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https://www.aimspress.com/journal/era

ERA, 33(6): 3571–3583.
DOI: 10.3934/era.2025159
Received: 28 September 2024
Revised: 27 February 2025
Accepted: 04 June 2025
Published: 11 June 2025

Research article

On global existence for the stochastic nonlinear Schrödinger equation with time-dependent linear loss/gain

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Abstract: This paper was focused on global existence for the stochastic nonlinear Schrödinger equation with time-dependent loss/gain, which read $idu + (\Delta u + \lambda|u|^{\alpha}u + ia(t)u)dt = dW$. We proved the global existence and uniqueness of the solution in $H^1(\mathbb{R}^N)$ through the uniform boundedness of the momentum and energy functionals. The global existence result of the solution for this type of equation depended on the ranges of time-dependent loss/gain coefficient.

Keywords: stochastic nonlinear Schrödinger equation; global existence; time-dependent loss/gain; additive noise; energy space

1. Introduction

The nonlinear Schrödinger equation with time-dependent coefficient, as one of the basic models for optics and Bose-Einstein condensates (BECs), has gained widespread attention in recent years (see [1, 2] and references therein). It is logical to account for random effects disturbing the system. A standard approach in physics involves considering the Gaussian space-time white noise. Nevertheless, space-time white noise cannot be theoretically handled in mathematics, thus the noise is considered white in time and colored in space (see [3–5]).

In this paper, we are concerned with the global existence of the solution for the following stochastic nonlinear Schrödinger equation with time-dependent linear loss/gain

$$idu + (\Delta u + \lambda|u|^{\alpha}u + ia(t)u)dt = dW, \quad (t, x) \in [0, \infty) \times \mathbb{R}^N \tag{1.1}$$

with

$$u(0, x) = u_0(x), \quad x \in \mathbb{R}^N,$$

where $u_0(x) \in H^1(\mathbb{R}^N)$, $\lambda = 1$, or $\lambda = -1$ denotes that the nonlinearity is focusing or defocusing, $0 < \alpha < \frac{4}{N-2}$ if $N \geq 3$ or $\alpha > 0$ if $N = 1, 2$, W denotes the complex valued Wiener process, $a(t)$ is a real valued function defined on the interval $[0,\infty)$, and $a(t) > 0$ or $a(t) < 0$ describes the strength of loss or gain. For example, the time-dependent linear loss/gain term is described in the theory of BECs, where it represents the mechanism of continuously loading external atoms into the BECs (gain) by optical pumping or continuous depletion (loss) of atoms from the BECs (see [6]). From a phenomenological perspective, the time-dependent linear loss/gain term is used to explain the interaction of atomic clouds or thermal clouds (see [7]).

Recently, the global existence for the stochastic nonlinear Schrödinger equation has been extensively studied and many important results have been achieved. For example, in [4], it is proved that the classical stochastic nonlinear Schrödinger equation with additive or multiplicative noise admits the global existence in $H^1(\mathbb{R}^N)$, respectively. In [5], it is showed that the defocusing energy-critical stochastic nonlinear Schrödinger equation with an additive noise admits the global existence by atomic spaces machinery and probabilistic perturbation argument. The authors in [8] study the global existence for the stochastic nonlinear Schrödinger equation with nonlinear Stratonovich noise in subcritical case, based on the deterministic and stochastic Strichartz's estimates. [9] investigates the defocusing mass critical nonlinear Schrödinger equation with a small multiplicative noise, it shows the global space-time bound by the decomposition of the solution. [10] proves that the solution of the stochastic nonlinear Schrödinger equation with linear multiplicative noise is global when defocusing, $\alpha = \frac{4}{N}$ (mass-critical), $N \geq 1$ or $\alpha = \frac{4}{N-2}$ (energy-critical), and $N \geq 3$ by rescaling transformation and the stability results. When $a(t) = a > 0$, Equation (1.1) reduces to the weakly damped stochastic nonlinear Schrödinger equation. The global existence for this type of stochastic nonlinear Schrödinger equation driven by an additive noise is obtained (see [11, 12]). [13] is devoted to the global existence for the stochastic nonlinear Schrödinger equation with weak damping driven by a multiplicative noise in mass-critical case. [14] uses the conservation of the $L^2(\mathbb{R}^N)$ norm and iteration argument to study the global existence for the random nonlinear Schrödinger equation with white noise dispersion and nonlinear time-dependent loss/gain in $L^2(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$.

