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*Research article*

# New comparison theorems for testing the oscillation of solutions of fourth-order differential equations with a variable argument

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**Abstract:** In this study, we establish new oscillation criteria for the fourth-order delay differential equation. Our main objective is to build upon recent advancements in the study of the oscillatory behavior of second-order equations and extend these findings to higher-order equations. Although fourth-order equations have numerous applications, their study presents significant analytical challenges due to the complex nature of their solutions, which we will discuss in this study. We use the comparison technique with first-order equations in several approaches. Our results show an improvement in the oscillation test due to the development of some monotonic and asymptotic properties of positive solutions. We present a comparison of the new criteria to test their effectiveness, as well as a comparison with previous studies to illustrate the novelty.

**Keywords:** differential equations; oscillatory behavior; fourth-order equations; comparison theorem

## 1. Introduction

In the modeling of complex systems, such as those with effects like elasticity and bending, fourth-order differential equations play a crucial role. These equations are often used in structural mechanics to understand the behavior of beams and slabs under varying loading conditions, as demonstrated in [1]. They also appear in fluid dynamics and quantum mechanics, where they are employed to study wave propagation and energy distribution with the utmost accuracy, as noted in [2]. The significance of these equations in analyzing complex systems highlights the importance of studying their solutions and understanding their behavior, both from theoretical and practical perspectives.

A particularly important aspect of fourth-order differential equations, as well as delay differential equations (DDEs), is their oscillatory behavior. Given that the exact solution of the fourth-order differential equations equation is known, it can serve as a benchmark solution in numerical simulations of fluid dynamics. Many authors have devoted themselves to studying high-order numerical method for the equations; the authors can refer to a significant amount of theoretical and numerical analysis, e.g., [3].

All of these research topics highlight the necessity of further investigation into oscillations and solution behaviors in both DDEs and fourth-order differential equations. The capacity to look for and understand these oscillatory behaviors is essential for developing technology and expanding scientific understanding, regardless of whether it is applied to mechanical structures, fluid dynamics, or quantum systems.

So, in this study, we aim to provide improved and developed oscillation criteria in more than one way for the fourth-order DDE

$$x^{(4)}(t) + q(t) F(x(g(t))) = 0, \tag{1.1}$$

where $t \geq t_0$. During this study, we assume that $q,\ g \in \mathbf{C}([t_0, \infty))$, $q(t) \geq 0$, $g(t) \leq t$, $g'(t) \geq 0$, $\lim_{t\to\infty} g(t) = \infty$, $F \in \mathbf{C}(\mathbb{R})$, and $\frac{F(x)}{x} \geq k > 0$ for $x \neq 0$.

A solution to the aforementioned equation is defined as the function $x$ that possesses the following properties: $x \in C^4([t_0, \infty), (0, \infty))$, $x$ satisfies (1.1) for $t$ sufficiently large enough, and $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq t_0$. A solution $x$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is said to be nonoscillatory. If every solution to the equation oscillates, then the equation itself is considered to be oscillatory.

## 2. Literature review

The oscillatory behavior of even-order differential equations has attracted significant attention. Foundational results on nonlinear and delayed systems, including those with retarded arguments and small forcing terms, have been extensively explored in the literature. Dahiya and Bhagat [4] studied the even-order half-linear DDE

$$x^{(n)}(t) + q(t) x^{\beta}(t - g(t)) = 0,$$

for $n$ be an even integer, $n \geq 2$, and $\beta$ be a quotient of any two odd positive integers. Their work was limited just for $\beta \in (0, 1)$ or $\beta \in (1, \infty)$, which means that they neglected the linear case and failed to provide a single condition that guarantees the oscillation of the equation for all $\beta$ cases, other than restricting the delay function to a single case $(t - g(t))$. Then, Grace and Lalli [5] used the weighted integral method to study the oscillatory behavior of the nth even-order nonlinear DDE (1.1) and its damped form

$$x^{(n)}(t) + p(t) \left|x^{(n-1)}(t)\right|^{\beta} x^{(n-1)}(t) + q(t) F(x(g(t))) = 0,$$

for $\beta \geq 0$. They aimed to extend some earlier famous oscillation criteria of (1.1) in its second-order form (i.e., at $n = 2$) to $n \geq 2$ by ensuring that the condition

$$\int_{t_0}^{\infty} \sigma'(\varsigma) \sigma^{n-2}(\varsigma) \left( \int_{\varsigma}^{\infty} q(\xi) \, \mathrm{d}\xi \right) \mathrm{d}\varsigma = \infty$$

are satisfied, for $\sigma \in C([t_0, \infty), \mathbb{R}^+)$ be a twice continuously differentiable function, $\sigma \leq g$, and $\sigma' > 0$, $\lim_{t\to\infty} \sigma(t) = \infty$. After that, improvements were made to the oscillation criteria of (1.1) in works such as [6], where its oscillation was proven for $\sigma(t) = \min\{t, g(t)\}$ if one of the following criteria was met:

$$\int_{t_0}^{\infty} \sigma^{n-1}(\varsigma)\, q(\varsigma)\, \mathrm{d}\varsigma = \infty,$$

for $\beta > 1$, or

$$\int_{t_0}^{\infty} \sigma^{\epsilon n-\epsilon}(\varsigma)\, q(\varsigma)\, \mathrm{d}\varsigma = \infty,$$

for some $\epsilon > 0$ and $\beta = 1$, or

$$\int_{t_0}^{\infty} \sigma^{\beta n-\beta}(\varsigma)\, q(\varsigma)\, \mathrm{d}\varsigma = \infty,$$

for $0 < \beta < 1$. Whatever, these criteria required that $F$ be either nondecreasing or have bounded variation locally. So, in his paper [7], Grace aimed to overcome these limitations by providing the oscillation conditions for any $\ell = 1, 3, 5, ..., n-1$, as

