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Research article

Cheng-Yau type gradient estimates for $\Delta_f v^\tau + \lambda(x)v^l = 0$ on smooth metric measure spaces

Fanqi Zeng[1], **Cheng Jin**[1], **Peilong Dong**[2,*] **and Xinying Jiang**[1]

[1] School of Mathematics and Statistics, Xinyang Normal University, Xinyang 464000, China
[2] School of Mathematics and Statistics, Zhengzhou Normal University, Zhengzhou 450044, China

* **Correspondence:** Email: dpl2021@163.com.

Abstract: In this paper, by using the Saloff-Coste Sobolev-type inequality and Nash-Moser iteration, we proved a local gradient estimate of Cheng-Yau type for positive solutions to the equation

$$\Delta_f v^\tau + \lambda(x)v^l = 0$$

on metric measure spaces with m-Bakry-Emery Ricci curvature bounded from below. Here $\tau > 0$ and l were constants, and $\lambda(x)$ was allowed to change sign. As applications, we also obtained a Liouville-type result and Harnack's inequality. Compared with previous works, this paper did not need to suppose the positive solutions are bounded and extended the ranges of τ and l.

Keywords: nonlinear elliptic equation; gradient estimate; Liouville theorem; Harnack inequality

1. Introduction

Gradient estimation serves as a fundamental analytical tool in the investigation of partial differential equations within the framework of Riemannian geometry [1–3]. This method enables researchers to establish Liouville-type results [4–6] and further facilitates the derivation of Harnack inequalities through its systematic application [7–9]. Many mathematicians have been attracted by the topic; see for example, [10–12].

In this paper, we improve some gradient estimates and Liouville-type results previously obtained in Wang [13] and Wang et al. [14] for solutions to the equation

$$\Delta_f v^\tau + \lambda(x)v^l = 0 \tag{1.1}$$

on a complete smooth metric measure space $(M^n, g, e^{-f}dv)$ with m-Bakry-Émery Ricci curvature bounded from below, where $\tau > 0$ and l are constants, and $\lambda(x)$ is a smooth function. The

m-Bakry-Émery Ricci tensor is defined as

$$\mathrm{Ric}_f^m = \mathrm{Ric} + \nabla^2 f - \frac{1}{m-n}\mathrm{d}f \otimes \mathrm{d}f, \qquad m > n. \tag{1.2}$$

$\Delta_f \cdot = \Delta - \langle \nabla f, \nabla \cdot \rangle$ is the weighted Laplacian for some smooth potential function f. For smooth metric measure spaces, see Wei and Wylie [15].

When $\lambda(x) = \lambda$ is a constant and $l = 1$. Equation (1.1) becomes

$$\Delta_f v^\tau + \lambda v = 0. \tag{1.3}$$

Wang [13] established gradient estimates for positive solutions to the Eq (1.3) on a smooth metric measure space whose m-Bakry-Émery Ricci tensor is bounded from below by $-(m-1)\kappa$ ($\kappa \geq 0$). We state his main results as below.

Theorem 1.1. *([13]) Suppose that $(M^n, g, e^{-f}dv)$ is a smooth metric measure space with $\mathrm{Ric}_f^m \geq -(m-1)\kappa g$ for some $\kappa \geq 0$ and $m > 2$. Let v be a smooth positive solution to the Eq (1.3) and $u = \frac{\tau}{\tau-1}v^{\tau-1}$.*

(1) If $\tau > 1$, then

$$\lambda \leq C_1\left(m, \tau, \kappa, \sup_M u\right), \qquad \frac{|\nabla u|^2}{u} \leq C_2\left(m, \tau, \lambda, \kappa, \sup_M u\right), \tag{1.4}$$

where $C_1 > 0$ and $C_2 \geq 0$ are explicit.

(2) If $1 - \frac{2}{m} < \tau < 1$, then

$$\lambda \leq \overline{C}_1\left(m, \tau, \kappa, \sup_M u\right), \qquad \frac{|\nabla u|^2}{u} \leq -\overline{C}_2\left(m, \tau, \lambda, \kappa, \sup_M u\right), \tag{1.5}$$

where $\overline{C}_1 > 0$ and $\overline{C}_2 \geq 0$ are explicit and $u < 0$.

As a corollary, Wang [13] also derived a uniform bound for positive solutions to the Eq (1.3).

Later, when f is a constant and $\tau > 1$ in Eq (1.1), Wang et al. [14] found the following gradient estimate.

Theorem 1.2. *([14]) Suppose that (M^n, g) is a complete Riemannian manifold without boundary. Suppose that $B_R(o)$ is a geodesic ball of radius R around $o \in M$ and* Ric $\geq -\kappa$ *($\kappa \geq 0$). Let $v(x)$ be a positive solution of the Eq (1.1), where f is a constant and $\tau > 1$. Suppose that $u = \frac{\tau}{\tau-1}v^{\tau-1}$ and $|\nabla\lambda|^2 \leq K|\lambda|^2$ for a positive constant K. If $\lambda(x) \geq 0$ and $l \leq \frac{(n+1)(\tau+1)}{2(n-1)}$ or $\lambda(x) \leq 0$ and $l \geq \frac{(n+1)(\tau+1)}{2(n-1)}$, then we have*

$$\sup_{B_R(o)} \frac{|\nabla u|^2}{u} \leq \tilde{C}_1(n)\left[\frac{1}{R^2}(1 + \sqrt{\kappa}R) + 2\kappa + K\right] \sup_{x \in B_R(o)} u. \tag{1.6}$$

Applying (1.6), Wang et al. [14] derived the following Liouville-type theorem as $\lambda(x)$ is a constant.

Theorem 1.3. *([14]) Suppose that (M^n, g) is a complete Riemannian manifold without boundary. Suppose that* Ric ≥ 0*. Let $v(x)$ is a positive solution of the Eq (1.1), where f is constant, $\tau > 1$ and $\lambda(x)$ is a constant. If $\lambda \geq 0$ and $l \leq \frac{(n+1)(\tau+1)}{2(n-1)}$ or $\lambda \leq 0$ and $l \geq \frac{(n+1)(\tau+1)}{2(n-1)}$, then v is a constant.*

The methods employed in [13, 14] are the maximum principle and cutoff functions, i.e., Bernstein and Yau's method. Except for Bernstein and Yau's method, Nash-Moser iteration provides us with an elegant way to study gradient estimates for positive solutions to linear elliptic and parabolic equations on a Riemannian manifold [16–18]. A huge literature exists on these; we refer to the classical and recent papers [19, 20] for interested readers.

A natural question is, "Could we obtain some Cheng-Yau-type gradient estimates for positive solutions to (1.1) through the Nash-Moser iteration ?"

In the present paper, we try to answer the above questions, i.e., adapting the main idea from the works in [21–23], we can prove a Cheng-Yau type gradient estimate for positive solutions of Eq (1.1) on any smooth metric measure spaces with $\mathrm{Ric}_f^m \geq -(m-1)\kappa g$ $(\kappa \geq 0)$.

So as to state the main results of this paper, we need some notations; see Song and Wu [21]. We let the smooth function $\lambda(x)$ be not identically zero on $B_R(o) \subset M$. We define

$$\Omega_0 = \{x \in B_R(o) | \lambda(x) = 0\}.$$

We need the assumption that

$$\nabla\lambda(x) = 0, \forall x \in \Omega_0, \tag{1.7}$$

if $\Omega_0 \neq \emptyset$.

