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https://www.aimspress.com/journal/era

ERA, 33(7): 4363–4381.
DOI: 10.3934/era.2025199
Received: 29 April 2025
Revised: 09 July 2025
Accepted: 21 July 2025
Published: 30 July 2025

Research article

Global existence and exponential stability of a fractionally damped plate equation with past history

Muhammad Fahim Aslam[1,2,*], **Jianghao Hao** [1], **Salah Boulaaras** [3,*] **and Luqman Bashir**[1]

[1] School of Mathematics and Statistics, Shanxi University, Taiyuan 030006, China
[2] Department of Mathematics, University of Kotli Azad Jammu and Kashmir (UOKAJK), Kotli, Pakistan
[3] Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia

* **Correspondence:** Email: fahim.sihaab@gmail.com, s.boulaaras@qu.edu.sa; Tel: +8618874152802, +966559618327.

Abstract: This paper is an extension of our earlier research [1]. In this article, we discussed the fractionally damped plate equation with infinite memory. We proved that solutions are global under different initial and boundary conditions. Subsequently, we investigated the exponential stability of the solutions through Lyapunov functionals in order to establish stability conditions sufficient for stability analysis.

Keywords: plate equation; general decay; fractional derivatives; infinite memory; global existence; nonlinear equations

1. Introduction

The paper in hand focuses on the following plate equation:

$$(W)\begin{cases} \mathscr{Y}_{tt} + \Delta^2\mathscr{Y} - \int_0^{+\infty} \rho(s)\Delta^2\mathscr{Y}(t-s)ds + \partial_t^{\eta,\alpha}\mathscr{Y}(t) = \mathscr{Y}|\mathscr{Y}|^{p-2}\ln|\mathscr{Y}|, & \text{in } \Omega\times(0,\infty),\\ \frac{\partial\mathscr{Y}}{\partial\nu} = \mathscr{Y} = 0, \quad \text{on } \partial\Omega\times(0,\infty), & \\ \mathscr{Y}(x,0) = \mathscr{Y}_0(x), \quad \mathscr{Y}_t(x,0) = \mathscr{Y}_1(x), \quad \text{in } \Omega, & \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, ρ is a function that will be discussed later, and $p > 2$. The modified Caputo fractional derivative, denoted by $\partial_t^{\eta,\alpha}$, is defined by (see [2,3]):

$$\partial_t^{\eta,\alpha}\mathscr{Y}(t) = \frac{1}{\Gamma(1-\eta)}\int_0^t (t-\tau)^{-\eta}e^{-\alpha(t-\tau)}\mathscr{Y}_\tau(\tau)d\tau, \qquad 0<\eta<1, \alpha\geq 0.$$

In diverse scientific domains like engineering, materials science, and applied mathematics, the viscoelastic equation that integrates fractional damping, infinite memory, and a logarithmic nonlinear source term plays a key role in modeling complex physical phenomena. This type of research is significant in both theoretical and practical applications, as these equations employ a complex framework to portray real-world situations characterized by intricate temporal interactions.

Such equations are valuable for anticipating failure modes in viscoelastic materials like beams and plates when subjected to dynamic loads. Understanding how these materials react to varying loads enables engineers to design structures that offer enhanced safety and efficiency. This model describes damping as a condition in which the damping force is not merely related to velocity but is determined by a fractional derivative. Such damping is predominantly found in viscoelastic materials, which react based on their past and present states. Consequently, fractional derivatives are employed to effectively represent these materials' memory characteristics, yielding improved models for their dynamic behavior.

The developed model's integral term for unlimited memory permits the current state to be influenced by all of its past. In materials like polymers or biological tissues, it becomes significant when it is established that previous stress or strain affects present behavior. In the mathematical formulation, convolution integrals and a relaxation function that expresses the influence of previous states on the dynamics of the current state are commonly used. A novel form of nonlinearity, which may present analytical challenges but is sufficiently compelling for consideration in qualitative analyses, is introduced through the inclusion of logarithmic nonlinearity in the equation. This particular nonlinearity is commonly employed to replicate the behavior of real materials under extreme conditions by simulating scenarios where material strength diminishes over time due to ongoing stress or deformation. Fractional calculus finds application in numerous technical and scientific domains such as biological population modeling, fluid dynamics, electrochemistry, optics, signal processing, viscoelasticity, and electromagnetism. It has been utilized to depict physical and technical phenomena that are effectively described by fractional differential equations. Traditional theories have been observed to be inadequate in the explanation of phenomena concerning unnormal kinetics by means of integer-order derivatives in recent years. This has increased the interest in fractional differential equations, and fractional differential equations can accurately model the dynamics of structures by means of the simulation of the behavior of integer-order systems. Fractional derivatives can accurately model the dynamics of structures reasonably. While the integer-order differential operator is local. The fractional-order differential operator is non-local because it takes into account the current state and previous states. Fractional-order systems are increasingly becoming better known to be realistic in nature, especially in most of the scientific and physical models with memory. Since fractional-order partial equations are fundamental to most physical phenomena in areas such fluid dynamics, quantum physics, electricity, and ecological systems, it has become increasingly important to grasp both traditional and modern methods for solving these equations and understanding their applications. Furthermore, the mathematical modeling of real-world issues frequently results in the development of fractional partial differential equations, along with various special functions in mathematical physics and their generalizations and extensions to multiple variables. The studies highlighted in [4–6] demonstrate that partial differential equations with fractional derivatives have garnered significant interest from mathematicians, biologists, and scientists in the physical sciences. In recent years, these equations have found extensive applications in fields such as electronics, relaxation vibrations, and viscoelasticity, as noted in sources like [7, 8].

Researchers Benaissa and Benkheda examined the asymptotic profile of fractional boundary dissipation [9]. Aounallah et al. [10] recently investigated the wave equation with a fractional derivative on the system boundary. The identical system with finite memory $\int_0^t \rho(t-s)\mathscr{Y}(s)\,ds$ was examined by Boulaaras et al. in [11]. The authors in [10, 11] utilized an augmented system to formulate the problem and investigate the existence and decay properties of the necessary solutions when fractional derivatives are present at the system's boundary. Additionally, Mbodje explored wave energy decay through fractional derivative controls in [12]. Furthermore, in [13], Doudi et al. examined blow-up, general decay, and global existence problems related to the logarithmic problem:

$$\begin{cases} \mathscr{Y}_{tt} - \Delta\mathscr{Y} + a\mathscr{Y}_t = \mathscr{Y}|\mathscr{Y}|^{p-1}\ln|\mathscr{Y}|, & x \in \Omega, t > 0, \\ \frac{\partial\mathscr{Y}}{\partial\nu} = -b\partial_t^{\alpha,\eta}\mathscr{Y}, & x \in \Gamma_0, t > 0, \\ \mathscr{Y} = 0, & x \in \Gamma_1, t > 0, \\ \mathscr{Y}(x,0) = \mathscr{Y}_0(x), \quad \mathscr{Y}_t(x,0) = \mathscr{Y}_1(x), & x \in \Omega. \end{cases} \tag{1.1}$$

The analysis of blow-up phenomena and the asymptotic behavior in a system featuring internal fractional time was undertaken by Aounallaha et al., as cited in [14]. Aslam and Hao [15] studied the blow-up behavior in a particular nonlinear wave equation, with fractional as well as strong damping with infinite memory. Recently, Aslam and his colleagues discussed exponential decay and blow-up of solutions in [16] when the fractional derivative was at a boundary. Subsequently, Hajjej [17] analyzed the asymptotic profile and blow-up dynamics of suspension bridges under fractional time delays, providing a detailed description of these dynamics.