$$\int_{t_0}^{\infty} t^{n-\beta-\ell} \left(g^{\ell-1}(\varsigma)\, \sigma(\varsigma)\right)^{\beta} q(\varsigma)\, \mathrm{d}\varsigma = \infty,$$

for $\beta > 1$, or

$$\limsup_{t\to\infty} \left[\int_{\sigma(t)}^{t} (\varsigma - \sigma(t))^{n-\ell-1} \sigma'(\varsigma)\, q(\varsigma)\, \mathrm{d}\varsigma + \sigma(t) \int_{t}^{\infty} (\varsigma - \sigma(t))^{n-\ell-1} \sigma^{\ell-1}(\varsigma)\, q(\varsigma)\, \mathrm{d}\varsigma\right] > (n-\ell-1)!\,(\ell)!,$$

for $\beta = 1$, or for $\beta < 1$ under satisfying of

$$\int_{t_0}^{\infty} \varsigma^{\beta} \left(\int_{\varsigma}^{\infty} (\xi - \varsigma)^{n-\ell-2} \left(g^{\ell-1}(\xi)\, \frac{\sigma(\xi)}{\xi}\right)^{\beta} q(\xi)\, \mathrm{d}\xi\right) \mathrm{d}\varsigma = \infty,$$

for $\ell = 1, 3, 5, ..., n-3$ and

$$\int_{t_0}^{\infty} \left(g^{n-2}(\varsigma)\, \sigma(\varsigma)\right)^{\beta} q(\varsigma)\, \mathrm{d}\varsigma = \infty,$$

for $\ell = n-1$. Subsequently, Zafer [8] investigated the oscillatory behavior of the nonlinear NDDE

$$[x(t) + a(t)\, x(h(t))]^{(n)} + F(t, x(t), x(g(t))) = 0. \tag{2.1}$$

He establishes the following oscillation conditions for (2.1):

$$\liminf_{t\to\infty} \int_{g(t)}^{t} g^{n-1}(t)\,(1 - a(g(\varsigma)))\, q(\varsigma)\, \mathrm{d}\varsigma > \frac{(n-1)\, 2^{(n-1)(n-2)}}{\mathrm{e}} \tag{2.2}$$

or

$$\limsup_{t\to\infty} \int_{g(t)}^{t} g^{n-1}(t)\,(1 - a(g(\varsigma)))\, q(\varsigma)\, \mathrm{d}\varsigma > (n-1)\, 2^{(n-1)(n-2)}$$

which addresses the earlier ones by incorporating both the linear case of the equation and the effect of the neutral delay term. Then Agarwal et al. [9] ensure the oscillation of (2.1) for $\pi_j \in C^1([t_0, \infty), \mathbb{R}^+)$, $j = 1, 2$, if

$$\int_{t_1}^{\infty} \pi_1(\nu)\, q(\nu)\,(1 - a(g(\nu)))\, \frac{g^3(\nu)}{\nu^3} - \frac{1}{2\epsilon} \frac{\left(\pi_1'(\nu)\right)^2}{\nu^2 \pi_1(\nu)} \mathrm{d}\nu = \infty \tag{2.3}$$

and

$$\int_{t_1}^{\infty} \pi_2(\nu)\left(\int_{\nu}^{\infty}(s-\nu)\, q(s)(1-a(g(s)))\frac{g(s)}{s}\mathrm{d}s\right)-\frac{\left(\pi_2'(\nu)\right)^2}{4\pi_2(\nu)}\mathrm{d}\nu=\infty \tag{2.4}$$

holds. While, Zhang et al. [10] followed a different approach to introduce the following oscillation criteria, for $r\in\mathbb{Z}^+$, $\rho_0(t)=\max\{\sigma(\varepsilon), t_1\le\varepsilon\le t\}$ and

$$\Psi_0(t)=\frac{g^3(t)(1-a(g(t)))\, q(t)}{6}.$$

If

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t}\Psi_r(\nu)\,\mathrm{d}\nu>\frac{1}{\mathrm{e}^r}, \tag{2.5}$$

where

$$\Psi_1(t)=\int_{\rho_0(t)}^{t}\Psi_0(\nu)\,\mathrm{d}\nu$$

for $t\in[\rho_{-1}(t),\infty)$, $\rho_{-1}(t)=\sup\{\rho_0(\varepsilon)=t:\varepsilon\ge t_1\}$ and

$$\Psi_{r+1}(t)=\int_{\rho_0(t)}^{t}\Psi_0(\nu)\,\Psi_r(t)\,\mathrm{d}\nu$$

for $t\in[\rho_{-(r+1)}(t),\infty)$, $\rho_{-(r+1)}(t)=\rho_{-1}(\rho_{-r}(t))$.

Following this, Zhang and Yan [11] generalized these criteria using the following lemmas for NDDE:

$$[x(t)+a(t)\,x(h(t))]^{(n)}+q(t)\,x(g(t))=0, \tag{2.6}$$

which was introduced by Philos [12] and widely used in subsequent studies.

