singular induced bifurcation. Additionally, the method that eliminates singularity induced bifurcation will be explored. This focus is crucial because the biological interpretation of a stable internal equilibrium is closely tied to the sustainable coexistence of prey and predator populations, along with the impact of commercial harvest effort on the predator population.

If $M = 0$, then system (2.5) becomes

$$\begin{cases} \frac{1}{\sigma^{1-\alpha}}\mathrm{D}^{\alpha}x(t) = rx\left(1 - \frac{x}{K}\right) - a_1xy - \omega_1xw - \omega_2xv, \\ \frac{1}{\sigma^{1-\alpha}}\mathrm{D}^{\alpha}y(t) = sy\left(1 - \frac{y}{L}\right) - a_2xy - \frac{\omega_3wy}{m+y} - \omega_4vy, \\ \frac{1}{\sigma^{1-\alpha}}\mathrm{D}^{\alpha}w(t) = n_1\omega_1xw + \frac{n_2\omega_3yw}{m+y} - b_1wv - d_1w, \\ \frac{1}{\sigma^{1-\alpha}}\mathrm{D}^{\alpha}v(t) = n_3\omega_2xv + n_4\omega_4yv - b_2wv - d_2v - zv, \\ 0 = z(pv - c), \end{cases} \tag{3.1}$$

then, the interior equilibrium is : $Y^* = (x^*, y^*, w^*, v^*, z^*)$, where

$$x^* = \frac{cmb_1 + pmd_1 + pd_1y^* + cb_1y^* - pn_2\omega_3y^*}{pn_1\omega_1(m + y^*)}, \quad w^* = \frac{rp - pa_1y^* - c\omega_2}{p\omega_1} - \frac{rx^*}{K\omega_1},$$
$$v^* = \frac{c}{p}, \quad z^* = \frac{Kb_2(c\omega_2 - pr) - Kpd_2\omega_2 + (rpb_2 + pKn_3\omega_1\omega_2)x^* + (pKa_1b_1 + pK\omega_1n_4\omega_4)y^*}{Kp\omega_1},$$

and y^* is a positive root of the

$$B_1y^3 + B_2y^2 + B_3y + B_4 = 0, \tag{3.2}$$

where

$$\begin{aligned}
B_1 &= spKn_1\omega_1^2 > 0,\\
B_2 &= 2spKmn_1\omega_1^2 + LKcn_1\omega_4\omega_1^2 + LKa_2\omega_1(pd_1 + cb_1 - pn_2\omega_3) - pLKa_1n_1\omega_1\omega_3 - spLKn_1\omega_1^2,\\
B_3 &= spKn_1m^2\omega_1^2 + mLKa_2\omega_1(2pd_1 + 2cb_1 - pn_2\omega_3) + LKn_1\omega_1\omega_3(pr - mpa_1 - c\omega_2)\\
&\quad -rL\omega_3(pd_1 + cb_1 - pn_2\omega_3) - 2mLK\omega_1^2n_1(sp - c\omega_4),\\
B_4 &= LKmn_1\omega_1\omega_3(rp - c\omega_2) - rmL\omega_3(cb_1 + pd_1) - LK\omega_1m^2(spn_1\omega_1 - ca_2b_1 - pa_2d_1 - cn_1\omega_1\omega_4).
\end{aligned}$$

Obviously, Eq (3.2) has the discriminant as $\Delta = 18B_1B_2B_3B_4 - 4B_2^3B_4 + B_2^2B_3^2 - 4B_1B_3^3 - 27B_1^2B_4^2$. In Eq (3.2), we have

(1) all three roots are real and distinct when $\Delta > 0$,

(2) there is one real root and a pair of imaginary roots when $\Delta < 0$.

Taking M as the bifurcation parameter, Theorem 2 will investigate the existence of singularity-induced bifurcation (SIB) arising from variations in M.

Theorem 2. *System* (3.1) *undergoes a singularity induced bifurcation at equilibrium Y^* as the bifurcation parameter M increases through zero. Besides, the stability of the equilibrium varies from stable to unstable.*

Proof. According to [44], system (3.1) undergoes a SIB at the equilibrium as the bifurcation parameter M increases through zero, provided that the following three conditions are satisfied. Here, D denotes the differential operator. Denote

$$\begin{aligned}
f_1(x(t), y(t), w(t), v(t), z(t), M) &= rx\left(1 - \frac{x}{K}\right) - a_1xy - \omega_1xw - \omega_2xv,\\
f_2(x(t), y(t), w(t), v(t), z(t), M) &= sy\left(1 - \frac{y}{L}\right) - a_2xy - \frac{\omega_3wy}{m + y} - \omega_4vy,\\
f_3(x(t), y(t), w(t), v(t), z(t), M) &= n_1\omega_1xw + \frac{n_2\omega_3yw}{m + y} - b_1wv - d_1w,\\
f_4(x(t), y(t), w(t), v(t), z(t), M) &= n_3\omega_2xv + n_4\omega_4yv - b_2wv - d_2v - zv,\\
f_5(x(t), y(t), w(t), v(t), z(t), M) &= z(pv - c).
\end{aligned}$$

(1) Let $f_6(x(t), y(t), w(t), v(t), z(t), M) = D_zf_5(x(t), y(t), w(t), v(t), z(t), M) = pv - c$. It has a simple zero eigenvalue, and

$$\text{trace}\left[\begin{pmatrix} D_zf_1\\ D_zf_2\\ D_zf_3\\ D_zf_4 \end{pmatrix} \text{adj}(\mathrm{D_zf_5}) \begin{pmatrix} D_xf_5 & D_yf_5 & D_wf_5 & D_vf_5 \end{pmatrix}\right]_{\mathrm{Y^*}} = -pv^*z^* < 0. \tag{3.3}$$

(2)

$$\begin{vmatrix} D_xf_1 & D_yf_1 & D_wf_1 & D_vf_1 & D_zf_1 \\ D_xf_2 & D_yf_2 & D_wf_2 & D_vf_2 & D_zf_2 \\ D_xf_3 & D_yf_3 & D_wf_3 & D_vf_3 & D_zf_3 \\ D_xf_4 & D_yf_4 & D_wf_4 & D_vf_4 & D_zf_4 \\ D_xf_5 & D_yf_5 & D_wf_5 & D_vf_5 & D_zf_5 \end{vmatrix}_{Y^*} = \begin{vmatrix} C_1 & -a_1x & -\omega_1 x & -\omega_2 x & 0 \\ -a_2y & C_2 & -\dfrac{\omega_3 y}{m+y} & -\omega_4 y & 0 \\ n_1\omega_1 w & \dfrac{mn_2\omega_3 w}{(m+y)^2} & C_3 & -b_1 w & 0 \\ n_3\omega_2 v & n_4\omega_4 v & -b_2 v & C_4 & -v \\ 0 & 0 & 0 & pz & pv-c \end{vmatrix} \tag{3.4}$$

$$\begin{aligned} &= -\left[csrKm^2d_1d_2 + crsKd_1d_2(y^*)^2 + Kcrsm^2z + 2crm^2a_2d_1d_2(x^*)^2\right] \\ &\quad -\left[2rspd_1d_2xv(y^*)^2 + 2cra_2b_2d_1wx^2y^2\right] < 0, \end{aligned}$$

where

$$C_1 = r - \frac{2rx}{K} - a_1y - \omega_1 w - \omega_2 v, \quad C_2 = s - \frac{2sy}{L} - a_2x - \frac{m\omega_3 w}{(m+y)^2} - \omega_4 v,$$
$$C_3 = n_1\omega_1 x + \frac{n_2\omega_3 y}{m+y} - b_1v - d_1, \quad C_4 = n_3\omega_2 x + n_4\omega_4 y - b_2 w - d_2 - z.$$

(3)

$$\begin{vmatrix} D_xf_1 & D_yf_1 & D_wf_1 & D_vf_1 & D_zf_1 & D_Mf_1 \\ D_xf_2 & D_yf_2 & D_wf_2 & D_vf_2 & D_zf_2 & D_Mf_2 \\ D_xf_3 & D_yf_3 & D_wf_3 & D_vf_3 & D_zf_3 & D_Mf_3 \\ D_xf_4 & D_yf_4 & D_wf_4 & D_vf_4 & D_zf_4 & D_Mf_4 \\ D_xf_5 & D_yf_5 & D_wf_5 & D_vf_5 & D_zf_5 & D_Mf_5 \\ D_xf_6 & D_yf_6 & D_wf_6 & D_vf_6 & D_zf_6 & D_Mf_6 \end{vmatrix}_{Y^*} = \begin{vmatrix} C_1 & -a_1x & -\omega_1 x & -\omega_2 x & 0 & 0 \\ -a_2y & C_2 & -\dfrac{\omega_3 y}{m+y} & \omega_4 y & 0 & 0 \\ n_1\omega_1 w & \dfrac{mn_2\omega_3 w}{(m+y)^2} & C_3 & -b_1 w & 0 & 0 \\ n_3\omega_2 v & n_4\omega_4 v & -b_2 v & C_4 & -v & 0 \\ 0 & 0 & 0 & pz & pv-c & -1 \\ 0 & 0 & 0 & 0 & p & 0 \end{vmatrix}$$

$$= p\left[2rm^2a_2d_1d_2(x^*)^2 + rsKd_1z^*(y^*)^2 + 2rmd_1d_2\omega_3w^*x^* + Ka_1d_1d_2\omega_4v^*(y^*)^3\right] \neq 0.$$

Thus, by combining Theorem 3 in [44], we can obtain that system (3.1) has a SIB around Y^*, with the bifurcation value being $M = 0$.

Based on the results obtained from the above analysis, we can derive the following conclusions:

$$N_1 = -\text{trace}\left[\begin{pmatrix} D_zf_1 \\ D_zf_2 \\ D_zf_3 \\ D_zf_4 \end{pmatrix} \text{adj}(\mathrm{D_z f_5}) \begin{pmatrix} D_xf_5 & D_yf_5 & D_wf_5 & D_vf_5 \end{pmatrix}\right]_{\mathrm{Y^*}} = pv^*z^*,$$

$$
\begin{aligned}
N_2 &= \left[D_M f_6 - \begin{pmatrix} D_x f_6 & D_y f_6 & D_w f_6 & D_v f_6 & D_z f_6 \end{pmatrix} C_5^{-1} \begin{pmatrix} D_M f_1 \\ D_M f_2 \\ D_M f_3 \\ D_M f_4 \\ D_M f_5 \end{pmatrix} \right]_{Y^*} \\
&= pv^* \left[rKd_1 n_4 \omega_4^2 (y^*)^3 + mKb_1 b_2 \omega_1 (w^*)^3 + 2rmb_1 b_2 \omega_3 x^* (w^*)^2 \right],
\end{aligned}
$$

where

$$
C_5 = \begin{pmatrix} D_x f_1 & D_y f_1 & D_w f_1 & D_v f_1 & D_z f_1 \\ D_x f_2 & D_y f_2 & D_w f_2 & D_v f_2 & D_z f_2 \\ D_x f_3 & D_y f_3 & D_w f_3 & D_v f_3 & D_z f_3 \\ D_x f_4 & D_y f_4 & D_w f_4 & D_v f_4 & D_z f_4 \\ D_x f_5 & D_y f_5 & D_w f_5 & D_v f_5 & D_z f_5 \end{pmatrix}.
$$

It is evident that $N_1 > 0, N_2 > 0$, then $\frac{N_1}{N_2} > 0$. As M surpasses zero, the eigenvalues of the system transit from the left half of the plane to the right half of the plane, eventually diverging into infinity. Hence, when the economic profit rises and crosses zero, the stability of system (3.1) undergoes a transformation at the equilibrium. □

Remark 2. *Recent research has identified three primary categories of codimension-one local bifurcations in differential-algebraic systems: SIB, Hopf bifurcations, and saddle-node bifurcations. Notably, SIB does not occur in ordinary differential equation systems but has been demonstrated to arise in singular systems. For a more detailed discussion on singularity induced bifurcations, readers are referred to [44].*

To eliminate the SIB near the equilibrium point, we introduce a feedback controller to regulate the system's stability. For this feedback controller, the following analysis needs to be performed:

Let $M = 0$. Based on E in (2.6) and $J(Y^*)$ in (2.8), it is easy to calculate rank (J(Y*), EJ(Y*), E^{2}J(Y*), E^{3}J(Y*), E^{4}J(Y*)) = 5. Then, it can be demonstrated that system (3.1) is locally controllable around Y^* by applying Theorem 2.2.1 in [45]. Thus, a feedback controller can be applied to regulate the stability of system (3.1).

