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Research article

Global existence of mild solutions for impulsive $\psi-$Caputo fractional parabolic equations

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Abstract: This paper investigated a class of nonlinear $\psi-$Caputo fractional parabolic equations with impulsive. We reformulated the fractional parabolic equations into abstract evolution equations. By using the nonlinear analysis method and fixed point theorems, we obtained the existence and uniqueness of the global of mild solutions for the problem.

Keywords: $\psi-$Caputo fractional derivative; impulsive; global existence; mild solution

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a suffciently smooth boundary $\partial\Omega$. Consider the following $\psi-$Caputo fractional parabolic equations with impulsive:

$$\begin{cases} {}^cD_t^{\alpha,\psi} x(z,t) + A(z,D)x(z,t) = g(z,t,x(z,t)), \quad z \in \Omega, \\ \qquad t \in [0,\ b],\ t \neq t_i, \\ \Delta x(z,t_i) = I_i(x(z,t_i)), \quad z \in \Omega,\ i = 1,2,\cdots,p, \\ x(z,0) = \varphi(z), \quad z \in \Omega, \end{cases} \tag{1.1}$$

where ${}^cD_t^{\alpha,\psi}$ is the $\psi-$Caputo fractional derivatives of order $\frac{1}{2} < \alpha \leq 1$, $g : \overline{\Omega} \times [0,\ b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $J = [0,\ b]$, $0 = t_0 < t_1 < t_2 < \cdots < t_p < b = t_{p+1}$, I_i is an impulsive function, $i = 1,2,\cdots,p$, $\Delta x(z,t_i) = x(z,t_i^+) - x(z,t_i^-)$ for $z \in \Omega$, $x(z,t_i^+)$, and $x(z,t_i^-)$ denotes the right and the left limit of t at t_i, respectively. $\varphi \in L^2(\Omega)$, $A(z,D)$ is the following strongly elliptic differential operator of second-order:

$$A(z,D)x = -\sum_{i,j=1}^{N} \frac{\partial}{\partial z_i}\Big(a_{ij}(z)\frac{\partial x}{\partial z_j}\Big),$$

where the coefficients $a_{ij}(z) = a_{ji}(z)$, and there exists a constant $\upsilon > 0$, such that

$$\sum_{i,j=1}^{n} a_{ij}(z)\vartheta_i\vartheta_j \geq \upsilon|\vartheta|^2, \quad \forall\ \vartheta = (\vartheta_1, \vartheta_2, \cdots, \vartheta_n) \in \mathbb{R}^N,\ z \in \overline{\Omega}.$$

It is widely known that fractional differential equations are found to be better tools in the modeling of many phenomena in various fields of science and engineering than their corresponding integer-order counterparts. Many researchers have paid more attention to the theory of ordinary and partial differential equations involving fractional derivatives; see the monographs of Podlubny [1], Kilbas et al. [2] and Miller et al. [3]. The existence theory of solutions for the Cauchy problem of fractional evolution equations has been investigated by many authors [4–6]. Additionally, recent literature [7–9] has employed numerical simulation methods to study several classes of fractional differential equations, yielding some meaningful results. To enhance the applicability of fractional calculus, Almeida [10] developed the $\psi-$Caputo fractional derivative in 2017, which relies on an increasing function ψ. The $\psi-$Caputo fractional derivative is indispensable in systems with complex memory kernels, such as biological transport, heterogeneous materials, or financial time series. Its mathematical structure inherently satisfies physical conservation laws, making it the preferred method for multi-physics field systems with historical dependence (see Definitions 2.1–2.3 and Remark 2.1 in Section 2). Differential equations with the $\psi-$Caputo derivative were studied in [11–13]. The authors of [14–16] have also been concerned with $\psi-$Caputo fractional abstract evolution equations. In 2022, Liang et al. [14] analyzed a class of integro-differential equations involving the $\psi-$Caputo fractional derivative in Banach spaces. By employing generalized Laplace transform techniques, they derived explicit expressions for mild solutions. In 2023, Ding et al. [15] proved the existence and finite-approximate controllability of mild solutions for impulsive evolution equations involving the $\psi-$Caputo fractional derivative by using the fixed point theorem and multiple approximation techniques.

The fractional parabolic equations as a generalization of the integer order parabolic equations has been extensively studied. In [17], the authors investigated the null-controllability of a parabolic equation involving fractional powers of Laplace operators. In 2021, [18] studied the boundedness and Hölder continuity of local weak solutions to the fractional parabolic p-Laplace operator of $s-$order. In recent literatures [19–21], the authors investigated the existence and uniqueness of solutions for several classes of fractional parabolic equations. Huang et al. [22] studied the existence and approximate controllability of mild solutions for a class of fractional parabolic equations by utilizing the theory of semigroup of the linear operator and fixed point theorem. Tuan et al. [23] established the local well-posedness theory for a class of initial boundary value problems of nonlinear Caputo time-fractional pseudo-parabolic equations with fractional Laplacian. Gal et al. [24] established the existence of finite-dimensional global attractors, and also derived basic conditions for blow-up for a class of semilinear parabolic and elliptic problems with fractional dynamic boundary conditions.

The $\psi-$Caputo fractional parabolic equation provides a unified framework for the nonlocal time-varying dynamics of complex systems. Its applications, ranging from drug transport at the nanoscale to groundwater pollution prediction at the kilometer scale, have promoted the deep integration of computational mathematics and engineering physics, demonstrating irreplaceable value in solving multi-physics field coupling and historical path-dependence problems. So far as is known to the authors, the study of $\psi-$Caputo fractional parabolic equations with impulsive has not been carried out. Motivated by above results, the subject of this paper is to study the blowup and global existence of mild solutions to the parabolic equations (1.1). For this purpose, we suppose that the following hypotheses are satisfied:

(H1) $\psi \in C^2(J, \mathbb{R})$ and $\psi'(t) > 0$, $\psi''(t) \geq 0$ for $t \in J$.

(H2) The nonlinear function $g : \overline{\Omega} \times J \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $\phi \in L^2(\Omega \times J)$ such that

$$|g(z, t, \omega)| \leq \phi(z,\ t), \quad (z, t) \in \Omega \times J, \ \ \omega \in \mathbb{R}.$$

(H3) Let $I_i : \mathbb{R} \to \mathbb{R}$ be a continuous function for every $i = 1, 2, \cdots, p$, and there exist constants $\kappa_1 > 0$ and $\kappa_2 > 0$ such that

$$|I_i(\omega) - I_i(\nu)| \leq \kappa_1 |\omega - \nu|, \qquad |I_i(\omega)| \leq \kappa_2 |\omega|$$

for any $\omega,\ \nu \in \mathbb{R},\ i = 1, 2, \cdots, p$.