The plate equation is a fundamental component of viscoelastic wave equation system studies and a robust mathematical foundation for studying plate-type structures composed of viscoelastic materials and examining their dynamic characteristics. Its applicability spans many areas of engineering disciplines, including aerospace, civil engineering, material science, and structural dynamics. In his paper, Lagnese [18] discussed a viscoelastic plate equation being utilized in the analysis of the plate, displaying how energy decays to zero as time is approaching infinity using an energy dissipation mechanism upon integration at the system boundary. Rivera et al. [19] noted that for an exponentially decaying memory kernel, first and second-order energies for the solutions to the viscoelastic plate equation also decay to zero exponentially. Komornik [20] treated energy dissipation in a plate model under moderate growth conditions. The following system was investigated by Messaoudi [21]:

$$\begin{cases} \mathscr{Y}_{tt} + \Delta^2\mathscr{Y} + a\,\mathscr{Y}_t|\mathscr{Y}_t|^{m-2} = \mathscr{Y}\,|\mathscr{Y}|^{p-2}, & \text{in } \Omega\times(0,\infty), \\ \mathscr{Y} = \frac{\partial\mathscr{Y}}{\partial\nu} = 0, & \text{on } \Omega\times(0,\infty), \\ \mathscr{Y}(x,0) = \mathscr{Y}_0(x), \quad \mathscr{Y}_t(x,0) = \mathscr{Y}_1(x), & x \in \Omega, \end{cases} \tag{1.2}$$

and demonstrated that the solution exists everywhere and explodes in a finite time. They also established a result regarding existence, which was subsequently refined by Chen and Zhou [22]. Messaoudi and Mukiawa examined the following fourth-order viscoelastic plate equation in [23]:

$$\mathscr{Y}_{tt} + \Delta^2 \mathscr{Y} - \int_0^t \rho(t-s)\Delta^2 \mathscr{Y}(s)ds = 0. \tag{1.3}$$

The following equation was addressed by Cavalcanti et al. [24]:

$$\mathscr{Y}_{tt} - \int_0^t \rho(t-s)\Delta^2 \mathscr{Y}(s)ds + \Delta^2 \mathscr{Y} + \gamma \Delta \mathscr{Y}_{tt} + a(t)\mathscr{Y}_t = 0 \tag{1.4}$$

to talk about decay in $\Omega \times (0, \infty)$. The identical formula's energy decay was demonstrated by Rivera et al. [19]. In the study presented in [25], Mukiawa explored the following equation:

$$\mathscr{Y}_{tt} - \int_0^t \rho(t-s)\Delta^2 \mathscr{Y}(s)ds + \Delta^2 \mathscr{Y} + \omega_1 \mathscr{Y}_t + \omega_2 \mathscr{Y}_t(t-s) = 0 \tag{1.5}$$

with the idea of a delay. The author obtained the decay of solutions for this equation. In [26], Mustafa and Kafini examined the following equation:

$$\mathscr{Y}_{tt} + \omega_1 \mathscr{Y}_t - \int_0^\infty \rho(t-s)\Delta^2 \mathscr{Y}(s)ds + \Delta^2 \mathscr{Y} + \omega_2 \mathscr{Y}_t(t-s) = \mathscr{Y}\,|\mathscr{Y}|^\gamma. \tag{1.6}$$

They also showed that solutions generally undergo decay. For a more comprehensive understanding of the plate equation, we direct readers to the works [27–31]. These works of fractional derivatives as well as the plate equation system motivate us to study system (W). One such interesting and rapidly developing research area that remains poorly explored by researchers is the use of fractional derivatives in describing a plate equation for viscoelasticity. It could prove to be considerably more useful to formulate a fractional-order plate equation model because it would offer some highly useful benefits over traditional integer-order models. Development and investigation of a fractional-order plate equation model would prove to be practical and cutting-edge for the topic. It would be very helpful to advance this study direction in conjunction with specialists in these fields. By using Lyapunov functionals, the authors hope to first examine the world-wide existence of solutions to this equation and subsequently their exponential stability. The structure of the paper is as follows. The presumptions required to establish the primary findings are given in Section 2. The existence of a global solution is discussed in Section 3. Exponential stability is demonstrated using a meticulously designed Lyapunov functional in Section 4.

2. Preliminaries

In this section, the system (W) will be converted to an augmented system (M), and some definitions and results will be stated for use in different proofs. We use the standard Lebesgue space $L^2(\Omega)$ and the Sobolev space $H_0^1(\Omega)$ with their usual scalar products and norms. For instance, the following claims must be made:

(w_1) $\rho : \Re_+ \to \Re_+$ is a function C^1 such that

$$\rho(0) > 0, \quad \rho_0 = \int_0^{\infty} \rho(s)ds = 1 - \varsigma > 0;$$

(w_2) When σ is a positive constant:

$$\rho'(t) \leq -\sigma\rho(t), \quad t \geq 0.$$

Lemma 2.1. *[1] Regarding a function ρ, the following inequality holds:*

$$\int_\Omega \left[\int_0^{+\infty} \rho(s)\Delta\mathscr{Y}(s)ds \right]^2 dx \leq (1 - \varsigma) \int_0^{+\infty} \rho(s)\|\Delta\mathscr{Y}(s)\|_2^2 \, ds.$$

Lemma 2.2. *[12] Consider ζ to be the function: $\zeta(\gamma) = |\gamma|^{\frac{(2\eta-1)}{2}}, \quad \gamma \in \rho, 0 < \eta < 1$, and $b = \frac{\sin(\eta\pi)}{\pi}$. Then the connection between the system's input U and output O:*

$$\begin{cases} \partial_t\beta(\gamma, t) + (\gamma^2 + \alpha)\beta(\gamma, t) - U(x, t)\zeta(\gamma) = 0, \quad \gamma \in \rho, t > 0, \alpha \geq 0, \\ \beta(\gamma, 0) = 0, \\ O(t) := b \int_{-\infty}^{+\infty} \beta(\gamma, t)\zeta(\gamma)d\gamma \end{cases} \tag{2.1}$$

is given by

$$O := I^{1-\eta,\alpha} U,$$

where

$$I^{\eta,\alpha}u(t) := \frac{1}{\Gamma(\eta)} \int_0^t (t - \tau)^{\eta-1} e^{-\alpha(t-\tau)} u(\tau)d\tau.$$

Lemma 2.3. *[9] For all $\varsigma \in D_\alpha = \mathbb{C} \backslash \,]-\infty, -\alpha]$, we have*

$$A_\varsigma := \int_{-\infty}^{+\infty} \frac{\zeta^2(\gamma)}{\varsigma + \alpha + \gamma^2} d\gamma = \frac{\pi}{sin(\eta\pi)} (\varsigma + \alpha)^{\eta-1}.$$