By applying Theorem 3.1.2 in [45], we can design a feedback controller to be $u(t) = k(z(t) - z^*)$, where k is a feedback gain. Therefore, a controlled system will be:

$$
\begin{cases}
\frac{1}{\sigma^{1-\alpha}} \mathrm{D}^\alpha x(t) = rx\left(1 - \frac{x}{K}\right) - a_1 xy - \omega_1 xw - \omega_2 xv, \\
\frac{1}{\sigma^{1-\alpha}} \mathrm{D}^\alpha y(t) = sy\left(1 - \frac{y}{L}\right) - a_2 xy - \frac{\omega_3 wy}{m+y} - \omega_4 vy, \\
\frac{1}{\sigma^{1-\alpha}} \mathrm{D}^\alpha w(t) = n_1 \omega_1 xw + \frac{n_2 \omega_3 yw}{m+y} - b_1 wv - d_1 w, \\
\frac{1}{\sigma^{1-\alpha}} \mathrm{D}^\alpha v(t) = n_3 \omega_2 xv + n_4 \omega_4 yv - b_2 wv - d_2 v - zv, \\
0 = z(pv - c) + k(z - z^*).
\end{cases} \tag{3.5}
$$

Theorem 3. *If the feedback gain k satisfies the following conditions,*

(1) $H_1 = l_1 > 0$,

(2) $H_2 = \begin{vmatrix} l_1 & l_3 \\ 1 & l_2 \end{vmatrix} > 0$,

(3) $H_3 = \begin{vmatrix} l_1 & l_3 & 0 \\ 1 & l_2 & l_4 \\ 0 & l_1 & l_3 \end{vmatrix} > 0$,

(4) $H_4 = \begin{vmatrix} l_1 & l_3 & 0 & 0 \\ 1 & l_2 & l_4 & 0 \\ 0 & l_1 & l_3 & 0 \\ 0 & 1 & l_2 & l_4 \end{vmatrix} > 0$,

(5) $k > \max\left\{-\frac{G_{11}}{G_{12}}, -\frac{G_{21}}{G_{22}}, -\frac{G_{31}}{G_{32}}, -\frac{G_{41}}{G_{42}}\right\}$,

where the definitions of $l_i (i = 1, 2, 3, 4)$ *and* $G_{ij}(i, j = 1, 2, 3, 4)$ *are given in the proof, then system* (3.5) *is stable around* Y^*.

Proof. According to Theorem 1, the characteristic equation of system (3.5) around Y^* is det|Eλ − J| = 0, or, equivalently,

$$\lambda^4 + l_1\lambda^3 + l_2\lambda^2 + l_3\lambda + l_4 = 0, \tag{3.6}$$

where

$$\begin{aligned}
l_1 &= \frac{G_{11} + kG_{12}}{-k + c - pv^*}, \quad l_2 = \frac{G_{21} + kG_{22}}{-k + c - pv^*}, \quad l_3 = \frac{G_{31} + kG_{32}}{-k + c - pv^*}, \quad l_4 = \frac{G_{41} + kG_{42}}{-k + c - pv^*}, \\
G_{11} &= -(c - pv^*)G_{12}, \\
G_{12} &= A_1 + A_2 + A_3 + A_4, \\
G_{21} &= -(c - pv^*)G_{22} - pv^*z^*(A_1 + A_2 + A_3), \\
G_{22} &= -A_1A_2 - A_1A_3 - A_1A_4 - A_2A_3 - A_2A_4 - A_3A_4 + a_1a_2x^*y^* + b_1b_2v^*w^* - n_3\omega_2^2v^*x^* \\
&\quad -n_1\omega_1^2w^*x^* - n_4\omega_4^2v^*y^* + \frac{n_2\omega_3^2w^*(y^*)^2}{(m + y^*)^2} - \frac{n_2\omega_3^2w^*y^*}{(m + y^*)^3}, \\
G_{31} &= -(c - pv^*)G_{32} \\
&\quad +pv^*z^*\left[\frac{n_2\omega_3^2w^*y^*}{(m + y^*)^2} - \frac{n_2\omega_3^2w^*(y^*)^2}{(m + y^*)^3} + A_1A_2 + A_1A_3 + A_2A_3 - a_1a_2x^*y^* + n_1\omega_1^2x^*w^*\right], \\
G_{32} &= A_1A_2A_3 + A_1A_2A_4 + A_1A_3A_4 + A_2A_3A_4 + n_4\omega_4^2y^*v^*(A_1 + A_3) + n_3\omega_2^2x^*v^*(A_2 + A_3) \\
&\quad +n_1\omega_1^2x^*w^*(A_2 + A_4) - b_1b_2v^*w^*(A_1 + A_2) - a_1a_2x^*y^*(A_3 + A_4) \\
&\quad +\omega_1\omega_2x^*v^*w^*(n_1b_2 + n_3b_1) + \omega_2\omega_4x^*v^*y^*(a_1n_3 + a_2n_4) \\
&\quad +\frac{\omega_3w^*y^*}{m + y^*}\left[\omega_4v^*(b_2n_2 + b_1n_4) + \omega_1x^*(a_1n_1 + a_2n_2)\right] - \frac{n_2\omega_3^2w^*(y^*)^2}{(m + y^*)^3}(A_1 + A_4)
\end{aligned}$$

$$
\begin{aligned}
& +\frac{n_2\omega_3 w^* y^*}{(m+y^*)^2}\left[A_1\omega_3 - A_4\omega_3 - b_2\omega_4 v^* y^* - a_2\omega_1 x^* y^*\right], \\
G_{41} \;=\; & -(c-pv^*)G_{42} + pv^* z^*\left[-A_1A_2A_3 - A_2 n_1\omega_1^2 w^* x^* + a_1 a_2 A_3 x^* y^*\right] \\
& +pv^* z^*\left[-\frac{\omega_1\omega_3 w^* x^* y^*}{m+y^*}(a_1 n_1 + a_2 n_2) - \frac{n_2\omega_3 w^* y^*}{(m+y^*)^2}(A_1\omega_3 - a_2\omega_1 x^* y^*) + \frac{A_1 n_2\omega_3^2 w^*(y^*)^2}{(m+y^*)^3}\right], \\
G_{42} \;=\; & -A_1A_2A_3A_4 + A_1A_3 n_4\omega_4^2 v^* y^* - A_2A_3 n_3\omega_2^2 v^* x^* - A_2A_4 n_1\omega_1^2 w^* x^* \\
& -a_1A_3 n_3\omega_2\omega_4 x^* v^* y^* - a_2 A_3 n_4\omega_2\omega_4 x^* v^* y^* - b_2 A_2 n_1\omega_1\omega_2 x^* v^* w^* \\
& -b_1 A_2 n_3\omega_1\omega_2 x^* v^* w^* - a_1 a_2 b_1 b_2 x^* v^* w^* y^* - a_1 b_2 n_1\omega_1\omega_4 x^* v^* w^* y^* \\
& -a_2 b_1 n_4\omega_1\omega_4 x^* v^* w^* y^* - n_1 n_4\omega_1^2\omega_4^2 x^* v^* w^* y^* + A_1A_2 b_1 b_2 v^* w^* + A_3A_4 a_1 a_2 x^* y^* \\
& +\frac{\omega_3 w^* y^*}{m+y^*}\left[-A_1 b_1 n_4\omega_4 v^* - A_1 b_2 n_2\omega_4 v^* + (n_1 n_4 + n_2 n_3)\omega_1\omega_2\omega_4 v^* x^*\right] \\
& +\frac{\omega_3 w^* y^*}{m+y^*}\left[-(a_1 n_1 + a_2 n_2)A_4\omega_1 x^* - (a_1 b_1 n_3 + a_2 b_2 n_2)\omega_2 v^* x^*\right] \\
& +\frac{n_2\omega_3 w^* y^*}{(m+y^*)^2}\left[a_2 A_4\omega_1 x^* y^* + a_2 b_2\omega_2 v^* x^* y^* - n_3\omega_2^2\omega_3 v^* x^* - n_3\omega_1\omega_2\omega_4 v^* x^* y^*\right] \\
& +\frac{n_2\omega_3 w^* y^*}{(m+y^*)^2}(-A_1A_4\omega_3 + A_1 b_2\omega_4 v^* y^*) + \frac{n_2\omega_3^2 w^*(y^*)^2}{(m+y^*)^3}\left(A_1A_4 + n_3\omega_2^2 v^* x^*\right).
\end{aligned}
$$

By using the Routh-Hurwitz criterion [46], it can be concluded that system (3.5) is stable at its internal equilibrium Y^* provided that k satisfies condition: $k > \max\left\{-\frac{G_{11}}{G_{12}}, -\frac{G_{21}}{G_{22}}, -\frac{G_{31}}{G_{32}}, -\frac{G_{41}}{G_{42}}\right\}$.

The proof is complete. □

4. Optimal control problem

In the field of renewable resource commercialization, a critical economic issue from a professional perspective lies in striking the optimal balance between immediate profitability and long-term sustainability. This delicate equilibrium not only influences current economic gains but also has profound implications for the enduring utilization of resources and the stability of ecological-economic systems.

This section will focus on analyzing the economic dynamics within the predator-prey model. Empirical observations from real-world markets reveal a consistent trend: as biomass increases, market prices tend to decline, highlighting an inverse relationship between biomass levels and unit prices. To maximize the total discounted net revenue, we have formulated this problem as an optimal control model, as detailed below:

$$\widetilde{J}(u(t)) = \int_{t_0}^{t_f} e^{-\beta t}\left[(p - Au(t))u(t) - \frac{cu(t)}{v}\right]\mathrm{d}t, \tag{4.1}$$

where c denotes the cost per unit of effort, p is the prices of a unit of the harvested biomass, A is an economic constant, and β stands for the instantaneous annual discount rate.

To solve (4.1), it is essential to apply Pontryagin's Maximum Principle, taking into account the characteristics of system (2.5) and the control constraint $u(t) \in [0,1]$. To ensure the existence of an optimal control, we can combine the convexity of the objective function with respect to $u(t)$, the linearity of the differential equation with regard to the control, and the compactness of the state variable's domain. This integrated analytical approach not only confirms the existence of an optimal control but also provides a rigorous theoretical foundation for its determination.

