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Research article

Global well-posedness and large-time behavior for anisotropic inhomogeneous Navier-Stokes equations with mixed partial dissipation

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Abstract: In this paper, we consider the Cauchy problem for the two-dimensional anisotropic inhomogeneous Navier-Stokes equations with mixed partial dissipation, where the density at infinity is in the state $\widetilde{\rho} > 0$. We obtain the global well-posedness of strong solutions and the large-time behavior of the velocity field under the condition that $\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}$ is sufficiently small. It is worth noting that our results include the case of vacuum.

Keywords: inhomogeneous Navier-Stokes equations; global strong solutions; partial dissipation; decay rates; vacuum

1. Introduction

In geophysical fluid dynamics, Earth's rotation and the stratification of the atmosphere and oceans introduce pronounced anisotropic features into fluid flows. These anisotropies, prevalent in fluid mechanics, significantly shape flow dynamics. For example, in turbulent flows, the interplay between inertial and viscous forces results in distinct behaviors across different spatial directions, thereby influencing flow structure and the transport of momentum and heat. For further background, see [1–3]. To study these anisotropic behaviors, the anisotropic Navier-Stokes equations are widely employed, as shown below:

$$\begin{cases} \rho_t + \mathrm{div}(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) - \mu_1\Delta_h\mathbf{u} - \mu_2\partial_z^2\mathbf{u} + \nabla P = 0, \\ \mathrm{div}\mathbf{u} = 0, \\ (\rho, \rho\mathbf{u})|_{t=0} = (\rho_0, \rho_0\mathbf{u}_0), \end{cases} \tag{1.1}$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_n)(x,t)$ with $x \in \mathbb{R}^n$ represents the fluid velocity field, $\rho = \rho(x,t)$, *and* $P = P(x,t)$ are the density and pressure of the flow, respectively. The operator $\Delta_h \triangleq \sum_{j=1}^{n-1}\partial_j^2$ is defined as the horizontal Laplacian, while $\partial_z^2 \triangleq \partial_n^2$ represents the partial derivative with respect to the n-th coordinate. The coefficients μ_1 and μ_2 are viscosity coefficients associated with the fluid dynamics.

When $\mu_1 = \mu_2 > 0$, the system reduces to the Navier-Stokes equations with standard dissipation, whose well-posedness and large-time behavior of solutions have been extensively studied. When away from vacuum, Kažihov [4] proved the global existence of three-dimensional weak solutions. Subsequently, Antontsev et al. [5] established the local existence and uniqueness of three-dimensional strong solutions, which were later extended to global solutions by Ladyzhenskaya and Solonnikov [6] and Salvi [7]. In the presence of vacuum, the problem becomes more complex. Initially, Simon [8] established the global existence of weak solutions. Choe and Kim [9] proposed a compatibility condition to prove the local existence and uniqueness of strong solutions in three-dimensions. Later, Liang [10] established a blow-up criterion to prove the local existence of strong solutions for the two-dimensional Cauchy problem, and Lü et al. [11] extended these local solutions to global ones. In succession, Huang and Wang [12] proved the global strong solutions in a two-dimensional bounded domain with Dirichlet boundary conditions. Most recently, Liu [13] established the optimal decay estimates for the two-dimensional inhomogeneous Navier-Stokes equations. For more related literature, see [14–16] and the references therein.

When the viscosity coefficients in system (1.1) are unequal or partially vanish, it is frequently used to simulate geophysical flows and has produced numerous intriguing results. In particular, when $\mu_2 = 0$ and the density ρ is constant, this three-dimensional model garners considerable attention from scholars. The seminal work was conducted by Chemin et al. [17], who established the existence of local solutions and the existence of global solutions under the smallness condition of $\|\mathbf{u}_0\|_{H^{0,s}}$ $(s > \frac{1}{2})$. Subsequently, Paicu [18] demonstrated the existence of global solutions in the anisotropic Besov space $B^{0,\frac{1}{2}}$ within the L^2 framework. Building on this, Chemin and Zhang [19] and Zhang and Fang [20] extended these results to the more general L^p framework of anisotropic Besov spaces $B_p^{-1+\frac{2}{p},\frac{1}{2}}$ for $2 \leq p < \infty$. For more related results, we refer to [21, 22]. More recently, Liu and Zhang [23] showed that the system admits a unique global solution as long as the one directional derivative of the initial velocity is sufficiently small in some scaling-invariant framework. They also established the existence of global large solutions when the initial data varies slowly. Additionally, when the density is constant, some interesting results have also been obtained in the two-dimensional case. Regmi and Wu [24] established the global well-posedness of classical solutions for three distinct types of partially dissipative systems with large initial data. Shang and Zhou [25] demonstrated the stability and large-time behavior of solutions for a 2D mixed partially dissipative system. Dong et al. [26] examined the stability and exponential decay of solutions for a system with $\mu_2 = 0$ in the domain $\mathbb{T} \times \mathbb{R}$, where $\mathbb{T}$ denotes a one-dimensional periodic box.

When the density is non-constant, the well-posedness and large-time behavior of solutions to system (1.1) are relatively limited in the existing literature. Hao [27] established the global well-posedness of strong solutions to the three-dimensional Cauchy problem under the conditions that the viscosity coefficient μ_2 is significantly larger than the initial data and μ_1, and the initial density is near a positive constant. Recently, Paicu and Zhu [28] extended the results of [23] to the inhomogeneous Navier-Stokes equations. However, to the best of our knowledge, the well-posedness of nonhomogeneous systems with partial dissipation is still an open question. Therefore, we aim to address this issue.

We consider the two-dimensional anisotropic Navier-Stokes equations with mixed partial dissipation, which are given by the following system:

$$\begin{cases} \rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \\ \rho u_{1t} + \rho\mathbf{u}\cdot\nabla u_1 - \mu\partial_2^2 u_1 + \partial_1 P = 0, \\ \rho u_{2t} + \rho\mathbf{u}\cdot\nabla u_2 - \mu\partial_1^2 u_2 + \partial_2 P = 0, \\ \operatorname{div}\mathbf{u} = 0, \\ (\rho, \rho\mathbf{u})|_{t=0} = (\rho_0, \rho_0\mathbf{u}_0), \end{cases} \tag{1.2}$$

with far field state satisfying

$$\mathbf{u}(x,t) \to 0, \quad \rho(x,t) \to \widetilde{\rho} > 0, \text{ as } |x| \to \infty. \tag{1.3}$$

In this paper, we investigate the global well-posedness and the large-time behavior of strong solutions to systems (1.2) and (1.3).

Before stating the theorem, we first introduce some notation conventions throughout this paper. We assume $\mu = 1$ without loss of generality. We denote by L^m the usual scalar-valued and vector-valued L^m space over $\mathbb{R}^2$. For $k \geq 0, 1 \leq m \leq \infty$, we use the following simplified notations for the standard Sobolev spaces:

$$\begin{cases} D^{k,m} = \{u \in L^1_{loc}(\mathbb{R}^2) : \|\nabla^k u\|_{L^m} < \infty\}, \\ \quad W^{k,m} = L^m \cap D^{k,m}, \quad H^k = W^{k,2}. \end{cases}$$

For all $f \in L^1 \cap L^2(\mathbb{R}^2)$, the Fourier transform of f is defined as

$$\mathcal{F} f(x) = \hat{f}(x) = \int_{\mathbb{R}^2} f(\xi) e^{-ix\cdot\xi}\, d\xi,$$

and it holds that $\|\hat{f}\|^2_{L^2} = \|f\|^2_{L^2}$. Additionally, for $a \lesssim b$, this notation indicates that there exists a uniform constant C, which may vary from line to line, such that $a \leq Cb$.

Now, the main theorems are presented below.

Theorem 1.1. *Assume that* $(\rho_0, \mathbf{u}_0)$ *satisfies for some* $q > 2$*,*

$$\begin{gathered} 0 \leq \rho_0 \leq \bar{\rho}, \ \ \rho_0 - \widetilde{\rho} \in H^1 \cap W^{1,q}(\mathbb{R}^2), \\ \sqrt{\rho_0}\mathbf{u}_0 \in L^2(\mathbb{R}^2), \ \ \nabla\mathbf{u}_0 \in H^1(\mathbb{R}^2), \ \ \operatorname{div}\mathbf{u}_0 = 0, \end{gathered} \tag{1.4}$$

and the compatibility condition

$$\begin{cases} -\mu\partial_2^2 u_{10} + \partial_1 P_0 = \sqrt{\rho_0} g_1, \\ -\mu\partial_1^2 u_{20} + \partial_2 P_0 = \sqrt{\rho_0} g_2, \end{cases} \tag{1.5}$$

for some $P_0 \in H^1(\mathbb{R}^2)$ *and* $\mathbf{g} = (g_1, g_2) \in L^2(\mathbb{R}^2)$*. Then there exist a time* $T_0 > 0$ *and a unique strong solution* $(\rho, \mathbf{u})$ *to the initial boundary value problems* (1.2) *and* (1.3) *satisfying*

$$\begin{cases} \rho - \widetilde{\rho} \in C([0, T_0]; H^1 \cap W^{1,q}(\mathbb{R}^2)), \\ \nabla\mathbf{u} \in L^\infty(0, T_0; H^1(\mathbb{R}^2)) \cap L^2(0, T_0; H^1(\mathbb{R}^2)), \\ \sqrt{\rho}\mathbf{u}_t \in L^\infty(0, T_0; L^2(\mathbb{R}^2)) \cap L^2(0, T_0; L^2(\mathbb{R}^2)), \\ \nabla P \in L^\infty(0, T_0; L^2(\mathbb{R}^2)) \cap L^2(0, T_0; L^2(\mathbb{R}^2)), \\ \nabla\mathbf{u}_t \in L^2(0, T_0; L^2(\mathbb{R}^2)). \end{cases} \tag{1.6}$$

Notably, the presence of nonlinearity and vacuum in the inhomogeneous Navier-Stokes system, combined with partial dissipation in (1.2) and (1.3), significantly complicates the problem. To establish local well-posedness, inspired by [25], we first derive the estimates for the anisotropic Stokes problem by leveraging the inequality $\|\nabla^{k+1}\mathbf{u}\|_{L^2} \le C\|\nabla^k \omega\|_{L^2}$, where $\omega = \partial_2 u_1 - \partial_1 u_2$ represents the vorticity. For the detailed process, see Lemma 2.4. Then, utilizing Lemma 2.2 and (1.3), we proceed to the proof, with specific details provided in Section 3.