Physically speaking, the nature of the function $a(t)$ will have a significant impact on the behavior of the solution. [1] and [2] show the global existence result of the solution depending on the ranges of the time-dependent coefficient $a(t)$. Inspired by the articles above, our main goal in this paper is to study the global existence of solution for Eq (1.1) with $a(t)$ being time-dependent. Because of the loss of energy, the energy functional no longer satisfies the conservation law. In order to overcome the difficulty, setting $u(t,x) = e^{-\int_0^t a(s)ds} v(t,x)$ in Eq (1.1), $v(t,x)$ satisfies

$$idv + (\Delta v + \lambda e^{-\alpha \int_0^t a(s)ds} |v|^\alpha v)dt = e^{\int_0^t a(s)ds} dW, \quad (t,x) \in [0,\infty) \times \mathbb{R}^N \tag{1.2}$$

with

$$v(0,x) = u_0(x), \quad x \in \mathbb{R}^N.$$

Therefore, in order to obtain the global existence of the solution for Eq (1.1), we only need to study the global existence of the solution for Eq (1.2). We use the uniform boundedness of the momentum and energy functionals to obtain the global existence for Eq (1.2). Our main theorem is as follows.

Theorem 1.1. *Let $0 < \alpha < \frac{4}{N-2}$ if $N \geq 3$ or $\alpha > 0$ if $N = 1, 2$, and $\phi \in L_2^{0,1}$ and v_0 is an $\mathcal{F}_0$-measurable random variable with values in $H^1(\mathbb{R}^N)$. Assume that*

(1) *either $\lambda = 1$, $0 < \alpha < \frac{4}{N}$, $a(t) \in L^1(0,\infty)$, $a(t) < 0$,*

(2) *or $\lambda = -1$, $a(t) \in L^1_{loc}(0,\infty)$, $a(t)$ permits sign-changing,*

then for every v_0, there exists a unique global solution of Eq (1.2) in $H^1(\mathbb{R}^N)$, i.e., $\tau^*(v_0) = +\infty$.

According to Theoerm 1.1, we find that the global existence result of the solution depends on the ranges of the time-dependent loss/gain coefficient. In the absence of the time-dependent loss/gain term, i.e., $a(t) = 0$, the conclusion of Theoerm 1.1 reduces to the well-established result presented in Theorem 3.4 in [4], thereby demonstrating the consistency of our result within the existing theoretical framework.

This paper is organized as follows. In Section 2, we show the local existence for Eq (1.2) and study the evolution laws of the momentum and energy. In Section 3, under certain assumptions on λ, α, and $a(t)$, we prove the global existence for Eq (1.2).

2. Preliminaries

In this section, we first introduce some mathematical spaces and estimates. Then, through using the method of [4], the local existence for Eq (1.2) is proved. Finally, we give the evolution laws of the momentum and energy.

Throughout this paper, we use the following notations (see [15]). For $p \geq 1$, $L^p(\mathbb{R}^N)$ denotes the Lebesgue space of p-integrable complex valued functions on $\mathbb{R}^N$. The Hilbert space $L^2(\mathbb{R}^N)$ is endowed with the norm and inner product

$$\|u\|_{L^2(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |u(x)|^2 dx\right)^{\frac{1}{2}},$$

$$(u, v) = Re \int_{\mathbb{R}^N} u(x)\bar{v}(x)dx, \quad u, v \in L^2(\mathbb{R}^N).$$

For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^N)$ of tempered distribution $u \in S'(\mathbb{R}^N)$ whose Fourier transform $\hat{v}$ satisfies $(1 + |\xi|^2)^{\frac{s}{2}} \hat{v}(\xi) \in L^2(\mathbb{R}^N)$. For a Banach space B, $T > 0$ and $p \geq 1$, and $L^p(0, T; B)$ denotes the space of functions from $[0, T]$ into B with p-integrable over $[0, T]$.

Definition 2.1. (See [15, 16].) The pair (r, q) is said to be admissible if $\frac{2}{r} = N(\frac{1}{2} - \frac{1}{q})$ and $2 \leq q \leq \frac{2N}{N-2}$ when $N \geq 3$, or $2 \leq r \leq \infty$ when $N = 1, 2$.