Given that $u^*(t)$ is an optimal control associated with the states $x^*(t), y^*(t), w^*(t)$, and $v^*(t)$, our objective is to determine $u^*(t)$ such that $\widetilde{J}(u^*(t)) = \max \widetilde{J}(u(t))$.

Similar to [47], the Hamiltonian function is

$$\begin{aligned} H = \ & (p - Au)u - \frac{cu}{v} + \lambda_1 \left[rx(1 - \frac{x}{K}) - a_1xy - \omega_1 xw - \omega_2 xv \right] \\ & + \lambda_2 \left[sy(1 - \frac{y}{L}) - a_2xy - \frac{\omega_3 wy}{m + y} - \omega_4 vy \right] + \lambda_3 \left[n_1\omega_1 xw + \frac{n_2\omega_3 yw}{m + y} - b_1wv - d_1w \right] \\ & + \lambda_4 \left[n_3\omega_2 xv + n_4\omega_4 yv - b_2wv - d_2v - u \right], \end{aligned}$$

with adjoint variables $\lambda_i(t)$, $i = 1, 2, 3, 4$.

According to [47], the following adjoint equations and transversality conditions can be obtained:

$$\begin{cases} \mathrm{D}^\alpha \lambda_1(t) = & \lambda_1 \left[\beta - r + \frac{2rx}{K} + a_1y + \omega_1 w + \omega_2 v \right] + \lambda_2 a_2 y - \lambda_3 n_1 \omega_1 w - \lambda_4 n_3 \omega_2 v, \\ \mathrm{D}^\alpha \lambda_2(t) = & \lambda_1 a_1 x + \lambda_2 \left[\beta - s + \frac{2sy}{L} + a_2 x + \frac{m\omega_3 w}{(m + y)^2} + \omega_4 v \right] - \lambda_3 \frac{mn_2\omega_3 w}{(m + y)^2} - \lambda_4 n_4 \omega_4 v, \\ \mathrm{D}^\alpha \lambda_3(t) = & \lambda_1 \omega_1 x + \lambda_2 \frac{\omega_3 y}{m + y} + \lambda_3 \left[\beta - n_1\omega_1 x - \frac{n_2\omega_3 y}{m + y} + b_1 v + d_1 \right] + \lambda_4 b_2 v, \\ \mathrm{D}^\alpha \lambda_4(t) = & -\frac{cu}{v^2} + \lambda_1 \omega_2 x + \lambda_2 \omega_2 y + \lambda_3 b_1 w + \lambda_4 \left[\beta - n_3\omega_2 x + n_4\omega_4 y + b_2 w + d_2 \right], \end{cases} \tag{4.2}$$

with the transversal conditions

$$\lambda_i(t_f) = 0, \ i = 1, 2, 3, 4.$$

The optimality conditions give

$$\frac{\partial H}{\partial u} = p - 2Au - \frac{c}{v} - \lambda_4 = 0,$$

from which we get

$$u^*(t) = \frac{p - \frac{c}{v} - \lambda_4}{2A}.$$

Therefore, we arrive at the following theorem.

Theorem 4. *There exists an optimal control $u^*(t)$ with corresponding optimal state trajectories $x^*(t), y^*(t), w^*(t)$, and $v^*(t)$ that maximize the objective functional $\widetilde{J}(u(t))$. Additionally, there exist adjoint variables $\lambda_1^*(t), \lambda_2^*(t), \lambda_3^*(t)$, and $\lambda_4^*(t)$ satisfying equations* (16) *with the transversality condition $\lambda_i(t_f) = 0$ for $i = 1, 2, 3, 4$. Furthermore, the optimal control is characterized by $u^*(t) = \dfrac{p - \frac{c}{v} - \lambda_4}{2A}$.*

Proof. The proof is similar to that in [30], so we omit it. □

5. Examples and numerical simulations

To verify and elaborate on the theoretical results derived from system (2.5), numerical simulations are carried out. System parameters are partially based on [13, 30, 32], preserving methodological continuity while being adapted to our study's requirements. These simulations aim to provide a comprehensive understanding of the system's behaviors. Then, we use MATLAB software to perform the simulation experiments.

Using Gordon's economic theory of natural resource utilization [40] as a basis, we conduct numerical simulation analyses from two cases, $M = 0$ and $M > 0$. For $M < 0$, this is a short term phenomenon and will eventually return to the zero economic profit state. Therefore, we do not investigate it further here.

5.1. Zero economic profit (M=0)

Example 1. The parameter values for system (3.1) are selected as follows:

$r = 0.85$, $K = 6000$, $L = 2000$, $s = 0.7$, $a_1 = 0.000001$, $a_2 = 0.0009$, $b_1 = 0.002$, $b_2 = 0.0005$, $d_1 = 0.04$, $d_2 = 0.03$, $m = 1200$, $n_1 = 0.007$, $n_2 = 0.0104$, $n_3 = 0.001$, $n_4 = 0.0093$, $\omega_1 = 0.013$, $\omega_2 = 0.005$, $\omega_3 = 0.02$, $\omega_4 = 0.03$, $\alpha = 0.9$, $p = 1.5$, and $c = 7$.

(i) Using these parameter values, we can obtain that Eq (3.2) has a positive root $y^* = 528.2100$. Meanwhile, when the remaining conditions are satisfied, we have derived the internal equilibrium of system (3.1) as follows: $(528.2100,\ 290.773,\ 58.0016,\ 4.1736,\ 0.0028)$.

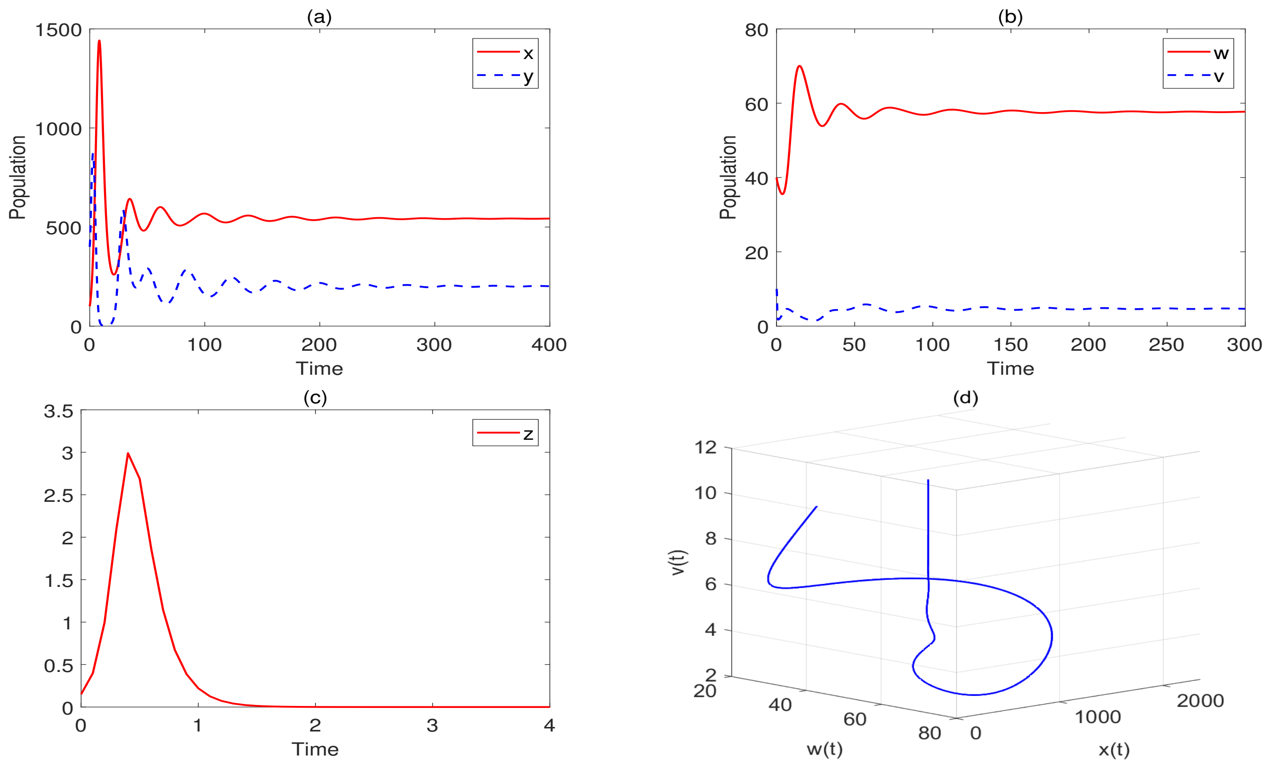

Figure 1. (a)–(c) are time series of system (3.1). (d) is the phase portrait of system (3.1).

(ii) This equilibrium is at the singularity. It is noteworthy that the eigenvalues of the system are $\lambda_1 = -0.2660$, $\lambda_2 = -0.0324$, $\lambda_3 = -0.0152+0.3963i$ and $\lambda_4 = -0.0152-0.3963i$ when $M = -0.0001$, whereas they become $\lambda_1 = 4.6765$, $\lambda_2 = 0.0494$, $\lambda_3 = -0.0744+0.1459i$ and $\lambda_4 = -0.0744-0.1459i$ when $M = 0.0001$. Clearly, as M increases past zero, the signs of two eigenvalues shift—they move from the left half-plane to the right half-plane along the real axis, ultimately diverging to infinity. This observation is consistent with the conclusion of Theorem 2.

(iii) According to Theorem 3, if the feedback gain is $k > \max\{-5,\ 1.324,\ 2.513,\ 2.314\}$, then the state feedback controller $u(t) = 3(z(t) - 0.0028)$ can be constructed to stabilize system (3.1) at Y^* for $M = 0$.

(iv) Figure 1 presents the numerical values of this equilibrium and its phase portrait. Simultaneously, Figure 1 also reveals that various preys and predators can coexist and reproduce effectively within the same environment.

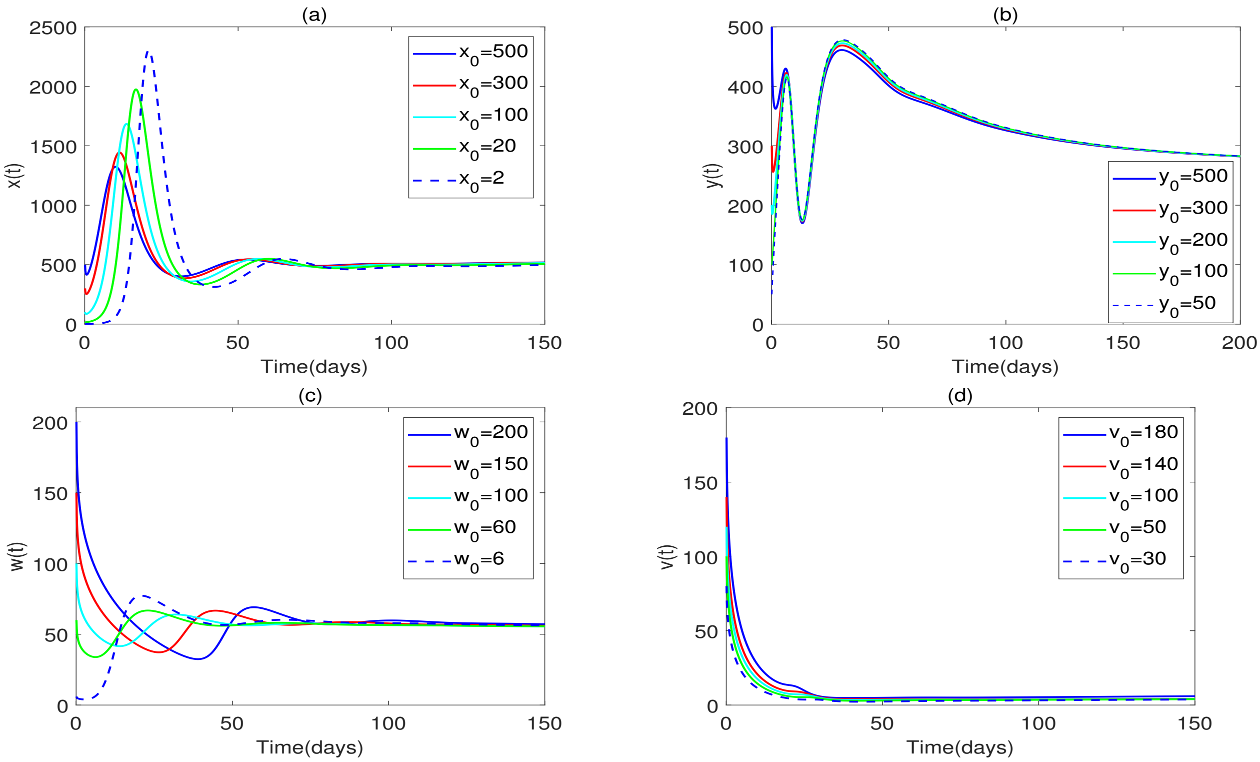

Figure 2. Dynamic behaviors of system (3.1) for different initial values.