Building upon the local solution, we establish the global well-posedness and the large-time behavior of the solution as follows:

Theorem 1.2. *Assume that initial data* $(\rho_0, \mathbf{u}_0)$ *satisfies* (1.4) *and* (1.5) *with* $q \in (2,4)$*, in addition to* $\mathbf{u}_0 \in \dot{H}^{-\sigma}(\mathbb{R}^2)$ *with* $\sigma \in (\frac{q-2}{q}, \frac{1}{2}]$*. Then, there exists a sufficiently small constant* η*, which depends only on* μ*,* $\bar{\rho}$*,* $\|\nabla \mathbf{u}_0\|_{H^1(\mathbb{R}^2)}$*, and* $\|\rho_0 - \widetilde{\rho}\|_{H^1\cap W^{1,q}(\mathbb{R}^2)}$*, such that if*

$$\|\sqrt{\rho_0}\mathbf{u}_0\|^2_{L^2(\mathbb{R}^2)} \le \eta, \tag{1.7}$$

then the systems (1.2) *and* (1.3) *admit a unique global strong solution* $(\rho, \mathbf{u})$ *satisfying that for* $\forall\ 0 < \tau < T < \infty$*,*

$$\begin{cases} \rho - \widetilde{\rho} \in C([0,T]; H^1 \cap W^{1,q}(\mathbb{R}^2)), \\ \nabla \mathbf{u} \in L^\infty(0,T; H^1(\mathbb{R}^2)) \cap L^2(0,T; H^1(\mathbb{R}^2)), \\ \sqrt{\rho}\mathbf{u}_t \in L^2(0,T; L^2(\mathbb{R}^2)) \cap L^\infty(0,T; L^2(\mathbb{R}^2)), \\ \nabla P \in L^\infty(0,T; L^2(\mathbb{R}^2)) \cap L^2(0,T; L^2(\mathbb{R}^2)), \\ t\nabla \mathbf{u} \in L^\infty(\tau,T; H^1(\mathbb{R}^2)),\ t^2\sqrt{\rho}\mathbf{u}_t \in L^\infty(\tau,T; L^2(\mathbb{R}^2)), \\ t^2\nabla \mathbf{u}_t \in L^2(\tau,T; L^2(\mathbb{R}^2)),\ t^2\nabla P \in L^\infty(0,T; L^2(\mathbb{R}^2)). \end{cases} \tag{1.8}$$

Moreover, there exists t_1 *such that the following decay rates hold when* $t \ge t_1$*:*

$$\begin{aligned} &\|\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} \le C(1+t)^{-\sigma+\varepsilon}, \\ &\|\nabla \mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} \le C(1+t)^{-1-\sigma+\varepsilon}, \\ &\|\nabla^2 \mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} \le C(1+t)^{-2-\sigma+\varepsilon}, \end{aligned} \tag{1.9}$$

where C *depends only on* $\widetilde{\rho}$*,* $\bar{\rho}$*,* $\|\rho_0 - \widetilde{\rho}\|_{H^1\cap W^{1,q}(\mathbb{R}^2)}$*, and* $\|\mathbf{u}_0\|_{H^2(\mathbb{R}^2)}$*, and* ε *is a sufficiently small positive constant.*

It is well-known that establishing the boundedness of $\nabla \mathbf{u}$ in $L^1(0,T;L^\infty)$ is essential to proving global well-posedness.

Unfortunately, for partially dissipative systems, the elliptic estimates do not yield for $p > 2$,

$$\|\nabla^2 \mathbf{u}\|_{L^p} + \|\nabla P\|_{L^p} \le C\|\rho \mathbf{u}_t\|_{L^p} + C\|\rho \mathbf{u}\cdot\nabla \mathbf{u}\|_{L^p}. \tag{1.10}$$

By invoking Lemma 2.4 and using interpolation, we find that the estimate $\|\nabla\rho\|_{L^p}$ is unavoidable. From the mass equation, we observe that $\|\nabla\rho\|_{L^p}$ and $\int_0^T \|\nabla \mathbf{u}\|_{L^\infty}\, dt$ are interdependent and from (3.31)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho\|_{L^q} &\lesssim \|\nabla \mathbf{u}\|_{L^\infty}\|\nabla\rho\|_{L^q} \\ &\lesssim \|\nabla \mathbf{u}\|^{\frac{1}{2}}_{L^2}\|\nabla^3 \mathbf{u}\|^{\frac{1}{2}}_{L^2}\|\nabla\rho\|_{L^q} \\ &\lesssim \|\nabla\rho\|^{\frac{3}{2}}_{L^q}\|\nabla \mathbf{u}\|^{\frac{1}{2}}_{L^2}\|\mathbf{u}_t\|^{\frac{q-2}{2q}}_{L^2}\|\nabla \mathbf{u}_t\|^{\frac{1}{q}}_{L^2} + \|\nabla \mathbf{u}\|^{\frac{1}{2}}_{L^2}\|\nabla \mathbf{u}_t\|^{\frac{1}{2}}_{L^2}\|\nabla\rho\|_{L^q} + \cdots \end{aligned} \tag{1.11}$$

It is noted that the exponents in the higher-order estimates of $\|\nabla\mathbf{u}_t\|_{L^2}$ in (1.11) are relatively small, which poses difficulties in achieving the boundedness of $\int_1^T \|\nabla\mathbf{u}\|_{L^\infty}\, dt$ through the usual time-weighting. In light of the aforementioned difficulties, we need to propose new ideas to close the estimates.

To overcome these difficulties, we impose smallness on $\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}$. Subsequently, through energy estimates and weighted-t estimates for $\|\nabla\mathbf{u}\|_{L^2}$, we obtain the smallness of $\int_0^T \|\nabla\mathbf{u}\|_{L^2}^2 dt$ and $\sup_{t\in[0,T]} t\|\nabla\mathbf{u}\|_{L^2}^2$, thereby avoiding the smallness requirement on $\|\nabla\rho_0\|_{L^p}$. Meanwhile, inspired by Schonbek's work [29] on utilizing frequency decomposition to obtain the large-time behavior of the velocity field, we assume that $\mathbf{u}_0 \in \dot{H}^{-\sigma}$ and we simultaneously employ this method to derive the decay rate of $\|\sqrt{\rho}\mathbf{u}\|_{L^2}$. Based on this analysis, we then leverage the decay of the velocity field to update the decay rates of its derivatives, thereby effectively overcoming difficulties. For the detailed process, please refer to Section 4.

Remark 1. It should be noted that the similar ideas in this paper can be applied to the following magnetohydrodynamic equations with mixed dissipation, as well as to analogous systems.

$$\begin{cases} \rho_t + \mathrm{div}(\rho\mathbf{u}) = 0, \\ \rho u_{1t} + \rho\mathbf{u}\cdot\nabla u_1 - \mu\partial_2^2 u_1 + \partial_1 P = \mathbf{b}\cdot\nabla b_1, \\ \rho u_{2t} + \rho\mathbf{u}\cdot\nabla u_2 - \mu\partial_1^2 u_2 + \partial_2 P = \mathbf{b}\cdot\nabla b_2, \\ \partial_t b_1 + \mathbf{u}\cdot\nabla b_1 - \nu\partial_2^2 b_1 = \mathbf{b}\cdot\nabla u_1, \\ \partial_t b_2 + \mathbf{u}\cdot\nabla b_2 - \nu\partial_1^2 b_2 = \mathbf{b}\cdot\nabla u_2, \\ \mathrm{div}\,\mathbf{u} = 0, \quad \mathrm{div}\,\mathbf{b} = 0. \end{cases} \tag{1.12}$$

Remark 2. In this paper, to establish global well-posedness, we impose a compatibility condition to derive the decay rate of $\|\sqrt{\rho}\mathbf{u}\|_{L^2}$. It is natural to study the global strong solution without (1.7), which we leave for future research.

The rest of this paper is organized as follows: In Section 2, we gather some basic inequalities and Stokes estimates that will be used in subsequent analysis. In Section 3, we focus on establishing the local well-posedness of the solution. Ultimately, in Section 4, we obtain the global well-posedness of the solution, thereby completing the proof of Theorem 1.2.

At the end of this section, we explain the notation and conventions throughout this paper. We assume $\mu = 1$ without loss of generality. We denote by L^m the usual scalar-valued and vector-valued L^m space over $\mathbb{R}^2$. Additionally, for $a \lesssim b$, this notation indicates that there exists a uniform constant C, which may vary from line to line, such that $a \le Cb$.

2. Preliminaries

In this section, we present several frequently used inequalities and conclusions that will be employed in subsequent proofs.

Lemma 2.1. ([30])*(Gagliardo-Nirenberg inequalities) For $q \in [2,\infty)$, $r \in (2,\infty)$, and $s \in (1,\infty)$, there exists some generic constant $C > 0$ that may depend on q, r, and s such that for $f \in H^1$ and $g \in L^s \cap D^{1,r}$, we have*

$$\|f\|_{L^q}^q \le C\|f\|_{L^2}^2\|\nabla f\|_{L^2}^{q-2}, \tag{2.1}$$

$$\|g\|_{L^\infty} \le C\|g\|_{L^s}^{s(r-2)/(2r+s(r-2))}\|\nabla g\|_{L^r}^{2r/(2r+s(r-2))}. \tag{2.2}$$

Lemma 2.2. ([31]) *Suppose that* $0 \leq \rho \leq \bar{\rho}$ *and* $\mathbf{u} \in H^1$*, then we have*

$$\| \sqrt{\rho}\mathbf{u}\|_{L^4}^2 \leqslant C(\bar{\rho})(1 + \| \sqrt{\rho}\mathbf{u}\|_{L^2})\|\mathbf{u}\|_{H^1} \sqrt{\ln(2 + \|\mathbf{u}\|_{H^1}^2)}, \tag{2.3}$$

where and in what follows, we sometimes use $C(f)$ *to emphasize the dependence on* f.

Lemma 2.3. *Let* $\mathbf{u}$ *be a divergence-free vector function, then* $\forall\ k \geq 0$,

$$\|\nabla^{k+1}\mathbf{u}\|_{L^2} \lesssim \|\nabla^k \omega\|_{L^2}, \tag{2.4}$$

where $\omega = \partial_2 u_1 - \partial_1 u_2$.

Proof. We can derive the result by employing the properties of the Riesz operator, along with the facts that the velocity divergence is zero and $-\Delta\mathbf{u} = -\nabla\mathrm{div}\mathbf{u} - \nabla^{\perp}\omega$, where $\nabla^{\perp} = (\partial_2, -\partial_1)$. □

The last lemma gives the uniform regularity estimates for solutions to the anisotropic Stokes equations. The proof is similar to the classical results, and we provide a brief proof here.