**Lemma 1.** *Suppose that x be an eventually positive and $n^{th}$ times-differentiable solution of (2.6) over $(t_0,\infty)$. Then, there exists an integer $\kappa\in[0,n-1]$, n even,*

$$x^{(i)}>0,$$

*for $i=1,2,...,\kappa-1$, $\kappa>1$, and*

$$(-1)^{\kappa+i}x^{(i)}>0,$$

*for $i=\kappa,\kappa+1,...n-1$, eventually.*

**Lemma 2.** *Suppose that x be an eventually positive and $n^{th}$ times-differentiable solution of (2.6) over $(t_0,\infty)$. If $x\nrightarrow 0$ as $t\to\infty$, then there is $t\ge t_\nu\ge t_0$ for every $\nu\in(0,1)$, such that*

$$x(t)\ge\frac{\nu}{(n-1)!}t^{n-1}x^{(n-1)}(t).$$

These relationships enabled them to obtain criteria that give better results in some scenarios, such as,

$$\liminf_{t\to\infty}\int_{g(t)}^{t} g^{n-1}(t)(1-a(g(\varsigma)))\, q(\varsigma)\,\mathrm{d}\varsigma>\frac{(n-1)!}{\mathrm{e}}$$

or

$$\limsup_{t\to\infty}\int_{g(t)}^{t} g^{n-1}(t)(1-a(g(\varsigma)))\, q(\varsigma)\,\mathrm{d}\varsigma>(n-1)!$$

under assuming that $g'>0$. Then, Kiguradze and Chanturia [13] provided refined forms for Lemmas 1 and 2 as follows:

**Lemma 3.** *Suppose that x be an eventually positive and $n^{th}$ times-differentiable solution of*

$$x^{(n)}(t) + q(t)\,x(g(t)) = 0 \tag{2.7}$$

*over* $(t_0, \infty)$. *Then, there exists a positive integer such that*

$$x^{(i)} > 0,$$

*for* $i = 0, 1, ...\kappa$, *and*

$$x^{(\kappa+1)} < 0,$$

*eventually.*

**Lemma 4.** *Suppose that x be as in Lemma 3; then there is* $t \geq t_\nu \geq t_0$ *for every* $\nu \in (0, 1)$, *such that*

$$\binom{\kappa}{j} x(t) \geq \nu \frac{t^j x^{(j)}(t)}{(j)!},$$

*for any* $j = 1, 2, ..., \kappa$.

For half-linear DDE, Agarwal and Grace [14] studied the equation

$$\left[\left(x^{(n-1)}(t)\right)^{\alpha}\right]' + q(t)\,F(x(g(t))) = 0,$$

for $\alpha$ be a quotient of any two odd positive integers, and denote that every solution of the above equation oscillates if

$$\liminf_{t\to\infty} \int_{g(t)}^{t} g^{n\alpha-\alpha}(t)\,q(\varsigma)\,\mathrm{d}\varsigma > [(n-1)!]^{\alpha},$$

for $\alpha = \beta$, or

$$\int_{t_0}^{\infty} g^{n\beta-\beta}(t)\,q(\varsigma)\,\mathrm{d}\varsigma = \infty,$$

for $\alpha > \beta$.

Despite significant progress, there remains a gap in the literature regarding higher-order even DEs, and further exploration is still needed.

So, the motivation for writing this paper is to address this gap by obtaining new and improved oscillation criteria of fourth-order DE (1.1). The key differences that distinguish the oscillation criteria we aim to derive from those proposed in previous works are as follows:

- Relying on various approaches, particularly those aimed at enhancing the monotonic properties of all possible cases of the positive solutions and their derivatives by structuring them in an iterative form, ensures their long-term effectiveness, even when initial conditions are not satisfied.
- Emphasis on continuous improvement, ensuring that the proposed criteria are distinguished by their progressive enhancement over time.
- Apply the well-known comparison method, based on these new relationships, which in turn ensures the derivation of improved criteria and takes advantage of the broader scope of sharp criteria for lower-order equations.

- Deriving absolute oscillation criteria, rather than those constrained within a specific interval, ensures a broader coverage and greater applicability.

This approach will allow for the refinement of several of the previously mentioned criteria, some of which impose more stringent conditions, while others involve complexities that challenge their practical applicability. Additionally, certain criteria rely on intricate derivations that limit the range of their applicability in real-world scenarios.

The main results are organized into three subsections. In Subsection 3.1, we establish key lemmas regarding the monotonic behavior of the solution and its derivatives, which are then used to apply the classical comparison theorem and derive an initial oscillation criterion. In Subsection 3.2, these relations are refined into an iterative form, leading to an improved criterion using the same approach. Subsection 3.3 introduces a different method by applying the $\liminf(\cdot)$ concept to some fundamental functions and coefficients, yielding further enhancements in certain cases. Finally, Section 4 presents the conclusion, a practical application, and a comparative analysis of the proposed criteria.

## 3. Main results

Here, we define class $\mathbb{S}^+$ as the class of all eventually positive solutions to Eq (1.1). Moreover, we use the symbols $\downarrow$ and $\uparrow$ to indicate that the function is decreasing and increasing, respectively. The following symbols are also defined for convenience: $u(t) = x''(t)/t$, $w(t) = x(t)/t$, and

$$\widehat{q}(t) := \int_t^\infty \int_\xi^\infty q(\varsigma)\, d\varsigma d\xi.$$

For classification of positive solutions to Eq (1.1), we present the following lemma, which is directly inferred from [12]. So

**Lemma 5.** *If $x \in \mathbb{S}^+$, then $x$ satisfies one of the following cases:*
$(\overline{\mathrm{P}})$ $x^{(j)} \uparrow$ *for* $j = 0, 1, 2$ *and* $x^{(3)} \downarrow$;
$(\underline{\mathrm{P}})$ $x^{(2j)} \uparrow$ *and* $x^{(2j+1)} \downarrow$ *for* $j = 0, 1$.

### *3.1. Oscillation criterion*

The following results infer the monotonicity and asymptotic properties of solutions in classes $(\overline{\mathrm{P}})$ and $(\underline{\mathrm{P}})$ in Lemma 5.

First, we derive conditions under which both $u$ and $w$ are decreasing and tend to zero, as shown in Lemmas 6 and 7. This behavior is crucial for understanding the asymptotic nature of the solution and provides explicit relationships between the solution and its derivatives.