Example 2. The following parameters are assigned fixed values:

$r = 0.85$, $K = 6000$, $L = 2000$, $s = 0.7$, $a_1 = 0.000001$, $a_2 = 0.0009$, $b_1 = 0.002$, $b_2 = 0.0005$, $d_1 = 0.04$, $d_2 = 0.03$, $m = 1200$, $n_1 = 0.007$, $n_2 = 0.0104$, $n_3 = 0.001$, $n_4 = 0.0093$, $\omega_1 = 0.013$, $\omega_2 = 0.005$, $\omega_3 = 0.02$, $\omega_4 = 0.03$, $p = 1.5$, and $c = 7$.

The different initial values are taken as $Y_0 = [500,\ 500,\ 200,\ 180,\ 0.15]$, $[300,\ 300,\ 150,\ 140,\ 0.055]$, $[100,\ 200,\ 100,\ 100,\ 0.1]$, $[20,\ 100,\ 60,\ 50,\ 0.15]$, and $[2,\ 50,\ 6,\ 30,\ 0.055]$, respectively, and $\alpha = 0.9$. It can be observed from Figure 2 that varying initial values do not affect the stability of the equilibrium Y^* in system (3.1).

Remark 3. *Figures* 1–2 *are used to demonstrate the existence and stability of the equilibrium under the condition of zero economic benefit. The initial values do not affect the stability of system* (3.1)*. This demonstrates that the ecosystem possesses regulatory capacity, capable of buffering external disturbances through complex interactions, thereby providing a scientific basis for ecological conservation.*

Example 3. The following parameters are assigned fixed values:

$r = 0.85$, $K = 6000$, $L = 2000$, $s = 0.7$, $a_1 = 0.000001$, $a_2 = 0.0009$, $b_1 = 0.002$, $b_2 = 0.0005$, $d_1 = 0.04$, $d_2 = 0.03$, $m = 1200$, $n_1 = 0.007$, $n_2 = 0.0104$, $n_3 = 0.001$, $n_4 = 0.0093$, $\omega_1 = 0.013$, $\omega_2 = 0.005$, $\omega_3 = 0.02$, $\omega_4 = 0.03$, $p = 1.5$, and $c = 7$.

(i) The initial value is $Y_0 = [300,\ 40,\ 60,\ 5,\ 0.15]$, and α has different values (1, 0.99, 0.98, 0.97, 0.96). As depicted in Figure 3, the internal equilibrium Y^* exhibits asymptotic stability when the fractional order α lies within $[0.96,\ 1]$.

(ii) The initial value is $Y_0 = [300,\ 40,\ 60,\ 5,\ 0.15]$, and α has different values (1, 0.96, 0.88, 0.8, 0.75). Figure 4 shows that Y^* is unstable when α lies within $[0.75,\ 0.96]$.

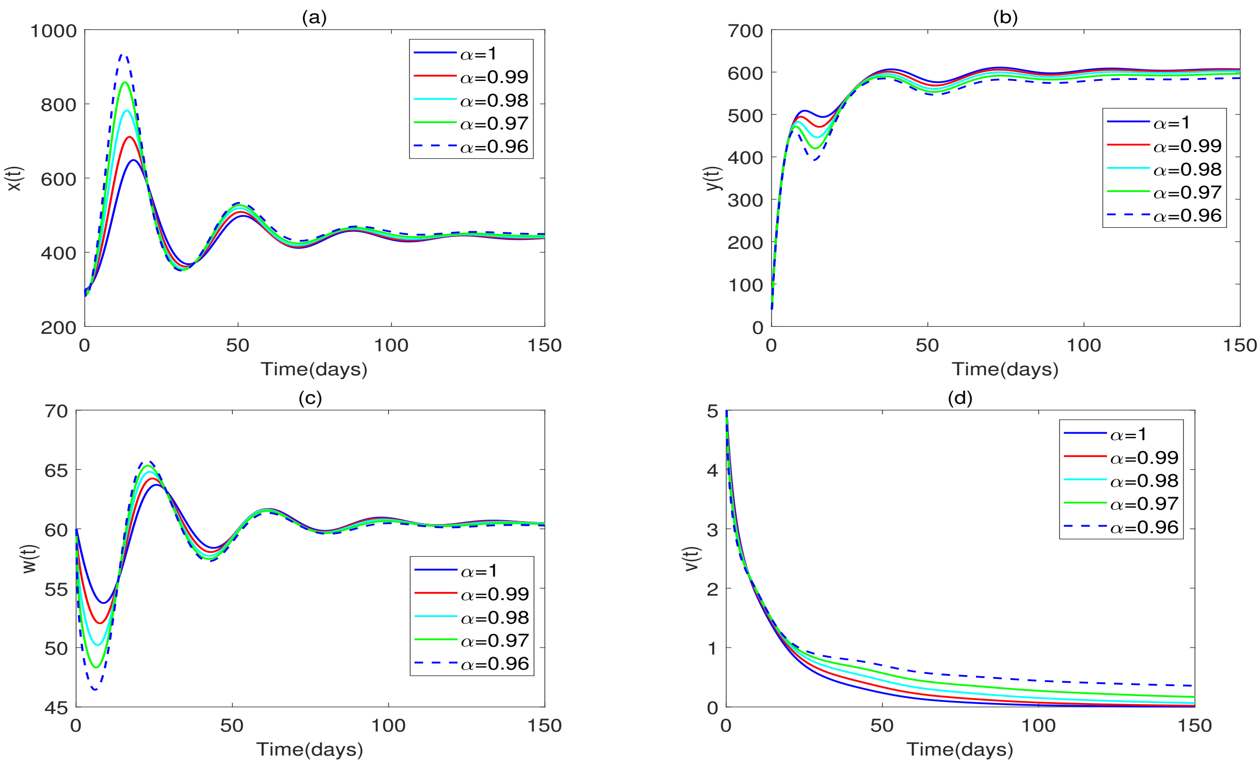

Figure 3. Dynamic behaviors of system (3.1) for different values of α ($\alpha = 1,\ 0.99,\ 0.98,\ 0.97,\ 0.96$).

Remark 4. *The following conclusions are derived from Figures* 3 *and* 4*:*

(i) Figures 3–4 *are used to demonstrate the influence of the order on system* (3.1)*.*

(ii) When the value of α is relatively large, the order does not affect the stability of system (3.1)*, but α influences the speed at which it converges to stability.*

(iii) When the value of α is relatively small, the order affects the stability of system (3.1)*, and the*

value of y temporarily turns negative and then increases to a certain value. This phenomenon cannot be explained by integer order.

In conclusion, the fractional order effect is significantly more specific compared to the integer order case. Moreover, lower-order systems are more sensitive to environmental changes and other factors, which can easily lead to population collapse and are detrimental to ecosystem development.

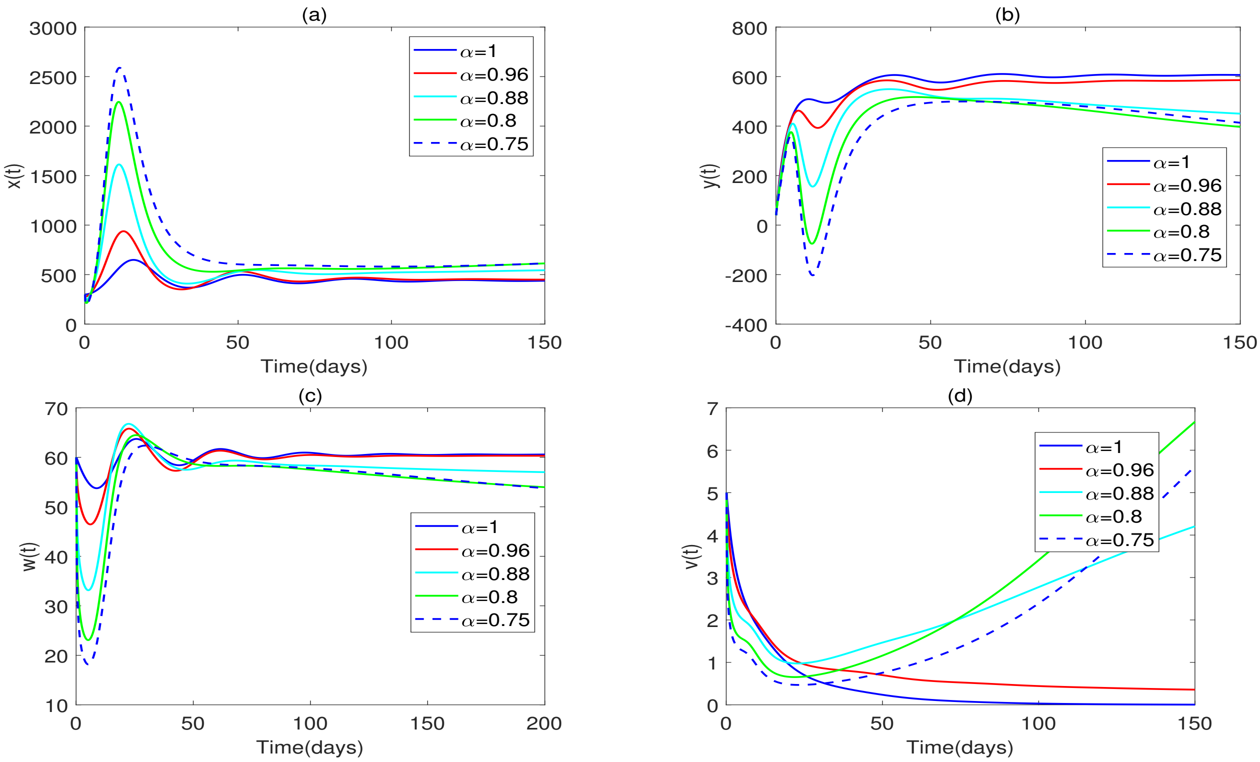

Figure 4. Dynamic behaviors of system (3.1) for different values of α ($\alpha = 1,\ 0.96,\ 0.88,\ 0.8,\ 0.75$).