Lemma 2.2. *(Strichartz's estimates). (See [15, 16].) Let (r, q), (r_1, q_1), and (r_2, q_2) be admissible pairs; $S(t) = e^{it\Delta}$ denotes the linear Schrödinger propagator, $T > 0$, then the following properties hold,*

(i) for every $g \in L^2(\mathbb{R}^N)$, there exists a constant C such that

$$\|S(\cdot)g\|_{L^r(0,T;L^q(\mathbb{R}^N))} \leq C\|g\|_{L^2(\mathbb{R}^N)},$$

(ii) for every $G \in L^{r_2'}(0, T; L^{q_2'}(\mathbb{R}^N))$, there exists a constant C such that

$$\left\|\int_0^T S(t-s)G(s)ds\right\|_{L^{r_1}(0,T;L^{q_1}(\mathbb{R}^N))} \leq C\|G\|_{L^{r_2'}(0,T;L^{q_2'}(\mathbb{R}^N))},$$

where r_2' and q_2' are the conjugates of r_2 and q_2.

In order to state precisely Eqs (1.1) and (1.2), we consider the probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$. Let $\{\beta_k\}_{k\in\mathbb{N}}$ be a sequence of independent real valued Brownian motions associated to $\{\mathcal{F}_t\}_{t\geq 0}$, and let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^N)$. We consider the complex valued Wiener process

$$W(t, x, \omega) = \sum_{k\in\mathbb{N}} \beta_k(t, \omega)\phi e_k(x), \quad t \geq 0, \quad x \in \mathbb{R}^N, \quad \omega \in \Omega,$$

where $\phi \in L_2^{0,s}$, which is the space of the Hilbert-Schmidt operator from $L^2(\mathbb{R}^N)$ into $H^s(\mathbb{R}^N)$. The corresponding norm is then given by

$$\|\phi\|^2_{L_2^{0,s}} = tr(\phi^*\phi) = \sum_{k\in\mathbb{N}} \|\phi e_k\|^2_{H^s(\mathbb{R}^N)}.$$

Next, we study the local existence for Eq (1.2).

Theorem 2.3. *Let $0 < \alpha < \frac{4}{N-2}$ if $N \geq 3$ or $\alpha > 0$ if $N = 1, 2$, $a(t) \in L^1(0,\infty)$, $\phi \in L_2^{0,1}$, and the initial data v_0 is an $\mathcal{F}_0$-measurable random variable with values in $H^1(\mathbb{R}^N)$. Then, there exists a unique solution v to Eq (1.2) with continuous $H^1(\mathbb{R}^N)$ valued paths, such that $v(0) = v_0$. This solution is defined on a random interval $[0, \tau^*(v_0))$, where $\tau^*(v_0)$ is a stopping time such that*

$$\tau^*(v_0) = +\infty \quad or \lim_{t\to\tau^*(v_0)} \|v(t)\|_{H^1(\mathbb{R}^N)} = +\infty.$$

Proof. We use the mild form of Eq (1.2), that is,

$$v(t) = S(t)v_0 + i\lambda \int_0^t S(t-s)\left(e^{-\alpha\int_0^s a(m)dm}|v(s)|^\alpha v(s)\right)ds - i\int_0^t S(t-s)e^{\int_0^s a(m)dm}dW(s). \tag{2.1}$$

Set

$$z(t) = i\int_0^t S(t-s)e^{\int_0^s a(m)dm}dW(s).$$

By similar analysis to [4], we just need to prove $z \in L^r(0,T; W^{1,\alpha+2}(\mathbb{R}^N))$ almost surely for any $T > 0$, where $(r, \alpha+2)$ is an admissible pair and $r = \frac{4(\alpha+2)}{N\alpha}$. Since z is a Gaussian process and $r > 2$, we have

$$\begin{aligned}
\mathbb{E}\left(\int_0^T \|z(s)\|^r_{L^{\alpha+2}(\mathbb{R}^N)}ds\right) &= \int_0^T \mathbb{E}\left(\|z(s)\|^r_{L^{\alpha+2}(\mathbb{R}^N)}\right)ds \\
&\leq c_1 \int_0^T \left(\mathbb{E}(\|z(s)\|^{\alpha+2}_{L^{\alpha+2}(\mathbb{R}^N)})\right)^{\frac{r}{\alpha+2}} ds \\
&= c_1 \int_0^T \left(\int_{\mathbb{R}^N} \mathbb{E}(z|s,x|^{\alpha+2})dx\right)^{\frac{r}{\alpha+2}} ds \\
&\leq c_2 \int_0^T \left(\int_{\mathbb{R}^N} \left(\mathbb{E}(|z(s,x)|^2)\right)^{\frac{\alpha+2}{2}} dx\right)^{\frac{r}{\alpha+2}} ds.
\end{aligned} \tag{2.2}$$