Following the approach in [32, 33], we now present a new variable:

$$\omega(x, s) = \mathscr{Y}(x, t) - \mathscr{Y}(x, t - s). \tag{2.2}$$

The variable ω signifies the relative history of $\mathscr{Y}$ which meets the following equation:

$$\omega_t(x, s) - \mathscr{Y}_t(x, t) + \omega_s(x, s) = 0, \quad x \in \Omega, \ \ t, s > 0. \tag{2.3}$$

With the help of Lemma 2.2 and Eq (2.2), the system (W) can be formulated as follows:

$$(M)\begin{cases} \mathscr{Y}_{tt} + \varsigma\Delta^2\mathscr{Y}(t) + \int_0^{+\infty}\rho(s)\Delta^2\omega(x,s)ds \\ +b\int_{-\infty}^{+\infty}\beta(\gamma,t)\zeta(\gamma)d\gamma = \mathscr{Y}|\mathscr{Y}|^{p-2}\ln|\mathscr{Y}|, & x\in\Omega, t>0, \\ \partial_t\beta(\gamma,t) + (\gamma^2+\alpha)\beta(\gamma,t) - \mathscr{Y}_t(x,t)\zeta(\gamma) = 0, & \gamma\in\rho, t>0, \alpha\geq 0, \\ \omega_t(x,s) + \omega_s(x,s) = \mathscr{Y}_t(x,t), & x\in\Omega, t,s>0, \\ \mathscr{Y} = \omega(x,s) = \frac{\partial\mathscr{Y}}{\partial\nu} = 0, & x\in\partial\Omega, t,s>0, \\ \mathscr{Y}(x,0) = \mathscr{Y}_0(x), \quad \mathscr{Y}_t(x,0) = \mathscr{Y}_1(x), & x\in\Omega, \\ \omega(x,0) = 0, \quad \omega^0(x,s) = \mathscr{Y}_0(x) - \mathscr{Y}_0(x,-s), & x\in\Omega, t,s>0, \\ \beta(\gamma,0) = 0, & x\in\Omega, \gamma\in\rho. \end{cases}$$

Lemma 2.4. *[1] The energy associated with the system (M) is given by:*

$$\begin{aligned} E(t) := \frac{1}{2}\|\mathscr{Y}_t(t)\|_2^2 + \frac{b}{2}\int_\Omega\int_{-\infty}^{+\infty}|\beta(\gamma,t)|^2 d\gamma dx + \frac{\varsigma}{2}\|\Delta\mathscr{Y}(t)\|_2^2 \\ + \frac{1}{p^2}\|\mathscr{Y}(t)\|_p^p - \frac{1}{p}\int_\Omega \mathscr{Y}^p\ln|\mathscr{Y}|dx + \frac{1}{2}\int_0^{+\infty}\rho(s)\|\Delta\omega(s)\|_2^2 ds, \end{aligned} \tag{2.4}$$

which satisfies

$$\frac{dE(t)}{dt} = \frac{1}{2}\int_0^{+\infty}\rho'(s)\|\Delta\omega(s)\|_2^2 ds - b\int_\Omega\int_{-\infty}^{+\infty}(\gamma^2+\alpha)|\beta(\gamma,t)|^2 d\gamma dx \leq 0. \tag{2.5}$$

Here, $\mathscr{H}$ is the energy space defined by

$$\mathscr{H} = H_0^1(\Omega)\times L^2(\Omega)\times L^2(\Omega,\mathbb{R})\times L^2_\rho(\mathbb{R}_+, H_0^1(\Omega)),$$

such that $$L^2_\rho(\mathbb{R}_+, H_0^1(\Omega)) = \left\{\omega : \mathbb{R}_+ \to H_0^1(\Omega), \int_0^{+\infty}\rho(s)\|\Delta w(s)\|_2^2 ds < \infty\right\},$$

where the inner product in the space $L^2_\rho(\mathbb{R}_+, H_0^1(\Omega))$ is defined as follows:

$$\langle w_1, w_2\rangle_{L^2_\rho(\mathbb{R}_+,H_0^1(\Omega))} = \int_0^{+\infty}\rho(s)\int_\Omega \Delta w_1(s)\Delta w_2(s)dxds.$$

Any U and $\bar{U}$ have an inner product defined in $\mathscr{H}$ as:

$$\langle U,\bar{U}\rangle_{\mathscr{H}} = \int_\Omega[\varsigma\Delta\mathscr{Y}\Delta\bar{\mathscr{Y}} + u\bar{u}]dx + b\int_\Omega\int_{-\infty}^{+\infty}\beta\bar{\beta}d\xi dx$$

$$+ \int_0^{+\infty}\rho(s)\int_\Omega\Delta\omega(s)\Delta\bar{\omega}(s)dxds,$$

where $U = (\mathscr{Y}, u, \beta, \omega)^T \in \mathscr{H}$ and $\bar{U} = (\bar{\mathscr{Y}}, \bar{u}, \bar{\beta}, \bar{\omega})^T \in \mathscr{H}$.
For the vector function $U = (\mathscr{Y}, \mathscr{Y}_t, \beta, \omega)^T$, suppose $u = \mathscr{Y}_t$. Then the problem (M) can be written as:

$$(\mathrm{M}^1)\begin{cases} U_t(t) + AU(t) = J(U(t)), \\ U(0) = U_0. \end{cases}$$

The operator $A : D(A) \to \mathscr{H}$ is defined by

$$AU = \begin{pmatrix} -u \\ \varsigma\Delta^2\mathscr{Y} + \int_0^{+\infty}\rho(s)\Delta^2\omega(x,s)ds + b\int_{-\infty}^{+\infty}\beta(x,\gamma,t)\zeta(\gamma)d\gamma \\ (\gamma^2+\alpha)\beta - u(x)\zeta(\gamma) \\ \omega_s(s) - u \end{pmatrix},$$

$$J(U) = \left(0, |\mathscr{Y}|^{p-2}\mathscr{Y}\ln|\mathscr{Y}|, 0, 0\right)^T. \tag{2.6}$$

The domain of A is given by

$$D(A) = \left\{ \begin{array}{l} U = (\mathscr{Y}, u, \beta, \omega)^T \in \mathscr{H}; \mathscr{Y} \in H^2(\Omega); u \in H_0^1(\Omega); \\ (\gamma^2+\alpha)\beta - u\zeta(\gamma) \in L^2(\Omega,\mathbb{R}); \\ |\gamma|\beta \in L^2(\Omega,\mathbb{R}); \omega_s \in L^2_\rho(\mathbb{R}_+, H_0^1(\Omega)) \end{array} \right\}.$$

Theorem 2.5. *[1] Let $T > 0$. The system (M) possesses a unique solution $U = (\mathscr{Y}, u, \beta, \omega)^T$ that satisfies the following conditions:*

1) *If $U_0 \in \mathcal{H}$, then $U \in C([0,T];\mathcal{H})$.*
2) *If $U_0 \in D(A)$, then $U \in C^1([0,T];\mathcal{H}) \cap C([0,T];D(A))$.*