Lemma 2.4. *Assume that* $(\mathbf{u}, P)$ *is a smooth solution to the following anisotropic Stokes system:*

$$\begin{cases} -\partial_2^2 u_1 + \partial_1 P = f_1, & x \in \mathbb{R}^2, \\ -\partial_1^2 u_2 + \partial_2 P = f_2, & x \in \mathbb{R}^2, \\ \mathrm{div}\, \mathbf{u} = 0, & x \in \mathbb{R}^2. \end{cases} \tag{2.5}$$

Then for $\forall\ \mathbf{f} = (f_1, f_2) \in H^1$*, it holds that*

$$\|\nabla^2\mathbf{u}\|_{L^2} + \|\nabla P\|_{L^2} \leq C\|\mathbf{f}\|_{L^2}, \tag{2.6}$$

and

$$\|\nabla^3\mathbf{u}\|_{L^2} + \|\nabla^2 P\|_{L^2} \leq C\|\nabla\mathbf{f}\|_{L^2}. \tag{2.7}$$

Proof. To begin, we take the inner product of $(2.5)_1$ with $\partial_1^2 u_1$ and the inner product of $(2.5)_2$ with $\partial_1^2 u_2$. Adding these two results together and applying integration by parts, we obtain

$$\|\partial_1\partial_2 u_1\|_{L^2}^2 + \|\partial_1^2 u_2\|_{L^2}^2 \leq C\|\mathbf{f}\|_{L^2}\|\nabla^2\mathbf{u}\|_{L^2}, \tag{2.8}$$

where we have used

$$\int_{\mathbb{R}^2} \partial_1 P \partial_1^2 u_1 dx + \int_{\mathbb{R}^2} \partial_2 P \partial_1^2 u_2 dx = 0. \tag{2.9}$$

Similarly, we can also obtain

$$\|\partial_2^2 u_1\|_{L^2}^2 + \|\partial_1\partial_2 u_2\|_{L^2}^2 \leq C\|\mathbf{f}\|_{L^2}\|\nabla^2\mathbf{u}\|_{L^2}. \tag{2.10}$$

Based on the fact (2.4), we can therefore obtain

$$\|\nabla^2\mathbf{u}\|_{L^2}^2 \leq 2(\|\partial_2^2 u_1\|_{L^2}^2 + \|\partial_1\partial_2 u_2\|_{L^2}^2 + \|\partial_1\partial_2 u_1\|_{L^2}^2 + \|\partial_1^2 u_2\|_{L^2}^2) \leq C\|\mathbf{f}\|_{L^2}^2. \tag{2.11}$$

Applying ∂_1 to $(2.5)_1$ and ∂_2 to $(2.5)_2$, and then adding them together, we can derive that

$$\Delta P = \mathrm{div}\mathbf{f} + \partial_2^2\partial_1 u_1 + \partial_2\partial_1^2 u_2, \tag{2.12}$$

i.e.,

$$P=-(-\Delta)^{-1}\mathrm{div}\mathbf{f}-(-\Delta)^{-1}\partial_2^2\partial_1 u_1-(-\Delta)^{-1}\partial_2\partial_1^2 u_2. \tag{2.13}$$

By classical estimates in harmonic analysis, we have the following conclusions:

$$\|\nabla P\|_{L^2}\le C(\|\mathbf{f}\|_{L^2}+\|\nabla^2\mathbf{u}\|_{L^2}). \tag{2.14}$$

Therefore, by combining (2.11), we arrive at (2.6). For (2.7), it is similar to (2.6), and we omit the details here. □

3. Local well-posedness

In this section, our objective is to demonstrate the local well-posedness of the system. We begin in Section 3.1 by obtaining uniform a priori estimates for smooth solutions. Subsequently, employing the standard linearization process, we proceed to establish the proof of the local solutions in Section 3.2.

3.1. A priori estimates

To begin with, since $(1.2)_1$ is a transport equation, following the standard procedure, we have that for all $0\le t\le T_0$,

$$0\le\rho\le\bar{\rho},\quad \|\rho-\widetilde{\rho}\|_{L^q}=\|\rho_0-\widetilde{\rho}\|_{L^q},\ \ 2\le q<\infty. \tag{3.1}$$

In what follows, we shall establish the fundamental energy inequalities:

Lemma 3.1. *It holds that*

$$\sup_{t\in[0,T_0]}\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+\int_0^{T_0}\|\nabla\mathbf{u}\|_{L^2}^2dt\le\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2. \tag{3.2}$$

Proof. Multiplying $(1.2)_{1,2}$ by $\mathbf{u}$ and integrating by parts over $\mathbb{R}^2$ yield that $\forall\ 0\le t\le T_0$,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\rho|\mathbf{u}|^2\,dx+\int_{\mathbb{R}^2}|\partial_1 u_2|^2\,dx+\int_{\mathbb{R}^2}|\partial_2 u_1|^2\,dx=0. \tag{3.3}$$

Observe that

$$\|\nabla\mathbf{u}\|_{L^2}^2\le\int_{\mathbb{R}^2}(\partial_1 u_2-\partial_2 u_1)^2\,dx\le 2(\|\partial_2 u_1\|_{L^2}^2+\|\partial_1 u_2\|_{L^2}^2). \tag{3.4}$$

Thus, (3.3) and (3.4) imply (3.2), and the proof of Lemma 3.1 is completed. □

Next, we aim to derive uniform estimates and weighted-in-time estimates for $\|\nabla\mathbf{u}\|_{L^\infty(0,T;L^2)}$.

Lemma 3.2. *There exists a constant C depending only on* $\|\nabla\mathbf{u}_0\|_{H^1},\mu,\bar{\rho}$, *and* $\widetilde{\rho}$, *such that for* $0\le t\le T_0$ *and* $i=0,1$,

$$\sup_{t\in[0,T_0]}(t^i\|\nabla\mathbf{u}\|_{L^2}^2)+\int_0^{T_0}t^i\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2dt\le C. \tag{3.5}$$

Proof. First, multiplying $(1.2)_{1,2}$ by $\partial_t\mathbf{u}$ and integrating over $\mathbb{R}^2$ leads to

$$\frac{1}{2}\frac{d}{dt}\Big(\int_{\mathbb{R}^2}|\partial_1 u_2|^2dx+\int_{\mathbb{R}^2}|\partial_2 u_1|^2dx\Big)+\int_{\mathbb{R}^2}\rho|\mathbf{u}_t|^2dx=-\int_{\mathbb{R}^2}(\rho\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u}_t dx. \tag{3.6}$$

Using Young's inequality, (3.2), and (2.1), we obtain

$$
\begin{aligned}
\left|\int_{\mathbb{R}^2}(\rho\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u}_t dx\right| &\lesssim \|\sqrt{\rho}\mathbf{u}_t\|_{L^2}\|\sqrt{\rho}\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}\\
&\leq \frac{1}{2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + C\|\sqrt{\rho}\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^4}^2\\
&\leq \frac{1}{2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + C\|\sqrt{\rho}\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^2}\|\nabla^2\mathbf{u}\|_{L^2}.
\end{aligned}
\tag{3.7}
$$

Substituting (3.7) into (3.6), we have

$$
\frac{1}{2}\frac{d}{dt}(\|\partial_1 u_2\|_{L^2}^2 + \|\partial_2 u_1\|_{L^2}^2) + \|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 \leq C\|\sqrt{\rho}\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^2}\|\nabla^2\mathbf{u}\|_{L^2}. \tag{3.8}
$$

Utilizing Lemma 2.4, we derive from (3.2) that

$$
\begin{aligned}
\|\nabla^2\mathbf{u}\|_{L^2} + \|\nabla P\|_{L^2} &\leq C\left(\|\rho\mathbf{u}_t\|_{L^2} + \|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}\right)\\
&\leq C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2} + C\|\sqrt{\rho}\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}\|_{L^2}^{\frac{1}{2}}\|\nabla^2\mathbf{u}\|_{L^2}^{\frac{1}{2}}\\
&\leq C\|\sqrt{\rho}\mathbf{u}_t\|_{L^2} + C\|\sqrt{\rho}\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^2} + \frac{1}{2}\|\nabla^2\mathbf{u}\|_{L^2}.
\end{aligned}
\tag{3.9}
$$

Substituting (3.9) into (3.8) yields

$$
\frac{1}{2}\frac{d}{dt}(\|\partial_1 u_2\|_{L^2}^2 + \|\partial_2 u_1\|_{L^2}^2) + \|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 \leq C\|\sqrt{\rho}\mathbf{u}\|_{L^4}^4\|\nabla\mathbf{u}\|_{L^2}^2.
$$

Considering (1.3) and (3.1), we can obtain

$$
\begin{aligned}
\tilde{\rho}\int_{\mathbb{R}^2}|\mathbf{u}|^2 dx &= \int_{\mathbb{R}^2}\rho|\mathbf{u}|^2 dx + \int_{\mathbb{R}^2}(\tilde{\rho}-\rho)|\mathbf{u}|^2 dx\\
&\leq \|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + C\|\rho-\tilde{\rho}\|_{L^2}\|\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}\\
&\leq \|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}^2 + \frac{\widetilde{\rho}}{2}\|\mathbf{u}\|_{L^2}^2,
\end{aligned}
\tag{3.10}
$$

and we combine with (3.2) and (2.3) to yield

$$
\frac{d}{dt}(\|\partial_1 u_2\|_{L^2}^2 + \|\partial_2 u_1\|_{L^2}^2) + \|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 \lesssim \|\nabla\mathbf{u}\|_{L^2}^2(1+\|\nabla\mathbf{u}\|_{L^2}^2)\ln(C+\|\nabla\mathbf{u}\|_{L^2}^2). \tag{3.11}
$$

Set

$$
g(t) \triangleq C + \|\partial_1 u_2\|_{L^2}^2 + \|\partial_2 u_1\|_{L^2}^2, \quad (C>2)
$$

then from (3.11) and (2.4), we can deduce that

$$
g'(t) \leq C\|\nabla\mathbf{u}\|_{L^2}^2 g(t)\ln g(t), \tag{3.12}
$$

i.e.,

$$
(\ln g(t))' \leq C\|\nabla\mathbf{u}\|_{L^2}^2 \ln g(t).
$$

By applying Grönwall's inequality in conjunction with (3.2), we arrive at the conclusion that

$$\sup_{t\in[0,T_0]} \ln g(t) \le C.$$

Hence, we get that

$$\sup_{t\in[0,T_0]} \|\nabla \mathbf{u}\|_{L^2}^2 \le 2 \sup_{t\in[0,T_0]} (\|\partial_1 u_2\|_{L^2}^2 + \|\partial_2 u_1\|_{L^2}^2) \le C. \tag{3.13}$$

Integrating (3.11) with respect to t together with (3.13) leads to

$$\int_0^{T_0} \|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \|\nabla^2 \mathbf{u}\|_{L^2}^2 dt \le C. \tag{3.14}$$

Finally, applying the operator t to (3.8) and combining (2.1), we can obtain

$$\begin{aligned}\frac{d}{dt}(t\|\partial_1 u_2\|_{L^2}^2 + t\|\partial_2 u_1\|_{L^2}^2) + t\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 &\lesssim t\|\sqrt{\rho}\mathbf{u}\|_{L^4}^4\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2 \\ &\lesssim t\|\mathbf{u}\|_{L^2}^2\|\nabla \mathbf{u}\|_{L^2}^4 + \|\nabla \mathbf{u}\|_{L^2}^2.\end{aligned} \tag{3.15}$$

Combining (3.2), (2.4), and Grönwall's inequality, we obtain the weighted-t estimate. Thus, we complete the proof. □

Naturally, we have the following time-weighted estimates on $\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2$.