Since

$$\mathbb{E}\left(|z(s,x)|^2\right) = \sum_{k\in\mathbb{N}} \int_0^s |S(s-\tau)e^{\int_0^\tau a(m)dm}\phi e_k|^2 d\tau,$$

where $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}^N)$, by Minkowski's inequality and $r > 2$, we deduce

$$\begin{aligned}
\left(\int_{\mathbb{R}^N} \left(\mathbb{E}(|z(s,x)|^2)\right)^{\frac{\alpha+2}{2}} dx\right)^{\frac{2}{\alpha+2}} &\leq \sum_{k\in\mathbb{N}} \int_0^s \left(\int_{\mathbb{R}^N} |S(s-\tau)e^{\int_0^\tau a(m)dm}\phi e_k|^{\alpha+2}dx\right)^{\frac{2}{\alpha+2}} d\tau \\
&= \sum_{k\in\mathbb{N}} \|S(\cdot)e^{\int_0^\tau a(m)dm}\phi e_k\|^2_{L^2(0,s;L^{\alpha+2}(\mathbb{R}^N))}
\end{aligned}$$

$$\leq c_3 \sum_{k\in\mathbb{N}} \|S(\cdot)e^{\int_0^\tau a(m)dm}\phi e_k\|^2_{L^r(0,T;L^{\alpha+2}(\mathbb{R}^N))}$$
$$\leq c_3 \sum_{k\in\mathbb{N}} \|e^{\int_0^\tau a(m)dm}\phi e_k\|^2_{L^2(\mathbb{R}^N)},$$

where c_3 depends only on r, α, and T, and Strichartz's estimates are used in the last inequality. Because $a(t) \in L^1(0,\infty)$,

$$\left(\int_{\mathbb{R}^N} \left(\mathbb{E}(|z(s,x)|^2)\right)^{\frac{\alpha+2}{2}} dx\right)^{\frac{2}{\alpha+2}} \leq c_3 e^{\int_0^T a(m)dm} \sum_{k\in\mathbb{N}} \|\phi e_k\|^2_{L^2(\mathbb{R}^N)} = c_4\|\phi\|^2_{L_2^{0,0}}, \tag{2.3}$$

where c_4 depends only on r, α, T, and $a(t)$. Combining (2.2) and (2.3), we get

$$\mathbb{E}\left(\|z\|^r_{L^r\left(0,T;L^{\alpha+2}(\mathbb{R}^N)\right)}\right) \leq c_5\|\phi\|^2_{L_2^{0,0}},$$

where c_5 depends only on r, α, T, and $a(t)$. Since the spatial derivatives and $S(\cdot)$ commute, the same computation shows that

$$\mathbb{E}\left(\|z(\cdot)\|^r_{L^r\left(0,T;W^{1,\alpha+2}(\mathbb{R}^N)\right)}\right) \leq c_6\|\phi\|_{L_2^{0,1}},$$

which proves the Theorem 2.3. □

Remark 2.4. Suppose that $u(t,x) = e^{-\int_0^t a(s)ds}v(t,x)$, and we obtain the local existence for Eq (1.1).

Now, we give the evolution laws of the momentum

$$M(v) = \|v\|^2_{L^2(\mathbb{R}^N)}$$

and energy

$$H(v,t) = \frac{1}{2}\|\nabla v\|^2_{L^2(\mathbb{R}^N)} - \frac{\lambda}{\alpha+2}e^{-\alpha\int_0^t a(s)ds}\|v\|^{\alpha+2}_{L^{\alpha+2}(\mathbb{R}^N)}. \tag{2.4}$$

Proposition 2.5. *Let α, $a(t)$, ϕ, and v_0 be as in Theorem 2.3. Then, for any stopping time τ such that $\tau < \tau^*(v_0)$ a.s., we have*

$$M(v(\tau)) = M(v_0) - 2\,\mathrm{Im}\sum_{k\in\mathbb{N}} \int_0^\tau \int_{\mathbb{R}^N} v e^{\int_0^s a(m)dm}\overline{\phi e_k}dxd\beta_k(s) + \|\phi\|^2_{L_2^{0,0}} \int_0^\tau e^{2\int_0^s a(m)dm}ds. \tag{2.5}$$

Moreover, for any $p \in \mathbb{R}$ and $p \geq 1$, there exist constants $M_p \geq 0$, such that

$$\mathbb{E}\left(\sup_{t\in[0,\tau]} M^p(v(t))\right) \leq M_p\mathbb{E}\left(M^p(v_0)\right). \tag{2.6}$$