3. Global existence

Here, we aim to prove the global existence of a solution for the specified problem (M). First, we will define the following functionals:

$$I(t) = \varsigma\|\Delta\mathscr{Y}(t)\|_2^2 + b\int_\Omega\int_{-\infty}^{+\infty}|\beta(\gamma,t)|^2\,\mathrm{d}\gamma\mathrm{dx} + \int_0^{+\infty}\rho(s)\,\|\Delta\omega(t)\|_2^2\,ds - \int_\Omega \ln|\mathscr{Y}|\mathscr{Y}^p dx,$$

$$J(t) = \frac{\varsigma}{2}\|\Delta\mathscr{Y}(t)\|_2^2 + \frac{1}{p^2}\|\mathscr{Y}(t)\|_p^p - \frac{1}{p}\int_\Omega \ln|\mathscr{Y}|\mathscr{Y}^p dx + \frac{1}{2}\int_0^{+\infty}\rho(s)\,\|\Delta\omega(t)\|_2^2\,ds$$
$$+\frac{b}{2}\int_\Omega\int_{-\infty}^{+\infty}|\beta(\gamma,t)|^2\,\mathrm{d}\gamma\mathrm{dx}.$$

We have

$$E(t) = J(t) + \frac{1}{2}\,\|\mathscr{Y}_t(t)\|_2^2\,.$$

We denote by C^*_ρ the Sobolev embedding constant in $H_0^1(\Omega) \hookrightarrow L^\rho(\Omega)$, i.e.,

$$\|\mathscr{Y}\|_\rho^\rho \le C^*_\rho\|\Delta\mathscr{Y}\|_2^\rho,$$

for any $2 < \rho < \frac{2n}{n-2}$.

Lemma 3.1. *For any $U_0 \in \mathcal{H}$ satisfying*

$$\left\{ \begin{array}{l} \chi = \frac{2C^*_{p+l}}{\varsigma}\left(\frac{2p}{\varsigma(p-2)}E(0)\right)^{\frac{p-2+l}{2}} < 1 \\ I(0) > 0, \end{array} \right. \tag{3.1}$$

we have $I(t) > 0, \forall\, t > 0$.

Proof. Given the continuity of $\mathscr{Y}$ and the condition $I(0) > 0$, it follows that there exists a $T^* < T$ such that $I(t) \geq 0,\ \forall\, t \in [0, T^*]$. Besides, we have

$$J(t) = \varsigma\Big(\frac{p-2}{2p}\Big)\|\Delta\mathscr{Y}(t)\|_2^2 + \frac{1}{p^2}\|\mathscr{Y}(t)\|_p^p + b\Big(\frac{p-2}{2p}\Big)\int_\Omega\int_{-\infty}^{+\infty}|\beta(\gamma,t)|^2\,d\gamma\,dx$$
$$+\Big(\frac{p-2}{2p}\Big)\int_0^{+\infty}\rho(s)\left\|\Delta\omega^t(t)\right\|_2^2\,ds + \frac{1}{p}I(t).$$

Therefore,

$$\varsigma\|\Delta\mathscr{Y}(t))\|_2^2 \leq \frac{2p}{p-2}J(t) \leq \frac{2p}{p-2}E(t) \leq \frac{2p}{p-2}E(0). \tag{3.2}$$

By using the fact that $\ln|\mathscr{Y}| < |\mathscr{Y}|^l$, we get

$$\int_\Omega \ln|\mathscr{Y}|\mathscr{Y}^p dx \leq \int_\Omega |\mathscr{Y}|^{p+l}dx,$$

where l is chosen to be $\frac{1}{e} < l < \frac{2}{n-2}$, so that

$$p + l < \frac{2n-2}{n-2} + l < \frac{2n}{n-2}.$$

Therefore, by embedding $H_0^1(\Omega) \hookrightarrow L^{p+l}(\Omega)$, it holds that

$$\int_\Omega \ln|\mathscr{Y}|\mathscr{Y}^p dx \leq C_{p+l}^*\|\Delta\mathscr{Y}\|_2^{p+l} \leq \frac{2C_{p+l}^*}{\varsigma}\|\Delta\mathscr{Y}\|_2^{p+l-2}\left(\frac{\varsigma}{2}\|\Delta\mathscr{Y}\|_2^2\right)$$
$$\leq \frac{2C_{p+l}^*}{\varsigma}\left(\left(\frac{2p}{\varsigma(p-2)}E(0)\right)\right)^{\frac{p-2+l}{2}}\left(\frac{\varsigma}{2}\|\Delta\mathscr{Y}\|_2^2\right) < \frac{\varsigma}{2}\|\Delta\mathscr{Y}\|_2^2. \tag{3.3}$$

Thus, $I(t) > 0, \forall\, t \in [0, T^*]$. Repeating this process and taking into account the fact that

$$\lim_{t\to T^*}\frac{2C_{p+l}^*}{\varsigma}\left(\frac{2p}{\varsigma(p-2)}E(0)\right)^{\frac{p-2+l}{2}} < 1,$$

we have $T^* = T$. Furthermore, we have

$$\frac{1}{2}\|\mathscr{Y}_t(t)\|_2^2 + \frac{\varsigma(p-2)}{2p}\|\Delta\mathscr{Y}\|_2^2 \leq \frac{1}{2}\|\mathscr{Y}_t(t)\|_2^2 + J(t) = E(t) \leq E(0),$$

which implies that the solution of system (M) is both global in time and bounded.

4. Exponential stability

Let us define the functional $L(t)$ by

$$L(t) = \mathrm{N}E(t) + \mathrm{N}_1\varphi(t) + \mathrm{N}_1\varphi_1(t) + \Psi(t), \tag{4.1}$$

where N and N_1 are positive constants that will be fixed later, and

$$\varphi(t) = \int_\Omega \mathscr{Y}_t\mathscr{Y}\,dx,$$

$$\varphi_1(t) = \frac{b}{2}\int_\Omega \int_{-\infty}^{+\infty} \left(\gamma^2+\alpha\right)|\mathcal{M}(x,\gamma)|^2\, d\gamma\, dx,$$

$$\Psi(t) = -\int_\Omega \mathscr{Y}_t\left(\int_{-\infty}^{+\infty} \rho(s)\omega(s)\, ds\right) dx,$$

where

$$\mathcal{M}(x,\gamma) = \frac{\mathscr{Y}_0(x)\zeta(\gamma)}{(\gamma^2+\alpha)} + \int_0^t \beta(x,\gamma,s)ds.$$

Lemma 4.1. *Let* $(\mathscr{Y}, \mathscr{Y}_t, \beta, \omega)^T$ *be a regular solution of the problem* (M)*. Then, we have*

$$\begin{aligned}&\int_\Omega \int_{-\infty}^{+\infty} \left(\gamma^2+\alpha\right)\beta(\gamma,t)\mathcal{M}(x,\gamma)d\gamma dx = \\ &\int_\Omega \mathscr{Y}(x,t)\int_{-\infty}^{+\infty} \beta(\gamma,t)\zeta(\gamma)d\gamma dx - \int_\Omega \int_{-\infty}^{+\infty} |\beta(\gamma,t)|^2 d\gamma dx.\end{aligned} \tag{4.2}$$