**Lemma 6.** *Assume that $x \in \mathbb{S}^+$ and satisfies properties in $(\overline{\mathrm{P}})$. If*

$$\int_{t_0}^\infty g^3(\varsigma)\, q(\varsigma)\, d\varsigma = \infty, \tag{3.1}$$

*then $u \downarrow$ and converges to zero.*

*Proof.* Suppose $x \in \mathbb{S}^{+}$ and satisfies properties in $(\overline{\mathrm{P}})$. Since $x^{(j)} > 0$ for $j = 0, 1, 2, 3$, and $x^{(4)} < 0$, it follows from Lemmas 3 and 4 that

$$x > \frac{\nu}{6} t^2 x'', \tag{3.2}$$

for $\nu \in (0, 1)$ and $t \geq t_1 \geq t_0$, which with (1.1) yields

$$\begin{aligned} 0 &\geq x^{(4)} + k\, q\, (x \circ g) \\ &\geq x^{(4)} + \frac{k\nu}{6} q\, g^2 (x'' \circ g)\,. \end{aligned} \tag{3.3}$$

Next, we observe that

$$\begin{aligned} \left[t^2 u'\right]' &= [t\, x''' - x'']' \\ &= t\, x^{(4).} \end{aligned}$$

But the increasing monotonicity of $x'$, (3.2), and (3.3) implies that

$$\begin{aligned} \left[t^2 u'(t)\right]' &\leq -k\, t\, q(t)\ x(g(t)) \\ &\leq -\frac{k\nu}{6}\, t\, q(t)\ g^2(t)\, x''(g(t))\,. \end{aligned} \tag{3.4}$$

Now, using the fact that $x''$ is positive and integrating (3.4) over $[t_1, t]$, we obtain

$$t^2 u'(t) \leq -L - \frac{k\nu}{6} \int_{t_1}^{t} \varsigma q(\varsigma)\, g^2(\varsigma)\, x''(g(\varsigma))\, \mathrm{d}\varsigma, \tag{3.5}$$

where

$$L := -t_1^2 u'(t_1)\,. \tag{3.6}$$

Since $x'' \uparrow$, we have

$$t^2 u'(t) \leq -L - \frac{k\nu}{6} x''(g(t_1)) \int_{t_1}^{t} \varsigma g^2(\varsigma)\, q(\varsigma)\, \mathrm{d}\varsigma.$$

Using (3.1), we find that $u \downarrow$, and so $L$ is positive.

Since $u$ is positive, we get that $\lim_{t\to\infty} u(t) = u_0 \geq 0$. Suppose that $u_0 > 0$. Then, (3.5) becomes

$$u'(t) + \frac{k\nu}{6}\frac{1}{t^2} \int_{t_1}^{t} \varsigma g^3(\varsigma)\, q(\varsigma)\, u(g(\varsigma))\, \mathrm{d}\varsigma \leq -\frac{L}{t^2} < 0, \tag{3.7}$$

Integration of (3.7) over $[t_1, \infty]$, we arrive at

$$\begin{aligned} u(t_1) &> u_0 + \frac{k\nu}{6} \int_{t_1}^{\infty} \frac{1}{\zeta^2} \int_{t_1}^{\zeta} \varsigma g^3(\varsigma)\, q(\varsigma)\, u(g(\varsigma))\, \mathrm{d}\varsigma \mathrm{d}\zeta \\ &> u_0 + \frac{k\nu}{6} u_0 \int_{t_1}^{\infty} \frac{1}{\zeta^2} \int_{t_1}^{\zeta} \varsigma g^3(\varsigma)\, q(\varsigma)\, \mathrm{d}\varsigma \mathrm{d}\zeta \\ &= u_0 + \frac{k\nu}{6} u_0 \int_{t_1}^{\infty} g^3(\varsigma)\, q(\varsigma)\, \mathrm{d}\varsigma. \end{aligned}$$

In view of (3.1), we get a contradiction, and so $u_0 = 0$.

The proof is complete. □

**Lemma 7.** *Assume that $x \in \mathbb{S}^{+}$ and satisfies properties in ($\underline{P}$). If*

$$\int_{t_0}^{\infty} g(\varsigma)\widehat{q}(\varsigma)\,\mathrm{d}\varsigma = \infty, \tag{3.8}$$

*then $w \downarrow$ and converges to zero.*

*Proof.* Suppose $x \in \mathbb{S}^{+}$ and satisfies properties in ($\underline{P}$). Integration of (1.1) over $[t, \infty]$, we arrive at

$$-x'''(t) + k\int_{t}^{\infty} q(\varsigma)\,x(g(\varsigma))\,\mathrm{d}\varsigma \le -\lim_{t\to\infty} x'''(t) \le 0,$$

or

$$-x'''(t) + kx(g(t))\int_{t}^{\infty} q(\varsigma)\,\mathrm{d}\varsigma \le 0. \tag{3.9}$$

Integration of (3.9) over $[t, \infty]$, we arrive at

$$x''(t) + k\int_{t}^{\infty} x(g(\xi))\int_{t}^{\infty} q(\varsigma)\,\mathrm{d}\varsigma\mathrm{d}\xi \le 0,$$

which implies that

$$x'' + k\widehat{q}\,(x \circ g) \le 0. \tag{3.10}$$

Next, we observe that

$$\begin{aligned} \left[t^2 w'\right]' &= [tx' - x]' \\ &= tx'' \\ &\le -k\,t\widehat{q}\,(x \circ g)\,. \end{aligned} \tag{3.11}$$

Integration of (3.11) over $[t_0, t]$, we obtain

$$t^2 w'(t) \le -M - k\int_{t_0}^{t} \varsigma\widehat{q}(\varsigma)\,x(g(\varsigma))\,\mathrm{d}\varsigma, \tag{3.12}$$

where

$$M = -t_0^2 w'(t_0)\,.$$

Since $x \uparrow$, we have

$$t^2 w'(t) \le -M - kx(g(t_0))\int_{t_0}^{t} \varsigma\widehat{q}(\varsigma)\,\mathrm{d}\varsigma.$$

Using (3.8), we find that $w \downarrow$, and so $M$ is positive.