Lemma 3.3. *There exists a generic positive constant C only depending on $\bar{\rho}$, μ, $\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}$, and $\|\nabla \mathbf{u}_0\|_{H^1}$ such that for $j = 0, 1, 2$,*

$$\sup_{t\in[0,T_0]} (t^j\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2) + \int_0^{T_0} t^j\|\nabla \mathbf{u}_t\|_{L^2}^2 \mathrm{d}t \le C. \tag{3.16}$$

Proof. First, operating ∂_t to $(1.2)_2$ and $(1.2)_3$ yields that

$$\begin{aligned}\rho u_{1tt} + \rho\mathbf{u}\cdot\nabla u_{1t} - \partial_{22}u_{1t} + \partial_1 P_t &= -\rho_t u_{1t} - (\rho\mathbf{u})_t\cdot\nabla u_1, \\ \rho u_{2tt} + \rho\mathbf{u}\cdot\nabla u_{2t} - \partial_{11}u_{2t} + \partial_2 P_t &= -\rho_t u_{2t} - (\rho\mathbf{u})_t\cdot\nabla u_2.\end{aligned} \tag{3.17}$$

Multiplying the (3.17) by $\mathbf{u}_t$, we obtain after using integration by parts and $(1.2)_1$ that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} \rho|\mathbf{u}_t|^2 \mathrm{d}x + \int_{\mathbb{R}^2} |\partial_2 u_{1t}|^2 + |\partial_1 u_{2t}|^2 \mathrm{d}x \\ &= -2\int_{\mathbb{R}^2} \rho\mathbf{u}\cdot\nabla\mathbf{u}_t\cdot\mathbf{u}_t \mathrm{d}x - \int_{\mathbb{R}^2} \rho\mathbf{u}\cdot\nabla(\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t)\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^2} \rho\mathbf{u}_t\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t \mathrm{d}x \triangleq \sum_{i=1}^3 H_i.\end{aligned} \tag{3.18}$$

By employing (2.1) and (3.9), we proceed to estimate each term on the righthand side of (3.18) in the following:

$$\begin{aligned}|H_1| &\le 2\|\sqrt{\rho}\mathbf{u}\|_{L^4}\|\sqrt{\rho}\mathbf{u}_t\|_{L^4}\|\nabla\mathbf{u}_t\|_{L^2} \\ &\lesssim \|\nabla\mathbf{u}_t\|_{L^2}\|\sqrt{\rho}\mathbf{u}\|_{L^4}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac14}\|\mathbf{u}_t\|_{L^6}^{\frac34} \\ &\lesssim \|\nabla\mathbf{u}_t\|_{L^2}\|\sqrt{\rho}\mathbf{u}\|_{L^4}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac14}(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2} + \|\nabla\mathbf{u}_t\|_{L^2})^{\frac14}\|\nabla\mathbf{u}_t\|_{L^2}^{\frac12} \\ &\le \frac{1}{12}\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\|\sqrt{\rho}\mathbf{u}\|_{L^4}^4\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2,\end{aligned} \tag{3.19}$$

where we have used

$$\|\mathbf{u}_t\|_{L^2}^2 \le C(\tilde\rho)\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \|\rho-\tilde\rho\|_{L^2}^2\|\nabla\mathbf{u}_t\|_{L^2}^2\right) \\ \le C\left(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \|\nabla\mathbf{u}_t\|_{L^2}^2\right). \tag{3.20}$$

Similarly, we have

$$\begin{aligned}|H_2| &\le \int_{\mathbb{R}^2}\left(\rho|\mathbf{u}||\nabla\mathbf{u}|^2|\mathbf{u}_t| + \rho|\mathbf{u}|^2|\nabla^2\mathbf{u}||\mathbf{u}_t| + \rho|\mathbf{u}|^2|\nabla\mathbf{u}||\nabla\mathbf{u}_t|\right)dx \\ &\lesssim \|\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}\|_{L^4}^2\|\sqrt{\rho}\mathbf{u}_t\|_{L^4} + \|\mathbf{u}\|_{L^\infty}^2\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}_t\|_{L^2} + \|\mathbf{u}\|_{L^8}^2\|\nabla^2\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^4} \\ &\lesssim \|\mathbf{u}\|_{L^2}^{\frac12}\|\nabla\mathbf{u}\|_{L^2}^{\frac32}\|\nabla^2\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac12}\|\nabla\mathbf{u}_t\|_{L^2}^{\frac12} + \|\nabla\mathbf{u}_t\|_{L^2}\|\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}\|\nabla^2\mathbf{u}\|_{L^2} \\ &\quad + \|\mathbf{u}\|_{L^2}^{\frac12}\|\nabla\mathbf{u}\|_{L^2}^{\frac32}\|\nabla^2\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}_t\|_{L^2} \\ &\le \frac{1}{12}\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla^2\mathbf{u}\|_{L^2}^{\frac43}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac23} + C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla^2\mathbf{u}\|_{L^2}^2,\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}|H_3| &\lesssim \|\nabla\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 \\ &\lesssim \|\nabla\mathbf{u}\|_{L^2}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac12}(\|\sqrt{\rho}\mathbf{u}_t\|_{L^2} + \|\nabla\mathbf{u}_t\|_{L^2})^{\frac12}\|\nabla\mathbf{u}_t\|_{L^2} \\ &\le \frac16\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}^2\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2.\end{aligned} \tag{3.22}$$

Substituting (3.19), (3.21), and (3.22) into (3.18), gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\rho|\mathbf{u}_t|^2\mathrm{d}x + \int_{\mathbb{R}^2}|\partial_2 u_{1t}|^2 + |\partial_1 u_{2t}|^2\mathrm{d}x \\ \le C\|\nabla\mathbf{u}\|_{L^2}^2\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla^2\mathbf{u}\|_{L^2}^2. \tag{3.23}$$

From the compatibility condition (1.5), we define, for $j = 0, 1$,

$$\sqrt{\rho}\mathbf{u}_{jt}(x, t=0) = -\mathbf{g}_j - \sqrt{\rho_0}\mathbf{u}_0\cdot\nabla\mathbf{u}_{j0}. \tag{3.24}$$

By applying Grönwall's inequality along with (2.4) and (3.24), we can derive the case for $j = 0$ in (3.16).

Next, multiplying (3.23) by t and combining (3.9), it thus follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t\int_{\mathbb{R}^2}\rho|\mathbf{u}_t|^2\mathrm{d}x\right) + t\int_{\mathbb{R}^2}|\partial_2 u_{1t}|^2 + |\partial_1 u_{2t}|^2\mathrm{d}x \\ \le Ct\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}^2 + Ct\|\nabla\mathbf{u}\|_{L^2}^6 + \|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2. \tag{3.25}$$

By using Grönwall's inequality and combining with (3.5) leads to

$$\sup_{t\in[0,T_0]} t\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \int_0^{T_0} t\|\partial_2 u_{1t}\|_{L^2}^2 + t\|\partial_1 u_{2t}\|_{L^2}^2 dt \le C. \tag{3.26}$$

Similarly, we weigh (3.23) by t^2 again and then apply Grönwall's inequality to obtain the case for $j = 2$ in (3.16), thereby completing the proof. □

We will employ Lemmas 3.1–3.3 to establish bounds on the $L^1(0,T_0;L^\infty)$-norm of $\nabla\mathbf{u}$ and to estimate the gradient of the density.

Lemma 3.4. *There exists a generic positive constant C depending on $\bar\rho$, μ, $\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}$, and $\|\nabla\mathbf{u}_0\|_{H^1}$ such that $\forall q>2$*

$$\sup_{t\in[0,T_0]}\|\rho-\widetilde{\rho}\|_{H^1\cap W^{1,q}}+\int_0^{T_0}\|\nabla\mathbf{u}\|_{L^\infty}\mathrm{d}t\le C. \tag{3.27}$$

Proof. First, by using Lemma 2.4 and the Gagliardo-Nirenberg's inequality, we obtain the following:

$$\begin{aligned}
&\|\nabla^3\mathbf{u}\|_{L^2}+\|\nabla^2P\|_{L^2}\\
\lesssim\,&\|\nabla(\rho\mathbf{u}_t+\rho\mathbf{u}\cdot\nabla\mathbf{u})\|_{L^2}\\
\lesssim\,&\|\nabla\rho\mathbf{u}_t\|_{L^2}+\|\nabla\mathbf{u}_t\|_{L^2}+\|\nabla\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}+\|\nabla\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}+\|\mathbf{u}\cdot\nabla^2\mathbf{u}\|_{L^2}\\
\lesssim\,&\|\nabla\rho\|_{L^q}\|\mathbf{u}_t\|_{L^{\frac{2q}{q-2}}}+\|\nabla\mathbf{u}_t\|_{L^2}+\|\nabla\rho\|_{L^q}\|\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^{\frac{2q}{q-2}}}\\
&+\|\nabla\mathbf{u}\|_{L^4}^2+\|\mathbf{u}\|_{L^\infty}\|\nabla^2\mathbf{u}\|_{L^2}\\
\lesssim\,&\|\nabla\rho\|_{L^q}(\|\mathbf{u}_t\|_{L^2}^{\frac{q-2}{q}}\|\nabla\mathbf{u}_t\|_{L^2}^{\frac{2}{q}}+\|\mathbf{u}\|_{L^2}^{\frac12}\|\nabla\mathbf{u}\|_{L^2}^{\frac{q-2}{q}}\|\nabla^2\mathbf{u}\|_{L^2}^{\frac{4+q}{2q}})+\|\nabla\mathbf{u}_t\|_{L^2}\\
&+\|\nabla\mathbf{u}\|_{L^2}\|\nabla^2\mathbf{u}\|_{L^2}+\|\mathbf{u}\|_{L^2}^{\frac12}\|\nabla^2\mathbf{u}\|_{L^2}^{\frac32}.
\end{aligned} \tag{3.28}$$

Therefore, by the Gagliardo-Nirenberg inequality, we can obtain that for $\forall\ 0\le t\le T_0$,

$$\begin{aligned}
\|\nabla\mathbf{u}\|_{L^\infty}\le\,&C\|\nabla\mathbf{u}\|_{L^2}^{\frac12}\|\nabla^3\mathbf{u}\|_{L^2}^{\frac12}\\
\lesssim\,&\|\nabla\mathbf{u}\|_{L^2}^{\frac12}\|\nabla\rho\|_{L^q}^{\frac12}(\|\mathbf{u}_t\|_{L^2}^{\frac{q-2}{q}}\|\nabla\mathbf{u}_t\|_{L^2}^{\frac{2}{q}}+\|\mathbf{u}\|_{L^2}^{\frac12}\|\nabla\mathbf{u}\|_{L^2}^{\frac{q-2}{q}}\|\nabla^2\mathbf{u}\|_{L^2}^{\frac{4+q}{2q}})^{\frac12}\\
&+\|\nabla\mathbf{u}\|_{L^2}^{\frac12}\|\nabla\mathbf{u}_t\|_{L^2}^{\frac12}+\|\nabla\mathbf{u}\|_{L^2}\|\nabla^2\mathbf{u}\|_{L^2}^{\frac12}+\|\mathbf{u}\|_{L^2}^{\frac14}\|\nabla\mathbf{u}\|_{L^2}^{\frac12}\|\nabla^2\mathbf{u}\|_{L^2}^{\frac34}.
\end{aligned} \tag{3.29}$$

Since $(1.2)_1$, standard calculations show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho\|_{L^q}\le C\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\rho\|_{L^q}. \tag{3.30}$$