5.2. *Positive economic profit* ($M > 0$)

Example 4. Fix the following parameter values:

$r = 0.85$, $K = 6000$, $L = 2000$, $s = 0.7$, $a_1 = 0.000001$, $a_2 = 0.0009$, $b_1 = 0.002$, $b_2 = 0.0005$, $d_1 = 0.04$, $d_2 = 0.03$, $m = 1200$, $n_1 = 0.007$, $n_2 = 0.0104$, $n_3 = 0.001$, $n_4 = 0.0093$, $\omega_1 = 0.013$, $\omega_2 = 0.005$, $\omega_3 = 0.02$, $\omega_4 = 0.03$, $p = 1.5$, and $c = 1$.

The starting values are taken as $Y_0 = [500,\ 500,\ 200,\ 180,\ 0.00025]$, $[300, 300, 150, 140, 0.00025]$, $[100,\ 200,\ 100,\ 100,\ 0.00025]$, $[20,\ 100,\ 60,\ 50,\ 0.00025]$, and $[2,\ 50,\ 6,\ 30,\ 0.00025]$, while the fractional order α is set to 0.9.

(i) By using given parameter values, based on Theorem 2, it follows that system (2.5) is unstable around $\widetilde{Y}^* = (1130.54,\ 172.14,\ 46.71,\ 29.18)$ as $M = 0.0004$.

(ii) According to Theorem 3, if $k > \max\{-5,\ 32.4783,\ 11.9730,\ 15.7330\}$, then $u(t) = 33(z(t) - 29.18)$ can be designed such that system (2.5) moves toward $\widetilde{Y}^*$.

(iii) Numerous numerical simulations demonstrate the existence and stability of the equilibrium $\widetilde{Y}^*$, as can be observed from Figure 5. Furthermore, Figure 5 also indicates that the initial values does not affect the stability of the equilibrium.

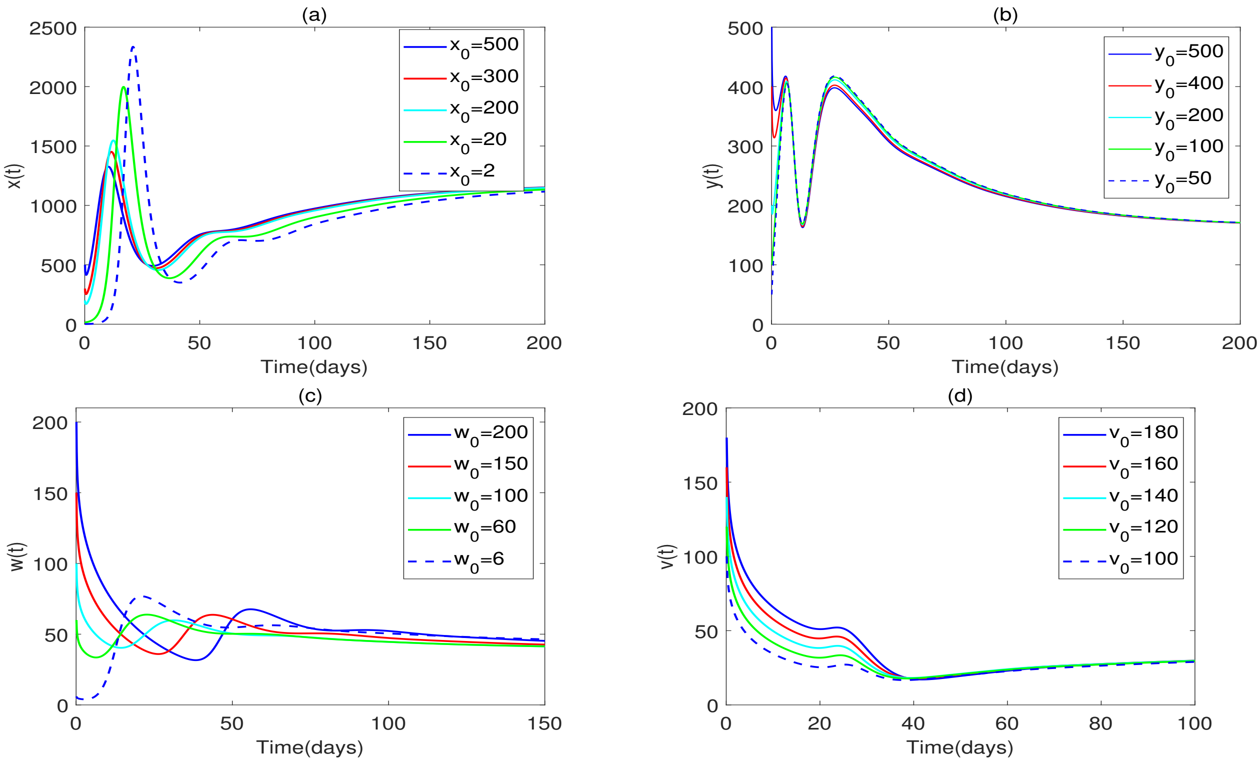

Figure 5. Dynamic behaviors of system (2.5) for different initial values.

Example 5. The following parameters are assigned fixed values:

$r = 0.85$, $K = 6000$, $L = 2000$, $s = 0.7$, $a_1 = 0.000001$, $a_2 = 0.0009$, $b_1 = 0.002$, $b_2 = 0.0005$, $d_1 = 0.04$, $d_2 = 0.03$, $m = 1200$, $n_1 = 0.007$, $n_2 = 0.0104$, $n_3 = 0.001$, $n_4 = 0.0093$, $\omega_1 = 0.013$, $\omega_2 = 0.005$, $\omega_3 = 0.02$, $\omega_4 = 0.03$, $p = 1.5$, and $c = 1$.

The initial value is fixed to $Y_0 = [200,\ 50,\ 60,\ 5,\ 0.00025]$, and α has different values (0.98, 0.94, 0.92, 0.85, 0.84). Figure 6 clearly shows that the internal equilibrium $\widetilde{Y}^*$ is an asymptotic stability when the fractional order α lies within the interval $[0.84,\ 0.98]$.

Remark 5. *The following conclusions are derived from Figures* 5 *and* 6*:*

(i) Figures 5–6 *show the existence and stability of the equilibrium under the condition of positive economic benefit.*

(ii) The initial values and order do not affect the stability of system (2.5)*, but α influences the speed at which it converges to stability.*

(iii) As α increases, the fractional derivative exhibits a damping effect on the model's oscillatory behavior, consequently improving system stability.

In conclusion, under economically profitable conditions, the fractional-order modeling approach demonstrates superior capacity to reconcile economic development with ecological conservation, thereby providing a theoretical foundation for sustainable resource management.

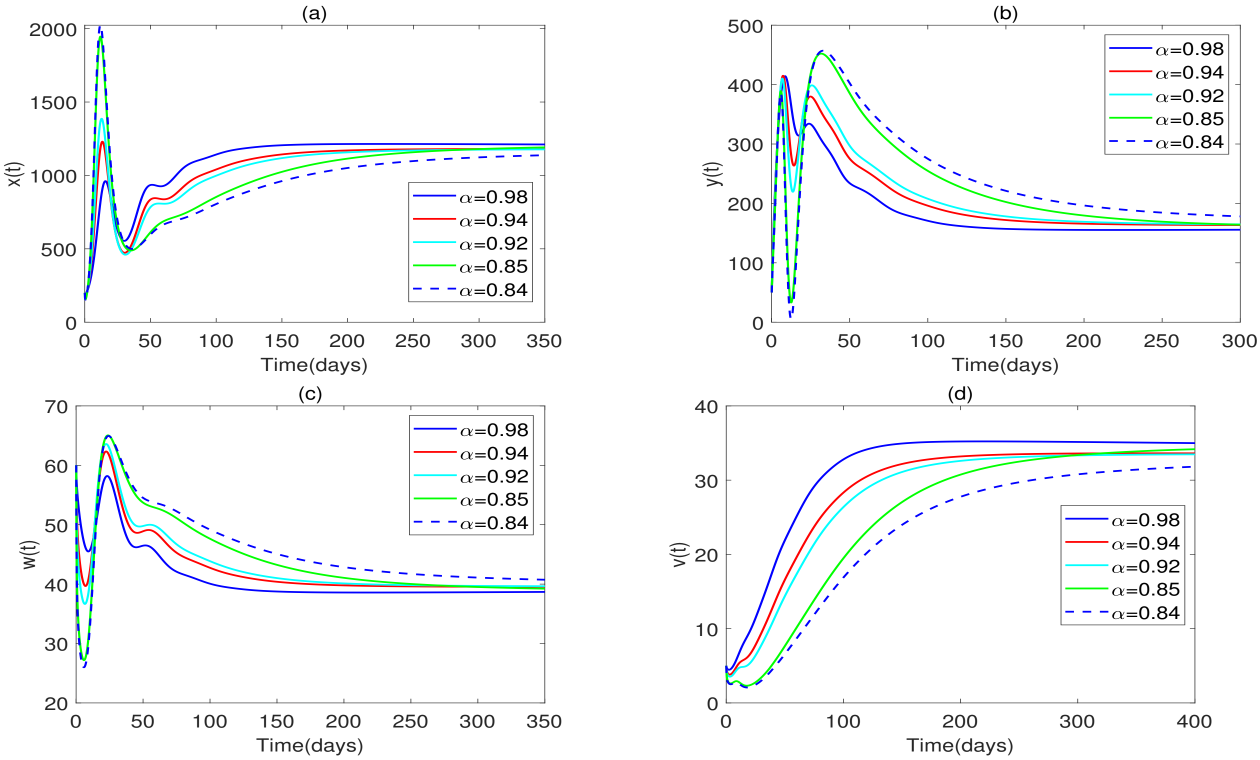

Figure 6. Dynamic behaviors of system (2.5) for different values of α (α = 0.98, 0.94, 0.92, 0.85, 0.84).

5.3. *Examples and numerical simulations for optimal control problem*

Example 6. Fix the values of the following parameters:

$r = 0.9$, $K = 6000$, $L = 2000$, $s = 0.8$, $a_2 = 0.0005$, $b_1 = 0.002$, $b_2 = 0.0005$, $d_1 = 0.04$, $d_2 = 0.03$, $m = 1200$, $n_1 = 0.077$, $n_2 = 7.5$, $n_3 = 0.01$, $n_4 = 0.0093$, $\omega_1 = 0.013$, $\omega_2 = 0.01$, $\omega_3 = 0.02$, $\omega_4 = 0.03$, $A = 0.125$, $\beta = 0.01$, $c = 5$, and $p = 1.5$.

(i) In Figure 7, α is set at 0.9 with initial condition $[x_0,\ y_0,\ w_0,\ v_0] = [28,\ 290,\ 58,\ 4]$, and different values of a_1 ($a_1 = 0.00001,\ 0.00002,\ 0.00003,\ 0.00004,\ 0.00005$) are considered to investigate its influence on system (2.5).

(ii) In Figure 8, α is set at 0.9 with initial condition $[x_0,\ y_0,\ w_0,\ v_0] = [28,\ 290,\ 58,\ 4]$, and different values of a_1 ($a_1 = 0.0001,\ 0.0002,\ 0.0003,\ 0.0004,\ 0.0005$) are considered to investigate its influence on system (2.5).