Since $w$ is positive, we have that $\lim_{t\to\infty} w(t) = w_0 \ge 0$. Suppose that $w_0 > 0$. So, (3.12) reduces to

$$w'(t) + k\frac{1}{t^2}\int_{t_0}^{t} \varsigma g(\varsigma)\widehat{q}(\varsigma)\,w(g(\varsigma))\,\mathrm{d}\varsigma \le -\frac{M}{t^2} < 0 \tag{3.13}$$

Integration of (3.13) over $[t_0, \infty]$, we arrive at

$$w(t_0) \;>\; w_0 + k\int_{t_0}^{\infty} \frac{1}{\xi^2}\int_{t_0}^{t} \varsigma g(\varsigma)\widehat{q}(\varsigma)\,w(g(\varsigma))\,\mathrm{d}\varsigma\mathrm{d}\xi$$

$$> \quad w_0 + kw_0 \int_{t_0}^{\infty} g(\varsigma)\, \widehat{q}(\varsigma)\, \mathrm{d}\varsigma,$$

which contradicts (3.8). So, $u_0 = 0$.

The proof is complete. □

Next, we apply the conditions derived in the previous lemmas to a well-known comparison theorem to reduce the order of (1.1) to first-order DDE. This allows us to utilize its sharp and numerous oscillatory criteria for this type of equation, ensuring we obtain improved oscillation criteria for our equations.

**Theorem 1.** *Assume that (3.1) and (3.8) hold. If*

$$\liminf_{t\to\infty} \int_{g(t)}^{t} \frac{1}{\theta^2} \int_0^{\theta} \varsigma g^3(\varsigma)\, q(\varsigma)\, \mathrm{d}\varsigma \mathrm{d}\theta > \frac{6}{k\nu \mathrm{e}} \tag{3.14}$$

*for any* $\nu \in (0,1)$*, and*

$$\liminf_{t\to\infty} \int_{g(t)}^{t} \frac{1}{\theta^2} \int_0^{\theta} \varsigma g(\varsigma)\, \widehat{q}(\varsigma)\, \mathrm{d}\varsigma \mathrm{d}\theta > \frac{1}{k\mathrm{e}} \tag{3.15}$$

*then Eq (1.1) is oscillatory.*

*Proof.* Suppose the contrary that $x \in \mathbb{S}^+$. From Lemma 5, we have cases $(\overline{\mathrm{P}})$ and $(\underline{\mathrm{P}})$.

Let $x$ satisfy properties in $(\overline{\mathrm{P}})$. It follows from Lemma 6 that (3.7) holds. Then,

$$\begin{aligned} u'(t) &\leq -\frac{1}{t^2}\left(L + \frac{k\nu}{6}\int_{t_1}^{t} \varsigma g^3(\varsigma)\, q(\varsigma)\, u(g(\varsigma))\, \mathrm{d}\varsigma\right) \\ &\leq -\frac{1}{t^2}\left(L + \frac{k\nu}{6} u(g(t)) \int_{t_1}^{t} \varsigma g^3(\varsigma)\, q(\varsigma)\, \mathrm{d}\varsigma\right) \\ &= -\frac{1}{t^2}\left(\frac{k\nu}{6} u(g(t)) \int_{0}^{t} \varsigma g^3(\varsigma)\, q(\varsigma)\, \mathrm{d}\varsigma + L - Ku(g(t))\right), \end{aligned} \tag{3.16}$$

where $L$ is defined as in (3.6), and

$$K := \frac{k\nu}{6} \int_0^{t_0} \varsigma g^3(\varsigma)\, q(\varsigma)\, \mathrm{d}\varsigma.$$

Since $u$ converges to zero, there is $t_1 \geq t_0$ such that $L - Ku(g(t)) \geq 0$. Hence, (3.16) becomes

$$u'(t) + \frac{k\nu}{6}\frac{1}{t^2} u(g(t)) \int_0^{t} \varsigma g^3(\varsigma)\, q(\varsigma)\, \mathrm{d}\varsigma \leq 0. \tag{3.17}$$

According to Theorem 2.1.1 in [15], the existence of a positive solution $u$ to inequality (3.17) contradicts condition (3.14).

Let $x$ satisfy properties in $(\underline{\mathrm{P}})$. It follows from Lemma 7 that (3.13) holds. Proceeding as in the previous case, we also get a contradiction with condition (3.15). Hence, all solutions are oscillatory. □

*3.2. Improved oscillation criterion I*

In this section, we work on improving the monotonic properties of the functions $u$ and $w$ and then see the effect of this on the oscillation criterion. The following symbols are also defined for convenience:

$$\mu_{\nu}(t) := \exp\left[\frac{k\nu}{6}\int_{t_0}^{t}\frac{1}{\theta^2}\int_{t_1}^{\theta}\varsigma g^3(\varsigma)\,q(\varsigma)\,\mathrm{d}\varsigma\mathrm{d}\theta\right]$$

and

$$\eta(t) := \exp\left[k\int_{t_0}^{t}\frac{1}{\theta^2}\int_{t_0}^{\theta}\varsigma g(\varsigma)\widehat{q}(\varsigma)\,\mathrm{d}\varsigma\mathrm{d}\theta\right].$$

The following lemma builds on the results from earlier lemmas by considering the recursive nature of the relationship between the solution and its derivatives.