By substituting (3.29) into (3.30), we can derive

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho\|_{L^q}\lesssim\,&\|\nabla\rho\|_{L^q}^{\frac32}\|\nabla\mathbf{u}\|_{L^2}^{\frac12}(\|\mathbf{u}_t\|_{L^2}^{\frac{q-2}{q}}\|\nabla\mathbf{u}_t\|_{L^2}^{\frac{2}{q}}+\|\mathbf{u}\|_{L^2}^{\frac12}\|\nabla\mathbf{u}\|_{L^2}^{\frac{q-2}{q}}\|\nabla^2\mathbf{u}\|_{L^2}^{\frac{4+q}{2q}})^{\frac12}\\
&+(\|\nabla\mathbf{u}\|_{L^2}^{\frac12}\|\nabla\mathbf{u}_t\|_{L^2}^{\frac12}+\|\nabla\mathbf{u}\|_{L^2}\|\nabla^2\mathbf{u}\|_{L^2}^{\frac12}+\|\mathbf{u}\|_{L^2}^{\frac14}\|\nabla\mathbf{u}\|_{L^2}^{\frac12}\|\nabla^2\mathbf{u}\|_{L^2}^{\frac34})\|\nabla\rho\|_{L^q}\\
\lesssim\,&\|\nabla\rho\|_{L^q}^2+\alpha(t),
\end{aligned} \tag{3.31}$$

where by utilizing Lemmas 3.1–3.3, we can readily obtain that $\alpha(t)\in L^1(0,T_0)$ and denote $\int_0^{T_0}\alpha(t)dt\triangleq C_1$. Then there exists $T_0=\frac{1}{4(C_1+\|\nabla\rho_0\|_{L^q})}$, for which we have

$$\sup_{t\in[0,T_0]}\|\nabla\rho(t)\|_{L^q}\le 2(C_1+\|\nabla\rho_0\|_{L^q}), \tag{3.32}$$

and from (3.32) and (3.29), we also have

$$\int_0^{T_0}\|\nabla\mathbf{u}\|_{L^\infty}\mathrm{d}t\le C(T_0). \tag{3.33}$$

Similarly, by using $(1.2)_1$, (3.33), and Grönwall's inequality, we can derive that

$$\begin{aligned}\sup_{t\in[0,T_0]}\|\nabla\rho\|_{L^2} &\le \|\nabla\rho_0\|_{L^2}\exp\left\{C\int_0^{T_0}\|\nabla\mathbf{u}\|_{L^\infty}\mathrm{d}t\right\}\\ &\le C\|\nabla\rho_0\|_{L^2}.\end{aligned} \tag{3.34}$$

Combining this with (3.1), we thus complete the proof of Lemma 3.4. □

3.2. *The proof of Theorem 1.1*

Based on the a priori estimates in Section 3.1, we present the proof of the local well-posedness of the systems (1.2) and (1.3).

Proof. Following the standard procedure, we linearize the system and construct a sequence of approximate solutions via successive iterations. Let $\epsilon > 0$ be a small parameter and function $\psi_\epsilon(x) = \epsilon^{-2}\psi(x/\epsilon)$ with ψ satisfying

$$\psi \geqslant 0, \quad \psi \in C_0^\infty(\mathbb{R}^2) \quad \text{and} \quad \int_{\mathbb{R}^2}\psi dx = 1.$$

We define

$$\mathbf{u}_{0\epsilon} = \psi_\epsilon * \mathbf{u}_0, \quad \rho_{0\epsilon} = \frac{\psi_\epsilon * \rho_0 + \epsilon\widetilde{\rho}}{1+\epsilon}, \tag{3.35}$$

and satisfy $\mathrm{div}\mathbf{u}_{0\epsilon} = 0$, $\lim_{\epsilon\to 0}\|\mathbf{u}_{0\epsilon} - \mathbf{u}_0\|_{H^2(\mathbb{R}^2)} = 0$, and $\lim_{\epsilon\to 0}\|\rho_{0\epsilon} - \rho_0\|_{H^1\cap W^{1,q}(\mathbb{R}^2)} = 0$. Next, we construct the approximate system for (1.2) and (1.3) by $\mathbf{u}^{(0)} = \mathbf{u}_{0\epsilon}$ as follows:

$$\begin{cases}\rho_t^{(k)} + \mathbf{u}^{(k-1)}\cdot\nabla\rho^{(k)} = 0,\\ \rho^{(k)}\partial_t u_1^{(k)} + \rho^{(k)}\mathbf{u}^{(k-1)}\cdot\nabla u_1^{(k)} + \partial_1 P^{(k)} - \partial_2^2 u_1^{(k)} = 0,\\ \rho^{(k)}\partial_t u_2^{(k)} + \rho^{(k)}\mathbf{u}^{(k-1)}\cdot\nabla u_2^{(k)} + \partial_2 P^{(k)} - \partial_1^2 u_2^{(k)} = 0,\\ \nabla\cdot\mathbf{u}^{(k)} = 0.\end{cases} \tag{3.36}$$

By combining the theory of linear evolution equations and Lemma 2.3, we can obtain the existence and uniqueness of the solution $(\rho^{(k)}, \mathbf{u}^{(k)})$ such that $(\rho^{(k)}, \mathbf{u}^{(k)}) \in C^\infty(\mathbb{R}^2 \times (0, T_0])$. Moreover, in conjunction with the a priori estimates from the previous section, it follows that up to extraction of a subsequence, it converges to some limit functions $(\rho, \mathbf{u})$ in the following weak sense:

$$\begin{cases}\rho^{(k)} - \widetilde{\rho} \overset{*}{\rightharpoonup} \rho - \widetilde{\rho}, \quad \text{in } L^\infty(0,T_0;H^1\cap W^{1,q}),\\ \nabla\mathbf{u}^{(k)} \overset{*}{\rightharpoonup} \nabla\mathbf{u}, \quad \text{in } L^\infty(0,T_0;H^1),\\ \nabla\mathbf{u}_t^{(k)} \rightharpoonup \nabla\mathbf{u}_t, \quad \sqrt{\rho^k}\mathbf{u}_t^{(k)} \rightharpoonup \sqrt{\rho}\mathbf{u}_t, \quad \text{in } L^2(0,T_0;L^2),\\ \sqrt{\rho^k}\mathbf{u}_t^{(k)} \overset{*}{\rightharpoonup} \sqrt{\rho}\mathbf{u}_t, \quad \text{in } L^2(0,T_0;L^2),\\ \nabla P^{(k)} \overset{*}{\rightharpoonup} \nabla P, \quad \text{in } L^\infty(0,T_0;L^2),\\ \nabla P^{(k)} \rightharpoonup \nabla P, \quad \text{in } L^2(0,T_0;L^2).\end{cases} \tag{3.37}$$

Therefore, following a standard procedure as in [9], we can obtain that the limit functions $(\rho, \mathbf{u})$ are strong solutions to the system (1.2) and (1.3).

The uniqueness of local solutions is similar to the method in [22], and thus we omit the details here. For specific details, please refer to [22]. Therefore, the local well-posedness of the solutions to the systems (1.2) and (1.3) is proven.

□

4. Global well-posedness

Based on the local well-posedness established in the previous section, we present the proof of the global well-posedness of the systems (1.2) and (1.3) and also obtain the decay rate of the solution.

In this section, the constant C is used to denote a generic constant that depends solely on $\mu, \bar{\rho}, \|\nabla \mathbf{u}_0\|_{H^1}$, and $\|\rho_0 - \widetilde{\rho}\|_{H^1 \cap W^{1,q}}$.

Proposition 4.1. Let $M \triangleq \|\nabla \rho_0\|_{L^q}$, where $q \in (2,4)$. Under the condition of Theorem 1.2, there exists a constant η such that if $(\rho, \mathbf{u})$ is a smooth solution to the systems (1.2) and (1.3) on $\mathbb{R}^2 \times (0,T)$ and satisfying the following inequality holds:

$$\sup_{t \in [0,T]} \|\nabla \rho\|_{L^q} \le 4M, \tag{4.1}$$

then it can be deduced that

$$\sup_{t \in [0,T]} \|\nabla \rho\|_{L^q} \le 2M, \tag{4.2}$$

provided $\| \sqrt{\rho_0} \mathbf{u}_0\|_{L^2}^2 \le \eta$.

Proof. The core of the proof of this proposition lies in utilizing the smallness condition to obtain a uniform norm estimate for $\nabla \mathbf{u}$ in $L^1(0,T;L^\infty)$ that is independent of time T. The proof of this Proposition can be easily derived from Lemma 4.6. □

Lemma 4.2. *There exists a constant C that depends only on μ, $\bar{\rho}$, and $\|\nabla \mathbf{u}_0\|_{H^1}$ such that for all $t > 0$, the following inequality holds:*

$$\sup_{t \in [0,T]} \| \sqrt{\rho} \mathbf{u}\|_{L^2}^2 + \int_0^T \|\nabla \mathbf{u}\|_{L^2}^2 dt \le \eta, \tag{4.3}$$

$$\sup_{t \in [0,T]} (t\|\nabla \mathbf{u}\|_{L^2}^2) + \int_0^T t\| \sqrt{\rho} \mathbf{u}_t\|_{L^2}^2 dt \le C\eta, \tag{4.4}$$

$$\sup_{t \in [0,T]} (t^2\| \sqrt{\rho} \mathbf{u}_t\|_{L^2}^2) + \int_0^T t^2\|\nabla \mathbf{u}_t\|_{L^2}^2 \mathrm{d}t \le C\eta, \tag{4.5}$$

$$\sup_{t \in [0,T]} \|\nabla \mathbf{u}\|_{L^2}^2 + \int_0^T \| \sqrt{\rho} \mathbf{u}_t\|_{L^2}^2 dt \le C, \tag{4.6}$$

and

$$\sup_{t \in [0,T]} (t^j\| \sqrt{\rho} \mathbf{u}_t\|_{L^2}^2) + \int_0^T t^j\|\nabla \mathbf{u}_t\|_{L^2}^2 \mathrm{d}t \le C, \quad (j = 0, 1) \tag{4.7}$$

under the assumption that $\| \sqrt{\rho_0} \mathbf{u}_0\|_{L^2}^2 \le \eta$.

Proof. First, we can directly obtain (4.3) from (3.2), under the assumption of condition (1.7). Thus, we can obtain $\int_0^T \|\nabla \mathbf{u}\|_{L^2}^2 \, dt \leq \eta$. We then combine this with (3.15) to derive (4.4). For (4.5), we use (3.23) and (4.4) to get (4.5). Finally, since the estimates from (3.5) and (3.16) are uniformly bounded and independent of T, it follows that (4.6) and (4.7) hold for $\forall\ t > 0$. □

Next, we provide the decay estimate for $\|\sqrt{\rho}\mathbf{u}\|_{L^2}$ and $\|\mathbf{u}\|_{L^2}$. Before doing so, we first present the following auxiliary lemma.

Lemma 4.3. *For $\forall\ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $t > 0$, it holds that*

$$
\begin{aligned}
|\hat{\mathbf{u}}(\xi, t)| \leq\ & e^{-a\min\{|\xi_1|^2, |\xi_2|^2\}t} |\hat{\mathbf{u}}_0(\xi, t)| \\
& + C \int_0^t e^{-a\min\{|\xi_1|^2, |\xi_2|^2\}(t-\tau)} (|\xi| \|\mathbf{u}\|_{L^2}^2 + \|\sqrt{\rho}\mathbf{u}_t\|_{L^2} + \|\nabla \mathbf{u}_t\|_{L^2} + \|\nabla \mathbf{u}\|_{L^2}^2) d\tau,
\end{aligned}
\tag{4.8}
$$

where $a = \frac{1}{\widetilde{\rho}} > 0$.