Remark 6. *(i) Figures* 7–8 *demonstrate the sensitivity analysis of parameter* a_1*. For Figure* 7*, where parameter* a_1 *assumes a relatively small value, it indicates that the impact of parameter* a_1 *on the two prey species and two predator species is negligible. As* a_1 *increases, the population of small prey rises sharply, while that of large prey declines. Consequently, the numbers of both predator species initially decrease, then increase, followed by damped oscillations that gradually subside over time. This dynamic aligns with the inherent resilience of ecological systems, ultimately achieving long-term stability. In contrast, Figure* 8*, with a larger* a_1 *value, its effect on predator-prey dynamics is substantial. Comparative analysis reveals that the former scenario is more aligned with the inherent* *resilience of ecological systems, ultimately achieving long-term stability.*

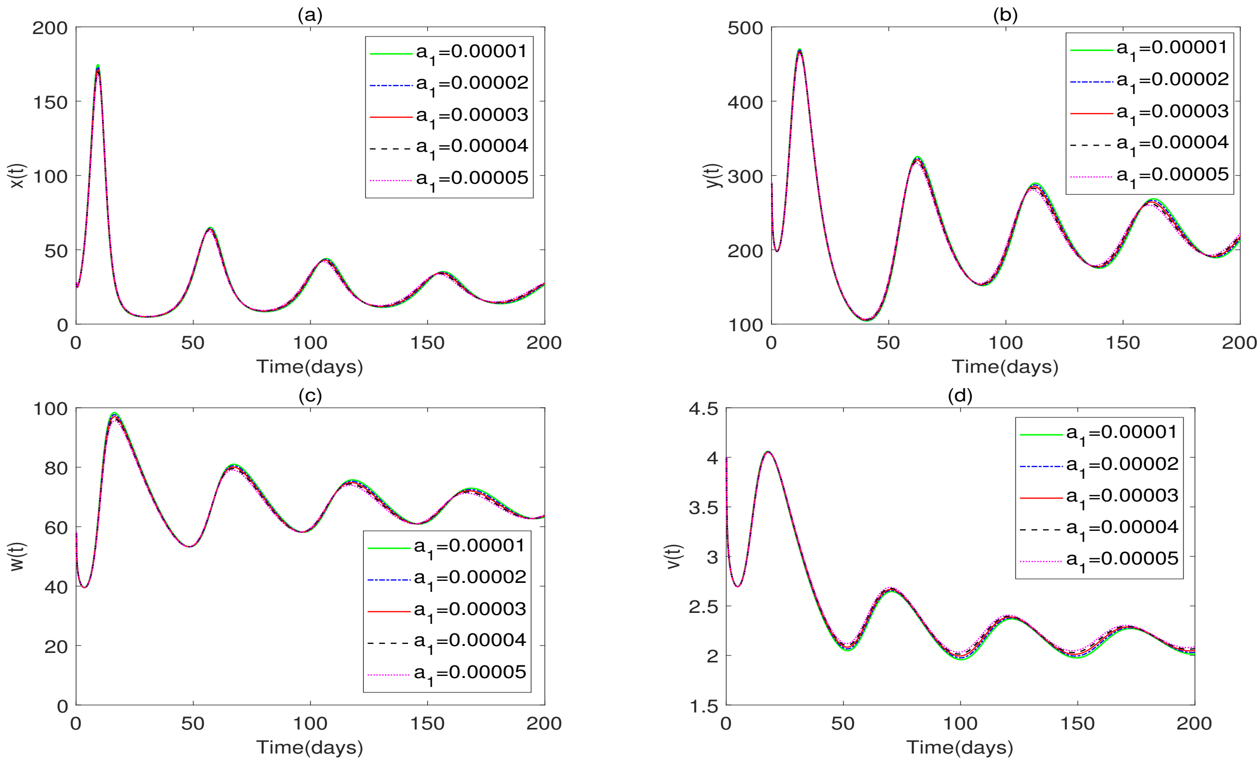

Figure 7. Dynamic behaviors of system (2.5) for different values values of a_1 (a_1 = 0.00001, 0.00002, 0.00003, 0.00004, 0.00005). Here, the values of a_1 are relatively small.

(ii) Figures 7 *and* 8 *show that when the two competition coefficients are asymmetric, the scenario where a_1 is smaller than a_2 is more conducive to maintaining the coexistence of the two prey species and preserving the stability of the ecosystem.*

Example 7. Set the values of the following parameters as fixed:

$r = 0.9$, $K = 6000$, $L = 2000$, $s = 0.8$, $a_1 = 0.00001$, $a_2 = 0.0005$, $b_2 = 0.0005$, $d_1 = 0.04$, $d_2 = 0.03$, $m = 1200$, $n_1 = 0.077$, $n_2 = 7.5$, $n_3 = 0.01$, $n_4 = 0.0093$, $\omega_1 = 0.013$, $\omega_2 = 0.01$, $\omega_3 = 0.02$, $\omega_4 = 0.03$, $A = 0.125$, $\beta = 0.01$, $c = 5$, and $p = 1.5$.

(i) In Figure 9, α is set at 0.9 with initial condition $[x_0, y_0, w_0, v_0] = [28, 290, 58, 4]$, and different values of b_1 ($b_1 = 0.001, 0.002, 0.003, 0.004, 0.005$) are considered to investigate its influence on system (2.5).

(ii) In Figure 10, α is set at 0.9 with initial condition $[x_0, y_0, w_0, v_0] = [28, 290, 58, 4]$, and different values of b_1 ($b_1 = 0.0005, 0.0006, 0.0007, 0.0008, 0.0009$) are considered to investigate its influence on system (2.5).

Remark 7. *Figures* 9–10 *illustrate the sensitivity analysis of parameter b_1. For Figure* 9*, where parameter b_1 assumes a relatively large value, its effect on predator-prey dynamics is substantial. As b_1 increases, small predator populations decline, while large predator populations rise significantly, a trend that adversely affects the population dynamics of small predator. In contrast, Figure* 10*, with a smaller b_1 value, exhibits less pronounced effects on predator and prey abundances. Comparative* *analysis reveals that the latter scenario is more favorable for sustaining ecological equilibrium and mitigates the risk of ecosystem collapse.*

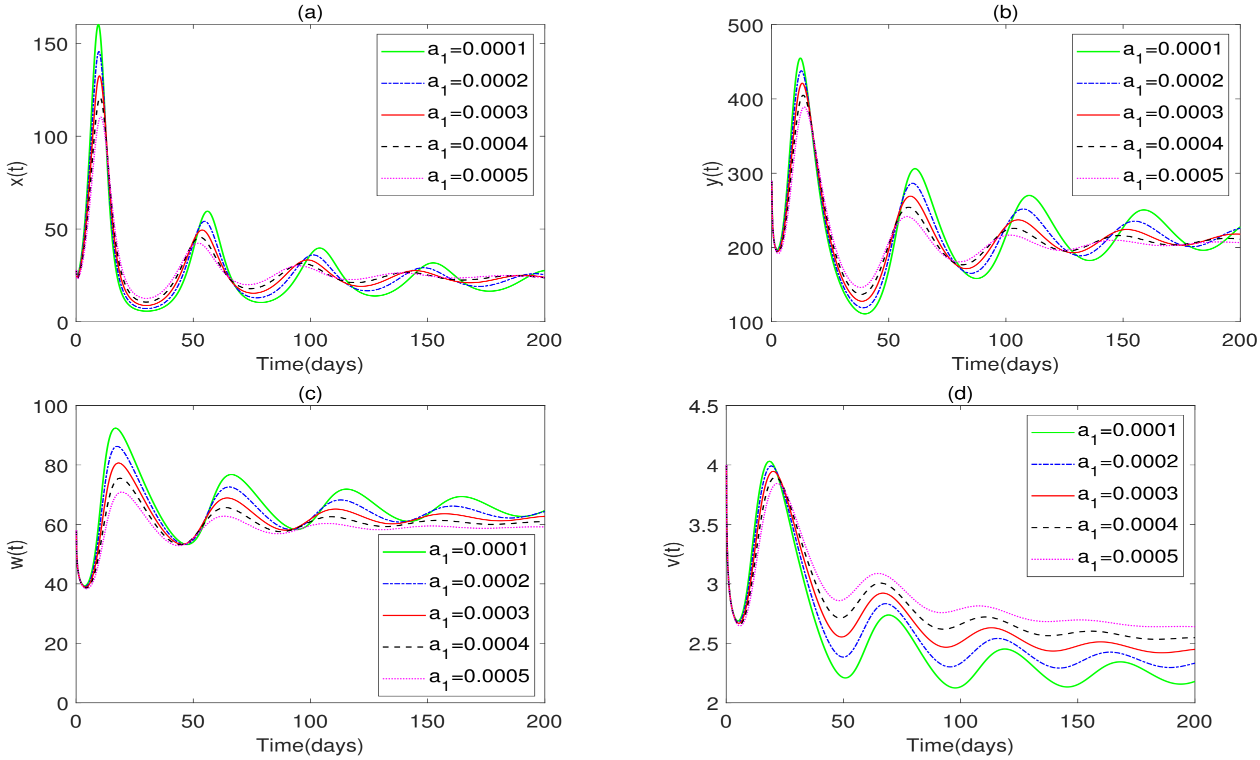

Figure 8. Dynamic behaviors of system (2.5) under varying values of a_1 (a_1 = 0.0001, 0.0002, 0.0003, 0.0004, 0.0005). Here, the values of a_1 are relatively large.

Example 8. Set the values of the following parameters as fixed:

$r = 0.9$, $K = 6000$, $L = 2000$, $s = 0.8$, $a_1 = 0.00001$, $a_2 = 0.0005$, $b_1 = 0.002$, $b_2 = 0.0005$, $d_1 = 0.04$, $d_2 = 0.03$, $m = 1200$, $n_1 = 0.077$, $n_2 = 7.5$, $n_3 = 0.01$, $n_4 = 0.0093$, $\omega_2 = 0.01$, $\omega_3 = 0.02$, $\omega_4 = 0.03$, $A = 0.125$, $\beta = 0.01$, $c = 5$, and $p = 1.5$.

In Figure 11, α is set at 0.9 with initial condition $[x_0, y_0, w_0, v_0] = [28, 290, 58, 4]$, and different values of ω_1 ($\omega_1 = 0.018, 0.023, 0.028, 0.033, 0.038$) are considered to investigate its influence on system (2.5).

Remark 8. *Figure* 11 *presents the sensitivity analysis of parameter* ω_1. *As* ω_1 *increases, the population sizes of the two prey species show certain fluctuations but tend to stabilize overall, while the population sizes of the two predator species fluctuate more significantly and eventually gradually stabilize. The results indicate that within this range of values,* ω_1 *does not disrupt the basic stability of the ecosystem, but it affects the speed at which populations converge to stability and the amplitude of fluctuations.*

Example 9. Set the values of the following parameters as fixed:

$r = 0.9$, $K = 6000$, $L = 2000$, $s = 0.8$, $a_1 = 0.00001$, $a_2 = 0.0005$, $b_1 = 0.002$, $b_2 = 0.0005$, $d_1 = 0.04$, $d_2 = 0.03$, $m = 1200$, $n_2 = 7.5$, $n_3 = 0.01$, $n_4 = 0.0093$, $\omega_1 = 0.013$, $\omega_2 = 0.01$, $\omega_3 = 0.02$, $\omega_4 = 0.03$, $A = 0.125$, $\beta = 0.01$, $c = 5$, and $p = 1.5$.