Proof. Clearly, by using the second equation of (M), we obtain

$$\left(\gamma^2+\alpha\right)\beta(\gamma,t) = \mathscr{Y}_t(x,t)\zeta(\gamma) - \partial_t\beta(\gamma,t), \quad \forall x \in \Omega. \tag{4.3}$$

Integrating (4.3) between 0 and t yields

$$\int_0^t \left(\gamma^2+\alpha\right)\beta(\gamma,s)ds = \mathscr{Y}(x,t)\zeta(\gamma) - \beta(\gamma,t) - \mathscr{Y}_0(x)\zeta(\gamma), \quad \forall x \in \Omega.$$

So,

$$\left(\gamma^2+\alpha\right)\left(\int_0^t \beta(\gamma,s)ds + \frac{\mathscr{Y}_0(x)\zeta(\gamma)}{(\gamma^2+\alpha)}\right) = \mathscr{Y}(x,t)\zeta(\gamma) - \beta(\gamma,t), \quad \forall x \in \Omega. \tag{4.4}$$

Multiplying (4.4) by β and integrating over $\Omega \times (-\infty, +\infty)$, we obtain (4.2).

Lemma 4.2. *Let* $(\mathscr{Y}, \mathscr{Y}_t, \beta, \omega)$ *be the solution of problem* (M)*. Therefore,*

$$|\varphi_1(t)| \le \frac{b}{\alpha}\int_\Omega \int_{-\infty}^{+\infty} |\beta(\gamma,t)|^2 d\gamma dx + bA_0C_*^2\|\Delta\mathscr{Y}\|_2^2.$$

Proof. Using (4.4), we get

$$\mathcal{M}(x,\gamma) = \frac{-\beta(\gamma,t)}{\gamma^2+\alpha} + \frac{\mathscr{Y}(x,t)\zeta(\gamma)}{\gamma^2+\alpha}, \forall x \in \Omega. \tag{4.5}$$

Then

$$(\mathcal{M}(x,\gamma))^2 = \frac{|\beta(\gamma,t)|^2}{(\gamma^2+\alpha)^2} + \frac{|\mathscr{Y}(x,t)|^2\zeta^2(\gamma)}{(\gamma^2+\alpha)^2} - 2\frac{\beta(\gamma,t)\mathscr{Y}(x,t)\zeta(\gamma)}{(\gamma^2+\alpha)^2}. \tag{4.6}$$

Multiplying (4.6) by $\frac{b}{2}(\gamma^2+\alpha)$ and integrating over $\Omega \times (-\infty, +\infty)$, we easily get

$$|\varphi_1(t)| \leq \frac{b}{2}\int_\Omega\int_{-\infty}^{+\infty}\frac{|\beta(\gamma,t)|^2}{\gamma^2+\alpha}d\gamma dx + \frac{b}{2}\int_\Omega|\mathscr{Y}(x,t)|^2\int_{-\infty}^{+\infty}\frac{\zeta^2(\gamma)}{\gamma^2+\alpha}d\gamma dx$$
$$+b\int_\Omega\int_{-\infty}^{+\infty}\frac{|\beta(\gamma,t)\mathscr{Y}(x,t)\zeta(\gamma)|}{\gamma^2+\alpha}d\gamma dx,$$

$$|\varphi_1(t)| \leq \frac{b}{2}\int_\Omega\int_{-\infty}^{+\infty}\frac{|\beta(\gamma,t)|^2}{\gamma^2+\alpha}d\gamma dx + \frac{bA_0}{2}\int_\Omega|\mathscr{Y}(x,t)|^2dx + b\int_\Omega\int_{-\infty}^{+\infty}\frac{|\beta(\gamma,t)\mathscr{Y}(x,t)\zeta(\gamma)|}{\gamma^2+\alpha}d\gamma dx, \tag{4.7}$$

where at $\varsigma = 0$, from Lemma 2.3:

$$\int_{-\infty}^{+\infty}\frac{\zeta^2(\gamma)}{\alpha+\gamma^2}d\gamma = A_0.$$

Using Young's inequality, we estimate the third term in (4.7) and get the following results:

$$b\int_\Omega\int_{-\infty}^{+\infty}\frac{|\beta(\gamma,t)\mathscr{Y}(x,t)\zeta(\gamma)|}{\gamma^2+\alpha}d\gamma dx = b\int_\Omega\int_{-\infty}^{+\infty}\frac{|\beta(\gamma,t)|}{(\gamma^2+\alpha)^{\frac{1}{2}}}\frac{|\mathscr{Y}(x,t)\zeta(\gamma)\,|}{(\gamma^2+\alpha)^{\frac{1}{2}}}d\gamma dx$$
$$\leq \frac{b}{2}\int_\Omega\int_{-\infty}^{+\infty}\frac{|\beta(\alpha,t)|^2}{\gamma^2+\alpha}d\gamma dx + \frac{b}{2}\int_\Omega|\mathscr{Y}|^2\int_{-\infty}^{+\infty}\frac{\zeta^2(\gamma)}{\gamma^2+\alpha}d\gamma dx$$
$$\leq \frac{b}{2}\int_\Omega\int_{-\infty}^{+\infty}\frac{|\beta(\alpha,t)|^2}{\gamma^2+\alpha}d\gamma dx + \frac{bA_0}{2}\|\mathscr{Y}\|_2^2.$$

So,

$$|\varphi_1(t)| \leq b\int_\Omega\int_{-\infty}^{+\infty}\frac{|\beta(\alpha,t)|^2}{\gamma^2+\alpha}d\gamma dx + bA_0\|\mathscr{Y}\|_2^2.$$

Using the fact that $\frac{1}{\gamma^2+\alpha} \leq \frac{1}{\alpha}$ and Poincare's inequality, with constant $C_*^2 > 0$, we get

$$|\varphi_1(t)| \leq \frac{b}{\alpha}\int_\Omega\int_{-\infty}^{+\infty}|\beta(\gamma,t)|^2d\gamma dx + bA_0C_*^2\|\Delta\mathscr{Y}\|_2^2. \tag{4.8}$$

Lemma 4.3. *Let $(\mathscr{Y},\mathscr{Y}_t,\beta,\omega)$ be the solution of problem (M). Then,*

$$|\varphi(t)| \leq \frac{1}{2}\|\mathscr{Y}_t\|_2^2 + \frac{C_*^2}{2}\|\Delta\mathscr{Y}\|_2^2. \tag{4.9}$$

Lemma 4.4. *Let $(\mathscr{Y},\mathscr{Y}_t,\beta,\omega)$ be the solution of problem (M). Therefore,*

$$|\Psi(t)| \leq \frac{1}{2}\,\|\mathscr{Y}_t(t)\|_2^2 + \frac{1}{2}(1-\varsigma)C_*^2\int_0^{+\infty}\rho(s)\,\|\omega(s)\|_2^2\,ds. \tag{4.10}$$

Proof. By Young's and Poincare's inequalities and Lemma 2.1,

$$|\Psi(t)| \leq \frac{1}{2}\,\|\mathscr{Y}_t(t)\|_2^2 + \frac{1}{2}\int_\Omega\left(\int_{-\infty}^{+\infty}\rho(s)\omega(s)ds\right)^2dx$$
$$\leq \frac{1}{2}\,\|\mathscr{Y}_t(t)\|_2^2 + \frac{1}{2}(1-\varsigma)C_*^2\int_0^{+\infty}\rho(s)\,\|\Delta\omega(s)\|_2^2\,ds.$$