Under the above assumptions we obtain the following results:

Theorem 1.1. *Let assumptions (H1)–(H3) be satisfied. Then, for any $\varphi \in L^2(\Omega)$, there exists a constant $0 < \tau < b$ such that the parabolic problem (1.1) has a mild solution $x(z,\ t)$ in $[0,\ \tau]$ for every $z \in \Omega$ provided that $p\kappa_1 < 1$.*

Theorem 1.2. *Assuming the conditions of Theorem 1.1 are satisfied, suppose $g : \overline{\Omega} \times [0,\ b] \times \mathbb{R} \to \mathbb{R}(\ 0 < b \leq \infty)$ is continuous and maps bounded subsets of $\overline{\Omega} \times [0,\ b] \times \mathbb{R}$ into bounded subsets of $\mathbb{R}$. Then, for each $\varphi \in L^2(\Omega)$, the parabolic problem (1.1) admits a mild solution $x(z,\ t)$ on a maximal interval $[0,\ t_{\max})$ for all $z \in \Omega$. If $t_{\max} < \infty$, then*

$$\lim_{t \to t_{\max}} \left(\int_\Omega |\ x(z,\ t)\ |^2\ dz \right)^{\frac{1}{2}} = \infty.$$

Furthermore, if we use the Lipschitz condition

(H4) The nonlinear function $g : \overline{\Omega} \times [0,\ b] \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a constant $\kappa_3 > 0$ such that for any $(z, t) \in \Omega \times [0,\ b]$ and $\omega_1,\ \omega_2 \in \mathbb{R}$,

$$|g(z, t, \omega_1) - g(z, t, \omega_2)| \leq \kappa_3 |\omega_1 - \omega_2|$$

instead of (H2), we obtain the following global uniqueness result:

Theorem 1.3. *Let assumptions (H1), (H3), and (H4) be satisfied. Then, for every $\varphi \in L^2(\Omega)$, the parabolic problem (1.1) has a unique mild solution $x(z,\ t)$ on a maximal interval $[0,\ t_{\max})$ for each $z \in \Omega$ provided that $p\kappa_1 < 1$. If $t_{\max} < \infty$, then*

$$\lim_{t \to t_{\max}} \left(\int_\Omega |\ x(z,\ t)\ |^2\ dz \right)^{\frac{1}{2}} = \infty.$$

Our discussion will be made in a frame of abstract space. Let $\mathbb{H} = L^2(\Omega)$ denote the Hilbert space equipped with the norm $\|\cdot\|$ defined as

$$\|x\| = \left(\int_\Omega |\ x(z)\ |^2\ dz \right)^{\frac{1}{2}}, \quad \forall\ x \in L^2(\Omega)$$

and inner product $(\cdot, \cdot)$ defined by

$$(u, v) = \int_\Omega u(z) v(z) dz, \quad \forall\ u,\ v\ \in L^2(\Omega).$$

Remark 1.1. *We have simultaneously considered the effects of both impulsive and the $\psi-$Caputo fractional derivative on the system. So, the Theorems 1.1–1.3 obtained here extend and complement the existing literature.*

Let $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ be an unbounded linear operator defined by

$$Ax = A(z, D)x, \quad \forall\ x \in D(A),$$

$$D(A) = \mathbb{H}^2(\Omega) \cap \mathbb{H}_0^1(\Omega).$$

It is well-known from [25] that the operator A is sectorial, and $-A$ generates an analytic, compact contraction semigroup $T(t)(t \geq 0)(i.e. \sup_{t\geq 0} \|T(t)\| \leq 1)$ on $\mathbb{H}$. Let $x(t) = x(\cdot, t)$, $f(t, x(t)) = g(\cdot, t, x(\cdot, t))$, $I_i(x(t_i)) = I_i(x(\cdot, t_i))$, $x_0 = \varphi(\cdot)$. We can reformulate the parabolic problem (1.1) in an abstract form as follows:

$$\begin{cases} {}^cD_t^{\alpha,\psi} x(t) + Ax(t) = f(t, x(t)), \quad t \geq 0,\ t \neq t_i, \\ \Delta x(t_i) = I_i(x(t_i)),\ i = 1, 2, \cdots, p, \\ x(0) = x_0. \end{cases} \tag{1.2}$$

For the abstract evolution equation (1.2), we obtain the following results:

Theorem 1.4. *Suppose that the assumptions (H1) and the following (H2)* and (H3)* hold.*

(H2)* The nonlinear function $f : J \times \mathbb{H} \to \mathbb{H}$ is continuous and there exists a function $\xi \in L^2(J,\ \mathbb{R}^+)$ such that

$$\|f(t,\ x)\| \leq \xi(t), \quad t \in J, \quad x \in \mathbb{H}.$$

(H3)* Let $I_i : \mathbb{H} \to \mathbb{H}$ be a continuous function for each $i = 1, 2, \cdots, p$, and there exist constants $\mu > 0$ and $\sigma > 0$ such that

$$\|I_i(x) - I_i(y)\|^2 \leq \mu\|x - y\|^2, \quad \|I_i(x)\| \leq \sigma\|x\|, \quad x,\ y \in \mathbb{H},\ i = 1, 2, \cdots, p.$$

Then, for any $x_0 \in \mathbb{H}$, there exists a $\tau = \tau(x_0)$ $(0 < \tau < b)$ such that the problem (1.2) has a mild solution $x(t)$ in $[0,\ \tau]$ provided that $p\mu < 1$.

Theorem 1.5. *Assuming the conditions of Theorem 1.4 are satisfied, let $f : [0,\ b] \times \mathbb{H} \to \mathbb{H}(\ 0 < b \leq \infty)$ be continuous and map bounded subsets of $[0,\ b] \times \mathbb{H}$ into bounded subsets of $\mathbb{H}$. Then, for each $x_0 \in \mathbb{H}$, the problem (1.2) has a mild solution $x(t)$ on a maximal interval $[0,\ t_{\max})$. If $t_{\max} < \infty$, then $\lim_{t\to t_{\max}} \|x(t)\| = \infty$.*

In addition, we also obtain the following result on global uniqueness:

Theorem 1.6. *Suppose that the assumptions (H1), (H3)* and the following (H4)* hold.*

(H4)* There exists a constant $\lambda > 0$ such that for any $u,\ v \in \mathbb{H}$ and $0 \leq t \leq b$,

$$\|f(t,\ u) - f(t,\ v)\| \leq \lambda\|u - v\|.$$

Then, for every $x_0 \in \mathbb{H}$, there exists a $t_{\max} \leq \infty$ such that the problem (1.2) has a unique mild solution $x(t)$ on $[0,\ t_{\max})$ provided that $p\mu < 1$. If $t_{\max} < \infty$, then $\lim_{t\to t_{\max}} \|x(t)\| = \infty$.