Proof. First, rewrite Eqs $(1.2)_2$–$(1.2)_4$ as follows:

$$
\begin{cases}
u_{1t} - a\partial_2^2 u_1 + a\partial_1 P = -\mathbf{u} \cdot \nabla u_1 - a(\rho - \widetilde{\rho})\mathbf{u} \cdot \nabla u_1 + a(\rho - \widetilde{\rho}) u_{1t}, \\
u_{2t} - a\partial_1^2 u_2 + a\partial_2 P = -\mathbf{u} \cdot \nabla u_2 - a(\rho - \widetilde{\rho})\mathbf{u} \cdot \nabla u_2 + a(\rho - \widetilde{\rho}) u_{2t}, \\
\mathrm{div}\mathbf{u} = 0.
\end{cases}
\tag{4.9}
$$

Applying the Fourier transform to (4.9), we obtain

$$
\begin{cases}
\hat{u}_{1t} + a|\xi_2|^2 \hat{u}_1 = -ai\xi_1 \hat{P} - \mathcal{F}(\mathbf{u} \cdot \nabla u_1) - a\mathcal{F}((\rho - \widetilde{\rho})\mathbf{u} \cdot \nabla u_1) + a\mathcal{F}((\rho - \widetilde{\rho}) u_{1t}), \\
\hat{u}_{2t} + a|\xi_1|^2 \hat{u}_2 = -ai\xi_2 \hat{P} - \mathcal{F}(\mathbf{u} \cdot \nabla u_2) - a\mathcal{F}((\rho - \widetilde{\rho})\mathbf{u} \cdot \nabla u_2) + a\mathcal{F}((\rho - \widetilde{\rho}) u_{2t}), \\
\xi_1 \hat{u}_1 + \xi_2 \hat{u}_2 = 0.
\end{cases}
\tag{4.10}
$$

Taking the inner products of $(4.10)_1$ and $(4.10)_2$ with $\bar{\hat{u}}_1$ and $\bar{\hat{u}}_2$, respectively, conjugating $(4.10)_1$ and $(4.10)_2$ and then multiplying them by $\hat{u}_1$ and $\hat{u}_2$, respectively, and finally summing them up, we obtain

$$
\begin{aligned}
\partial_t |\hat{u}_1|^2 + 2a|\xi_2|^2 |\hat{u}_1|^2 =\ & -ai\xi_1 \hat{P} \bar{\hat{u}}_1 + ai\xi_1 \bar{\hat{P}} \hat{u}_1 - \mathcal{F}(\mathbf{u} \cdot \nabla \mathbf{u}_1) \cdot \bar{\hat{u}}_1 - \overline{\mathcal{F}(\mathbf{u} \cdot \nabla \mathbf{u}_1)} \cdot \hat{u}_1 \\
& - a\mathcal{F}((\rho - \widetilde{\rho})\mathbf{u} \cdot \nabla u_1) \cdot \bar{\hat{u}}_1 - a\overline{\mathcal{F}((\rho - \widetilde{\rho})\mathbf{u} \cdot \nabla u_1)} \cdot \hat{u}_1 \\
& + a\mathcal{F}((\rho - \widetilde{\rho}) u_{1t}) \cdot \bar{\hat{u}}_1 + a\overline{\mathcal{F}((\rho - \widetilde{\rho}) u_{1t})} \cdot \hat{u}_1.
\end{aligned}
\tag{4.11}
$$

$$
\begin{aligned}
\partial_t |\hat{u}_2|^2 + 2a|\xi_1|^2 |\hat{u}_2|^2 =\ & -ai\xi_2 \hat{P} \bar{\hat{u}}_2 + ai\xi_2 \bar{\hat{P}} \hat{u}_2 - \mathcal{F}(\mathbf{u} \cdot \nabla \mathbf{u}_2) \cdot \bar{\hat{u}}_2 - \overline{\mathcal{F}(\mathbf{u} \cdot \nabla \mathbf{u}_2)} \cdot \hat{u}_2 \\
& - a\mathcal{F}((\rho - \widetilde{\rho})\mathbf{u} \cdot \nabla u_2) \cdot \bar{\hat{u}}_2 - a\overline{\mathcal{F}((\rho - \widetilde{\rho})\mathbf{u} \cdot \nabla u_2)} \cdot \hat{u}_2 \\
& + a\mathcal{F}((\rho - \widetilde{\rho}) u_{2t}) \cdot \bar{\hat{u}}_2 + a\overline{\mathcal{F}((\rho - \widetilde{\rho}) u_{2t})} \cdot \hat{u}_2.
\end{aligned}
\tag{4.12}
$$

Adding (4.11) and (4.12) together and based on $(4.10)_3$, we obtain

$$
\begin{aligned}
& \partial_t |\hat{u}|^2 + 2a\min\{|\xi_1|^2, |\xi_2|^2\} |\hat{u}|^2 \\
& \leq -\mathcal{F}(\mathbf{u} \cdot \nabla \mathbf{u}_i) \cdot \bar{\hat{u}}_i - \overline{\mathcal{F}(\mathbf{u} \cdot \nabla \mathbf{u}_i)} \cdot \hat{u}_i \\
& - a\mathcal{F}((\rho - \widetilde{\rho})\mathbf{u} \cdot \nabla u_i) \cdot \bar{\hat{u}}_i - a\overline{\mathcal{F}((\rho - \widetilde{\rho})\mathbf{u} \cdot \nabla u_i)} \cdot \hat{u}_i \\
& + a\mathcal{F}((\rho - \widetilde{\rho}) u_{it}) \cdot \bar{\hat{u}}_i + a\overline{\mathcal{F}((\rho - \widetilde{\rho}) u_{it})} \cdot \hat{u}_i \triangleq \sum_{i=l}^{12} K_l.
\end{aligned}
\tag{4.13}
$$

By using the fact that the divergence of the velocity field is zero, we can obtain

$$\begin{aligned} K_1 + K_2 + K_3 + K_4 &\lesssim (|\mathcal{F}(\mathbf{u}\cdot\nabla\mathbf{u}_1) + |\mathcal{F}(\mathbf{u}\cdot\nabla\mathbf{u}_2))||\bar{\hat{u}}_1| \\ &\lesssim |\xi|\|\mathbf{u}\mathbf{u}\|_{L^1}|\hat{u}| \lesssim |\xi|\|\mathbf{u}\|_{L^2}^2|\hat{u}|. \end{aligned} \tag{4.14}$$

Similarly, we have

$$\begin{aligned} K_5 + K_6 + K_7 + K_8 &\lesssim (|\mathcal{F}((\rho-\widetilde{\rho})\mathbf{u}\cdot\nabla u_2)| + |\mathcal{F}((\rho-\widetilde{\rho})\mathbf{u}\cdot\nabla u_1)|)|\hat{u}| \\ &\lesssim \|\widetilde{\rho}-\rho\|_{L^2}\|\nabla\mathbf{u}\|_{L^4}\|\mathbf{u}\|_{L^4}|\hat{u}| \\ &\lesssim \|\nabla^2\mathbf{u}\|_{L^2}^{\frac12}\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{u}\|_{L^2}^{\frac12}|\hat{u}| \\ &\lesssim (\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac12} + \|\nabla\mathbf{u}\|_{L^2})\|\nabla\mathbf{u}\|_{L^2}|\hat{u}|, \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} K_9 + K_{10} + K_{11} + K_{12} &\lesssim (|\mathcal{F}((\rho-\widetilde{\rho})u_{2t})| + |\mathcal{F}((\rho-\widetilde{\rho})u_{1t})|)|\hat{u}| \\ &\lesssim \|\widetilde{\rho}-\rho\|_{L^2}\|\mathbf{u}_t\|_{L^2}|\hat{u}| \\ &\lesssim (\|\sqrt{\rho}\mathbf{u}_t\|_{L^2} + \|\nabla\mathbf{u}_t\|_{L^2})|\hat{u}|. \end{aligned} \tag{4.16}$$

Inserting the bounds of K_1 through K_{12} into (4.13), it leads to

$$\partial_t|\hat{u}|^2 + 2a\min\{|\xi_1|^2, |\xi_2|^2\}|\hat{u}|^2 \le C(|\xi|\|\mathbf{u}\|_{L^2}^2 + \|\sqrt{\rho}\mathbf{u}_t\|_{L^2} + \|\nabla\mathbf{u}_t\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2}^2)|\hat{u}|. \tag{4.17}$$

Combining this with Grönwall's inequality, we arrive at the desired result (4.8). □

Subsequently, we employ Schonbek's method [29] to establish the decay rates.

Lemma 4.4. *Let $\mathbf{u}_0 \in \dot{H}^{-\sigma}$ with $\sigma \in (\frac{q-2}{q}, \frac12]$. Under the condition of Theorem 1.2, there exists a $t_1 > 1$ such that for any $t > t_1$, the following holds:*

$$\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + \|\mathbf{u}\|_{L^2}^2 \le C(1+t)^{-\sigma+\varepsilon}, \tag{4.18}$$

where ε is a sufficiently small positive constant.

Proof. Let

$$S(t) = \{\xi \in \mathbb{R}^2 | |\xi| \le M(t)\}, \tag{4.19}$$

with $M(t) > 0$ to be chosen later. By invoking Plancherel's theorem and employing a frequency decomposition, it holds that

$$\begin{aligned} \|\nabla\mathbf{u}\|_{L^2}^2 &\ge \int_{|\xi|\ge M(t)} |\xi|^2|\hat{u}(\xi)|^2 d\xi \\ &\ge M^2(t)\int_{|\xi|\ge M(t)} |\hat{u}(\xi)|^2 d\xi \\ &= M^2(t)\|\mathbf{u}\|_{L^2}^2 - M^2(t)\int_{|\xi|\le M(t)} |\hat{u}(\xi)|^2 d\xi \\ &= aM^2(t)\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + aM^2(t)\|(\tilde{\rho}-\rho)|\mathbf{u}|^2\|_{L^1} - M^2(t)\int_{|\xi|\le M(t)} |\hat{u}(\xi)|^2 d\xi. \end{aligned} \tag{4.20}$$

Substituting (4.20) into (3.2) and utilizing (4.8), we derive

$$
\begin{aligned}
&\frac{d}{dt}\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + aM^2(t)\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 \\
&\leq CM^2(t)\int_{|\xi|\leq M(t)} |\hat{u}(\xi)|^2 d\xi + CM^2(t)\|(\tilde{\rho}-\rho)|\mathbf{u}|^2\|_{L^1} \\
&\leq CM^2(t)\int_{|\xi|\leq M(t)} e^{-2a\min\{|\xi_1|^2,|\xi_2|^2\}t}|\hat{u}_0(\xi)|^2 d\xi + CM^2(t)\|\tilde{\rho}-\rho\|_{L^2}\|\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2} \\
&\quad + CM^2(t)\int_{|\xi|\leq M(t)}\Big(\int_0^t e^{-a\min\{|\xi_1|^2,|\xi_2|^2\}(t-\tau)}(|\xi|\|\mathbf{u}\|_{L^2}^2 + \|\sqrt{\rho}\mathbf{u}_t\|_{L^2})d\tau\Big)^2 d\xi \\
&\quad + CM^2(t)\int_{|\xi|\leq M(t)}\Big(\int_0^t e^{-a\min\{|\xi_1|^2,|\xi_2|^2\}(t-\tau)}(\|\nabla\mathbf{u}_t\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2}^2)d\tau\Big)^2 d\xi.
\end{aligned}
\tag{4.21}
$$