Lemma 4.5. *Assume $(\mathscr{Y}, \mathscr{Y}_t, \beta, \omega)$ is a solution to the problem (M), with Lemma 4.2, Lemma 4.3, and Lemma 4.9 being satisfied. It follows that there exist two positive constants α_1 and α_2 such that*

$$\alpha_1 \mathrm{E(t)} \le \mathrm{L(t)} \le \alpha_2 \mathrm{E(t)}.$$

Proof.

$$\begin{aligned}\|\mathrm{N}_1\varphi(t) + \mathrm{N}_1\varphi_1(t) + \psi(t)\| \le & \left(\frac{N_1+1}{2}\right)\|\mathscr{Y}_t\|_2^2 + \frac{N_1 b}{\alpha}\int_\Omega\int_{-\infty}^{+\infty}|\beta(\gamma,t)|^2\,d\gamma\,dx \\ & + N_1\left(\frac{C_*^2}{2} + bA_0C_*^2\right)\|\Delta\mathscr{Y}\|_2^2 \\ & + \frac{1}{2}(1-\varsigma)C_*^2\int_0^{+\infty}\rho(s)\,\|\Delta\omega(s)\|_2^2\,ds.\end{aligned} \tag{4.11}$$

Thus, by the definition of the energy (2.4), we can conclude the following for any $N > 0$:

$$\begin{aligned}\|\mathrm{N}_1\varphi(t) + \mathrm{N}_1\varphi_1(t) + \psi(t)\| \le & \, N\mathrm{E(t)} + \left(\frac{N_1+1}{2} - \frac{N}{2}\right)\|\mathscr{Y}_t\|_2^2 \\ & + \left(\frac{N_1 b}{\alpha} - \frac{Nb}{2}\right)\int_\Omega\int_{-\infty}^{+\infty}|\beta(\gamma,t)|^2 d\gamma dx \\ & - \frac{N}{p^2}\|\mathscr{Y}(x,t)\|_p^p + \left(\left(\frac{C_*^2 N_1(2bA_0+1)}{2}\right) - \frac{\varsigma N}{2}\right)\|\Delta\mathscr{Y}\|_2^2 \\ & + \left(\frac{1}{2}(1-\varsigma)C_*^2 - \frac{N}{2}\right)\int_0^{+\infty}\rho(s)\,\|\Delta\omega(s)\|_2^2\,ds \\ & + \frac{N}{p}\int_\Omega \ln|\mathscr{Y}|\mathscr{Y}^p dx.\end{aligned} \tag{4.12}$$

Using (3.3), we get

$$\begin{aligned}& \le N\mathrm{E(t)} + \left(\frac{N_1+1}{2} - \frac{N}{2}\right)\|\mathscr{Y}_t\|_2^2 \\ & + \left(\frac{N_1 b}{\alpha} - \frac{Nb}{2}\right)\int_\Omega\int_{-\infty}^{+\infty}|\beta(\gamma,t)|^2 d\gamma dx \\ & + \left(\left(\frac{C_*^2 N_1(2bA_0+1)}{2}\right) - \frac{\varsigma N(p-1)}{2p}\right)\|\Delta\mathscr{Y}\|_2^2 \\ & + \left(\frac{1}{2}(1-\varsigma)C_*^2 - \frac{N}{2}\right)\int_0^{+\infty}\rho(s)\,\|\Delta\omega(s)\|_2^2\,ds.\end{aligned} \tag{4.13}$$

If we choose

$$N > \max\left\{N_1+1, \frac{pC_*^2 N_1(2bA_0+1)}{\varsigma(p-1)}, \frac{2N_1}{\alpha}, (1-\varsigma)C_*^2\right\},$$

and ε_1 such that $\varepsilon_1 \ge N$, then the inequality (4.13) leads to

$$\alpha_1 \mathrm{E(t)} \le \mathrm{L(t)} \le \alpha_2 \mathrm{E(t)}.$$

Here $\alpha_1 = \varepsilon_1 - NN_1$ and $\alpha_2 = \varepsilon_1 + NN_1$. This completes the proof.

Lemma 4.6. *Let* $(\mathscr{Y}, \mathscr{Y}_t, \beta, \omega)$ *be the solution of* (M)*, and then we have*

$$\begin{aligned}\varphi'(t) &= \|\mathscr{Y}_t\|_2^2 - \frac{\varsigma}{4}\|\Delta\mathscr{Y}(t)\|_2^2 + \frac{(1-\varsigma)}{3\varsigma}\int_0^{+\infty}\rho(s)\,\|\Delta\omega(s)\|_2^2\,ds \\ &- b\int_\Omega \mathscr{Y}(x)\int_{-\infty}^{+\infty}\zeta(\gamma)\beta(\gamma,t)d\gamma dx + \int_\Omega \ln|\mathscr{Y}|\mathscr{Y}^p dx.\end{aligned} \tag{4.14}$$

Proof. Differentiating φ and using (M) yields

$$\begin{aligned}\varphi'(t) &= \|\mathscr{Y}_t\|_2^2 + \int_\Omega \mathscr{Y}\mathscr{Y}_{tt}dx \\ &= \|\mathscr{Y}_t\|_2^2 - \varsigma\|\Delta\mathscr{Y}(t)\|_2^2 - \int_\Omega \Delta\mathscr{Y}(t)\int_0^{+\infty}\rho(s)\Delta\omega(s)dsdx \\ &- b\int_\Omega \mathscr{Y}(x)\int_{-\infty}^{+\infty}\zeta(\gamma)\beta(\gamma,t)d\gamma dx + \int_\Omega \mathscr{Y}^p\ln|\mathscr{Y}|dx.\end{aligned} \tag{4.15}$$

The following is an estimate for the third term in (4.15):

$$\begin{aligned}&\left|\int_\Omega \Delta\mathscr{Y}(t)\int_0^{+\infty}\rho(s)\Delta\omega(s)dsdx\right| \\ &\le \frac{3\varsigma}{4}\|\Delta\mathscr{Y}(t)\|_2^2 + \frac{(1-\varsigma)}{3\varsigma}\int_0^{+\infty}\rho(s)\,\|\Delta\omega(s)\|_2^2\,ds.\end{aligned} \tag{4.16}$$

Combining (4.15) and (4.16), we get (4.14).

Lemma 4.7. *Let* $(\mathscr{Y}, \mathscr{Y}_t, \beta, \omega)$ *be the solution of* (M)*, and then we have*

$$\varphi_1'(t) = -b\int_\Omega\int_{-\infty}^{+\infty}|\beta(\gamma,t)|^2 d\gamma dx + b\int_\Omega \mathscr{Y}(x,t)\int_{-\infty}^{+\infty}\zeta(\gamma)\beta(\gamma,t)d\gamma dx. \tag{4.17}$$

Proof. By differentiating φ_1, we obtain

$$\varphi_1'(t) = b\int_\Omega\int_{-\infty}^{+\infty}\left(\gamma^2+\alpha\right)\beta(\gamma,t)\left(\int_0^t \beta(\gamma,s)ds + \frac{\mathscr{Y}_0(x)\zeta(\gamma)}{\gamma^2+\alpha}\right)d\gamma dx.$$

Using(4.5), we have (4.17).