Furthermore, we suppose that l and τ satisfy the following condition:

($G1$) If $\lambda(x) \geq 0$ for all $x \in B_R(o)$, suppose that

$$l \leq \frac{m+3}{m-1}\tau. \tag{1.8}$$

($G2$) If $\lambda(x) \leq 0$ for all $x \in B_R(o)$, suppose that

$$l \geq \tau. \tag{1.9}$$

($G3$) If $\lambda(x)$ changes sign in $x \in B_R(o)$, suppose that l and τ satisfy (1.8) and (1.9) simultaneously, i.e.,

$$\tau \leq l \leq \frac{m+3}{m-1}\tau. \tag{1.10}$$

Now we state:

Theorem 1.4. *Suppose that* $(M^n, g, e^{-f}dv)$ *is a complete smooth metric measure space of dimension* n *with* $\mathrm{Ric}_f^m \geq -(m-1)\kappa g$ $(\kappa \geq 0)$. *Let* $B_R(o) \subset M$ *be the open metric ball around a point* o *with radius* $R > 0$. *Let the smooth function* $\lambda(x)$ *not be identically zero on* $B_R(o)$. *Let* $\lambda(x)$ *satisfy condition* (1.7) *at* Ω_0. *Assume that there exists* $K > 0$ *and* $s \in (\frac{m}{2}, \infty]$ *such that*

$$\left(\fint_{B_R(o)\setminus\Omega_0} \left|\frac{\nabla\lambda(x)}{\lambda(x)}\right|^{2s} e^{-f}dv\right)^{\frac{1}{s}} \leq K. \tag{1.11}$$

Suppose τ, $l \in \mathbb{R}$ *satisfies either* ($G1$), ($G2$), *or* ($G3$). *Let* v *be a positive solution to Eq* (1.1) *on* $B_R(o)$, *then*

$$\sup_{B_{\frac{R}{2}}} \frac{|\nabla v|^2}{v^2} \leq C(m, s, \tau, l) e^{\frac{m}{2s-m}C_m(1+\sqrt{\kappa}R)} \frac{[1 + (\kappa + K)R^2]^{\frac{2s+m}{2s-m}}}{R^2}, \tag{1.12}$$

where $C(m, s, \tau, l)$ is a positive constant depending on m, s, τ, l, and C_m is a constant only depending on m.

Especially, suppose $K = 0$, i.e., λ is a constant, then

$$\sup_{B_{\frac{R}{2}}} \frac{|\nabla v|}{v} \le C(m, \tau, l)\frac{1 + \sqrt{\kappa}R}{R}. \tag{1.13}$$

Remark 1.1. 1) In Theorem 1.1 by Wang [13], τ is required to lie in $(1-\frac{2}{m}, 1)\cup(1, +\infty)$. In Theorem 1.2 by Wang et al. [14], τ is required to lie in $(1, +\infty)$. In our theorem, τ is required to lie in $(0, +\infty)$. Therefore, we extend the range of τ for the same problem as in [13, 14].

2) In Theorem 1.1 by Wang [13], l is required to be 1.

In Theorem 1.2 by Wang et al. [14], when $\lambda(x) \ge 0$, they obtained the estimate for the case $\tau > 1$ and

$$l \le \frac{(n+1)(\tau+1)}{2(n-1)} < \frac{(n+1)2\tau}{2(n-1)} = \frac{n+1}{n-1}\tau.$$

In our theorem, we obtained the estimate for the case $\tau > 0$ and

$$l \le \frac{n+3}{n-1}\tau.$$

On the other hand, in Theorem 1.2 by Wang et al. [14], when $\lambda(x) \le 0$, they obtained the estimate for the case $\tau > 1$ and

$$l \ge \frac{(n+1)(\tau+1)}{2(n-1)} > \frac{(n+1)2}{2(n-1)} = \frac{n+1}{n-1} > 1.$$

In our theorem, we obtained the estimate for the case $\tau > 0$ and

$$l > \tau > 0.$$

Therefore, we extend the range of l in [13, 14].

3) In our theorem, the gradient estimates on a positive solution v do not involve the bound of v. Meanwhile, the gradient estimates obtained in Theorem 1.1 and Theorem 1.2 concern the bound of v. Hence we improve the previous results in [13, 14].

4) Our condition (1.11) is weaker than the conditions on λ in [13, 14].

By letting $R \to \infty$ in (1.13), we obtain the following Liouville-type result.

Theorem 1.5. *Suppose that $(M^n, g, e^{-f}dv)$ is a complete smooth metric measure space of dimension n with $\mathrm{Ric}_f^m \ge 0$. If $\lambda(x) = \lambda \ne 0$ is a constant, and τ, l satisfy either (G1) or (G2), then $u(x)$ is a constant.*

Remark 1.2. We expand the ranges of the corresponding power τ and the constant λ for the same problem as in Theorem 1.3 of [14].

Another consequence of Theorem 1.4 can be stated as follows.

Corollary 1.6. *Let $(M^n, g, e^{-f}dv)$ be a complete smooth metric measure space of dimension n with $\mathrm{Ric}_f^m \ge -(m-1)\kappa g$ ($\kappa \ge 0$). Suppose that $\lambda(x) = \lambda$ is a nonzero constant, $\tau > 0$ and v is a smooth positive solution to the Eq (1.1), defined on a geodesic ball $B_R(o) \subset M$, with constants m, l, τ satisfying (G1) or (G2). Then, for any $x, y \in B_{R/2}(o)$ there holds*

$$v(x) \le e^{C(\tau,m,l)(1+\sqrt{\kappa}R)}v(y).$$

The structure of this paper is as follows. In Section 2, we obtain a pointwise differential inequality for $|\nabla \ln v|^2$. In Section 3, we prove Theorem 1.4. In Section 4, we provide the proofs of Theorem 1.5 and Corollary 1.6.

2. Preliminary

In this section we need some preliminary results to prove the gradient estimate (1.12) in Theorem 1.4.

We observe that Eq (1.1) is rewritten as follows:

$$\Delta_f v + (\tau - 1)v^{-1}|\nabla v|^2 + \tau^{-1}\lambda(x)v^{l-\tau+1} = 0. \tag{2.1}$$

Now, let $u = -\ln v$. Then with respect to (2.1), u satisfies the following nonlinear elliptic equation:

$$\Delta_f u - \tau|\nabla u|^2 - \tau^{-1}\lambda(x)e^{-(l-\tau)u} = 0. \tag{2.2}$$

Set $h = |\nabla u|^2$. Then we have

$$\Delta_f u - \tau h - \tau^{-1}\lambda(x)e^{-(l-\tau)u} = 0. \tag{2.3}$$

In what follows, we will consistently assume that $\lambda(x)$ satisfies (1.7) at zero point set Ω_0 (where Ω_0 might be an empty set) and that τ, m, l satisfy either (G1), (G2), or (G3). For the sake of convenience, at the zero points $x \in \Omega_0$, we define $\frac{|\nabla\lambda|}{\lambda}(x) = 0$. This makes the function $\frac{|\nabla\lambda|}{\lambda}(x)$ well-defined throughout the ball B_R, although it may be singular at Ω_0.

Lemma 2.1. *Suppose that $(M^n, g, e^{-f}dv)$ is a complete smooth metric measure space of dimension n with $\mathrm{Ric}_f^m \geq -(m-1)\kappa g$ ($\kappa \geq 0$). For any $\alpha > 1$ and $\varepsilon > 0$, the following inequality holds in $\{x : h(x) > 0\}$*

$$\begin{aligned}\frac{h^{1-\alpha}}{\alpha}\Delta_f h^\alpha \geq & -\left(2(m-1)\kappa + \frac{|\nabla\lambda|^2}{\varepsilon\lambda^2}\right)h - (2 - \frac{2}{m-1})\tau|\nabla h|h^{\frac{1}{2}} + 2(\frac{m+1}{m-1}\tau - l)hQ \\ & + \frac{2}{m-1}\tau^2h^2 + \left(\frac{2}{m-1} - \varepsilon - \frac{4}{\alpha(m-1)^2 + m^2 - 1}\right)Q^2,\end{aligned} \tag{2.4}$$

where $Q = \tau^{-1}\lambda(x)e^{-(l-\tau)u}$.

Proof. For any $\alpha > 1$, direct computation gives

$$\Delta_f h^\alpha = \alpha(\alpha-1)h^{\alpha-2}|\nabla h|^2 + \alpha h^{\alpha-1}\Delta_f h.$$

Using the Bochner formula

$$\frac{1}{2}\Delta_f h = |\nabla^2 u|^2 + \langle\nabla u, \nabla\Delta_f u\rangle + \mathrm{Ric}_f(\nabla u, \nabla u),$$

we obtain

$$\Delta_f h^\alpha = \alpha(\alpha-1)h^{\alpha-2}|\nabla h|^2 + 2\alpha h^{\alpha-1}\left(|\nabla^2 u|^2 + \langle\nabla u, \nabla\Delta_f u\rangle + \mathrm{Ric}_f(\nabla u, \nabla u)\right). \tag{2.5}$$

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal frame of TM on a domain with $h \neq 0$ such that $e_1 = \frac{\nabla u}{|\nabla u|}$. If we denote $\nabla u = \Sigma_{i=1}^n u_i e_i$, it is easy to see $u_1 = |\nabla u| = h^{1/2}$ and $u_i = 0$ for any $2 \leq i \leq n$. Then the following identities hold:

$$u_{11} = \frac{1}{2}h^{-1}\langle\nabla u, \nabla h\rangle \leq \frac{|\nabla h|}{2h^{\frac{1}{2}}}, \tag{2.6}$$

$$\sum_{i=2}^{n} u_{ii} = \tau h + \tau^{-1}\lambda(x)e^{-(l-\tau)u} - u_{11} + \langle \nabla f, \nabla u\rangle. \tag{2.7}$$