We apply the abstract results Theorems 1.4–1.6 to the parabolic equation (1.1). Obviously, when $g(z, t, \omega)$ satisfies the condition (H2), $I_i(\omega)$ satisfies the condition (H3), and the corresponding $f(t, x)$ and $I_i(x)$ satisfies the condition (H2)* and (H3)*, respectively, where

$$\xi(t) = \Big(\int_\Omega \phi^2(z,\ t)dz\Big)^{\frac{1}{2}},\ \ \mu = \kappa_1,\ \ \sigma = \kappa_2.$$

Hence, by Theorem 1.4, we obtain the local existence of mild solution in Theorem 1.1. By Theorem 1.5, we obtain the global existence of mild solution in Theorem 1.2. Similarly, when $g(z, t, \omega)$ satisfies the condition (H4), the corresponding $f(t, x)$ satisfies the conditions (H4)* of Theorem 1.5 with $\lambda = \kappa_3$. Hence, by Theorem 1.6, we obtain the conclusion of Theorem 1.3. The proofs of Theorems 1.4–1.6 will be given in Section 3.

2. Preliminaries

The space $C(J,\ \mathbb{H})$, consisting of all continuous functions from J to $\mathbb{H}$, is a Banach space under the supnorm $\|x\|_C = \sup_{t\in J} \|x(t)\|$.

$$\mathcal{PC}(J,\ \mathbb{H}) = \{x : J \to \mathbb{H}, x(t) \text{ is continuous at } t \neq t_i, \text{left continuous at } t = t_i, \text{ and the right limit } x(t_i^+) \text{ exists for } i = 1, 2, \cdots, p\}.$$

The space $\mathcal{PC}(J,\ \mathbb{H})$, consisting of all piecewise continuous $\mathbb{H}$-valued functions defined on J, is a Banach space by the norm $\|x\|_{\mathcal{PC}} = \sup_{t\in J} \|x(t)\|$.

Next we introduce the definitions of fractional integral and derivative based on an increasing function ψ.

Definition 2.1. *(see [10]) Let $\psi \in C^1[0,\ b]$ be a strictly increasing function with $\psi'(t) > 0$ for all $t \in J$. The ψ−Riemann-Liouville integral of an integrable function $y : [0,\ b] \to \mathbb{R}$ of order $\alpha > 0$ can be expressed as*

$$I_0^{\alpha,\psi} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1} y(s)\psi'(s)ds.$$

Definition 2.2. *(see [10]) Let $n-1 < \alpha < n$, $\psi \in C^1[0,\ b]$ be a strictly increasing function with $\psi'(t) > 0$ for all $t \in J$. The ψ−Riemann-Liouville fractional derivative of an integrable function $y : [0,\ b] \to \mathbb{R}$ of order $\alpha > 0$ can be expressed as*

$$\begin{aligned} D_0^{\alpha,\psi} y(t) &= \Big(\frac{1}{\psi'(t)}\frac{d}{dt}\Big)^n (I_0^{n-\alpha,\psi} y(t)) \\ &= \frac{1}{\Gamma(n-\alpha)}\Big(\frac{1}{\psi'(t)}\frac{d}{dt}\Big)^n \int_0^t (\psi(t) - \psi(s))^{n-\alpha-1} y(s)ds, \end{aligned}$$

where $n = [\alpha] + 1$.

Definition 2.3. *(see [10]) Let $n - 1 < \alpha < n$, and suppose $y, \psi \in C^n[0,\ b]$ where ψ is strictly increasing ($\psi'(t) > 0$). The ψ−Caputo fractional derivative of a function y of order $\alpha > 0$ is given by*

$$\begin{aligned} {}^cD_0^{\alpha,\psi} y(t) &= (I_0^{n-\alpha,\psi} y^{(n)})(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (\psi(t) - \psi(s))^{n-\alpha-1} y^{(n)}(s)\psi'(s)ds, \end{aligned}$$

where $n = [\alpha] + 1$.

Remark 2.1. *(i) Suppose $y \in C^n[0,\ b]$, and let $\alpha > 0$. Then,*

$$I_0^{\alpha,\psi}({}^cD_0^{\alpha,\psi}y(t)) = y(t) - \sum_{k=0}^{n-1} \frac{(\psi(t)-\psi(0))^k}{k!} y_\psi^{[k]}(0)$$

Specifically, when $0 < \alpha < 1$, it holds that $I_0^{\alpha,\psi}({}^cD_0^{\alpha,\psi}y(t)) = y(t) - y(0)$, where

$$y_\psi^{[k]}(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^k y(t).$$

Next, we prove this formula.

In fact, using the semigroup law and the integration by parts formula repeatedly, we get

$$\begin{aligned}
I_0^{\alpha,\psi}({}^cD_0^{\alpha,\psi}y(t)) =& I_0^{\alpha,\psi} I_0^{n-\alpha,\psi} y_\psi^{[n]}(x) = I_0^{n,\psi} y_\psi^{[n]}(x) \\
=& \frac{1}{(n-1)!}\int_0^t (\psi(t)-\psi(s))^{n-1} y_\psi^{[n]}(s)\psi'(s)ds \\
=& \frac{1}{(n-1)!}\int_0^t (\psi(t)-\psi(s))^{n-1} \frac{d}{dt} y_\psi^{[n-1]}(s)ds \\
=& \frac{1}{(n-2)!}\int_0^t (\psi(t)-\psi(s))^{n-2} \frac{d}{dt} y_\psi^{[n-2]}(s)ds \\
&- \frac{y_\psi^{[n-1]}(0)}{(n-1)!}(\psi(t)-\psi(0))^{n-1} \\
=& \frac{1}{(n-3)!}\int_0^t (\psi(t)-\psi(s))^{n-3} \frac{d}{dt} y_\psi^{[n-3]}(s)ds \\
&- \sum_{k=n-2}^{n-1} \frac{y_\psi^{[k]}(0)}{k!}(\psi(t)-\psi(0))^k \\
=& \cdots = \int_0^t \frac{d}{dt} y(s)ds - \sum_{k=1}^{n-1} \frac{y_\psi^{[k]}(0)}{k!}(\psi(t)-\psi(0))^k \\
=& y(t) - \sum_{k=0}^{n-1} \frac{y_\psi^{[k]}(0)}{k!}(\psi(t)-\psi(0))^k
\end{aligned}$$

(ii) The $\psi-$Caputo fractional derivative of a constant is equal to zero.