We first take

$$
M(t) = \Big(\frac{m}{a(e+t)\log(e+t)}\Big)^{\frac{1}{2}},
$$

with $m > 1$ in (4.19), and subsequently multiply (4.21) by $\log^m(e+t)$ to derive the following

$$
\begin{aligned}
&\frac{d}{dt}(\log^m(e+t)\|\sqrt{\rho}\mathbf{u}(t)\|_{L^2}^2) \\
&\lesssim \frac{\log^{m-1-\sigma}(e+t)}{(e+t)^{1+\sigma}}\|\mathbf{u}_0\|_{\dot{H}^{-\sigma}}^2 + \frac{\log^{m-3}(e+t)}{(e+t)^3}\Big(\int_0^t \|\mathbf{u}\|_{L^2}^2 d\tau\Big)^2 \\
&\quad + \frac{\log^{m-1}(e+t)}{(e+t)}\|\sqrt{\rho}\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2} + \frac{\log^{m-1}(e+t)}{(e+t)}\|\nabla\mathbf{u}\|_{L^2}^2 \\
&\quad + \frac{\log^{m-2}(e+t)}{(e+t)^2}\Big(\int_0^t \|\sqrt{\rho}\mathbf{u}_t\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{u}_t\|_{L^2} d\tau\Big)^2 \\
&\lesssim \frac{\log^{m-1-\sigma}(e+t)}{(e+t)^{1+\sigma}}\|\mathbf{u}_0\|_{\dot{H}^{-\sigma}}^2 + \frac{\log^{m-3}(e+t)}{(e+t)}\|\mathbf{u}_0\|_{L^2}^4 \\
&\quad + \log^m(e+t)\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2\frac{\log^{-1-\varepsilon}(e+t)}{(e+t)} + \frac{\log^{m-1+\varepsilon}(e+t)}{(e+t)^2} \\
&\quad + \frac{\log^{m-1}(e+t)}{(e+t)^2} + \frac{\log^{m-2}(e+t)}{(e+t)}\Big(\int_0^t \|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 + \|\nabla\mathbf{u}\|_{L^2}^4 + \|\nabla\mathbf{u}_t\|_{L^2}^2 d\tau\Big).
\end{aligned}
\tag{4.22}
$$

By employing Grönwall's inequality over $(0,t)$ and using (3.2) and (3.5), we can obtain

$$
\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 \leq C\log^{-\sigma}(e+t). \tag{4.23}
$$

Next, we update the decay rate by setting

$$
M(t) = \Big(\frac{m}{a(e+t)}\Big)^{\frac{1}{2}},
$$

with $m > 0$ in (4.19), and then multiplying (4.21) by $(e+t)^m$, we obtain the following

$$\begin{aligned}
&\frac{d}{dt}((e+t)^m\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2) \\
&\le C(e+t)^{m-1-\sigma}\|\mathbf{u}_0\|_{\dot{H}^{-\sigma}}^2 + C(e+t)^{m-2}\int_0^t \|\mathbf{u}\|_{L^2}^4 d\tau \\
&+ (e+t)^m\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 \frac{\log^{-1-\varepsilon}(e+t)}{e+t} + (e+t)^{m-1}\log^{1+\varepsilon}(e+t)\|\nabla\mathbf{u}\|_{L^2}^2 \\
&+ C(e+t)^{m-1}\|\nabla\mathbf{u}\|_{L^2}^2 + C(e+t)^{m-2}(\int_0^t \|\rho\mathbf{u}_t\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{u}_t\|_{L^2} d\tau)^2.
\end{aligned} \tag{4.24}$$

Using Grönwall's inequality and taking $m = \sigma - \varepsilon$ leads to

$$\begin{aligned}
&(e+t)^{\sigma-\varepsilon}\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 \\
&\le C(\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2 + \|\mathbf{u}_0\|_{\dot{H}^{-\sigma}}^2) + C\int_0^t (e+s)^{\sigma-2-\varepsilon}\int_0^s \|\mathbf{u}\|_{L^2}^4 d\tau ds + C(e+t)^{\sigma-\frac{1}{2}-\frac{1}{2}\varepsilon} \\
&+ C\int_0^t (e+s)^{\sigma-2-\varepsilon}(\int_0^s \|\rho\mathbf{u}_t\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{u}_t\|_{L^2} d\tau)^2 ds \\
&\lesssim 1 + (e+t)^{\sigma-1-\varepsilon}\int_0^t \log^{-\sigma}(e+\tau)(e+\tau)^{-\sigma+\varepsilon}(e+\tau)^{\sigma-\varepsilon}\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 d\tau + (e+t)^{\sigma-\frac{1}{2}-\frac{1}{2}\varepsilon} \\
&+ \int_0^t (e+s)^{\sigma-2-\varepsilon}s^{\varepsilon} ds(\int_0^t \tau^{1-\varepsilon}\|\rho\mathbf{u}_t\|_{L^2}^2 d\tau) + \int_0^t (e+s)^{\sigma-2-\varepsilon} ds \\
&+ \int_0^t (e+s)^{\sigma-2-\varepsilon}s^{\varepsilon} ds(\int_0^t \tau^{1-\varepsilon}\|\nabla\mathbf{u}_t\|_{L^2}^2 d\tau) \\
&\le C + C[(e+t)^{\sigma-1-\varepsilon} + \log^{-\sigma}(e+t)]\sup_{\tau\in[0,t]}\{(e+\tau)^{\sigma-\varepsilon}\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2\} + C(e+t)^{\sigma-\frac{1}{2}-\frac{1}{2}\varepsilon},
\end{aligned} \tag{4.25}$$

where ε is sufficiently small and we have used $\int_0^T (1+t)\|\nabla\mathbf{u}_t\|_{L^2}^2 dt + \int_0^T (1+t)\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2 dt \le C$ and $\log^2(e+t) \le (e+t)$, $\forall\, t \ge 0$.

Therefore, $\exists\, t_1$ s.t.

$$C[(e+t_1)^{\sigma-1-\varepsilon} + \log^{-\sigma}(e+t_1)] \le \frac{5}{6}, \tag{4.26}$$

and we have

$$\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 \le C(1+t)^{-\sigma+\varepsilon}. \tag{4.27}$$

Moreover, by using (3.10), we can obtain

$$\|\mathbf{u}\|_{L^2}^2 \le C(1+t)^{-\sigma+\varepsilon}. \tag{4.28}$$

Thus, the proof of Lemma 4.4 is completed. □

Building on that, we update the decay estimate for the derivatives of the velocity field on the interval $[t_1, T]$.

Lemma 4.5. *Under the condition of Theorem 1.2, it holds that for all $t_1 \leq t \leq T$,*

$$\sup_{t\in[t_1,T]}(t^{\sigma-2\varepsilon}\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2)+\int_{t_1}^{T}t^{\sigma-2\varepsilon}\|\nabla\mathbf{u}\|_{L^2}^2dt\leq C. \tag{4.29}$$

$$\sup_{t\in[t_1,T]}(t^{1+\sigma-2\varepsilon}\|\nabla\mathbf{u}\|_{L^2}^2)+\int_{t_1}^{T}t^{1+\sigma-2\varepsilon}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2dt\leq C. \tag{4.30}$$

$$\sup_{t\in[t_1,T]}(t^{2+\sigma-2\varepsilon}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2)+\int_{t_1}^{T}t^{2+\sigma-2\varepsilon}\|\nabla\mathbf{u}_t\|_{L^2}^2\mathrm{d}t\leq C. \tag{4.31}$$

where t_1, σ are defined in Lemma 4.4 and C is a constant independent of T.

Proof. Applying $t^{\sigma-2\varepsilon}$ to both sides of (3.3), we obtain

$$\frac{d}{dt}(t^{\sigma-2\varepsilon}\int_{\mathbb{R}^2}\rho|\mathbf{u}|^2\,dx)+t^{\sigma-2\varepsilon}\int_{\mathbb{R}^2}|\nabla\mathbf{u}|^2\,dx\leq Ct^{\sigma-1-2\varepsilon}\int_{\mathbb{R}^2}\rho|\mathbf{u}|^2\,dx. \tag{4.32}$$

Then, integrating (4.32) over the interval $[t_1, T]$, we can deduce that

$$\begin{aligned}\sup_{t\in[t_1,T]}(t^{\sigma-2\varepsilon}\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2)+\int_{t_1}^{T}t^{\sigma-2\varepsilon}\|\nabla\mathbf{u}\|_{L^2}^2\,dt&\leq t_1^{\sigma-2\varepsilon}\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+C\int_{t_1}^{T}t^{\sigma-1-2\varepsilon}\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2\,dt\\&\leq C+C\int_{t_1}^{T}t^{-1-\varepsilon}\,dt\leq C,\end{aligned} \tag{4.33}$$

which gives (4.29). Similarly, multiplying (3.8) by $t^{1+\sigma-2\varepsilon}$, we arrive at the following:

$$\begin{aligned}&\frac{d}{dt}(t^{1+\sigma-2\varepsilon}\|\nabla\mathbf{u}\|_{L^2}^2)+t^{1+\sigma-2\varepsilon}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2\\&\leq Ct^{1+\sigma-2\varepsilon}\|\sqrt{\rho}\mathbf{u}\|_{L^4}^4\|\nabla\mathbf{u}\|_{L^2}^2+Ct^{1+\sigma-2\varepsilon}\|\nabla\mathbf{u}\|_{L^2}^4+Ct^{\sigma-2\varepsilon}\|\nabla\mathbf{u}\|_{L^2}^2\\&\leq Ct^{1+\sigma-2\varepsilon}\|\nabla\mathbf{u}\|_{L^2}^4+Ct^{\sigma-2\varepsilon}\|\nabla\mathbf{u}\|_{L^2}^2.\end{aligned} \tag{4.34}$$

Then, integrating with respect to t and employing (4.33), we infer that

$$\begin{aligned}&\sup_{t\in[t_1,T]}(t^{1+\sigma-2\varepsilon}\|\nabla\mathbf{u}\|_{L^2}^2)+\int_{t_1}^{T}t^{1+\sigma-2\varepsilon}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2dt\\&\leq\exp\{C\int_{t_1}^{T}\|\nabla\mathbf{u}\|_{L^2}^2dt\}\Big(t_1^{1+\sigma-2\varepsilon}\|\nabla\mathbf{u}\|_{L^2}^2+C\int_{t_1}^{T}t^{\sigma-2\varepsilon}\|\nabla\mathbf{u}\|_{L^2}^2dt\Big)\leq C.\end{aligned} \tag{4.35}$$