Then rewrite Eq (2.7) as

$$\sum_{i=2}^{n} u_{ii} = \tau h + Q - u_{11} + \langle \nabla f, \nabla u\rangle. \tag{2.8}$$

Using the Cauchy inequality, we obtain

$$\begin{aligned} |\nabla^2 u|^2 \geq & \sum_{i=1}^{n} u_{1i}^2 + \sum_{i=2}^{n} u_{ii}^2 \\ \geq & u_{11}^2 + \frac{1}{n-1}\left(\sum_{i=2}^{n} u_{ii}\right)^2 . \end{aligned} \tag{2.9}$$

Then using the inequality

$$(a+b)^2 \geq \frac{a^2}{1+\xi} - \frac{b^2}{\xi}, \quad \xi = \frac{m-n}{n-1},$$

we can derive from (2.9) that

$$\begin{aligned} |\nabla^2 u|^2 \geq & u_{11}^2 + \frac{1}{n-1}\left(\tau h + Q - u_{11} + \langle \nabla f, \nabla u\rangle\right)^2 \\ \geq & u_{11}^2 + \frac{1}{n-1}\left(\frac{n-1}{m-1}(\tau h + Q - u_{11})^2 - \frac{n-1}{m-n}\langle \nabla f, \nabla u\rangle^2\right) \\ = & \frac{1}{m-1}\tau^2 h^2 + \frac{1}{m-1}Q^2 + \frac{m}{m-1}u_{11}^2 + \frac{2}{m-1}\tau h Q - \frac{2}{m-1}Q u_{11} \\ & - \frac{2}{m-1}\tau h u_{11} - \frac{1}{m-n}\langle \nabla f, \nabla u\rangle^2. \end{aligned} \tag{2.10}$$

As $\lambda(x)$ is supposed to satisfy condition (1.7), we have

$$\nabla Q = \left[\frac{\nabla \lambda}{\lambda} + (\tau - l)\nabla u\right] Q, \tag{2.11}$$

Then we can derive from (2.3) that

$$\begin{aligned} \langle \nabla u, \nabla \Delta_f u\rangle = & \langle \nabla u, \nabla(\tau h + Q)\rangle \\ = & \tau\langle \nabla u, \nabla h\rangle + \langle \frac{\nabla \lambda}{\lambda}, \nabla u\rangle Q + (\tau - l)hQ. \end{aligned} \tag{2.12}$$

Substituting (2.6), (2.10), (2.11), and (2.12) into (2.5), we obtain

$$
\begin{aligned}
\frac{h^{1-\alpha}}{\alpha}\Delta_f h^\alpha \geq&(\alpha-1)h^{-1}|\nabla h|^2-2(m-1)\kappa h+2\tau\langle\nabla u,\nabla h\rangle+2\langle\frac{\nabla\lambda}{\lambda},\nabla u\rangle Q\\
&+2(\tau-l)hQ+\frac{2}{m-1}\tau^2h^2+\frac{2}{m-1}Q^2+\frac{2m}{m-1}u_{11}^2+\frac{4}{m-1}\tau hQ\\
&-\frac{4}{m-1}Qu_{11}-\frac{4}{m-1}\tau hu_{11}\\
\geq&4(\alpha-1)u_{11}^2-2(m-1)\kappa h+(2-\frac{2}{m-1})\tau\langle\nabla u,\nabla h\rangle+2\langle\frac{\nabla\lambda}{\lambda},\nabla u\rangle Q\\
&+2(\frac{m+1}{m-1}\tau-l)hQ+\frac{2}{m-1}\tau^2h^2+\frac{2}{m-1}Q^2+\frac{2m}{m-1}u_{11}^2-\frac{4}{m-1}Qu_{11}\\
\geq&(4\alpha-\frac{2m-4}{m-1})u_{11}^2-2(m-1)\kappa h-(2-\frac{2}{m-1})\tau|\nabla h|h^{\frac{1}{2}}+2\langle\frac{\nabla\lambda}{\lambda},\nabla u\rangle Q\\
&+2(\frac{m+1}{m-1}\tau-l)hQ+\frac{2}{m-1}\tau^2h^2+\frac{2}{m-1}Q^2-\frac{4}{m-1}Qu_{11}.
\end{aligned}
\tag{2.13}
$$

Since for any $\varepsilon>0$,

$$
2\langle\frac{\nabla\lambda}{\lambda},\nabla u\rangle Q\leq\frac{|\nabla\lambda|^2h}{\varepsilon\lambda^2}+\varepsilon Q^2,
$$

so we have

$$
\begin{aligned}
\frac{h^{1-\alpha}}{\alpha}\Delta_f h^\alpha \geq&(4\alpha-\frac{2m-4}{m-1})u_{11}^2-\left(2(m-1)\kappa+\frac{|\nabla\lambda|^2}{\varepsilon\lambda^2}\right)h-(2-\frac{2}{m-1})\tau|\nabla h|h^{\frac{1}{2}}\\
&+2(\frac{m+1}{m-1}\tau-l)hQ+\frac{2}{m-1}\tau^2h^2+\left(\frac{2}{m-1}-\varepsilon\right)Q^2-\frac{4}{m-1}Qu_{11}.
\end{aligned}
\tag{2.14}
$$

We observe that

$$
\begin{aligned}
(4\alpha-\frac{2m-4}{m-1})u_{11}^2-\frac{4}{m-1}Qu_{11}\geq&(\alpha+\frac{m+1}{m-1})u_{11}^2-\frac{4}{m-1}Qu_{11}\\
\geq&-\frac{4}{\alpha(m-1)^2+m^2-1}Q^2,
\end{aligned}
\tag{2.15}
$$

then we obtain the desired inequality (2.4). □

Now, for each of the cases (G1)–(G3), we define a constant $\varepsilon_0>0$ as follows:

$$
\varepsilon_0:=\begin{cases}\min\left\{\frac{2}{m-1},\frac{m+3}{m-1}-\frac{l}{\tau}\right\}, & \text{in case of (G1)},\\ \min\left\{\frac{2}{m-1},\frac{l}{\tau}-1\right\}, & \text{in case of (G2)},\\ \min\left\{\frac{m+3}{m-1}-\frac{l}{\tau},\frac{l}{\tau}-1\right\}, & \text{in case of (G3)}.\end{cases}
\tag{2.16}
$$

Lemma 2.2. *For each of the cases* (*G*1)–(*G*3)*, let* ε_0 *be defined as in* (2.16)*. Then there exists a constant* $\alpha_0>1$ *that depends on m and* ε_0 *such that the following inequality holds in* $\{x:h(x)>0\}$

$$
\frac{h^{1-\alpha_0}}{\alpha_0}\Delta_f h^{\alpha_0}\geq-\left[2(m-1)\kappa+\frac{2|\nabla\lambda|^2}{\varepsilon_0\lambda^2}\right]h-2\tau|\nabla h|h^{\frac{1}{2}}+\varepsilon_0\tau^2h^2.
\tag{2.17}
$$

Proof. In order to estimate $\Delta_f h^\alpha$, we cut apart the arguments into three cases.