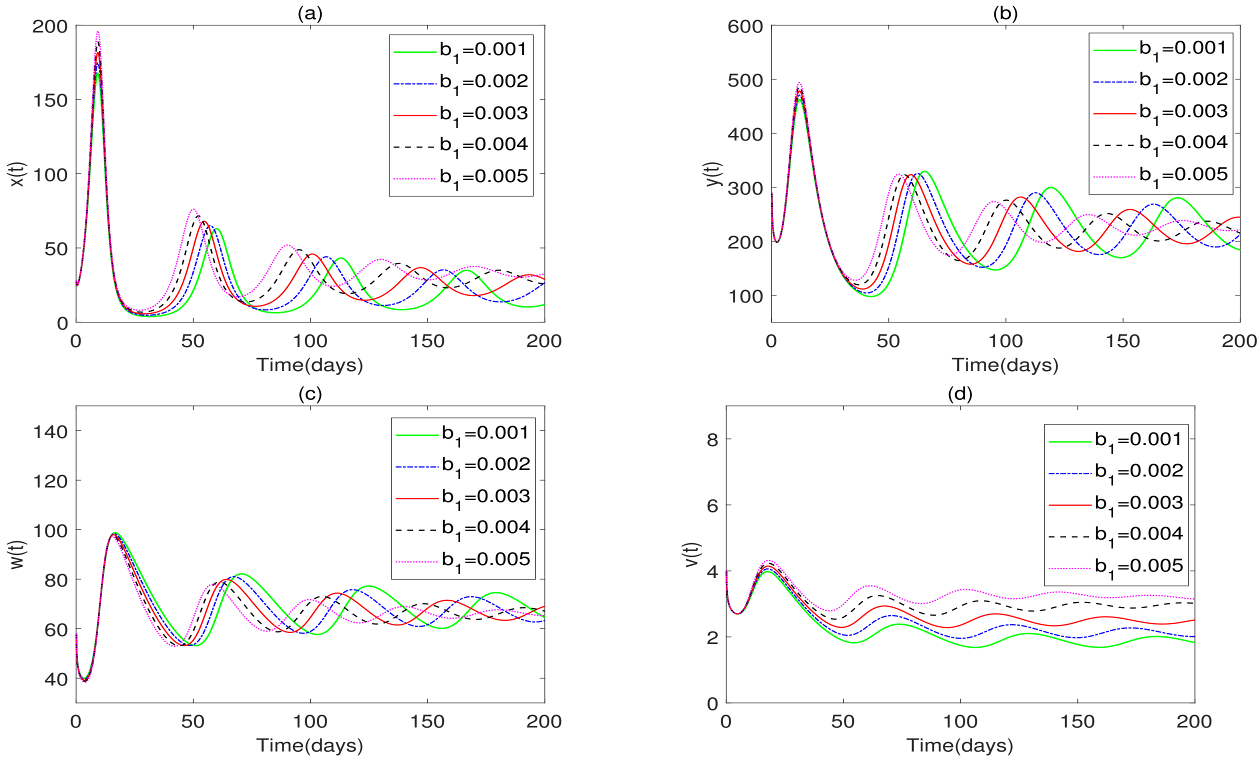

Figure 9. Dynamic behaviors of system (2.5) under varying values of b_1 (b_1 = 0.001, 0.002, 0.003, 0.004, 0.005). Here, the values of b_1 are relatively large.

In Figure 12, α is set at 0.9 with initial condition $[x_0, y_0, w_0, v_0] = [28, 290, 58, 4]$, and different values of n_1 ($n_1 = 0.054, 0.058, 0.062, 0.070, 0.077$) are considered to investigate its influence on system (2.5).

Remark 9. *Figure* 12 *shows the sensitivity analysis of parameter* n_1. *The data indicates that parameter* n_1 *has a relatively significant impact on large predators, while the rest of the influences are unobvious. As* n_1 *increases, the variation in the number of large predators becomes increasingly large. Such dynamics are detrimental to maintaining ecosystem integrity, underscoring the necessity of sustaining appropriate energy conversion efficiency to enhance ecological stability.*

Example 10. Set the values of the following parameters as fixed:

$r = 0.9$, $K = 6000$, $L = 2000$, $s = 0.8$, $a_1 = 0.000001$, $a_2 = 0.0009$, $b_1 = 0.002$, $b_2 = 0.0005$, $d_1 = 0.04$, $d_2 = 0.03$, $m = 1200$, $n_1 = 0.00091$, $n_2 = 0.000208$, $n_3 = 0.000005$, $n_4 = 0.000279$, $\omega_1 = 0.013$, $\omega_2 = 0.005$, $\omega_3 = 0.02$, $\omega_4 = 0.03$, $A = 0.125$, $\beta = 0.01$, and $p = 1.5$.

The initial value is set at $[x_0, y_0, w_0, v_0] = [200, 400, 40, 10]$. The parameter value c is set at 1, 2, 3, 4, 5. Figure 13 illustrates the sensitivity of the optimal control solution to c. As c gradually increases, the optimal control solution exhibits minor variations within the first approximately 10 days, after which it consistently converges to a stable state.

Example 11. Set the values of the following parameters as fixed:

$r = 0.85$, $K = 6000$, $L = 2000$, $s = 0.7$, $a_1 = 0.000001$, $a_2 = 0.0009$, $b_1 = 0.002$, $b_2 = 0.0005$,

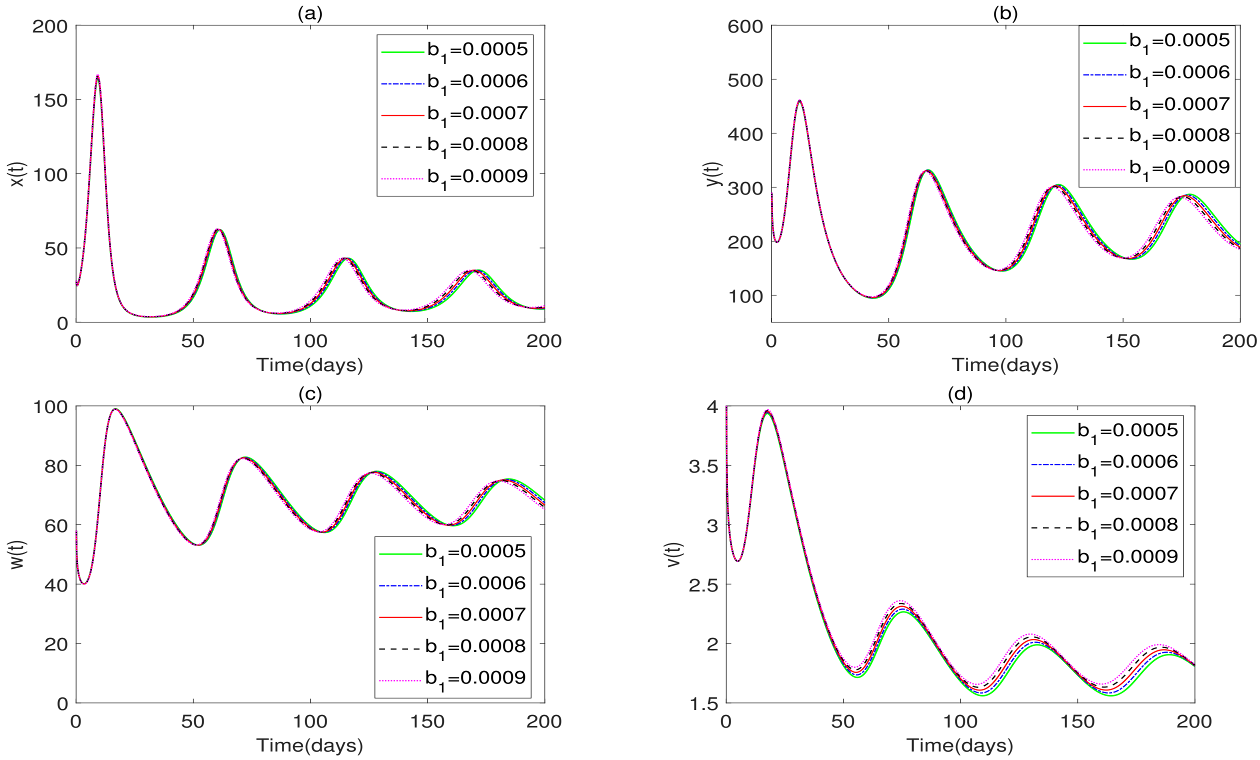

Figure 10. Dynamic behaviors of system (2.5) under varying values of b_1 (b_1 = 0.0005, 0.0006, 0.0007, 0.0008, 0.0009). Here, the values of b_1 are relatively small.

$d_1 = 0.04$, $d_2 = 0.03$, $m = 1200$, $n_1 = 0.00091$, $n_2 = 0.000208$, $n_3 = 0.000005$, $n_4 = 0.000279$, $\omega_1 = 0.013$, $\omega_2 = 0.005$, $\omega_3 = 0.02$, $\omega_4 = 0.03$, $A = 0.125$, $\beta = 0.01$, and $c = 1$.

The initial value is set at $[x_0, y_0, w_0, v_0] = [200, 400, 40, 10]$. The parameter value p is set at $0.5, 1, 1.5, 2, 2.5$. Figure 14 demonstrates the sensitivity analysis of the optimal control solution with respect to p. When p undergoes changes, the optimal control solution stabilizes after undergoing minor fluctuations.

Example 12. Set the values of the following parameters as fixed:

$r = 0.85$, $K = 6000$, $L = 2000$, $s = 0.7$, $a_1 = 0.000001$, $a_2 = 0.0009$, $b_1 = 0.002$, $b_2 = 0.0005$, $d_1 = 0.04$, $d_2 = 0.03$, $m = 1200$, $n_1 = 0.00091$, $n_2 = 0.000208$, $n_3 = 0.000005$, $n_4 = 0.000279$, $\omega_1 = 0.013$, $\omega_2 = 0.005$, $\omega_3 = 0.02$, $\omega_4 = 0.03$, $p = 1.5$, $\beta = 0.01$, and $c = 1$.

The initial value is set at $[x_0, y_0, w_0, v_0] = [200, 400, 40, 10]$. The parameter value A is set at $0.1, 0.15, 0.2, 0.25, 0.3$. Figure 15 demonstrates the sensitivity analysis of the optimal control solution with regard to A. When A undergoes changes, the optimal control solution stabilizes after undergoing minor fluctuations.