(iii) When the increasing function ψ reduces to the identity function $\psi(t) = t$, the $\psi-$Caputo fractional derivative coincides exactly with the classical Caputo derivative.

We use the concept of mild solution from [15] for system (1.2).

Definition 2.4. *We say $x \in \mathcal{PC}(J,\ \mathbb{H})$ is a mild solution of (1.2) on J when it satisfies the following integral equation:*

$$\begin{aligned}
x(t) =& \mathcal{T}_\alpha(\psi(t)-\psi(0))x_0 + \sum_{0<t_i<t} \mathcal{T}_\alpha(\psi(t)-\psi(t_i))I_i(x(t_i)) \\
&+ \int_0^t (\psi(t)-\psi(s))^{\alpha-1}\mathcal{S}_\alpha(\psi(t)-\psi(s))f(s,\ x(s))\psi'(s)ds,
\end{aligned}$$

where

$$\mathcal{T}_\alpha(t)x = \int_0^\infty \zeta_\alpha(\theta)T(t^\alpha\theta)xd\theta, \quad \mathcal{S}_\alpha(t)x = \alpha\int_0^\infty \theta\zeta_\alpha(\theta)T(t^\alpha\theta)xd\theta, \tag{2.1}$$

and

$$\zeta_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-1/\alpha}\rho_\alpha(\theta^{-1/\alpha}),$$

$$\rho_\alpha(\theta) = \frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \quad \theta \in (0,\infty). \tag{2.2}$$

$\zeta_\alpha(\theta)$ is a probability density function defined on $(0,+\infty)$, satisfying

$$\zeta_\alpha(\theta) \geq 0, \ \theta \in (0,\infty), \quad \int_0^\infty \zeta_\alpha(\theta)d\theta = 1, \quad \int_0^\infty \theta\zeta_\alpha(\theta)d\theta = \frac{1}{\Gamma(1+\alpha)}. \tag{2.3}$$

Lemma 2.1. *(see [4]) The operators' families $\mathcal{T}_\alpha(t)(t \geq 0)$ and $\mathcal{S}_\alpha(t)(t \geq 0)$ satisfy the following properties:*
(i) For each $t \geq 0$, $\mathcal{T}_\alpha(t)$ and $\mathcal{S}_\alpha(t)$ are bounded linear operators on $\mathbb{H}$, satisfying the norm estimates:

$$\| \mathcal{T}_\alpha(t)x \| \leq \| x \|, \quad \| \mathcal{S}_\alpha(t)x \| \leq \frac{\| x \|}{\Gamma(\alpha)}.$$

for all $x \in \mathbb{H}$.
(ii) For any fixed $x \in \mathbb{H}$, the mappings $t \to \mathcal{T}_\alpha(t)x$ and $t \to \mathcal{S}_\alpha(t)x$ are continuous from $[0,\infty)$ into $\mathbb{H}$.
(iii) For $t \geq 0$, $\mathcal{T}_\alpha(t)(t \geq 0)$ and $\mathcal{S}_\alpha(t)(t \geq 0)$ are strongly continuous.
(iv) If the semigroup $T(t)$ is compact for every $t \geq 0$, then $\mathcal{T}_\alpha(t)$ and $\mathcal{S}_\alpha(t)$ are also compact in $\mathbb{H}$ for every $t > 0$.

Lemma 2.2. *(Krasnoselskii's Fixed Point Theorem; see [4]). Let $\mathbb{X}$ be a Banach space and $Y \subset \mathbb{X}$ be a bounded, closed, convex subset. $F_1, \ F_2$ are maps of Y into $\mathbb{X}$ such that $F_1x + F_2y \in Y$ for every pair $x, \ y \in Y$. If F_1 is a contraction mapping and F_2 is completely continuous, then $F_1 + F_2$ admits a fixed point on Y.*

3. Proof of the main results

Proof of Theorem 1.4. For local solution considerations, we may take $b < \infty$. Take $t' > 0, \ r > 0$ such that $B_r(x_0) = \{x \in \mathbb{H} \mid \|x - x_0\| \leq r\}$ and $\|f(t, \ x)\| \leq C_0$ for $0 \leq t \leq t'$ and $x \in B_r(x_0)$. Let $t'' > 0$ be such that$\|\mathcal{T}_\alpha(t)(\psi(t) - \psi(0))x_0 - x_0\| < \frac{r}{2}$ for $0 < t < t''$. Let $\tau = \min\{t', t'', b\}$ and

$$\psi(\tau) - \psi(0) \leq \left[\frac{\Gamma(\alpha+1)(r - 2\sigma_1 p)}{2C_0}\right]^{\frac{1}{\alpha}}, \tag{3.1}$$

where $\sigma_1 = \sigma(r + \|x_0\|)$. Set

$$G = \{x \in \mathcal{PC}(J, \ \mathbb{H}) \mid x(0) = x_0, \ x(t) \in B_r(x_0), \ 0 \leq t \leq \tau\}.$$

By definition, G is a bounded, closed, convex subset of $\mathcal{PC}(J, \ \mathbb{H})$. For any $x \in G$, we define the operator F as follows:

$$\begin{aligned}(Fx)(t) &= \mathcal{T}_\alpha(\psi(t) - \psi(0))x_0 + \sum_{0<t_i<t}\mathcal{T}_\alpha(\psi(t) - \psi(t_i))I_i(x(t_i)) \\ &+ \int_0^t (\psi(t) - \psi(s))^{\alpha-1}\mathcal{S}_\alpha(\psi(t) - \psi(s))f(s, x(s))\psi'(s)ds, \ t \in [0, \ \tau],\end{aligned} \tag{3.2}$$

To facilitate our discussion, we split

$$(Fx)(t) = (F_1x)(t) + (F_2x)(t),$$

where

$$((F_1x)(t) = \mathcal{T}_\alpha(\psi(t)-\psi(0))x_0 + \sum_{0<t_i<t} \mathcal{T}_\alpha(\psi(t)-\psi(t_i))I_i(x(t_i)), \quad t \in [0,\ \tau],$$

$$(F_2x)(t) = \int_0^t (\psi(t)-\psi(s))^{\alpha-1}\mathcal{S}_\alpha(\psi(t)-\psi(s))f(s,x(s))\psi'(s)ds,\ t \in [0,\ \tau].$$

We will employ Krasnoselskii's fixed point theorem to show that F has a fixed point in G. To do so, we structure our proof as follows.