(i) If m, l, $\lambda(x)$, and τ satisfy $(G1)$, then

$$\begin{aligned}(\frac{m+1}{m-1}-\frac{l}{\tau})\tau hQ =&(\frac{m+3}{m-1}-\frac{2}{m-1}-\frac{l}{\tau})\tau h|Q|\\ \geq&(-\frac{2}{m-1}+\varepsilon_0)\tau h|Q|.\end{aligned}$$

(ii) If m, l, $\lambda(x)$, and τ satisfy $(G2)$, then

$$\begin{aligned}(\frac{m+1}{m-1}-\frac{l}{\tau})\tau hQ =&-(\frac{2}{m-1}+1-\frac{l}{\tau})\tau h|Q|\\ =&(-\frac{2}{m-1}+\frac{l}{\tau}-1)\tau h|Q|\\ \geq&(-\frac{2}{m-1}+\varepsilon_0)\tau h|Q|.\end{aligned}$$

(iii) If m, l, $\lambda(x)$, and τ satisfy $(G3)$, we let $\Omega_1=\{x\in\Omega_0|\lambda(x)\geq 0\}$, then

$$\begin{aligned}\left\{(\frac{m+1}{m-1}-\frac{l}{\tau})\tau hQ\right\}_{\Omega_0} =&\left\{(\frac{m+3}{m-1}-\frac{2}{m-1}-\frac{l}{\tau})\tau h|Q|\right\}_{\Omega_1}\\ &+\left\{-(\frac{2}{m-1}+1-\frac{l}{\tau})\tau h|Q|\right\}_{\Omega_0\backslash\Omega_1}\\ \geq&\left\{(-\frac{2}{m-1}+\varepsilon_0)\tau h|Q|\right\}_{\Omega_1}+\left\{(-\frac{2}{m-1}+\varepsilon_0)\tau h|Q|\right\}_{\Omega_0\backslash\Omega_1}\\ =&\left\{(-\frac{2}{m-1}+\varepsilon_0)\tau h|Q|\right\}_{\Omega_0}.\end{aligned}$$

Thus, we observe that, in either case of $(G1)$, $(G2)$, and $(G3)$, we have

$$(\frac{m+1}{m-1}-\frac{l}{\tau})\tau hQ\geq(-\frac{2}{m-1}+\varepsilon_0)\tau h|Q| \tag{2.18}$$

for all $x\in B_R$. Furthermore, we may let $\varepsilon=\frac{\varepsilon_0}{2}$ in (2.4) and sufficiently large $\alpha_0>1$ such that

$$\frac{4}{(m-1)^2\alpha_0+m^2-1}<\frac{\varepsilon_0}{2}. \tag{2.19}$$

Combining (2.18) and (2.19), we obtain

$$\begin{aligned}\frac{h^{1-\alpha_0}}{\alpha_0}\Delta_f h^{\alpha_0} \geq & -\left(2(m-1)\kappa + \frac{2|\nabla\lambda|^2}{\varepsilon_0\lambda^2}\right)h - \left(2 - \frac{2}{m-1}\right)\tau|\nabla h|h^{\frac{1}{2}} \\ & + \left(\frac{2}{m-1} - \varepsilon_0\right)(-2\tau h|Q| + Q^2) + \frac{2}{m-1}\tau^2 h^2 \\ \geq & -\left(2(m-1)\kappa + \frac{2|\nabla\lambda|^2}{\varepsilon_0\lambda^2}\right)h - \left(2 - \frac{2}{m-1}\right)\tau|\nabla h|h^{\frac{1}{2}} \\ & - \left(\frac{2}{m-1} - \varepsilon_0\right)\tau^2 h^2 + \frac{2}{m-1}\tau^2 h^2 \\ = & -\left(2(m-1)\kappa + \frac{2|\nabla\lambda|^2}{\varepsilon_0\lambda^2}\right)h - \left(2 - \frac{2}{m-1}\right)\tau|\nabla h|h^{\frac{1}{2}} + \varepsilon_0\tau^2 h^2 \\ \geq & -\left(2(m-1)\kappa + \frac{2|\nabla\lambda|^2}{\varepsilon_0\lambda^2}\right)h - 2\tau|\nabla h|h^{\frac{1}{2}} + \varepsilon_0\tau^2 h^2\end{aligned}$$

and the lemma is proved. □

To prove Theorem 1.4, we need the following.

Theorem 2.3. *([22, 24]) Let $(M^n, g, e^{-f}dv)$ be a complete smooth metric measure space of dimension n with $\mathrm{Ric}_f^m \geq -(m-1)\kappa g$ ($\kappa \geq 0$). Let $B_R \subset M$ be the open metric ball with radius $R > 0$. For $m > 2$, there exist some positive constants C_m depending only on m, such that for all $\phi \in C_0^\infty(B_R)$*

$$\left(\int_{B_R}|\phi|^{\frac{2m}{m-2}}e^{-f}dv\right)^{\frac{m-2}{m}} \leq e^{C_m(1+\sqrt{\kappa}R)}V_f^{-\frac{2}{m}}R^2\int_{B_R}\left(R^{-2}\phi^2 + |\nabla\phi|^2\right)e^{-f}dv,$$

where $V_f = \int_{B_R} e^{-f}dv$.

3. Proof of Theorem 1.4

In this section, we finish the proof of Theorem 1.4. We proceed with the following integral inequality on the solutions to Eq (1.1).

Suppose that $\lambda(x)$ satisfies (1.7) and τ, l, $\lambda(x)$, and m satisfy (G1), (G2), or (G3). Let ε_0 be defined according to (2.16) and α_0 satisfy the condition (2.19). According to Lemma 2.2, at non-zero points of h, the function h^{α_0} satisfies the inequality (2.17). We can rewrite this inequality as follows:

$$\Delta_f h^{\alpha_0} \geq -\alpha_0 F h^{\alpha_0} - 2\tau\alpha_0|\nabla h|h^{\alpha_0-\frac{1}{2}} + \alpha_0\varepsilon_0\tau^2 h^{\alpha_0+1}, \tag{3.1}$$

where

$$F := \left(2(m-1)\kappa + \frac{2|\nabla\lambda|^2}{\varepsilon_0\lambda^2}\right).$$

Given the assumption, there exist $s \in (\frac{m}{2}, \infty]$ and a positive constant K for which $\lambda(x)$ satisfies

$$\left(\fint_{B_R(o)}\left|\frac{\nabla\lambda(x)}{\lambda(x)}\right|^{2s}e^{-f}dv\right)^{\frac{1}{s}} \leq K. \tag{3.2}$$

We denote

$$\chi := \frac{m}{m-2}, \quad C_R = C_m(1+\sqrt{\kappa}R).$$

Lemma 3.1. *Suppose that $(M^n, g, e^{-f}dv)$ is a complete smooth metric measure space of dimension n with $\mathrm{Ric}_f^m \geq -(m-1)\kappa g$ $(\kappa \geq 0)$ and $B_R \subset M$ is an open metric ball. Let v be a positive solution to the Eq* (1.1) *with the constants $m > 2$, $\lambda(x)$, l, and τ, which satisfy the condition in Theorem 1.4. For all*

$$t \geq t_0 := 4\alpha_0 + \frac{8}{\varepsilon_0}, \tag{3.3}$$

some positive constant $C_1(m, s, \varepsilon_0)$ and smooth cutoff function $\eta \in C_0^\infty(B_R)$ there holds

$$\begin{aligned} &\frac{4}{5}\left(\int_{B_R} h^{t\chi}\eta^{2\chi}e^{-f}dv\right)^{\frac{1}{\chi}} + t\varepsilon_0\tau^2 e^{C_R}V_f^{-\frac{2}{m}}R^2\int_{B_R} h^{t+1}\eta^2 e^{-f}dv \\ \leq &te^{C_R}V_f^{-\frac{2}{m}}A(t)\int_{B_R}\eta^2 h^t e^{-f}dv + 8e^{C_R}V_f^{-\frac{2}{m}}R^2\int_{B_R}|\nabla\eta|^2 h^t e^{-f}dv, \end{aligned} \tag{3.4}$$

where

$$A(t) := C_1\left(te^{C_R}\right)^{\frac{m}{2s-m}}\left[1+(\kappa+K)R^2\right]^{\frac{2s}{2s-m}}. \tag{3.5}$$

Proof. Suppose that $\delta > 0$ is a small positive constant. Let $h_\delta := (h-\delta)_+$, which is a function that becomes zero in the vicinity of the zero-points of h. For any $t > \alpha_0$, we take the product of Eq (3.1) and a test function $\eta^2 h_\delta^{t-\alpha_0}$, and then perform integration by parts. This leads to the following inequality:

$$\begin{aligned} &\varepsilon_0\tau^2\alpha_0\int_{B_R} h^{\alpha_0+1}h_\delta^{t-\alpha_0}\eta^2 e^{-f}dv + \int_{B_R}\alpha_0(t-\alpha_0)h^{\alpha_0-1}h_\delta^{t-\alpha_0-1}|\nabla h|^2\eta^2 e^{-f}dv \\ \leq &\alpha_0\int_{B_R} Fh^{\alpha_0}h_\delta^{t-\alpha_0}\eta^2 e^{-f}dv - \int_{B_R} 2\alpha_0 h^{\alpha_0-1}h_\delta^{t-\alpha_0}\langle\nabla h, \nabla\eta\rangle\eta e^{-f}dv \\ &+ 2\tau\alpha_0\int_{B_R} h^{\alpha_0-\frac{1}{2}}h_\delta^{t-\alpha_0}|\nabla h|\eta^2 e^{-f}dv. \end{aligned} \tag{3.6}$$