Proof. We apply the Itô formula given in [3] to $M(v)$. Since $M(v)$ is Fréchet derivable, the derivatives of $M(v)$ along directions φ and (φ,ψ) are as follows,

$$DM(v)(\varphi) = 2\,\mathrm{Re}\int_{\mathbb{R}^N} v\overline{\varphi}dx, \quad D^2M(v)(\varphi,\psi) = 2\,\mathrm{Re}\int_{\mathbb{R}^N} \varphi\overline{\psi}dx.$$

Using the Itô formula yields

$$dM(v(\tau)) = DM(v)(dv) + \frac{1}{2}D^2M(v)(dv,dv). \tag{2.7}$$

For the first term of the righthand side of (2.7), we have

$$DM(v)(dv) = 2\,\mathrm{Re}\int_{\mathbb{R}^N} v\overline{dv}dx = -2\,\mathrm{Im}\sum_{k\in\mathbb{N}}\int_{\mathbb{R}^N} ve^{\int_0^s a(m)dm}\overline{\phi e_k}d\beta_k(s)dx.$$

For the second term of the righthand side of (2.7), we have

$$\frac{1}{2}D^2M(v)(dv,dv) = \mathrm{Re}\int_{\mathbb{R}^N} dv\overline{dv}dx = \sum_{k\in\mathbb{N}}\int_{\mathbb{R}^N} e^{2\int_0^s a(m)dm}|\phi e_k|^2 dsdx.$$

Integrating (2.7) over $[0,\tau]$, we get (2.5). We now prove (2.6). Applying the Itô formula to $M^p(v)$ yields

$$\begin{aligned}M^p(v(t)) =& M^p(v_0) - 2p\,\mathrm{Im}\sum_{k\in\mathbb{N}}\int_0^t M^{p-1}(v)\int_{\mathbb{R}^N} ve^{\int_0^s a(m)dm}\overline{\phi e_k}(x)dxd\beta_k(s)\\ &+p\|\phi\|^2_{L_2^{0,0}}\int_0^t e^{2\int_0^s a(m)dm}M^{p-1}(v)ds\\ &+2p(p-1)\int_0^t e^{2\int_0^s a(m)dm}M^{p-2}(v)\sum_{k\in\mathbb{N}}\mathrm{Re}\left(\int_{\mathbb{R}^N} v\overline{\phi e_k}(x)dx\right)^2 ds.\end{aligned}$$

Taking the supremum and using a martingale inequality, it yields

$$\begin{aligned}\mathbb{E}\left(\sup_{t\in[0,\tau]} M^p(v(t))\right) \le\;& \mathbb{E}\left(M^p(v_0)\right) + 6p\mathbb{E}\left(\left(\int_0^\tau M^{2(p-1)}(v)\|\phi^* ve^{\int_0^s a(m)dm}\|^2_{L^2(\mathbb{R}^N)}ds\right)^{\frac{1}{2}}\right)\\ &+ p\|\phi\|^2_{L_2^{0,0}}\mathbb{E}\left(\int_0^\tau e^{2\int_0^s a(m)dm}M^{p-1}(v)ds\right)\\ &+ 2p(p-1)\mathbb{E}\left(\int_0^\tau M^{p-2}(v)\,\|\phi^* ve^{\int_0^s a(m)dm}\|^2_{L^2(\mathbb{R}^N)}ds\right)\\ \le\;& \mathbb{E}\left(M^p(v_0)\right) + 6p\left(\int_0^\tau e^{2\int_0^s a(m)dm}ds\right)^{\frac{1}{2}}\|\phi\|^2_{L_2^{0,0}}\mathbb{E}\left(\sup_{t\in[0,\tau]} M^{p-\frac{1}{2}}(v)\right)\\ &+ p(2p-1)\int_0^\tau e^{2\int_0^s a(m)dm}ds\|\phi\|^2_{L_2^{0,0}}\mathbb{E}\left(\sup_{t\in[0,\tau]} M^{p-1}(v)\right).\end{aligned}$$

By using Hölder's and Young's inequalities in the second term of the righthand side and an induction argument, (2.6) holds. □

Then, we give the evolution law of the energy.