**Lemma 8.** *Assume that $x \in \mathbb{S}^+$, (3.1) and (3.8) hold. Then, $\mu_\nu u \downarrow$ and $\eta w \downarrow$, for $\nu \in (0,1)$.*

*Proof.* Suppose that $x \in \mathbb{S}^+$.

Let $x$ satisfy properties in $\left(\overline{\mathrm{P}}\right)$. It follows from Lemma 6 that (3.7) holds. Then,

$$u'(t) + \frac{k\nu}{6}\frac{1}{t^2}u(t)\int_{t_1}^{t}\varsigma g^3(\varsigma)\,q(\varsigma)\,\mathrm{d}\varsigma < 0,$$

or $(\mu_\nu u)' < 0$.

Let $x$ satisfy properties in $(\underline{\mathrm{P}})$. It follows from Lemma 7 that (3.13) holds. Proceeding as in the previous case, we get that $(\eta w)' < 0$. □

The following theorem is similar to Theorem 1, but we incorporate the newly derived recursive relations to improve the results. By applying these relations through a comparison method, we obtain more refined oscillation criteria for the solution and its derivatives.

**Theorem 2.** *Assume that (3.1) and (3.8) hold. If*

$$\liminf_{t\to\infty}\int_{g(t)}^{t}\frac{\mu_\nu(g(\theta))}{\theta^2}\int_{t_0}^{\theta}\frac{\varsigma g^3(\varsigma)\,q(\varsigma)}{\mu_\nu(g(\varsigma))}\mathrm{d}\varsigma\mathrm{d}\theta > \frac{6}{k\nu\mathrm{e}}, \tag{3.18}$$

*for any $\nu \in (0,1)$, and*

$$\liminf_{t\to\infty}\int_{g(t)}^{t}\frac{\eta(g(\theta))}{\theta^2}\int_{t_0}^{\theta}\frac{\varsigma g(\varsigma)\widehat{q}(\varsigma)}{\eta(g(\varsigma))}\mathrm{d}\varsigma\mathrm{d}\theta > \frac{1}{k\mathrm{e}} \tag{3.19}$$

*then Eq (1.1) is oscillatory.*

*Proof.* Suppose the contrary that $x \in \mathbb{S}^+$. From Lemma 5, we have cases $\left(\overline{\mathrm{P}}\right)$ and $(\underline{\mathrm{P}})$.

Let $x$ satisfy properties in $\left(\overline{\mathrm{P}}\right)$. It follows from Lemma 6 that (3.7) holds. Since $\mu_\nu u \downarrow$ , it follows from (3.7) that

$$u'(t) + \frac{k\nu}{6}\frac{\mu_\nu(g(t))}{t^2}u(g(t))\int_{t_1}^{t}\frac{\varsigma g^3(\varsigma)\,q(\varsigma)}{\mu_\nu(g(\varsigma))}\mathrm{d}\varsigma < 0 \tag{3.20}$$

According to Theorem 2.1.1 in [15], the existence of a positive solution $u$ to inequality (3.20) contradicts condition (3.18).

Let $x$ satisfy properties in $(\underline{\mathrm{P}})$. It follows from Lemma 7 that (3.13) holds. Proceeding as in the previous case, we also get a contradiction with condition (3.19). Hence, all solutions are oscillatory. □

### 3.3. Improved oscillation criterion II

In this part, we will employ a different strategy to enhance the monotonic features. We then evaluate the effect of the new properties on the oscillation criterion of Eq (1.1).

**Theorem 3.** *Equation (1.1) is oscillatory if*

$$\liminf_{t\to\infty}\int_{g(t)}^{t} q(\varsigma)\left[(g(\varsigma))^3+\frac{k\nu_1}{6}\int_{t_1}^{g(\varsigma)}\theta^3 q(\theta)\, g^3(\theta)\,\mathrm{d}\theta\right]\mathrm{d}\varsigma > \frac{6}{k\nu_2 \mathrm{e}} \tag{3.21}$$

*and*

$$\liminf_{t\to\infty}\int_{g(t)}^{t} \widehat{q}(\varsigma)\left[g(\varsigma)+k\nu_3\int_{t_1}^{g(\varsigma)}\theta g(\theta)\,\widehat{q}(\theta)\,\mathrm{d}\theta\right]\mathrm{d}\varsigma > \frac{1}{k\nu_3 \mathrm{e}}, \tag{3.22}$$

*for any $\nu_i \in (0,1)$ and $t_1 \geq t_0$ is large enough.*

*Proof.* Suppose the contrary that $x \in \mathbb{S}^+$. From Lemma 5, we have cases $(\overline{\mathrm{P}})$ and $(\underline{\mathrm{P}})$.

Let $x$ satisfy properties in $(\overline{\mathrm{P}})$. Let $x' = z$. Then, $z^{(j)} > 0$ for $j = 0, 1, 2$, and $z^{(3)} < 0$. Using Lemmas 3 and 4 twice, once for $x$ and once for $z$, we get

$$x > \frac{\nu_1}{6} t^3 x'''$$

and

$$\frac{1}{\nu_2} z > \frac{t^2}{2} z'' \rightarrow x' > \frac{\nu_2}{2} t^2 x''' \tag{3.23}$$

for all $\nu_i \in (0,1)$. Hence, Eq (1.1) reduces to

$$x^{(4)} \leq -\frac{k\nu_1}{6}\, q\, g^3\, (x''' \circ g). \tag{3.24}$$

From (3.23) and (3.24), we obtain

$$\begin{aligned}\left(x - \frac{\nu_2}{6} t^3 x'''\right)' &= x' - \frac{\nu_2}{2} t^2 x''' - \frac{\nu_2}{6} t^3 x^{(4)} \\ &> -\frac{\nu_2}{6} t^3 x^{(4)} \\ &> \frac{k\nu_1\nu_2}{36} t^3\, q\, g^3\, (x''' \circ g).\end{aligned} \tag{3.25}$$