Dividing both sides by α_0 and letting $\delta \to 0$, we have

$$\begin{aligned} &\varepsilon_0\tau^2\int_{B_R} h^{t+1}\eta^2 e^{-f}dv + \int_{B_R}(t-\alpha_0)h^{t-2}|\nabla h|^2\eta^2 e^{-f}dv \\ \leq &\int_{B_R} Fh^t\eta^2 e^{-f}dv - \int_{B_R} 2h^{t-1}\langle\nabla h, \nabla\eta\rangle\eta e^{-f}dv + 2\tau\int_{B_R} h^{t-\frac{1}{2}}|\nabla h|\eta^2 e^{-f}dv. \end{aligned} \tag{3.7}$$

Since

$$h^{t-1}\langle\nabla h, \nabla\eta\rangle \geq -h^{t-1}|\nabla h||\nabla\eta|, \tag{3.8}$$

inserting (3.8) into (3.7), we have

$$\begin{aligned} &\varepsilon_0\tau^2\int_{B_R} h^{t+1}\eta^2 e^{-f}dv + \int_{B_R}(t-\alpha_0)h^{t-2}|\nabla h|^2\eta^2 e^{-f}dv \\ \leq &\int_{B_R} Fh^t\eta^2 e^{-f}dv + \int_{B_R} 2h^{t-1}|\nabla h||\nabla\eta|\eta e^{-f}dv + 2\tau\int_{B_R} h^{t-\frac{1}{2}}|\nabla h|\eta^2 e^{-f}dv. \end{aligned} \tag{3.9}$$

Now applying Young's inequality, we obtain

$$
\begin{aligned}
2\tau h^{t-\frac{1}{2}}|\nabla h|\eta^2 &\le \frac{2}{\varepsilon_0}h^{t-2}|\nabla h|^2\eta^2 + \frac{\varepsilon_0\tau^2}{2}h^{t+1}\eta^2,\\
2h^{t-1}|\nabla h||\nabla\eta|\eta &\le \frac{t}{2}h^{t-2}|\nabla h|^2\eta^2 + \frac{2}{t}h^t|\nabla\eta|^2.
\end{aligned}
\tag{3.10}
$$

It can be concluded from (3.9) and (3.10) that

$$
\begin{aligned}
&\frac{\varepsilon_0\tau^2}{2}\int_{B_R} h^{t+1}\eta^2 e^{-f}dv + (\frac{t}{2} - \alpha_0 - \frac{2}{\varepsilon_0})\int_{B_R} h^{t-2}|\nabla h|^2\eta^2 e^{-f}dv\\
\le &\int_{B_R} Fh^t\eta^2 e^{-f}dv + \frac{2}{t}\int_{B_R} h^t|\nabla\eta|^2 e^{-f}dv.
\end{aligned}
\tag{3.11}
$$

Using (3.11) and

$$
\begin{aligned}
\left|\nabla\left(h^{\frac{t}{2}}\eta\right)\right|^2 &\le 2\left|\nabla h^{\frac{t}{2}}\right|^2\eta^2 + 2h^t|\nabla\eta|^2\\
&= \frac{t^2}{2}h^{t-2}|\nabla h|^2\eta^2 + 2h^t|\nabla\eta|^2,
\end{aligned}
\tag{3.12}
$$

we have

$$
\begin{aligned}
&\frac{\varepsilon_0\tau^2}{2}\int_{B_R} h^{t+1}\eta^2 e^{-f}dv + \frac{1}{t^2}\left(t - 2\alpha_0 - \frac{4}{\varepsilon_0}\right)\int_{B_R}\left|\nabla\left(h^{\frac{t}{2}}\eta\right)\right|^2 e^{-f}dv\\
\le &\int_{B_R} Fh^t\eta^2 e^{-f}dv + \frac{4}{t^2}\left(t - \alpha_0 - \frac{2}{\varepsilon_0}\right)\int_{B_R} h^t|\nabla\eta|^2 e^{-f}dv.
\end{aligned}
\tag{3.13}
$$

Suppose that there is a positive constant t large enough such that

$$
t \ge t_0 := 4\alpha_0 + \frac{8}{\varepsilon_0}.
\tag{3.14}
$$

It can be concluded from (3.13) and (3.14) that

$$
\begin{aligned}
&\varepsilon_0\tau^2 t\int_{B_R} h^{t+1}\eta^2 e^{-f}dv + \int_{B_R}\left|\nabla\left(h^{\frac{t}{2}}\eta\right)\right|^2 e^{-f}dv\\
\le &2t\int_{B_R} Fh^t\eta^2 e^{-f}dv + 8\int_{B_R} h^t|\nabla\eta|^2 e^{-f}dv.
\end{aligned}
\tag{3.15}
$$

Now letting $\phi = h^{\frac{t}{2}}\eta$ in the Saloff-Coste Sobolev-type inequality in Theorem 2.3, we have

$$
\begin{aligned}
&e^{-C_R}V_f^{\frac{2}{m}}R^{-2}\left(\int_{B_R}|h^{\frac{t}{2}}\eta|^{\frac{2m}{m-2}}e^{-f}dv\right)^{\frac{m-2}{m}}\\
\le &\int_{B_R}\left|\nabla\left(h^{\frac{t}{2}}\eta\right)\right|^2 e^{-f}dv + R^{-2}\int_{B_R} h^t\eta^2 e^{-f}dv.
\end{aligned}
$$

plugging the above inequality into (3.15), we obtain

$$
\begin{aligned}
&\left(\int_{B_R}|h^{\frac{t}{2}}\eta|^{2\chi}e^{-f}dv\right)^{\frac{1}{\chi}}\\
\le &e^{C_R}V_f^{-\frac{2}{m}}R^2\Big(2t\int_{B_R} Fh^t\eta^2 e^{-f}dv + 8\int_{B_R} h^t|\nabla\eta|^2 e^{-f}dv\\
&+ R^{-2}\int_{B_R} h^t\eta^2 e^{-f}dv - \varepsilon_0\tau^2 t\int_{B_R} h^{t+1}\eta^2 e^{-f}dv\Big).
\end{aligned}
\tag{3.16}
$$

Dividing both sides by $V_f^{\frac{1}{\chi}}$, we obtain

$$\begin{aligned}
&\left(\fint_{B_R}|h^{\frac{t}{2}}\eta|^{2\chi}e^{-f}dv\right)^{\frac{1}{\chi}}\\
\leq &e^{C_R}R^2\Big(2t\fint_{B_R}Fh^t\eta^2e^{-f}dv+8\fint_{B_R}h^t|\nabla\eta|^2e^{-f}dv\\
&+R^{-2}\fint_{B_R}h^t\eta^2e^{-f}dv-\varepsilon_0\tau^2t\fint_{B_R}h^{t+1}\eta^2e^{-f}dv\Big).
\end{aligned}\tag{3.17}$$

Eventually, we handle the leading term involving F in Eq (3.16).

$$\begin{aligned}
R^2\int_{B_R}Fh^t\eta^2e^{-f}dv=&2(m-1)\kappa R^2\int_{B_R}h^t\eta^2e^{-f}dv+\frac{2}{\varepsilon_0}R^2\int_{B_R}\left|\frac{\nabla\lambda}{\lambda}\right|^2h^t\eta^2e^{-f}dv\\
\leq&2(m-1)C_R^2\int_{B_R}h^t\eta^2e^{-f}dv+\frac{2}{\varepsilon_0}R^2\int_{B_R}\left|\frac{\nabla\lambda}{\lambda}\right|^2h^t\eta^2e^{-f}dv,
\end{aligned}\tag{3.18}$$

here $\frac{\nabla\lambda}{\lambda}$ meets the condition (3.2) for a certain $s\in(\frac{m}{2},\infty]$ and $K\geq 0$. When $K=0$, Eq (3.4) can be directly derived from (3.16), and the proof or analysis for this part is complete. Therefore, we can suppose $K\neq 0$ and consider two separate cases.