Proposition 2.6. *Let α, $a(t)$, ϕ, and v_0 be as in Theorem 2.3. Then, for any stopping time τ such that $\tau < \tau^*(v_0)$ a.s., we have*

$$H(v,\tau) = H(v_0) + \frac{\alpha\lambda}{\alpha+2}\int_0^\tau\int_{\mathbb{R}^N} a(s)e^{-\alpha\int_0^s a(m)dm}|v|^{\alpha+2}dxds$$

$$
\begin{aligned}
&-\operatorname{Im}\int_{\mathbb{R}^N}\int_0^\tau e^{\int_0^s a(m)dm}\left(\Delta\overline{v}+\lambda e^{-\alpha\int_0^s a(m)dm}|v|^\alpha\overline{v}\right)dWdx\\
&+\frac{1}{2}\sum_{k\in\mathbb{N}}\int_0^\tau\int_{\mathbb{R}^N} e^{2\int_0^s a(m)dm}|\nabla\phi e_k|^2\,dxds\\
&-\frac{\lambda}{2}\sum_{k\in\mathbb{N}}\int_0^\tau\int_{\mathbb{R}^N} e^{(2-\alpha)\int_0^s a(m)dm}\left(|v|^\alpha\,|\phi e_k|^2+\alpha|v|^{\alpha-2}\left(\operatorname{Im}\left(\bar{v}\phi e_k\right)\right)^2\right)dxds.
\end{aligned}
\tag{2.8}
$$

Proof. The proof is similar to Proposition 2.5. Since $H(v,t)$ is Fréchet derivable, the derivatives of $H(v,t)$ along directions φ and (φ,ψ) are as follows,

$$
DH(v,t)(\varphi)=\operatorname{Re}\int_{\mathbb{R}^N}\nabla v\nabla\overline{\varphi}dx-\lambda e^{-\alpha\int_0^t a(m)dm}\operatorname{Re}\int_{\mathbb{R}^N}|v|^\alpha v\overline{\varphi}dx,
$$

$$
D^2H(v,t)(\varphi,\psi)=\operatorname{Re}\int_{\mathbb{R}^N}\nabla\psi\nabla\overline{\varphi}dx-\lambda e^{-\alpha\int_0^t a(m)dm}\left(\operatorname{Re}\int_{\mathbb{R}^N}|v|^\alpha\psi\overline{\varphi}dx+\alpha\int_{\mathbb{R}^N}|v|^{\alpha-2}\operatorname{Re}(v\overline{\psi})\operatorname{Re}(v\overline{\varphi})\right)dx.
$$

Using the Itô formula yields

$$
dH(v,t)=\frac{\partial H(v,t)}{\partial t}dt+DH(v,t)(dv)+\frac{1}{2}D^2H(v,t)(dv,dv).
\tag{2.9}
$$

For the first term of the righthand side of (2.9), we have

$$
\frac{\partial H(v,t)}{\partial t}=\frac{\alpha\lambda}{\alpha+2}a(s)e^{-\alpha\int_0^t a(m)dm}\int_{\mathbb{R}^N}|v|^{\alpha+2}dx.
$$

For the second term of the righthand side of (2.9), we have

$$
\begin{aligned}
DH(v,t)(dv)&=\operatorname{Re}\int_{\mathbb{R}^N}\nabla v\nabla\overline{dv}dx-\lambda e^{-\alpha\int_0^t a(m)dm}\operatorname{Re}\int_{\mathbb{R}^N}|v|^\alpha v\overline{dv}dx\\
&=-\operatorname{Im}\int_{\mathbb{R}^N}e^{-\alpha\int_0^t a(m)dm}\Delta\overline{v}dWdx-\lambda e^{(1-\alpha)\int_0^t a(m)dm}\operatorname{Im}\int_{\mathbb{R}^N}|v|^\alpha\overline{v}dWdx.
\end{aligned}
$$

For the last term of the righthand side of (2.9), we have

$$
\begin{aligned}
\frac{1}{2}D^2H(v,t)(dv,dv)&=\frac{1}{2}\left(\operatorname{Re}\int_{\mathbb{R}^N}\nabla dv\nabla\overline{dv}dx-\lambda e^{-\alpha\int_0^t a(m)dm}\left(\operatorname{Re}\int_{\mathbb{R}^N}|v|^\alpha dv\overline{dv}dx\right.\right.\\
&\quad\left.\left.+\alpha\int_{\mathbb{R}^N}|v|^{\alpha-2}\operatorname{Re}(v\overline{dv})\operatorname{Re}(v\overline{dv})\right)dx\right)\\
&=\frac{1}{2}\sum_{k\in\mathbb{N}}\int_{\mathbb{R}^N}e^{2\int_0^t a(m)dm}|\nabla\phi e_k|^2dsdx-\frac{\lambda}{2}e^{(2-\alpha)\int_0^t a(m)dm}\sum_{k\in\mathbb{N}}\int_{\mathbb{R}^N}|v|^\alpha|\phi e_k|^2dsdx\\
&\quad-\frac{\lambda\alpha}{2}e^{(2-\alpha)\int_0^t a(m)dm}\sum_{k\in\mathbb{N}}\int_{\mathbb{R}^N}|v|^{\alpha-2}\left(\operatorname{Im}\left(\overline{v}\phi e_k\right)\right)^2dsdx.
\end{aligned}
$$