Integration of (3.25) over $[t_1, t]$, we arrive at

$$\begin{aligned}x(t) &> \frac{\nu_2}{6} t^3 x'''(t) + \frac{k\nu_1\nu_2}{36}\int_{t_1}^{t} \varsigma^3 q(\varsigma)\, g^3(\varsigma)\, x'''(g(\varsigma))\,\mathrm{d}\varsigma \\ &> \frac{\nu_2}{6} x'''(t)\left[t^3 + \frac{k\nu_1}{6}\int_{t_1}^{t}\varsigma^3 q(\varsigma)\, g^3(\varsigma)\,\mathrm{d}\varsigma\right].\end{aligned}$$

Setting $y := x'''$ and combining (1.1) with (3.26), we find that $y$ is a positive solution of

$$y'(t) + \frac{k\nu_2}{6} q(t)\left[(g(t))^3 + \frac{k\nu_1}{6}\int_{t_1}^{g(t)}\varsigma^3 q(\varsigma)\, g^3(\varsigma)\,\mathrm{d}\varsigma\right] y(g(t)) \leq 0.$$

Using Theorem 2.1.1 in [15], a contradiction arises with (3.21).

Let $x$ satisfy properties in $(\underline{\mathrm{P}})$. It follows from Lemma 7 that (3.10) holds. From Lemmas 3 and 4, we have that $x > \nu_3 t x'$ for $\nu_3 \in (0,1)$. Then, from (3.10), we arrive at

$$\begin{aligned}
(x - \nu_3 t x')' &= (1-\nu_3)\,x' - \nu_3 t x'' \\
&> -\nu_3 t x'' \\
&> k\nu_3\, t\,\widehat{q}\,(x \circ g) \\
&> k\nu_3^2\, t\, g\,\widehat{q}\,(x' \circ g)\,.
\end{aligned} \tag{3.26}$$

Integration of (3.26) over $[t_1, t]$, we obtain

$$\begin{aligned}
x(t) &> \nu_3 t x'(t) + k\nu_3^2 \int_{t_1}^{t} \varsigma g(\varsigma)\,\widehat{q}(\varsigma)\,x'(g(\varsigma))\,\mathrm{d}\varsigma \\
&> \nu_3 x'(t)\left[t + k\nu_3 \int_{t_1}^{t} \varsigma g(\varsigma)\,\widehat{q}(\varsigma)\,\mathrm{d}\varsigma\right].
\end{aligned}$$

Setting $z := x'$ and combining (1.1) with (3.10), we find that $z$ is a positive solution of

$$z'(t) + k\nu_3\,\widehat{q}(t)\left[g(t) + k\nu_3 \int_{t_1}^{g(t)} \varsigma g(\varsigma)\,\widehat{q}(\varsigma)\,\mathrm{d}\varsigma\right] z(g(t)) \le 0.$$

Using Theorem 2.1.1 in [15], a contradiction arises with (3.22). Hence, all solutions are oscillatory. □

Several techniques have been developed to improve the oscillation criteria of functional differential equations. One particularly effective approach is based on enhancing the monotonicity properties of positive solutions. In our earlier results, we used multiple methods to establish new monotonic relationships for the positive solution cases, namely, $(\overline{\mathrm{P}})$ and $(\underline{\mathrm{P}})$. These new relations served as the basis for the determination of the improved oscillation criteria presented in Theorems 1–3.

## 4. Examples and discussions

In what follows, we apply these results to a well-known form of Euler-type differential equations to evaluate their practical effectiveness and determine which criteria yield the sharpest bounds. In addition, we present a numerical comparison with the results of previous works to highlight the improvements achieved by our approach.

**Example 1.** *Consider the Euler DDE*

$$x^4(t) + \frac{q_0}{t^4} x(\delta t) = 0, \tag{4.1}$$

*where $q_0 > 0$ and $\delta \in (0,1]$. It is easy to check that*

$$\begin{aligned}
\widehat{q}(t) &= \frac{q_0}{6t^2}, \\
\mu_\nu(t) &= t^{\frac{\nu}{6}\delta^3 q_0}
\end{aligned}$$

*and*

$$\eta(t) = t^{\frac{1}{6}\delta q_0}.$$

*By simple calculations, we can verify that (3.1) and (3.8) are satisfied.*
*Using Theorem 1, Eq (4.1) is oscillatory if*

$$q_0 > \max\left\{\frac{6}{\mathrm{e}\delta^3 \ln(1/\delta)}, \frac{6}{\mathrm{e}\delta \ln(1/\delta)}\right\} = \frac{6}{\mathrm{e}\delta^3 \ln(1/\delta)}. \tag{4.2}$$

*From Theorem 2, Eq (4.1) is oscillatory if*

$$\begin{aligned} q_0 &> \frac{6}{\mathrm{e}\delta^3 \ln(1/\delta)}\left(1 - \tfrac{\nu}{6}\delta^3 q_0\right), \\ q_0 &> \frac{6}{\mathrm{e}\delta \ln(1/\delta)}\left(1 - \tfrac{1}{6}\delta q_0\right). \end{aligned} \tag{4.3}$$