Case 1. When $s=\infty$ and $|\frac{\nabla\lambda}{\lambda}|^2\leq K$, it can be concluded from (3.16) and (3.18) that

$$\begin{aligned}
&\left(\int_{B_R}|h^{\frac{t}{2}}\eta|^{2\chi}e^{-f}dv\right)^{\frac{1}{\chi}}\\
\leq &e^{C_R}V_f^{-\frac{2}{n}}\left(tA_1\int_{B_R}h^t\eta^2e^{-f}dv+8R^2\int_{B_R}h^t|\nabla\eta|^2e^{-f}dv-\varepsilon_0\tau^2tR^2\int_{B_R}h^{t+1}\eta^2e^{-f}dv\right),
\end{aligned}\tag{3.19}$$

where

$$A_1=4(m-1)C_R^2+\frac{4}{\varepsilon_0}KR^2+1.$$

Case 2. When $\frac{m}{2}<s<\infty$, then $\frac{1}{\chi}<\frac{s-1}{s}<1$. According to the Hölder inequality, we get

$$\begin{aligned}
&\fint_{B_R}\left|\frac{\nabla\lambda}{\lambda}\right|^2\eta^2h^te^{-f}dv\\
=&\fint_{B_R}\left(\left|\frac{\nabla\lambda}{\lambda}\right|^2(e^{-f})^{\frac{1}{s}}\right)\left(\eta^2h^t(e^{-f})^{\frac{s-1}{s}}\right)dv\\
\leq&\left(\fint_{B_R}\left(\left|\frac{\nabla\lambda}{\lambda}\right|^2(e^{-f})^{\frac{1}{s}}\right)^s dv\right)^{\frac{1}{s}}\left(\fint_{B_R}\left(\eta^2h^t(e^{-f})^{\frac{s-1}{s}}\right)^{\frac{s}{s-1}}dv\right)^{\frac{s-1}{s}}\\
\leq&K\left(\fint_{B_R}\left(\eta^2h^t\right)^{\frac{s}{s-1}}e^{-f}dv\right)^{\frac{s-1}{s}}.
\end{aligned}$$

Then, by the Hölder and Young's inequality, we have

$$
\begin{aligned}
&K\left(\fint_{B_R}\left(\eta^2h^t\right)^{\frac{s}{s-1}}e^{-f}d\nu\right)^{\frac{s-1}{s}}\\
=&K\left(\fint_{B_R}\left(\eta^2h^te^{-f}\right)^{\frac{2s-m}{2(s-1)}}\left((\eta^2h^t)^{\chi}e^{-f}\right)^{\frac{m-2}{2(s-1)}}d\nu\right)^{\frac{s-1}{s}}\\
\leq&K\left(\left(\fint_{B_R}\eta^2h^te^{-f}d\nu\right)^{\frac{2s-m}{2(s-1)}}\left(\fint_{B_R}(\eta^2h^t)^{\chi}e^{-f}d\nu\right)^{\frac{m-2}{2(s-1)}}\right)^{\frac{s-1}{s}}\\
=&K\left(\fint_{B_R}\eta^2h^te^{-f}d\nu\right)^{\frac{2s-m}{2s}}\left(\fint_{B_R}(\eta^2h^t)^{\chi}e^{-f}d\nu\right)^{\frac{m}{\chi 2s}}\\
\leq&K\epsilon\left(\fint_{B_R}(h^t\eta^2)^{\chi}e^{-f}d\nu\right)^{\frac{1}{\chi}}+KA(m,s)\epsilon^{-\frac{m}{2s-m}}\fint_{B_R}h^t\eta^2e^{-f}d\nu,
\end{aligned}
\tag{3.20}
$$

where $A(m,s)$ is some constant depending on s and m. Especially, we may let

$$
\epsilon=\frac{\epsilon_0}{20Kte^{C_R}R^2},
$$

then combining (3.17) and (3.20), we have

$$
\begin{aligned}
&\frac{4}{5}\left(\fint_{B_R}|h^{\frac{t}{2}}\eta|^{2\chi}e^{-f}d\nu\right)^{\frac{1}{\chi}}\\
\leq&e^{C_R}\left(tA_2\fint_{B_R}h^t\eta^2e^{-f}d\nu+8R^2\fint_{B_R}h^t|\nabla\eta|^2e^{-f}d\nu-\varepsilon_0\tau^2tR^2\fint_{B_R}h^{t+1}\eta^2e^{-f}d\nu\right),
\end{aligned}
$$

or equivalently,

$$
\begin{aligned}
&\frac{4}{5}\left(\int_{B_R}|h^{\frac{t}{2}}\eta|^{2\chi}e^{-f}d\nu\right)^{\frac{1}{\chi}}\\
\leq&e^{C_R}V_f^{-\frac{2}{m}}\left(tA_2\int_{B_R}h^t\eta^2e^{-f}d\nu+8R^2\int_{B_R}h^t|\nabla\eta|^2e^{-f}d\nu-\varepsilon_0\tau^2tR^2\int_{B_R}h^{t+1}\eta^2e^{-f}d\nu\right),
\end{aligned}
\tag{3.21}
$$

where

$$
A_2=4(m-1)C_R^2+\frac{4}{\varepsilon_0}A(m,s)KR^2\left(20t\varepsilon_0^{-1}e^{C_R}KR^2\right)^{\frac{m}{2s-m}}+1.
$$

Since for every $s\in(\frac{m}{2},\infty]$, the inequality

$$
max\{A_1,A_2\}\leq A(t):=C\left(m,s,\varepsilon_0\right)\left(te^{C_R}\right)^{\frac{m}{2s-m}}\left[1+(\kappa+K)R^2\right]^{\frac{2s}{2s-m}}
$$

holds, by referring to Eqs (3.19) and (3.21), we can infer that Eq (3.4) is valid. Thus completing the proof. □

The following lemma gives an initial integral bound for h.

Lemma 3.2. *Suppose that $(M^n, g, e^{-f}dv)$ is a complete smooth metric measure space of dimension n with $\mathrm{Ric}_f^m \geq -(m-1)\kappa g$ ($\kappa \geq 0$) and $B_R \subset M$ is an open metric ball. Let v be a positive solution to the Eq* (1.1) *with the constants $m > 2$, λ, and τ satisfying the conditions in Theorem 1.4. Suppose that $u = -\ln v$ and $h = |\nabla u|^2$. Then there exist constants $a_i (i = 1, 2, 3, \cdots)$ such that*

$$\left(\int_{B_R} h^{l_1}\eta^{2\chi}e^{-f}dv\right)^{\frac{1}{l_1}} \leq a_3 e^{\frac{C_R}{l_0}} V_f^{\frac{1}{l_1}} \left[\frac{A(l_0)^{1+\frac{1}{l_0}} + l_0^2}{\varepsilon_0\tau^2R^2}\right] \tag{3.22}$$

for all

$$l_0 \geq t_0 \quad \text{and} \quad l_1 := l_0\chi.$$

Proof. Let $t = l_0$ in (3.4), we have

$$\begin{aligned}&\frac{4}{5}e^{-C_R}V_f^{\frac{2}{m}}R^{-2}\left(\int_{B_R} h^{l_0\chi}\eta^{2\chi}e^{-f}dv\right)^{\frac{1}{\chi}} + l_0\varepsilon_0\tau^2\int_{B_R} h^{l_0+1}\eta^2e^{-f}dv\\ \leq& l_0R^{-2}A(l_0)\int_{B_R}\eta^2h^{l_0}e^{-f}dv + 8\int_{B_R}|\nabla\eta|^2h^{l_0}e^{-f}dv.\end{aligned} \tag{3.23}$$

Now we let $D = \left\{x \in B_R | h(x) \geq \frac{2A(l_0)}{\varepsilon_0\tau^2R^2}\right\}$. Hence, we have

$$\begin{aligned}&l_0R^{-2}A(l_0)\int_{B_R} h^{l_0}\eta^2e^{-f}dv\\ =&l_0R^{-2}A(l_0)\int_D h^{l_0}\eta^2e^{-f}dv + l_0R^{-2}A(l_0)\int_{B_R\setminus D} h^{l_0}\eta^2e^{-f}dv\\ \leq&\frac{l_0\varepsilon_0\tau^2}{2}\int_{B_R} h^{l_0+1}\eta^2e^{-f}dv + l_0\frac{A(l_0)}{R^2}\left(\frac{2A(l_0)}{\varepsilon_0\tau^2R^2}\right)^{l_0}V_f.\end{aligned} \tag{3.24}$$

Applying (3.23) to (3.24), we obtain

$$\begin{aligned}&\frac{l_0\varepsilon_0\tau^2}{2}\int_{B_R} h^{l_0+1}\eta^2e^{-f}dv + \frac{4}{5}e^{-C_R}V_f^{\frac{2}{m}}R^{-2}\left(\int_{B_R} h^{l_0\chi}\eta^{2\chi}e^{-f}dv\right)^{\frac{1}{\chi}}\\ \leq& l_0\frac{A(l_0)}{R^2}\left(\frac{2A(l_0)}{\varepsilon_0\tau^2R^2}\right)^{l_0}V_f + 8\int_{B_R}|\nabla\eta|^2h^{l_0}e^{-f}dv.\end{aligned} \tag{3.25}$$