Integrating (2.9) over $[0,\tau]$, we get (2.8). □

3. Global existence

In this section, our purpose is to prove the global existence for Eq (1.2), i.e., Theorem 1.1, via the uniform boundedness of the momentum and energy functionals. First, we have the following lemma.

Lemma 3.1. *Assume* $0 < \alpha < \frac{4}{N-2}$ *if* $N \geq 3$ *or* $\alpha > 0$ *if* $N = 1, 2$, *and we have*
(1) $\lambda = 1$, $0 < \alpha < \frac{4}{N}$, *then*

$$\|\nabla v\|^2_{L^2(\mathbb{R}^N)} \leq \frac{8}{3} H(v,t) + C\|v\|^{2+\frac{4\alpha}{4-N\alpha}}_{L^2(\mathbb{R}^N)},$$

(2) $\lambda = -1$, *then*

$$\|\nabla v\|^2_{L^2(\mathbb{R}^N)} \leq 2H(v,t).$$

Proof. Case (2) is obvious, so we only need to prove case (1). When $\lambda = 1$,

$$H(v,t) = \frac{1}{2}\|\nabla v\|^2_{L^2(\mathbb{R}^N)} - \frac{1}{\alpha+2} e^{-\alpha\int_0^t a(s)ds}\|v\|^{\alpha+2}_{L^{\alpha+2}(\mathbb{R}^N)}.$$

Using the Gagliardo-Nirenberg inequality and Young's inequality, the following estimation is obtained

$$\frac{1}{\alpha+2} e^{-\alpha\int_0^t a(s)ds}\|v\|^{\alpha+2}_{L^{\alpha+2}(\mathbb{R}^N)} \leq C\|v\|^{\alpha+2-\frac{N\alpha}{2}}_{L^2(\mathbb{R}^N)}\|\nabla v\|^{\frac{N\alpha}{2}}_{L^2(\mathbb{R}^N)} \leq \frac{1}{8}\|\nabla v\|^2_{L^2(\mathbb{R}^N)} + C\|v\|^{2+\frac{4\alpha}{4-N\alpha}}_{L^2(\mathbb{R}^N)}. \tag{3.1}$$

Substituting (3.1) into $H(v,t)$, we get

$$\frac{1}{2}\|\nabla v\|^2_{L^2(\mathbb{R}^N)} \leq H(v,t) + \frac{1}{8}\|\nabla v\|^2_{L^2(\mathbb{R}^N)} + C\|v\|^{2+\frac{4\alpha}{4-N\alpha}}_{L^2(\mathbb{R}^N)}.$$

Then, case (1) holds. □

Next, we begin to estimate $\mathbb{E}\left(\sup_{0\leq t\leq\tau}\|v(t)\|^2_{H^1(\mathbb{R}^N)}\right)$.

Lemma 3.2. *Let* α, ϕ, *and* v_0 *be as in Theorem 2.3, and assume that*
(1) *either* $\lambda = 1$, $0 < \alpha < \frac{4}{N}$, $a(t) \in L^1(0,\infty)$, $a(t) < 0$,
(2) *or* $\lambda = -1$, $a(t) \in L^1_{loc}(0,\infty)$, $a(t)$ *permits sign-changing,*
then for any given $T_0 > 0$ *and any stopping time* τ *with* $\tau < \inf(T_0, \tau^*(v_0))$ *a.s., we have*

$$\mathbb{E}\left(\sup_{0\leq t\leq\tau}\|v(t)\|^2_{H^1(\mathbb{R}^N)}\right) \leq C\left(T_0, \phi, a(t), \mathbb{E}(H(v_0)), \mathbb{E}\left(\|v_0\|^{2+\frac{4\alpha}{4-N\alpha}}_{L^2(\mathbb{R}^N)}\right)\right). \tag{3.2}$$

Proof. Supported by Proposition 2.5 and Lemma 3.1, we only need to prove the uniform boundedness of (2.8). Assume that $v_0 \in L^{2+\frac{4\alpha}{4-N\alpha}}(\Omega; L^2(\mathbb{R}^N)) \cap L^2(\Omega; H^1(\mathbb{R}^N))$ and that $\mathbb{E}(H(v_0))$ is finite.