Likewise, from (3.23), we obtain the following:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(t^{2+\sigma-2\varepsilon}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2)+t^{2+\sigma-2\varepsilon}\|\nabla\mathbf{u}_t\|_{L^2}^2\\&\leq Ct^{2+\sigma-2\varepsilon}\|\nabla\mathbf{u}\|_{L^2}^2\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+Ct^{2+\sigma-2\varepsilon}\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla^2\mathbf{u}\|_{L^2}^2\\&\quad+Ct^{2+\sigma-2\varepsilon}\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla^2\mathbf{u}\|_{L^2}^{\frac{4}{3}}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac{2}{3}}+Ct^{1+\sigma-2\varepsilon}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2,\end{aligned} \tag{4.36}$$

and integrating over $[t_1, T]$ in (4.36),

$$\begin{aligned}
&\sup_{t\in[t_1,T]}(t^{2+\sigma-2\varepsilon}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2)+\int_{t_1}^T t^{2+\sigma-2\varepsilon}\|\nabla\mathbf{u}_t\|_{L^2}^2dt\\
\le{}& \exp\{C\int_{t_1}^T\|\nabla\mathbf{u}\|_{L^2}^2dt\}(t_1^{2+\sigma-2\varepsilon}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+C\int_{t_1}^T t^{2+\sigma-2\varepsilon}\|\nabla\mathbf{u}\|_{L^2}^2\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2dt\\
&+C\int_{t_1}^T t^{2+\sigma-2\varepsilon}\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla^2\mathbf{u}\|_{L^2}^2dt+\int_{t_1}^T t^{1+\sigma-2\varepsilon}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2dt)\le C.
\end{aligned}\tag{4.37}$$

Thus, we complete the proof of the Lemma 4.5. □

Lemma 4.6. *Under the condition of Proposition 4.1, there exists a generic positive constant C depending only on $\bar{\rho}, \mu, \|\nabla\mathbf{u}_0\|_{H^1}$, and $\|\rho_0-\widetilde{\rho}\|_{H^1\cap W^{1,q}}$ such that $\forall\ q\in(2,4)$ and $\sigma\in(\frac{q-2}{q},\frac{1}{2}]$. The following formulas hold:*

$$\int_0^T\|\nabla\mathbf{u}\|_{L^\infty}\mathrm{d}t\le C\eta^{\frac14},\tag{4.38}$$

$$\sup_{t\in[0,T]}\|\nabla\rho\|_{L^q}\le 2M,\tag{4.39}$$

$$\sup_{t\in[0,T]}\|\rho-\widetilde{\rho}\|_{H^1\cap W^{1,q}}\le C,\tag{4.40}$$

provided that $\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2\le\eta$.

Proof. We can derive from (3.29) and (3.20) that

$$\begin{aligned}
\int_0^T\|\nabla\mathbf{u}\|_{L^\infty}dt\lesssim{}&\int_0^T\|\nabla\mathbf{u}\|_{L^2}^{\frac12}\|\nabla\rho\|_{L^q}^{\frac12}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^{\frac{q-2}{2q}}\|\nabla\mathbf{u}_t\|_{L^2}^{\frac1q}dt+\int_0^T\|\nabla\mathbf{u}\|_{L^2}^{\frac12}\|\nabla\mathbf{u}_t\|_{L^2}^{\frac12}dt\\
&+\int_0^T\|\nabla\mathbf{u}\|_{L^2}^{\frac12}\|\nabla\rho\|_{L^q}^{\frac12}\|\nabla\mathbf{u}_t\|_{L^2}^{\frac12}dt\\
&+\int_0^T\|\nabla\mathbf{u}\|_{L^2}^{\frac12}\|\nabla\rho\|_{L^q}^{\frac12}\|\mathbf{u}\|_{L^2}^{\frac14}\|\nabla\mathbf{u}\|_{L^2}^{\frac{q-2}{2q}}\|\nabla^2\mathbf{u}\|_{L^2}^{\frac{4+q}{4q}}dt\\
&+\int_0^T\|\nabla\mathbf{u}\|_{L^2}\|\nabla^2\mathbf{u}\|_{L^2}^{\frac12}dt+\int_0^T\|\mathbf{u}\|_{L^2}^{\frac14}\|\nabla\mathbf{u}\|_{L^2}^{\frac12}\|\nabla^2\mathbf{u}\|_{L^2}^{\frac34}dt\\
\triangleq{}&P_1+P_2+P_3+P_4+P_5+P_6.
\end{aligned}\tag{4.41}$$

Next, we estimate P_1 through P_6 in sequence. We utilize (4.1) and Lemma 4.5, thereby obtaining

$$\begin{aligned}
P_1\lesssim{}&\int_0^{t_1}\|\nabla\mathbf{u}\|_{L^2}^{\frac12}\|\sqrt{\rho}\mathbf{u}_s\|_{L^2}^{\frac{q-2}{2q}}\|\nabla\mathbf{u}_s\|_{L^2}^{\frac1q}ds+\int_{t_1}^T\|\nabla\mathbf{u}\|_{L^2}^{\frac12}\|\sqrt{\rho}\mathbf{u}_s\|_{L^2}^{\frac{q-2}{2q}}\|\nabla\mathbf{u}_s\|_{L^2}^{\frac1q}ds\\
\lesssim{}&(\int_0^{t_1}\|\nabla\mathbf{u}\|_{L^2}^2ds)^{\frac14}(\int_0^{t_1}\|\sqrt{\rho}\mathbf{u}_s\|_{L^2}^2ds)^{\frac{q-2}{4q}}(\int_0^{t_1}\|\nabla\mathbf{u}_s\|_{L^2}^2ds)^{\frac{1}{2q}}\\
&+\sup_{s\in[t_1,T]}(s\|\nabla\mathbf{u}\|_{L^2}^2)^{\frac14}\sup_{s\in[t_1,T]}(s^{2+\sigma-2\varepsilon}\|\sqrt{\rho}\mathbf{u}_s\|_{L^2}^2)^{\frac{q-2}{4q}}\\
&\cdot(\int_{t_1}^T s^{\sigma+2-2\varepsilon}\|\nabla\mathbf{u}_s\|_{L^2}^2ds)^{\frac{1}{2q}}(\int_{t_1}^T s^{-(\frac34+\frac{\sigma}{4}-\frac{\varepsilon}{2})\frac{2q}{2q-1}}ds)^{\frac{2q-1}{2q}}\le C\eta^{\frac14},
\end{aligned}\tag{4.42}$$

where $\sigma > \frac{q-2}{q} + 2\varepsilon$ ensures that the last integral is integrable and ε is a sufficiently small positive number.

Analogously, we estimate the following terms by using Lemmas 4.2 and 4.5:

$$\begin{aligned} P_2 + P_3 &\le \int_0^{t_1} \|\nabla \mathbf{u}\|_{L^2}^{\frac{1}{2}} \|\nabla \mathbf{u}_s\|_{L^2}^{\frac{1}{2}} ds + \int_{t_1}^{T} \|\nabla \mathbf{u}\|_{L^2}^{\frac{1}{2}} \|\nabla \mathbf{u}_s\|_{L^2}^{\frac{1}{2}} ds \\ &\lesssim \Big(\int_0^{t_1} \|\nabla \mathbf{u}\|_{L^2}^2 ds\Big)^{\frac{1}{4}} \Big(\int_0^{t_1} \|\nabla \mathbf{u}_s\|_{L^2}^2 ds\Big)^{\frac{1}{4}} \\ &\quad + \sup_{s\in[t_1,T]} (s\|\nabla \mathbf{u}\|_{L^2}^2)^{\frac{1}{4}} \Big(\int_{t_1}^{T} s^{2+\sigma-2\varepsilon} \|\nabla \mathbf{u}_s\|_{L^2}^2 ds\Big)^{\frac{1}{4}} \Big(\int_{t_1}^{T} s^{-1-\frac{\sigma}{3}+\frac{2\varepsilon}{3}} ds\Big)^{\frac{3}{4}} \\ &\le C\eta^{\frac{1}{4}}, \end{aligned} \tag{4.43}$$

and

$$\begin{aligned} P_4 &\le \int_0^{t_1} \|\nabla \mathbf{u}\|_{L^2}^{\frac{q-1}{q}} \|\nabla^2 \mathbf{u}\|_{L^2}^{\frac{4+q}{4q}} ds + \int_{t_1}^{T} \|\nabla \mathbf{u}\|_{L^2}^{\frac{q-1}{q}} \|\nabla^2 \mathbf{u}\|_{L^2}^{\frac{4+q}{4q}} \|\mathbf{u}\|_{L^2}^{\frac{1}{4}} ds \\ &\lesssim \Big(\int_0^{t_1} \|\nabla \mathbf{u}\|_{L^2}^2 ds\Big)^{\frac{q-1}{2q}} \Big(\int_0^{t_1} \|\nabla^2 \mathbf{u}\|_{L^2}^2 ds\Big)^{\frac{4+q}{8q}} + \sup_{s\in[t_1,T]} (s\|\nabla \mathbf{u}\|_{L^2}^2)^{\frac{q-1}{2q}} \\ &\quad \cdot \Big(\sup_{s\in[t_1,T]} s^{\sigma-\varepsilon} \|\mathbf{u}\|_{L^2}^2\Big)^{\frac{1}{8}} \Big(\sup_{s\in[t_1,T]} s^{2+\sigma-2\varepsilon} \|\nabla^2 \mathbf{u}\|_{L^2}^2\Big)^{\frac{4+q}{8q}} \int_{t_1}^{T} s^{-\frac{3q+2}{4q}-\frac{2+q}{4q}\sigma+\frac{8+3q}{8q}\varepsilon} ds \\ &\le C\eta^{\frac{q-1}{2q}}, \end{aligned} \tag{4.44}$$

where we used $\sigma \in (\frac{q-2}{q}, \frac{1}{2}]$.

$$\begin{aligned} P_5 &\le \int_0^{t_1} \|\nabla \mathbf{u}\|_{L^2} \|\nabla^2 \mathbf{u}\|_{L^2}^{\frac{1}{2}} dt + \int_{t_1}^{T} \|\nabla \mathbf{u}\|_{L^2} \|\nabla^2 \mathbf{u}\|_{L^2}^{\frac{1}{2}} ds \\ &\lesssim \Big(\int_0^{t_1} \|\nabla \mathbf{u}\|_{L^2}^2 ds\Big)^{\frac{1}{2}} \Big(\int_0^{t_1} \|\nabla^2 \mathbf{u}\|_{L^2}^2 ds\Big)^{\frac{1}{4}} \\ &\quad + \sup_{t\in[t_1,T]} (s\|\nabla \mathbf{u}\|_{L^2}^2)^{\frac{1}{2}} \sup_{s\in[t_1,T]} (s^{2+\sigma-2\varepsilon} \|\nabla^2 \mathbf{u}\|_{L^2}^2)^{\frac{1}{4}} \int_{t_1}^{T} s^{-1-\frac{\sigma}{4}+\frac{\varepsilon}{2}} ds \\ &\le C\eta^{\frac{1}{2}}, \end{aligned} \tag{4.45}$$

and

$$P_6 \le C\eta^{\frac{1}{4}}. \tag{4.46}$$

Therefore, by combining (4.42) through (4.46), and substituting into (4.41), we obtain (4.38).

By combining (3.30) and (4.38), we can obtain the following:

$$\|\nabla \rho(t)\|_{L^q} \le 2\|\nabla \rho_0\|_{L^q}, \tag{4.47}$$

assuming that

$$\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2 \le \eta \triangleq (C^{-1}\ln 2)^4.$$

Finally, we obtain (4.40) from (3.34) and (3.1). □

Proof of Theorem 1.2. Based on the a priori estimates established in Lemmas 4.2–4.6, we can derive the existence and uniqueness of a global strong solution to the systems (1.2) and (1.3) through a standard procedure, the details of which can be found in [31]. To avoid redundancy, we omit the detailed proof here. Moreover, the decay rate of the velocity field can be readily obtained from Lemmas 4.4 and 4.5. □

Use of AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares that there is no conflict of interest.

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