We choose $\zeta \in C_0^\infty(B_R(o))$ satisfying

$$\begin{cases}0 \leq \zeta(x) \leq 1, \quad |\nabla\zeta(x)| \leq \frac{C}{R}, & \forall x \in B_R(o);\\ \zeta(x) \equiv 1, & \forall x \in B_{3R/4}(o),\end{cases}$$

and choose $\eta = \zeta^{l_0+1}$. Then, we obtain

$$8R^2|\nabla\eta|^2 \leq 8C^2(l_0+1)^2\eta^{\frac{2l_0}{l_0+1}} \leq a_1(l_0+1)^2\eta^{\frac{2l_0}{l_0+1}}. \tag{3.26}$$

According to (3.26) and Hölder and Young's inequality, we obtain

$$
\begin{aligned}
&8R^2\int_{B_R} h^{l_0}|\nabla\eta|^2 e^{-f}dv\\
\le& a_1(l_0+1)^2\int_{B_R} h^{l_0}\eta^{\frac{2l_0}{l_0+1}}e^{-f}dv\\
=& a_1(l_0+1)^2\int_{B_R}\left((h^{l_0+1}\eta^2e^{-f})^{\frac{l_0}{l_0+1}}(e^{-f})^{\frac{1}{l_0+1}}\right)dv\\
\le& a_1(l_0+1)^2\left(\int_{B_R} h^{l_0+1}\eta^2e^{-f}dv\right)^{\frac{l_0}{l_0+1}} V_f^{\frac{1}{l_0+1}}\\
\le&\frac{\varepsilon_0\tau^2(l_0+1)R^2}{2}\frac{l_0}{l_0+1}\left(\left(\int_{B_R} h^{l_0+1}\eta^2e^{-f}dv\right)^{\frac{l_0}{l_0+1}}\right)^{\frac{l_0+1}{l_0}}\\
&+\left(\frac{\varepsilon_0\tau^2(l_0+1)R^2}{2}\right)^{-l_0}\frac{1}{l_0+1}\left(a_1(l_0+1)^2V_f^{\frac{1}{l_0+1}}\right)^{l_0+1}\\
=&\frac{l_0\varepsilon_0\tau^2R^2}{2}\int_{B_R} h^{l_0+1}\eta^2e^{-f}dv+\frac{2^{l_0}a_1^{l_0+1}(l_0+1)^{l_0+1}}{\varepsilon_0^{l_0}\tau^{2l_0}R^{2l_0}}V_f.
\end{aligned}
\tag{3.27}
$$

Notice that there exists a constant a_2 such that

$$
\begin{aligned}
\frac{2^{l_0}a_1^{l_0+1}(l_0+1)^{l_0+1}}{\varepsilon_0^{l_0}\tau^{2l_0}R^{2l_0}}\le&\frac{2^{l_0}a_1^{l_0+1}(2l_0)^{2l_0}}{\varepsilon_0^{l_0}\tau^{2l_0}R^{2l_0}}\\
=&\frac{2^{3l_0}a_1^{l_0+1}(l_0)^{2l_0}}{\varepsilon_0^{l_0}\tau^{2l_0}R^{2l_0}}\\
\le&\frac{a_2^{l_0}l_0^{2l_0}}{\varepsilon_0^{l_0}\tau^{2l_0}R^{2l_0}}.
\end{aligned}
\tag{3.28}
$$

Hence, by (3.25), (3.27), and (3.28), we have

$$
\begin{aligned}
&\left(\int_{B_R} h^{l_0\chi}\eta^{2\chi}e^{-f}dv\right)^{\frac{1}{\chi}}\\
\le&\frac{5}{4}e^{C_R}V_f^{1-\frac{2}{m}}R^2\left[l_0\frac{A(l_0)}{R^2}\left(\frac{2A(l_0)}{\varepsilon_0\tau^2R^2}\right)^{l_0}+\frac{a_2^{l_0}l_0^{2l_0}}{\varepsilon_0^{l_0}\tau^{2l_0}R^{2l_0+2}}\right].
\end{aligned}
\tag{3.29}
$$

Taking the $\left(\frac{1}{l_0}\right)$-power of the both sides of (3.29) yields

$$
\left(\int_{B_R} h^{l_1}\eta^{2\chi}e^{-f}dv\right)^{\frac{1}{l_1}}\le a_3e^{\frac{C_R}{l_0}}V_f^{\frac{1}{l_1}}\left[\frac{A(l_0)^{1+\frac{1}{l_0}}+l_0^2}{\varepsilon_0\tau^2R^2}\right].
\tag{3.30}
$$

Since $\eta\equiv 1$ in $B_{3R/4}(o)$, we obtain (3.22) and the lemma is proved. □

Now we show Theorem 1.4 by applying the Nash-Moser iteration method. Throughout, C represents a universal constant, possibly different each time it appears.

Proof. Now we let

$$\left\|h^{\frac{t}{2}}\eta\right\|^2_{L_f^{\frac{2m}{m-2}}(B_R)} := \left(\int_{B_R} |h^{\frac{t}{2}}\eta|^{\frac{2m}{m-2}} e^{-f} dv\right)^{\frac{m-2}{m}}.$$

Suppose that $\lambda(x)$ satisfies (1.7) and (1.11), and τ, l satisfy (G1), (G2), or (G3), and v is a positive solution to the Eq (1.1) on a complete smooth metric measure space $(M^n, g, e^{-f}dv)$ of dimension n. We note that for any cutoff function $\eta \in C_0^\infty(B_R)$, $t \geq t_0$ and $A(t)$ given by (3.5). we get rid of the second term in (3.4), and there exists a constant a_4 such that

$$\begin{aligned}
&\left(\int_{B_R} h^{t\chi}\eta^{2\chi} e^{-f} dv\right)^{\frac{1}{\chi}} \\
\leq &\frac{5}{4} t e^{C_R} V_f^{-\frac{2}{m}} A(t) \int_{B_R} \eta^2 h^t e^{-f} dv + 10 e^{C_R} V_f^{-\frac{2}{m}} R^2 \int_{B_R} |\nabla\eta|^2 h^t e^{-f} dv \\
\leq &a_4 e^{C_R} V_f^{-\frac{2}{m}} \left(tA(t) \int_{B_R} \eta^2 h^t e^{-f} dv + R^2 \int_{B_R} |\nabla\eta|^2 h^t e^{-f} dv\right).
\end{aligned} \tag{3.31}$$

Then, we take an increasing sequence $\{l_k\}_{k=1}^\infty$ such that

$$l_k = l_{k-1}\chi \quad \text{and} \quad r_k = \frac{R}{2} + \frac{R}{4^k}, \quad k = 1, 2, \cdots.$$

Let $\eta_k \in C_0^\infty(B_{r_k})$ be cut-off functions satisfying

$$\eta_k \equiv 1 \quad \text{on} \quad B_{r_{k+1}} \quad \text{and} \quad |\nabla\eta_k| \leq \frac{C4^k}{R}.$$

For each k substituting $t = l_k$ and $\eta = \eta_k$ in (3.31), we obtain

$$\begin{aligned}
&\left(\int_{B_{r_k}} h^{l_k\chi}\eta_k^{2\chi} e^{-f} dv\right)^{\frac{1}{\chi}} \\
\leq &\left(a_4 e^{C_R} V_f^{-\frac{2}{m}}\right)\left(l_k A(l_k) + C^2 16^k\right) \int_{B_{r_k}} h^{l_k} e^{-f} dv \\
= &\left(a_4 e^{C_R} V_f^{-\frac{2}{m}}\right)\left(l_k^{\frac{2s}{2s-m}} e^{\frac{mC_R}{2s-m}} C_1 \left[1 + (\kappa + K)R^2\right]^{\frac{2s}{2s-m}} + C^2 16^k\right) \int_{B_{r_k}} h^{l_k} e^{-f} dv \\
\leq &\left(a_5 e^{2C_R} V_f^{-\frac{2}{m}}\right)\left(\left[1 + (\kappa + K)R^2\right]^{\frac{2s}{2s-m}} + C^2 16^k\right) \int_{B_{r_k}} h^{l_k} e^{-f} dv \\
\leq &\left(a_5 e^{2C_R} V_f^{-\frac{2}{m}}\right)\left(16^k \left[1 + (\kappa + K)R^2\right]^{\frac{2s}{2s-m}} + C^2 16^k\right) \int_{B_{r_k}} h^{l_k} e^{-f} dv.
\end{aligned}$$