Lemma 4.8. *Let* $(\mathscr{Y}, \mathscr{Y}_t, \beta, \omega)$ *be the solution of* (M)*, and then we have*

$$\begin{aligned}\Psi'(t) &\le \left(k_1 - \int_0^{+\infty}\rho(s)ds\right)\|\mathscr{Y}_t\|_2^2 + k_1\left(\varsigma + c\right)\|\Delta\mathscr{Y}\|_2^2 \\ &- \frac{\rho(0)(1-\varsigma)C_*^2}{4k_1}\int_0^{+\infty}\rho'(s)\,\|\Delta\omega(s)\|_2^2\,ds \\ &+ k_1 b\int_\Omega\int_{-\infty}^{+\infty}\left(\gamma^2+\alpha\right)|\beta(\gamma,t)|^2 d\gamma dx \\ &+ (1-\varsigma)\left(\frac{\varsigma + A_0 bC_*^2 + C_*^2}{4k_1} + 1\right)\int_0^{+\infty}\rho(s)\,\|\Delta\omega(s)\|_2^2\,ds,\end{aligned} \tag{4.18}$$

where k_1 *is a positive constant.*

Proof. By differentiating Ψ, we obtain

$$\Psi'(t) = -\int_\Omega \mathscr{Y}_{tt}(t)\int_0^{+\infty}\rho(s)\omega(s)dsdx - \left(\int_0^{+\infty}\rho(s)ds\right)\|\mathscr{Y}_t(t)\|_2^2 - \int_\Omega \mathscr{Y}_t(t)\int_0^{+\infty}\rho'(s)\omega(s)dsdx. \tag{4.19}$$

Using problem (M) and additionally, by employing integration by parts over Ω, we get

$$\begin{aligned}\Psi'(t) = &-\left(\int_0^{+\infty}\rho(s)ds\right)\|\mathscr{Y}_t(t)\|_2^2 - \underbrace{\int_\Omega \mathscr{Y}_t(t)\int_0^{+\infty}\rho'(s)\omega(s)dsdx}_{J_1}\\ &+\varsigma\underbrace{\int_\Omega \Delta\mathscr{Y}(t)\cdot\int_0^{+\infty}\rho(s)\Delta\omega(s)dsdx}_{J_2}\\ &\underbrace{-\int_\Omega\left\{\int_0^{+\infty}\rho(s)\Delta\omega(s)ds\right\}^2 dx}_{J_3}\\ &+b\underbrace{\int_\Omega\int_{-\infty}^{+\infty}\zeta(\gamma)\beta(\gamma,t)d\gamma\left\{\int_0^{+\infty}\rho(s)\omega(s)ds\right\}dx}_{J_4}\\ &\underbrace{-\int_\Omega|\mathscr{Y}|^{p-2}\mathscr{Y}\ln|\mathscr{Y}|\int_0^{+\infty}\rho(s)\omega(s)dsdx}_{J_5}.\end{aligned} \tag{4.20}$$

Let us estimate the terms J_1, J_2, J_3, J_4, J_5 in (4.20).
Applying (w_1) and (w_2), along with Young's and Poincare's inequalities, we obtain

$$|J_1| \le k_1\|\mathscr{Y}_t(t)\|_2^2 - \frac{(1-\varsigma)\rho(0)C_*^2}{4k_1}\int_0^{+\infty}\rho'(s)\|\Delta\omega(s)\|_2^2 ds, \forall k_1 > 0. \tag{4.21}$$

Similarly,

$$\begin{aligned}|J_2| &\le k_1\|\Delta\mathscr{Y}(t)\|_2^2 + \frac{(1-\varsigma)}{4k_1}\int_0^{+\infty}\rho(s)\|\Delta\omega(s)\|_2^2 ds,\\ |J_4| &\le \frac{1}{4k_1}\int_\Omega\left\{\int_{-\infty}^{+\infty}\frac{\zeta^2(\gamma)}{\gamma^2+\alpha}d\gamma\right\}\left\{\int_0^{+\infty}\rho(s)\omega(s)ds\right\}dx\\ &\quad + k_1\int_\Omega\int_{-\infty}^{+\infty}\left(\gamma^2+\alpha\right)|\beta(\gamma,t)|^2 d\gamma dx\\ &\le \frac{A_0C_*^2(1-\varsigma)}{4k_1}\int_0^{+\infty}\rho(s)\|\Delta\omega(s)\|_2^2 ds\\ &\quad + k_1\int_\Omega\int_{-\infty}^{+\infty}\left(\gamma^2+\alpha\right)|\beta(\gamma,t)|^2 d\gamma dx.\end{aligned}$$

Using the proof in [34, Lemma 3.7] and Young's and Poincare's inequalities, the final term is calculated as follows:

$$|J_5| \leq k_1 C \|\Delta \mathscr{Y}(t)\|_2^2 + \frac{C_*^2(1-\varsigma)}{4k_1} \int_0^{+\infty} \rho(s)\, \|\Delta\omega(s)\|_2^2\, ds.$$

Inserting all estimates in (4.20), we will get (4.18).

Theorem 4.9. *Assume that* (w_1) *and* (w_2) *are true. In that case, the global solution of problem (M) fulfills the existence of positive constants m and K.*

$$\mathrm{E(t)} \leq K e^{-mt}. \tag{4.22}$$

Proof. Given that ρ is continuous and positive and that $\rho(0) > 0$, we obtain

$$\int_0^{+\infty} \rho(s) ds = \rho_0 > 0.$$

After differentiating (4.1), use (2.5), (4.14), (4.17), and (4.18) to get

$$\begin{aligned}
L'(t) &\leq \left(-\frac{\rho(0)(1-\varsigma)}{4k_1} C_*^2 + \frac{N}{2}\right) \int_0^{\infty} \rho'(s)\, \|\Delta\omega(s)\|_2^2\, ds \\
&- (N - k_1) b \int_\Omega \int_{-\infty}^{+\infty} (\gamma^2 + \alpha)\, |\beta(\gamma, t)|^2\, d\gamma\, dx \\
&+ (1-\varsigma)\left(\frac{N_1}{3\varsigma} + \left(\frac{\varsigma + A_0 b C_*^2 + C_*^2}{4k_1} + 1\right)\right) \int_0^{\infty} \rho(s)\, \|\Delta\omega(s)\|_2^2\, ds \\
&- ((-k_1 + \rho_0) - N_1)\, \|\mathscr{Y}_t\|_2^2 \\
&- \left[\frac{N_1 \varsigma}{4} - k_1(\varsigma + c)\right] \|\Delta \mathscr{Y}\|_2^2 \\
&- N_1 b \int_\Omega \int_{-\infty}^{+\infty} |\beta(\gamma, t)|^2\, d\gamma\, dx + N_1 \int_\Omega \mathscr{Y}^p \ln |\mathscr{Y}| dx.
\end{aligned} \tag{4.23}$$