Step 1. We show that $F_1x + F_2y \in G$ whenever $x,\ y \in G$.

In fact, by Lemma 2.1 and (3.1), for $x,\ y \in G,\ t \in [0,\ \tau]$, we have

$$\begin{aligned}
&\|(F_1x)(t) + (F_2y)(t) - x_0\| \\
\le\ & \|\mathcal{T}_\alpha(\psi(t)-\psi(0))x_0 - x_0\| + \sum_{0<t_i<t} \left\|\mathcal{T}_\alpha(\psi(t)-\psi(t_i))I_i(x(t_i))\right\| \\
&+ \left\|\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\mathcal{S}_\alpha(\psi(t)-\psi(s))f(s,y(s))\psi'(s)ds\right\| \\
\le\ & \frac{r}{2} + p\sigma_1 + \frac{C_0(\psi(\tau)-\psi(0))^\alpha}{\Gamma(\alpha+1)} \\
\le\ & r.
\end{aligned}$$

Thus, $F_1x + F_2y \in G$ for any $x,\ y \in G$.

Step 2. F_1 is a contraction mapping on G.

For any $x,\ y \in G$, it follows from (H2) that

$$\begin{aligned}
&\|(F_1x)(t) - (F_1y)(t)\| \\
&\le \sum_{0<t_i<t} \left\|\mathcal{T}_\alpha(\psi(t)-\psi(t_i))(I_i(x(t_i)) - I_i(y(t_i)))\right\| \\
&\le p\mu\|x-y\|_{\mathcal{PC}},
\end{aligned}$$

which implies that

$$\|F_1x - F_1y\|_{\mathcal{PC}} \le p\mu\|x-y\|_{\mathcal{PC}}.$$

Since $p\mu < 1$, we easily see that F_1 is a contraction mapping on G.

Step 3. F_2 is a completely continuous operator.

First, we show that the mapping F_2 is continuous on G. For this purpose, let $x_m \to x$ in G, then we have

$$f(t,\ x_m(t)) \to f(t,\ x(t)), \quad (m \to \infty).$$

Moreover, by the Lebesgue dominated convergence theorem and Hölder inequality , we can get

$$\begin{aligned}&\left\|\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\mathcal{S}_\alpha(\psi(t)-\psi(s))[f(s,\ x_m(s))-f(s,\ x(s))]\psi'(s)ds\right\|\\ \leq&\frac{1}{\Gamma(\alpha)}\Big(\int_0^t[(\psi(\tau)-\psi(s))^{\alpha-1}\psi'(s)]^2ds\Big)^{\frac{1}{2}}\Big(\int_0^t\|f(s,\ x_m(s))-f(s,\ x(s))\|^2ds\Big)^{\frac{1}{2}}\\ \leq&\frac{(\psi(\tau)-\psi(0))^{2\alpha-1}(\psi'(\tau))^{\frac{1}{2}}}{(2\alpha-1)\Gamma(\alpha)}\Big(\int_0^t\|f(s,\ x_m(s))-f(s,\ x(s))\|^2ds\Big)^{\frac{1}{2}}\\ \rightarrow&0\ (m\rightarrow\infty).\end{aligned}$$

Hence, we obtain that

$$\begin{aligned}&\|F_2(x_m)-F_2(x)\|_{\mathcal{PC}}\\ \leq&\left\|\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\mathcal{S}_\alpha(\psi(t)-\psi(s))[f(s,\ x_m(s))-f(s,\ x(s))]\psi'(s)ds\right\|\\ \rightarrow&\ 0\ (m\rightarrow\infty),\end{aligned}$$

which means that F_2 is continuous on G.

Second, we demonstrate that for every $t\in[0,\ \tau]$, the set $V(t)=\{F_2(x)(t)\ :\ x\in G\}$ is relatively compact in $\mathbb{H}$. It is obvious that $V(0)$ is relatively compact in $\mathbb{H}$. Let $0<t\leq\tau$ be given. For any $\epsilon\in(0,\ t)$ and $\nu>0$, define an operator $F^{\epsilon,\nu}$ on G by

$$\begin{aligned}&(F^{\epsilon,\nu}x)(t)\\ =&\alpha\int_0^{t-\epsilon}\int_\nu^\infty\theta\zeta_\alpha(\theta)(\psi(t)-\psi(s))^{\alpha-1}T((\psi(t)-\psi(s))^\alpha\theta)f(s,x(s))\psi'(s)d\theta ds\\ =&T(\epsilon^\alpha\nu)\alpha\int_0^{t-\epsilon}\int_\nu^\infty\theta\zeta_\alpha(\theta)(\psi(t)-\psi(s))^{\alpha-1}T((\psi(t)-\psi(s))^\alpha\theta-\epsilon^\alpha\nu)\\ &\times f(s,x(s))\psi'(s)d\theta ds.\end{aligned}$$

Then, relative compactness of $\{(F^{\epsilon,\nu}x)(t):\ x\in G\}$ in $\mathbb{H}$ follows from the compactness of $T(\epsilon^\alpha\nu)$.