We can find some constant a_6 such that

$$\left(\int_{B_{r_k}} h^{l_k\chi}\eta^{2\chi} e^{-f} dv\right)^{\frac{1}{\chi}} \leq a_6 e^{2C_R} V_f^{-\frac{2}{m}} 16^k \left[1 + (\kappa + K)R^2\right]^{\frac{2s}{2s-m}} \int_{B_{r_k}} h^{l_k} e^{-f} dv. \tag{3.32}$$

Then, taking the power of $1/l_k$ of both sides of (3.32), we have

$$\left(\int_{B_{r_k}} h^{l_k\chi}\eta^{2\chi}e^{-f}dv\right)^{\frac{1}{l_k\chi}} \le \left(a_6 e^{2C_R}V_f^{-\frac{2}{m}}\right)^{\frac{1}{l_k}} 16^{\frac{k}{l_k}}\left[1+(\kappa+K)R^2\right]^{\frac{2s}{l_k(2s-m)}}\left(\int_{B_{r_k}} h^{l_k}e^{-f}dv\right)^{\frac{1}{l_k}}. \tag{3.33}$$

Thus,

$$\left(\int_{B_{r_{k+1}}} h^{l_{k+1}}e^{-f}dv\right)^{\frac{1}{l_{k+1}}} \le \left(a_6 e^{2C_R}V_f^{-\frac{2}{m}}\right)^{\frac{1}{l_k}} 16^{\frac{k}{l_k}}\left[1+(\kappa+K)R^2\right]^{\frac{2s}{l_k(2s-m)}}\left(\int_{B_{r_k}} h^{l_k}e^{-f}dv\right)^{\frac{1}{l_k}}, \tag{3.34}$$

which implies that

$$\|h\|_{L_f^{l_{k+1}}(B_{r_{k+1}})} \le \left(a_6 e^{2C_R}V_f^{-\frac{2}{m}}\right)^{\frac{1}{l_k}} 16^{\frac{k}{l_k}}\left[1+(\kappa+K)R^2\right]^{\frac{2s}{l_k(2s-m)}}\|h\|_{L_f^{l_k}(B_{r_k})}. \tag{3.35}$$

By iteration we have

$$\|h\|_{L_f^{l_{k+1}}(B_{r_{k+1}})} \le \left(a_6 e^{2C_R}V_f^{-\frac{2}{m}}\right)^{\sum_{k=1}^{\infty}\frac{1}{l_k}} 16^{\sum_{k=1}^{\infty}\frac{k}{l_k}}\left[1+(\kappa+K)R^2\right]^{\sum_{k=1}^{\infty}\frac{2s}{l_k(2s-m)}}\|h\|_{L_f^{l_1}(B_{3R/4}(o))}. \tag{3.36}$$

We note that

$$\sum_{k=1}^{\infty}\frac{1}{l_k}=\frac{m}{2l_1} \quad \text{and} \quad \sum_{k=1}^{\infty}\frac{k}{l_k}=\frac{m^2}{4l_1},$$

then, by letting $k\to\infty$ in (3.36), we obtain the following inequality:

$$\|h\|_{L_f^{\infty}(B_{R/2}(o))} \le a_7 e^{\frac{mC_R}{l_1}}V_f^{-\frac{1}{l_1}}\left[1+(\kappa+K)R^2\right]^{\frac{sm}{l_1(2s-m)}}\|h\|_{L_f^{l_1}(B_{3R/4}(o))}. \tag{3.37}$$

According to (3.22), we have

$$\|h\|_{L_f^{\infty}(B_{R/2}(o))} \le a_3 e^{\frac{C_R}{l_0}}V_f^{\frac{1}{l_1}}\left[\frac{A(l_0)^{1+\frac{1}{l_0}}+l_0^2}{\varepsilon_0\tau^2R^2}\right]a_7 e^{\frac{mC_R}{l_1}}V_f^{-\frac{1}{l_1}}\left[1+(\kappa+K)R^2\right]^{\frac{sm}{l_1(2s-m)}}. \tag{3.38}$$

Then, considering the fact that for all $x>0$, the function $x^{\frac{1}{x}}\le C$, we can observe that $e^{\frac{mC_R}{l_1}}\le e^{\frac{mC_R}{l_0}}\le C(m)$ and

$$\left[1+(\kappa+K)R^2\right]^{\frac{sm}{l_1(2s-m)}} \le \left[1+(\kappa+K)R^2\right]^{\frac{sm}{l_0(2s-m)}} \le C(m,s).$$

Moreover,

$$A(l_0)^{\frac{1}{l_0}} = \left(C_1\left(l_0e^{C_R}\right)^{\frac{m}{2s-m}}\left[1+(\kappa+K)R^2\right]^{\frac{2s}{2s-m}}\right)^{\frac{1}{l_0}} \le C(m,s,\varepsilon_0).$$

Finally, we set

$$l_0 = t_0 + \sqrt{\kappa+K}R.$$

Thus, we can derive from (3.38) that

$$\begin{aligned}
&\|h\|_{L^\infty_f\left(B_{\frac{R}{2}}\right)}\\
\leq & C\left(m,s,\varepsilon_0\right)\frac{A\left(l_0\right)+l_0^2}{\varepsilon_0\tau^2R^2}\\
\leq & C\left(m,s,\varepsilon_0\right)\frac{e^{\frac{m}{2s-m}C_m(1+\sqrt{\kappa}R)}\left[t_0+\sqrt{\kappa+K}R\right]^{\frac{m}{2s-m}}\left[1+(\kappa+K)R^2\right]^{\frac{2s}{2s-m}}+t_0^2+(\kappa+K)R^2}{\varepsilon_0\tau^2R^2}\\
\leq & C\left(m,s,\varepsilon_0\right)\frac{e^{\frac{m}{2s-m}C_m(1+\sqrt{\kappa}R)}\left[1+(\kappa+K)R^2\right]^{\frac{2s+m}{2s-m}}+t_0^2+(\kappa+K)R^2}{\varepsilon_0\tau^2R^2}\\
\leq & C\left(m,s,\tau,\varepsilon_0\right)e^{\frac{m}{2s-m}C_m(1+\sqrt{\kappa}R)}\frac{\left[1+(\kappa+K)R^2\right]^{\frac{2s+m}{2s-m}}}{R^2}.
\end{aligned}$$

□

4. Proofs of Theorem 1.5 and Corollary 1.6

Proof of Theorem 1.5. Under the conditions in Theorem 1.5, by (1.13) of Theorem 1.4, we have that for any $x \in B_{R/2}(o) \subset M$,

$$\frac{|\nabla v(x)|}{v(x)} \leq \sup_{B_{R/2}(o)}\frac{|\nabla v|}{v} \leq \frac{C(m,\tau,l)}{R}. \tag{4.1}$$

Letting $R \to \infty$ in (4.1), we get

$$\nabla v(x) = 0, \quad \forall x \in M.$$

Therefore, v is a positive constant on M.

Proof of Corollary 1.6. Applying the same conditions as Theorem 1.5, let $x, y \in B_{R/2}(o)$ be any two points with minimal geodesic γ connecting them. Then using the gradient estimate (1.13) and the fact that length$(\gamma) \leq R$, we have

$$\begin{aligned}
\ln v(x) - \ln v(y) &= \int_\gamma |\nabla \ln v| dt\\
&\leq \int_\gamma C(\tau,m,l)\frac{1+\sqrt{\kappa}R}{R}dt\\
&\leq C(\tau,m,l)(1+\sqrt{\kappa}R).
\end{aligned}$$

Thus, for any $x, y \in B_{R/2}(o)$ we obtain

$$v(x) \leq e^{C(\tau,m,l)(1+\sqrt{\kappa}R)}v(y).$$

5. Conclusions

Using the Moser iteration strategy and the Saloff-Coste Sobolev-type inequality, we proved a local gradient estimate of Cheng-Yau type for positive solutions to the semilinear elliptic equation $\Delta_f v^\tau +$

$\lambda(x)v^l = 0$ on a complete smooth metric measure space with m-Bakry-Émery Ricci curvature bounded from below. As an application, the nonexistence of positive solutions was obtained. Compared with previous works, this paper extended the ranges of τ and l.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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