Remark 10. *Figures* 13–15 *show that changing the values of p, c, and A only causes minor disturbances to the solution of the optimal control. This indicates that the prices of harvest biomass, the cost of capture capacity, and economic discount factor have negligible influence on the optimal solution of $u(t)$. This demonstrates that ecosystem's self-regulatory capacity can mitigate the impacts of market fluctuations, ultimately achieving the dual objectives of ecological conservation and economic sustainability.*

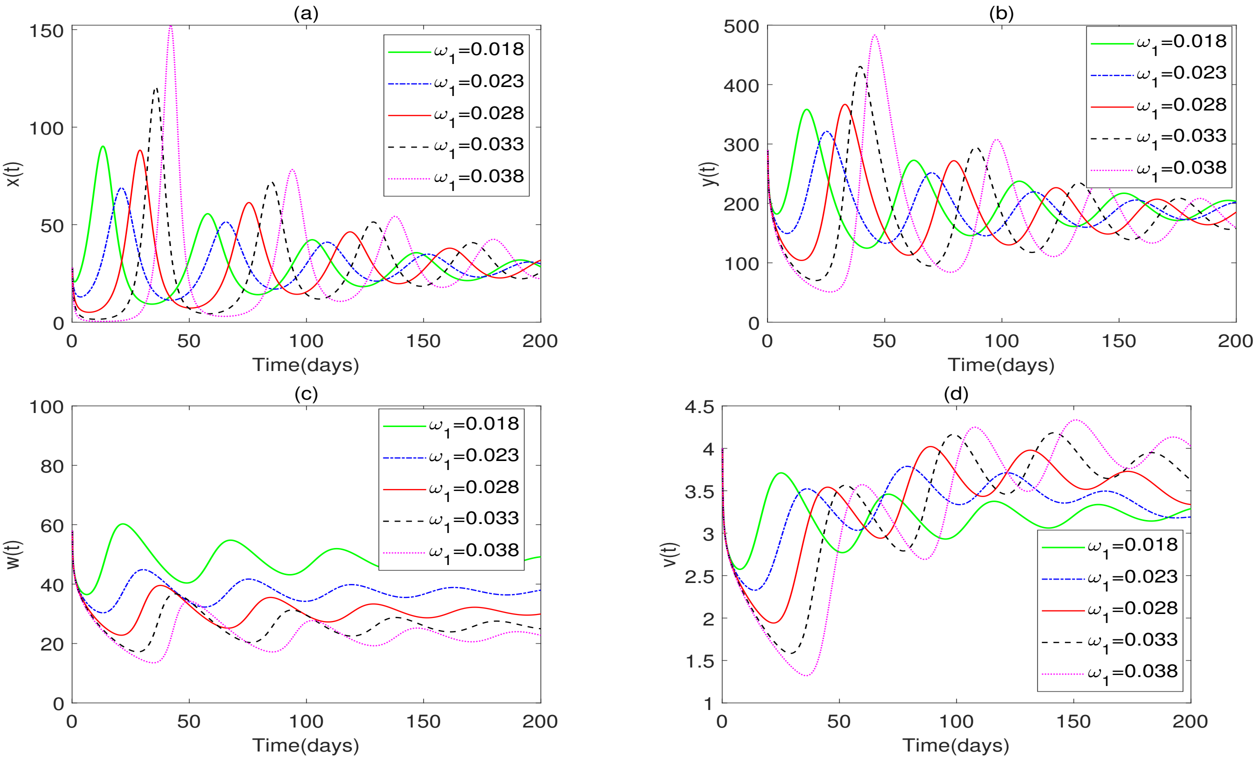

Figure 11. Dynamic behaviors of system (2.5) under varying values of ω_1 ($\omega_1 = 0.018$, 0.023, 0.028, 0.033, 0.038).

6. Conclusions

A fractional-order singular model with two preys and two predators is proposed in this paper to investigate how system dynamics are influenced by economic benefits. The study is systematically organized into two distinct scenarios: zero economic benefit and positive economic benefit. Through rigorous analysis, we examine the existence conditions of SIB and design a state feedback controller which eliminate singularity induced bifurcation. Additionally, a net profit maximization control problem is formulated, and its optimal solution is successfully derived through analytical approaches in this paper.

The following conclusion is derived from the numerical simulations.

Figure 1 indicates that two predators and two preys can coexist, demonstrating species diversity and contributing to the sustainable development of the ecosystem.

Figures 2 and 5 prove that different initial values do not affect the stability of the equilibrium. This shows that the ecosystem can maintain biodiversity.

Figure 4 shows that when $M = 0$, for relatively small values of α, the predator and prey populations undergo significant fluctuations. Notably, the prey population temporarily reaches negative values, which is unrealistic and leads to ecosystem collapse.

Figures 3 and 6 show that changing the order of the fractional order does not affect the stability of the equilibrium. Whether it is positive economic profit or zero economic profit, increasing the

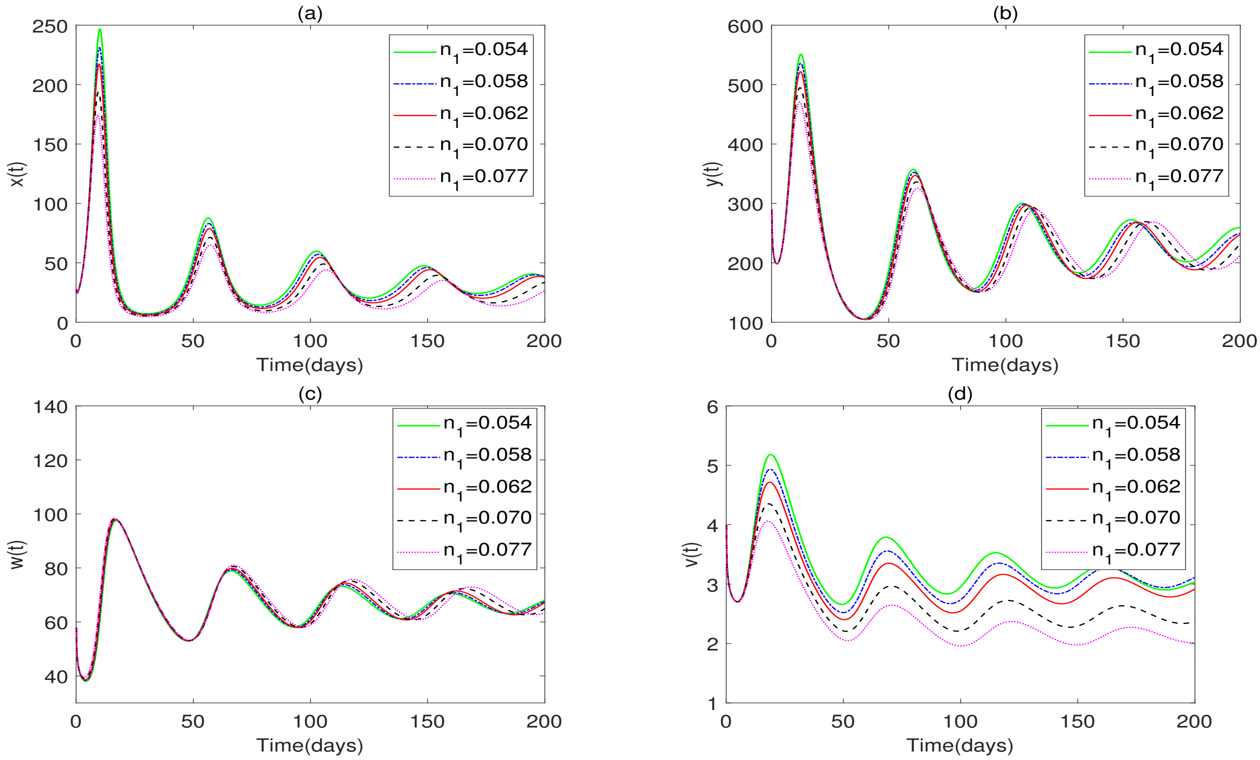

Figure 12. Dynamic behaviors of system (2.5) for different values of n_1 (n_1 = 0.054, 0.058, 0.062, 0.070, 0.077).

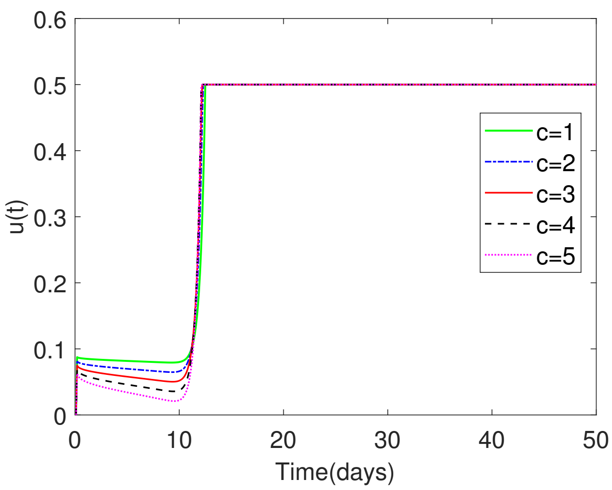

Figure 13. Time series for the optimal control problem under varying values of c (c = 1, 2, 3, 4, 5).

fractional order α directly reduces the population stabilization time. This indicates that the ecosystem can adapt to the complexity of environmental changes.

Figures 7–12 illustrate the sensitivity analysis of system parameters. The analysis results contribute

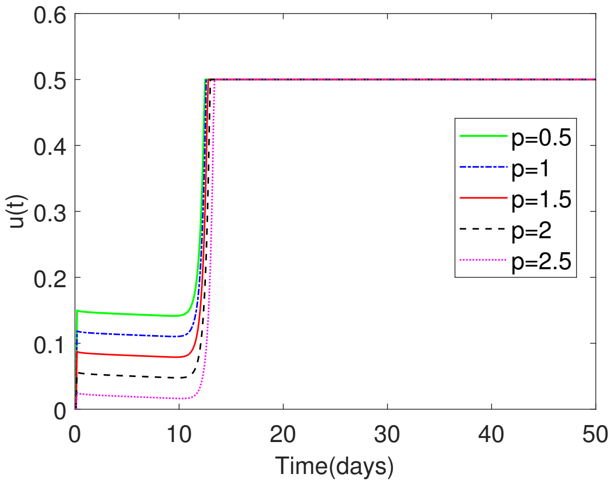

Figure 14. Time series for the optimal control problem under varying values of p (p = 0.5, 1, 1.5, 2, 2.5).

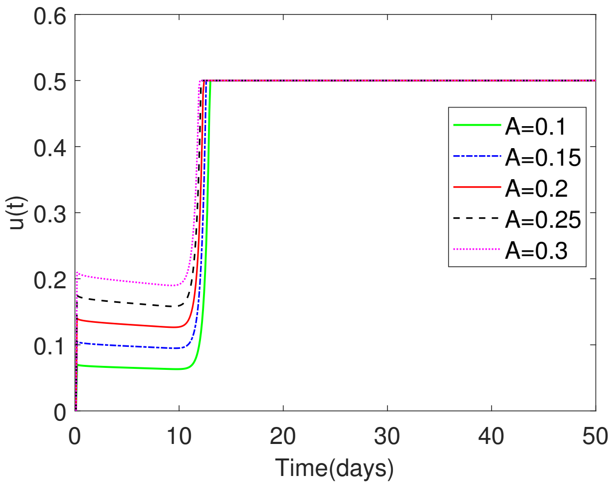

Figure 15. Time series for the optimal control problem under varying values of A (A = 0.1, 0.15, 0.2, 0.25, 0.3).

to the development of the ecological system.

The results of Figures 13–15 can be observed from an economic perspective. Neither the unit price of biomass, the fishing harvest cost, nor the economic discount factor has an impact on the optimal solution for $u(t)$, a situation that is beneficial to the sustainable development of enterprises.

Compared with existing studies [13, 30, 32], a more comprehensive investigation of species diversity is provided in this paper through the employment of fractional-order derivatives, which better captures the complexity of real-world ecological systems. Furthermore, the study systematically examines system dynamics under varying economic benefit scenarios, offering valuable insights with significant economic implications. Specifically, this article focuses more on long-term equilibrium and, thus, does not examine cases where economic profits are negative.

However, in this paper, a relatively simple controller is used in the study of singularity-induced bifurcations, without employing more complex controllers or conducting a comparative analysis between this controller and similar ones. Next, this paper presents a deterministic model that does not consider the impact of noise in a random environment. Therefore, in future research directions, the model can be extended by incorporating other more complex controllers, conducting a comparative analysis between the two types of controllers, and introducing factors such as time delay and random fluctuations, so as to obtain more robust and practically significant analysis results.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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