By (H1), (H2)*, Lemma 2.1, and the Hölder inequality, we have that

$$
\begin{aligned}
&\|(F_2x)(t)-(F^{\epsilon,\nu}x)(t)\| \\
\leq &\left\|\alpha\int_0^t\int_0^\nu \theta\zeta_\alpha(\theta)(\psi(t)-\psi(s))^{\alpha-1}T((\psi(t)-\psi(s))^\alpha\theta)f(s,x(s))\psi'(s)d\theta ds\right\| \\
+&\left\|\alpha\int_{t-\epsilon}^t\int_\nu^\infty \theta\zeta_\alpha(\theta)(\psi(t)-\psi(s))^{\alpha-1}T((\psi(t)-\psi(s))^\alpha\theta)f(s,x(s))\psi'(s)d\theta ds\right\| \\
\leq &\alpha\int_0^t\int_0^\nu \theta\zeta_\alpha(\theta)(\psi(t)-\psi(s))^{\alpha-1}\xi(s)\psi'(s)d\theta ds \\
&+\alpha\int_{t-\epsilon}^t\int_\nu^\infty \theta\zeta_\alpha(\theta)(\psi(t)-\psi(s))^{\alpha-1}\xi(s)\psi'(s)d\theta ds \\
\leq &\alpha\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\xi(s)\psi'(s)ds\Big(\int_0^\nu\theta\zeta_\alpha(\theta)d\theta\Big) \\
&+\frac{\alpha}{\Gamma(1+\alpha)}\int_{t-\epsilon}^t(\psi(t)-\psi(s))^{\alpha-1}\xi(s)\psi'(s)ds \\
\leq &\frac{\alpha(\psi(\tau)-\psi(0))^{2\alpha-1}(\psi'(\tau))^{\frac{1}{2}}\|\xi\|_{L^2}}{2\alpha-1}\Big(\int_0^\nu\theta\zeta_\alpha(\theta)d\theta\Big) \\
&+\frac{\alpha(\psi'(\tau))^{\frac{1}{2}}\|\xi\|_{L^2}}{(2\alpha-1)\Gamma(1+\alpha)}(\psi(\tau)-\psi(\tau-\epsilon))^{2\alpha-1} \\
\to &\, 0\ (\epsilon,\nu\to 0).
\end{aligned}
$$

Hence, there are relatively compact sets arbitrarily close to the set $V(t)(t>0)$ in $\mathbb{H}$. Therefore, the set $V(t)$ is relatively compact in $\mathbb{H}$.

Third, we prove that $F_2(G)=\{F_2x : x\in G\}$ is equicontinuous on $[0,\ \tau]$.

For any $x\in G$ and $0\leq\tau_1<\tau_2\leq\tau$, we have

$$
\begin{aligned}
&\|(F_2x)(\tau_2)-(F_2x)(\tau_1)\| \\
=&\left\|\int_{\tau_1}^{\tau_2}(\psi(\tau_2)-\psi(s))^{\alpha-1}\mathcal{S}_\alpha(\psi(\tau_2)-\psi(s))f(s,\ x(s))\psi'(s)ds\right\| \\
&+\Big\|\int_0^{\tau_1}[(\psi(\tau_2)-\psi(s))^{\alpha-1}\mathcal{S}_\alpha(\psi(\tau_2)-\psi(s)) \\
&-(\psi(\tau_1)-\psi(s))^{\alpha-1}\mathcal{S}_\alpha(\psi(\tau_1)-\psi(s))]f(s,\ x(s))\psi'(s)ds\Big\| \\
\leq&\left\|\int_{\tau_1}^{\tau_2}(\psi(\tau_2)-\psi(s))^{\alpha-1}\mathcal{S}_\alpha(\psi(\tau_2)-\psi(s))f(s,\ x(s))\psi'(s)ds\right\| \\
&+\Big\|\int_0^{\tau_1}[(\psi(\tau_2)-\psi(s))^{\alpha-1}-(\psi(\tau_1)-\psi(s))^{\alpha-1}]\mathcal{S}_\alpha(\psi(\tau_2)-\psi(s)) \\
&\times f(s,\ x(s))\psi'(s)ds\Big\|
\end{aligned}
$$

$$
\begin{aligned}
&+\left\|\int_0^{\tau_1}(\psi(\tau_1)-\psi(s))^{\alpha-1}[\mathcal{S}_\alpha(\psi(\tau_2)-\psi(s))-\mathcal{S}_\alpha(\psi(\tau_1)-\psi(s))]\right.\\
&\left.\times f(s,\ x(s))\psi'(s)ds\right\|\\
=&I_1+I_2+I_3.
\end{aligned}
$$

In order to prove $\|(F_2x)(\tau_2)-(F_2x)(\tau_1)\|\to 0(\tau_2-\tau_1\to 0)$, we only need to show $I_i\to 0$ independently when $\tau_2-\tau_1\to 0$ for $i=1,2,3$.

For I_1, we obtain by (H1), (H2)* and Lemma 2.1 that

$$
\begin{aligned}
I_1=&\left\|\int_{\tau_1}^{\tau_2}(\psi(\tau_2)-\psi(s))^{\alpha-1}\mathcal{S}_\alpha(\psi(\tau_2)-\psi(s))f(s,\ x(s))\psi'(s)ds\right\|\\
\le&\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\psi(\tau_2)-\psi(s))^{\alpha-1}\xi(s)\psi'(s)ds\\
\le&\frac{\|\xi\|_{L^2}(\psi'(\tau))^{\frac{1}{2}}(\psi(\tau_2)-\psi(\tau_1))^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}\Gamma(\alpha)}\\
\to& 0\ (\tau_2-\tau_1\to 0).
\end{aligned}
$$

For I_2, we get

$$
\begin{aligned}
I_2=&\left\|\int_0^{\tau_1}[(\psi(\tau_2)-\psi(s))^{\alpha-1}-(\psi(\tau_1)-\psi(s))^{\alpha-1}]\mathcal{S}_\alpha(\psi(\tau_2)-\psi(s))\right.\\
&\left.\times f(s,\ x(s))\psi'(s)ds\right\|\\
\le&\frac{\|\xi\|_{L^2}}{\Gamma(\alpha)}\left(\int_0^{t_1}[(\psi(\tau_2)-\psi(s))^{\alpha-1}-(\psi(\tau_1)-\psi(s))^{\alpha-1}]^2(\psi'(s))^2ds\right)^{\frac{1}{2}}\\
\to& 0\ (\tau_2-\tau_1\to 0).
\end{aligned}
$$