*Applying Theorem 3 confirms the oscillation of (4.1) when*

$$\begin{aligned} q_0\left(1 + \tfrac{\nu_1}{18}\delta^3 q_0\right) &> \frac{6}{\mathrm{e}\delta^3 \ln(1/\delta)}, \\ q_0\left(1 + \tfrac{\nu_3}{6}\delta q_0\right) &> \frac{6}{\mathrm{e}\delta \ln(1/\delta)}. \end{aligned} \tag{4.4}$$

**Remark 1.** *To test the adequacy of the conditions, we consider the special case of (4.1) when* $\delta = 1/e$. *Conditions (4.2) and (4.4) reduce, respectively, to* $q_0 > 6e^2$ *and* $q_0 \gtrapprox 39.925$. *Therefore, Theorem 3 improves Theorem 1 and provides a better criterion for testing the oscillation. On the other hand, conditions (4.3) becomes*

$$6\frac{e^3}{e+1} < q_0 < 6e^3 \tag{4.5}$$

*and*

$$\frac{6e}{e+1} < q_0 < 6e. \tag{4.6}$$

*We note that conditions (4.5) and (4.6) cannot be satisfied together. Therefore, Theorem 2 does not provide a criterion that guarantees the oscillation of all solutions. However, this theorem can be used to exclude one type of positive solution. For example, we find that condition (4.5) guarantees that there are no positive solutions that satisfy* $x''(t) > 0$.

**Remark 2.** *Applying the last example to the criterias (2.2)–(2.5) in literature, we obtain tha criterion (2.2) reduces to*

$$q_0 > \frac{192}{\mathrm{e}\delta^3 \ln(1/\delta)}, \tag{4.7}$$

*while, (2.3) and (2.4) becomes*

$$q_0 > \max\left\{\frac{9}{2\delta^3(1-a)}, \frac{3}{2\delta(1-a)}\right\}, \tag{4.8}$$

*and for* $r \in \mathbb{Z}^+$, *(2.5) yields*

$$q_0 > \left(\frac{1}{\mathrm{e}^r(1-\delta)}\right)^{1/r+1} \cdot \frac{6}{\delta^3 \ln\left(\frac{1}{\delta}\right)(1-a)}. \tag{4.9}$$

In order to complement the preceding qualitative analysis, we now provide a numerical comparison highlighting the performance of our proposed criteria relative to these existing results. The following table summarizes the minimum values of $q_0$ that guarantee oscillation across different test cases with $\nu_1 = \nu_3 = 0.9$, $r = 10$, and $a = 0$, thereby quantitatively illustrating the improvement achieved.

**Table 1.** Numerical comparison of minimum $q_0$ values for oscillation across different cases.

| | Our results | | Previous results | | |
|---|---|---|---|---|---|
| Special case | (4.2) | (4.4) | (4.7) | (4.8) | (4.9) |
| $\delta = 1/e$ | 44.334 | 39.925 | 1418.7 | 90.385 | 50.620 |
| $\delta = 0.5$ | 25.475 | 22.087 | 815.213 | 36 | 29.714 |

Based on this comparison, we conclude that our criterion (4.4) significantly improves the previous work of Zafer [8], Agarwal et al. [9], and Zhang et al. [10].

To further support the numerical findings presented in Table 1, we now provide a graphical comparison that illustrates the variation in the minimum values of $q_0$ that ensures oscillation, in a continuous range of $\delta$.

It should be noted that the criterion (4.7) does not appear in the Figure 1, as its values significantly exceed the plotted range, while the other criteria yield results within the displayed range.

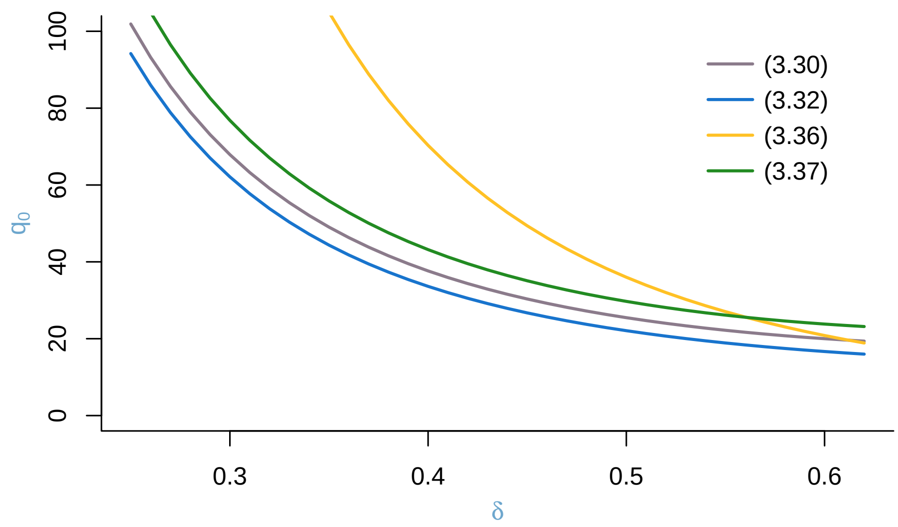


**Figure 1.** Oscillation criteria behavior across varying $\delta$ values.

## 5. Conclusions

In conclusion, this work provides a detailed analysis of the oscillation criteria for fourth-order delay differential equations (1.1) by proposing new theorems, namely Theorems 1–3, which led to the formulation of enhanced oscillation criteria. These new criteria were examined through a series of relations derived in Lemmas 5–8. The effectiveness of the new criteria was evaluated through numerical comparisons of the values of $q_0$ in various scenarios, presented in Table 1, and through graphical comparisons shown in Figure 1, contrasting our results with those of previous works such as Zafer [8], Agarwal et al. [9], and Zhang et al. [10]. The findings demonstrate that our criteria are sharper and more general, offering significant improvements in predicting oscillation behavior across various test cases.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare that there is no conflict of interest.

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