Also,

$$\begin{aligned}
L'(t) &\leq \left(-\frac{\rho(0)(1-\varsigma)}{4k_1} C_*^2 + \frac{N}{2}\right) \int_0^{\infty} \rho'(s)\, \|\Delta\omega(s)\|_2^2\, ds \\
&- (N - k_1) b \int_\Omega \int_{-\infty}^{+\infty} (\gamma^2 + \alpha)\, |\beta(\gamma, t)|^2\, d\gamma\, dx \\
&+ (1-\varsigma)\left(\frac{N_1}{3\varsigma} + \left(\frac{\varsigma + A_0 b C_*^2 + C_*^2}{4k_1} + 1\right)\right) \int_0^{\infty} \rho(s)\, \|\Delta\omega(s)\|_2^2\, ds \\
&- ((-k_1 + \rho_0) - N_1)\, \|\mathscr{Y}_t\|_2^2 + \frac{1}{p^2} \|\mathscr{Y}(t)\|_p^p \\
&- \left[\frac{N_1 \varsigma}{4} - k_1(\varsigma + c)\right] \|\Delta \mathscr{Y}\|_2^2 - \frac{1}{p^2} \|\mathscr{Y}(t)\|_p^p \\
&- N_1 b \int_\Omega \int_{-\infty}^{+\infty} |\beta(\gamma, t)|^2\, d\gamma\, dx + N_1 \int_\Omega \mathscr{Y}^p \ln |\mathscr{Y}| dx.
\end{aligned} \tag{4.24}$$

The term $\|\mathscr{Y}\|_p^p$ can be estimated as follows:

$$\|\mathscr{Y}\|_p^p \le C_*^p \|\Delta\mathscr{Y}\|_2^{p-2}\|\Delta\mathscr{Y}\|_2^2 \le \left(\|\Delta\mathscr{Y}\|_2^2\right)^{\frac{p-2}{2}} \|\Delta\mathscr{Y}\|_2^2. \tag{4.25}$$

Now using (3.2), we get

$$\|\mathscr{Y}\|_p^p \le C_*^p \left(\frac{2p}{\varsigma(p-2)}E(0)\right)^{\frac{p-2}{2}} \|\Delta\mathscr{Y}\|_2^2. \tag{4.26}$$

$$\begin{aligned} L'(t) \le{}& \left(-\frac{\rho(0)(1-\varsigma)}{4k_1}C_*^2 + \frac{N}{2}\right)\int_0^\infty \rho'(s)\,\|\Delta\omega(s)\|_2^2\,ds \\ &- (N-k_1)b\int_\Omega\int_{-\infty}^{+\infty}(\gamma^2+\alpha)\,|\beta(\gamma,t)|^2\,d\gamma\,dx \\ &+ (1-\varsigma)\left(\frac{N_1}{3\varsigma} + \left(\frac{\varsigma + A_0bC_*^2 + C_*^2}{4k_1} + 1\right)\right)\int_0^\infty \rho(s)\,\|\Delta\omega(s)\|_2^2\,ds \\ &- ((-k_1+\rho_0) - N_1)\,\|\mathscr{Y}_t\|_2^2 \\ &- \left[\frac{N_1\varsigma}{4} - k_1(\varsigma + c) - \left(\frac{C_p^*}{p^2}\left(\frac{2pE(0)}{\varsigma(p-2)}\right)^{\frac{p-2}{2}}\right)\right]\|\Delta\mathscr{Y}\|_2^2 \\ &- N_1b\int_\Omega\int_{-\infty}^{+\infty}|\beta(\gamma,t)|^2\,d\gamma\,dx + N_1\int_\Omega \mathscr{Y}^p\ln|\mathscr{Y}|dx - \frac{1}{p^2}\|\mathscr{Y}(t)\|_p^p. \end{aligned} \tag{4.27}$$

Therefore, by constant $\sigma > 0$, $\rho'(t) \le -\sigma\rho(t), \quad t \ge 0$.

$$\begin{aligned} L'(t) \le{}& -(N-k_1)b\int_\Omega\int_{-\infty}^{+\infty}(\gamma^2+\alpha)\,|\beta|^2\,d\gamma\,dx \\ &- \underbrace{\left[\sigma\left(-\frac{\rho(0)(1-\varsigma)}{4k_1}C_*^2 + \frac{N}{2}\right) - (1-\varsigma)\left(\frac{N_1}{3\varsigma} + \left(\frac{\varsigma + A_0bC_*^2 + C_*^2}{4k_1} + 1\right)\right)\right]}_{s_1}\int_0^\infty \rho(s)\,\|\Delta\omega(s)\|_2^2\,ds \\ &- \underbrace{((-k_1+\rho_0) - N_1)}_{s_2}\,\|\mathscr{Y}_t\|_2^2 \\ &- \underbrace{\left[\frac{N_1\varsigma}{4} - k_1(\varsigma + c) - \left(\frac{C_p^*}{p^2}\left(\frac{2pE(0)}{\varsigma(p-2)}\right)^{\frac{p-2}{2}}\right)\right]}_{s_3}\|\Delta\mathscr{Y}\|_2^2 \\ &- N_1b\int_\Omega\int_{-\infty}^{+\infty}|\beta|^2\,d\gamma\,dx + N_1\int_\Omega \mathscr{Y}^p\ln|\mathscr{Y}|dx - \frac{1}{p^2}\|\mathscr{Y}(t)\|_p^p. \end{aligned} \tag{4.28}$$

If we fix k_1, then it is obvious that $s_2 > 0$ and $s_3 > 0$.
Also, we can choose N and N_1 so small that (4.28) and Lemma 4.5 remain valid and $s_1 > 0$.

$$\begin{aligned} L'(t) \le{}& -s_1\int_0^\infty \rho(s)\,\|\Delta\omega(s)\|_2^2\,ds - s_2\,\|\mathscr{Y}_t\|_2^2 - s_3\,\|\Delta\mathscr{Y}\|_2^2 \\ &- N_1b\int_\Omega\int_{-\infty}^{+\infty}|\beta|^2\,d\gamma\,dx + N_1\int_\Omega \mathscr{Y}^p\ln|\mathscr{Y}|dx - \frac{1}{p^2}\|\mathscr{Y}(t)\|_p^p. \end{aligned} \tag{4.29}$$

Lemma 4.5 and Eq (4.28) thus give us

$$\begin{cases} \alpha_1 E(t) \leq L(t) \leq \alpha_2 E(t) \\ L'(t) \leq -s_4 E(t) \leq \frac{-s_4}{\alpha_2} L(t), \text{ for all t} \geq 0, \end{cases} \tag{4.30}$$

where s_4 is a positive constant having the following property

$$\max\left(2s_1, 2s_2, \frac{2s_3}{\varsigma}, 2N_1\right) < s_4 < pN_1. \tag{4.31}$$

Integrate $(4.30)_2$ over $(0, t)$ to get (4.22).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The researchers would like to thank the Deanship of Graduate Studies and Scientific Research at the Qassim University for financial support (QU-APC-2025).

Conflict of interest

The authors would like to affirm that there are no conflicts of interest to declare, ensuring the integrity and transparency of this work.

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