Further, for I_3, given a sufficiently small ε such that $0<\varepsilon<\tau_1$, the following inequalities hold:

$$
\begin{aligned}
I_3=&\left\|\int_0^{\tau_1}(\psi(\tau_1)-\psi(s))^{\alpha-1}[\mathcal{S}_\alpha(\psi(\tau_2)-\psi(s))-\mathcal{S}_\alpha(\psi(\tau_1)-\psi(s))]\right.\\
&\left.\times f(s,\ x(s))\psi'(s)ds\right\|\\
\le&\int_0^{\tau_1-\varepsilon}(\psi(\tau_1)-\psi(s))^{\alpha-1}[\mathcal{S}_\alpha(\psi(\tau_2)-\psi(s))-\mathcal{S}_\alpha(\psi(\tau_1)-\psi(s))]\\
&\times f(s,\ x(s))\psi'(s)ds\Big\|\\
&+\left\|\int_{\tau_1-\varepsilon}^{t_1}(\psi(\tau_1)-\psi(s))^{\alpha-1}[\mathcal{S}_\alpha(\psi(\tau_2)-\psi(s))-\mathcal{S}_\alpha(\psi(\tau_1)-\psi(s))]\right.\\
&\left.\times f(s,\ x(s))\psi'(s)ds\right\|
\end{aligned}
$$

$$
\begin{aligned}
&\le \sup_{s\in[0,\tau_1-\varepsilon]} \|\mathcal{S}_\alpha(\psi(\tau_2)-\psi(s)) - \mathcal{S}_\alpha(\psi(\tau_1)-\psi(s))\| \\
&\quad \times \int_0^{\tau_1-\varepsilon} (\psi(\tau_1)-\psi(s))^{\alpha-1}\xi(s)\psi'(s)ds \\
&\quad + \frac{2}{\Gamma(\alpha)}\int_{\tau_1-\varepsilon}^{\tau_1} (\psi(\tau_1)-\psi(s))^{\alpha-1}\xi(s)\psi'(s)ds \\
&= \sup_{s\in[0,\tau_1-\varepsilon]} \|\mathcal{S}_\alpha(\psi(t_2)-\psi(s)) - \mathcal{S}_\alpha(\psi(t_1)-\psi(s))\| \\
&\quad \times \frac{(\psi'(\tau))^{\frac{1}{2}}\|\xi\|_{L^2}}{2\alpha-1}((\psi(\tau_1)-\psi(0))^{2\alpha-1} - (\psi(\tau_1)-\psi(\tau_1-\varepsilon))^{2\alpha-1}) \\
&\quad + \frac{2(\psi'(\tau))^{\frac{1}{2}}\|\xi\|_{L^2}}{(2\alpha-1)\Gamma(\alpha)}(\psi(\tau_1)-\psi(\tau_1-\varepsilon))^{2\alpha-1} \\
&\to 0 \ (\tau_2-\tau_1\to 0 \text{ and } \varepsilon\to 0).
\end{aligned}
$$

Thus,

$$\|(F_2x)(\tau_2)-(F_2x)(\tau_1)\| \to \ 0 \ as \ (\tau_2-\tau_1\to 0).$$

This implies that $F_2(G) = \{F_2x : x \in G\}$ is equicontinuous.

It follows from the Arzela-Ascoli theorem that F_2 is a completely continuous operator. Thus, by Lemma 2.2, the operator $F_1 + F_2$ admits at least one fixed point $x \in G$, yielding a mild solution to (1.2). This proves Theorem 1.4.

Proof of Theorem 1.5. First, we observe that any mild solution $x(t)$ of (1.2) defined on a closed interval $[0,\ \tau]$ admits an extension to a larger interval $[0,\ \tau+h]$ for some $h > 0$, by defining $x(t+\tau) = y(t)$, where $y(t)$ is a mild solution of

$$
\begin{cases}
{}^cD_t^{\alpha,\psi}y(t) + Ay(t) = f(t, y(t)), \quad t\in[\tau,\ \tau+h], \\
\Delta y(t_i) = I_i(y(t_i)),\ i = 1,2,\cdots,p, \\
y(0) = x(\tau),
\end{cases}
\tag{3.3}
$$

The above procedure may be repeated to construct a mild solution defined on a maximal interval of existence denoted by $[0,\ t_{\max})$.

In what follows, we show that if $t_{\max} < \infty$, then $\lim\limits_{t\to t_{\max}} \|x(t)\| = \infty$. For this, we first prove that $t_{\max} < \infty$ implies $\limsup\limits_{t\to t_{\max}} \|x(t)\| = \infty$. In fact, if $t_{\max} < \infty$ and $\limsup\limits_{t\to t_{\max}} \|x(t)\| < \infty$, we may assume that $\sup\limits_{0<t<t_{\max}} \|x(t)\| < R_0$, where R_0 is a constant. Let

$$M_0 = \sup\{\|f(t, x(t))\| :\ \|x(t))\| \le R_0,\ 0\le t\le t_{\max}+1\}.$$

For $0 < t' < t'' < t_{\max}$, we have

$$\begin{aligned}
&\|x(t'') - x(t')\| \\
=&\Big\|(\mathcal{T}_\alpha(\psi(t'') - \psi(0)) - \mathcal{T}_\alpha(\psi(t') - \psi(0)))x_0\Big\| \\
&+ \sum_{0<t_i<t'} \Big\|(\mathcal{T}_\alpha(\psi(t'') - \psi(t_i)) - \mathcal{T}_\alpha(\psi(t') - \psi(t_i))I_i(x(t_i))\Big\| \\
&+ \sum_{t'<t_i<t''} \Big\|\mathcal{T}_\alpha(\psi(t'') - \psi(t_i))I_i(x(t_i))\Big\| \\
&+ \Big\| \int_{t'}^{t''} (\psi(t'') - \psi(s))^{\alpha-1}\mathcal{S}_\alpha(\psi(t'') - \psi(s))f(s,\ x(s))\psi'(s)ds\Big\| \\
&+ \Big\| \int_0^{t'} [(\psi(t'') - \psi(s))^{\alpha-1}\mathcal{S}_\alpha(\psi(t'') - \psi(s)) \\
&- (\psi(t') - \psi(s))^{\alpha-1}\mathcal{S}_\alpha((t') - \psi(s))]f(s,\ x(s))\psi'(s)ds\Big\| \\
\leq&\Big\|(\mathcal{T}_\alpha(\psi(t'') - \psi(0)) - \mathcal{T}_\alpha(\psi(t') - \psi(0)))x_0\Big\| \\
&+ \sigma_1 \sum_{0<t_i<t'} \Big\|(\mathcal{T}_\alpha(\psi(t'') - \psi(t_i)) - \mathcal{T}_\alpha(\psi(t') - \psi(t_i))\Big\| \\
&+ \sigma_1 \sum_{t'<t_i<t''} \Big\|\mathcal{T}_\alpha(\psi(t'') - \psi(t_i))\Big\| + \frac{M_0}{\Gamma(\alpha+1)}(\psi(t'') - \psi(t'))^\alpha \\
&+ M_0\psi'(\tau) \int_0^{t'} \Big\|(\psi(t'') - \psi(s))^{\alpha-1}\mathcal{S}_\alpha(\psi(t'') - \psi(s)) \\
&- (\psi(t') - \psi(s))^{\alpha-1}\mathcal{S}_\alpha((t') - \psi(s))\Big\|ds.
\end{aligned}$$

Since $\mathcal{T}_\alpha(t),\ \mathcal{S}_\alpha(t)$ are continuous in the uniform operator topology for $t > 0$, $\|x(t'') - x(t')\| \to 0$ as $t'',\ t' \to t_{\max}$. The Cauchy criteria guarantees the existence of the limit $\lim\limits_{t\to t_{\max}} x(t)$. We may suppose that $\lim\limits_{t\to t_{\max}} x(t) = \tilde{x}$, then there exists a constant $h > 0$ such that the equation

$$\begin{cases} {}^cD_t^{\alpha,\psi}x(t) + Ax(t) = f(t, x(t)), \quad t > t_{\max}, \\ \Delta x(t_i) = I_i(x(t_i)),\ i = 1, 2, \cdots, p, \\ x(t_{\max}) = \tilde{x} \end{cases} \tag{3.4}$$

has a mild solution on $[t_{\max},\ t_{\max} + h]$. This contradicts with the maximal interval $[0,\ t_{\max})$. Thus, $\limsup\limits_{t\to t_{\max}} \|x(t)\| = \infty$.

Now we show that $\lim\limits_{t\to t_{\max}} \|x(t)\| = \infty$. We argue by contradiction: If it is not true, then there is a sequence $\eta_n \uparrow t_{\max}$ and a constant K such that $\|x(\eta_n)\| \leq K$ for every n. Set

$$B = \sup\{\|f(t,\ x(t))\| :\ \|x(t)\| \leq K + 1,\ 0 \leq t \leq t_{\max}\}$$

and choose $\sigma < \frac{1}{2p(K+1)}$. The mapping $t \to \|x(t)\|$ being continuous and $\limsup_{t\to t_{\max}} \|x(t)\| = \infty$ implies the existence of a sequence $\{\delta_n\}$ possessing the following properties: $\delta_n \to 0$ as $n \to \infty$, $\|x(t)\| \leq K + 1$ for $\eta_n \leq t \leq \eta_n + \delta_n$ and $\|x(\eta_n + \delta_n)\| = K + 1$. However,

$$\begin{aligned}
K + 1 = \ & \|x(\eta_n + \delta_n)\| \leq \ \|\mathcal{T}_\alpha(\psi(\delta_n) - \psi(0))x(\eta_n)\| \\
& + \sum_{\eta_n < t_i < \eta_n + \delta_n} \left\| \mathcal{T}_\alpha(\psi(\eta_n + \delta_n) - \psi(t_i))I_i(x(t_i)) \right\| \\
& + \int_{\eta_n}^{\eta_n + \delta_n} (\psi(\eta_n + \delta_n) - \psi(s))^{\alpha - 1} \\
& \times \left\| \mathcal{S}_\alpha(\psi(\eta_n + \delta_n) - \psi(s)) f(s, x(s)) \right\| \psi'(s) ds \\
\leq \ & K + p\sigma(K + 1) + \frac{B}{\Gamma(\alpha + 1)}(\psi(\eta_n + \delta_n) - \psi(\eta_n))^\alpha \\
\leq \ & K + \frac{1}{2} + \frac{B}{\Gamma(\alpha + 1)}(\psi(\eta_n + \delta_n) - \psi(\eta_n))^\alpha \\
\to \ & K + \frac{1}{2} \ (n \to \infty),
\end{aligned}$$

which is a contradiction. Hence, we have $\lim_{t\to t_{\max}} \|x(t)\| = \infty$. This completes the proof of Theorem 1.5.

Proof of Theorem 1.6. Let τ be defined by (3.1), and we additionally suppose that

$$\psi(\tau) - \psi(0) \leq \left[\frac{\Gamma(\alpha + 1)(1 - p\mu)}{\lambda} \right]^{\frac{1}{\alpha}}. \tag{3.5}$$

For each $x \in G$, we define the operator $F : G \to G$ by (3.2). By (3.5), for $0 \leq t \leq b$, u, $v \in G$, we have

$$\begin{aligned}
& \|(Fu)(t) - ((Fv)(t)\| \\
\leq \ & \sum_{0 < t_i < t} \left\| \mathcal{T}_\alpha(\psi(t) - \psi(t_i))(I_i(u(t_i)) - I_i(v(t_i))) \right\| \\
& + \left\| \int_0^t (\psi(t) - \psi(s))^{\alpha - 1} \mathcal{S}_\alpha(\psi(t) - \psi(s))(f(s, u(s)) - f(s, v(s)))\psi'(s) ds \right\| \\
\leq \ & \left(p\mu + \frac{\lambda(\psi(\tau) - \psi(0))^\alpha}{\Gamma(\alpha + 1)} \right) \|u - v\|_{\mathcal{PC}}. \\
\leq \ & \|u - v\|_{\mathcal{PC}},
\end{aligned}$$

which means

$$\|Fu - Fv\|_{\mathcal{PC}} \leq \ \|u - v\|_{\mathcal{PC}}.$$

Therefore, the operator F is a contraction mapping. We conclude that a mild solution $x(t)$ to problem (1.2) exists on $[0,\ \tau]$. Similar to the method which we used in the proof of Theorem 1.5, one can prove the existence of $t_{\max} \leq \infty$ yielding a unique mild solution $x(t)$ to (1.2) with domain $[0,\ t_{\max})$. If $t_{\max} < \infty$ then $\lim_{t\to t_{\max}} \|x(t)\| = \infty$. This completes the proof of Theorem 1.6.

4. Conclusions

In this paper, the blowup and global existence of mild solutions for a class of impulsive $\psi-$Caputo fractional parabolic equations in a Banach space is studied. The $\psi-$Caputo fractional derivative is a generalization of the standard Caputo fractional derivative. When the increasing function ψ reduces to the identity function $\psi(t) = t$, the $\psi-$Caputo fractional derivative coincides exactly with the classical Caputo derivative. Therefore, the conclusions obtained in this paper constitute a generalization of existing literature, which primarily addresses the standard Caputo fractional derivative case. Moreover, impulsive effects have been integrated into our system analysis. In future research work, one can extend these results to stochastic differential equations of fractional order or arbitrary order with non-instantaneous impulses.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors have no relevant financial or non-financial interests to disclose.

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