Case (1): If $\lambda = 1$, we neglect the last term in (2.8) since they are nonpositive. Taking the expectation and using a martingale inequality to (2.8), we have

$$\mathbb{E}\left(\sup_{0\leq t\leq\tau} H(v,t)\right) \leq \mathbb{E}(H(v_0)) + \frac{\alpha}{\alpha+2}\mathbb{E}\left(\int_0^\tau\int_{\mathbb{R}^N}|a(s)|e^{-\alpha\int_0^s a(m)dm}|v|^{\alpha+2}dxds\right)$$

$$+3\mathbb{E}\left(\left(\int_0^\tau \left\|e^{\int_0^s a(m)dm}\phi^*(\Delta\bar{v}+e^{-\alpha\int_0^s a(m)dm}|v|^\alpha\bar{v})\right\|^2_{L^2(\mathbb{R}^N)}ds\right)^{\frac{1}{2}}\right)$$
$$+\frac{1}{2}\|\phi\|^2_{L_2^{0,1}}\int_0^\tau e^{2\int_0^s a(m)dm}ds. \tag{3.3}$$

For the second term of the righthand side of (3.3), using Hölder's inequality, we have

$$\begin{aligned}\frac{\alpha}{\alpha+2}\mathbb{E}\left(\int_0^\tau\int_{\mathbb{R}^N}|a(s)|e^{-\alpha\int_0^s a(m)dm}|v|^{\alpha+2}dxds\right)&\le\frac{\alpha}{\alpha+2}\mathbb{E}\left(\int_0^\tau|a(s)|e^{-\alpha\int_0^s a(m)dm}ds\sup_{0\le t\le\tau}\|v\|^{\alpha+2}_{L^{\alpha+2}(\mathbb{R}^N)}\right)\\&\le\frac{\alpha}{\alpha+2}\int_0^{T_0}|a(s)|e^{-\alpha\int_0^\infty a(m)dm}ds\,\mathbb{E}\left(\sup_{0\le t\le\tau}\|v\|^{\alpha+2}_{L^{\alpha+2}(\mathbb{R}^N)}\right).\end{aligned}\tag{3.4}$$

The validity of the last inequality in (3.4) depends critically on the condition $a(t)<0$. Using the Gagliardo-Nirenberg inequality and Young's inequality, we have

$$\frac{\alpha}{\alpha+2}\int_0^{T_0}|a(s)|e^{-\alpha\int_0^\infty a(m)dm}ds\|v\|^{\alpha+2}_{L^{\alpha+2}(\mathbb{R}^N)}\le C\|v\|^{\alpha+2-\frac{N\alpha}{2}}_{L^2(\mathbb{R}^N)}\|\nabla v\|^{\frac{N\alpha}{2}}_{L^2(\mathbb{R}^N)}\le\frac{1}{8}\|\nabla v\|^2_{L^2(\mathbb{R}^N)}+C\|v\|^{2+\frac{4\alpha}{4-N\alpha}}_{L^2(\mathbb{R}^N)}. \tag{3.5}$$

Note that in the last inequality of (3.5), it is crucial that $0<\alpha<\frac{4}{N}$. Substituting (3.5) into (3.4), and by Proposition 2.5, we get

$$\frac{\alpha}{\alpha+2}\mathbb{E}\left(\int_0^\tau\int_{\mathbb{R}^N}|a(s)|e^{-\alpha\int_0^s a(m)dm}|v|^{\alpha+2}dxds\right)\le C\left(\mathbb{E}\left(\|v_0\|^{2+\frac{4\alpha}{4-N\alpha}}_{L^2(\mathbb{R}^N)}\right)\right)+\frac{1}{8}\mathbb{E}\left(\sup_{0\le t\le\tau}\|\nabla v\|^2_{L^2(\mathbb{R}^N)}\right). \tag{3.6}$$

For the third term of the righthand side of (3.3), the operator ϕ^* is bounded from $H^{-1}(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ with the norm majorized by $\|\phi\|_{L_2^{0,1}}$. Furthermore, $H^1(\mathbb{R}^N)$ is embedded into $L^{\alpha+2}(\mathbb{R}^N)$ and ϕ^* is also bounded from $L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$. We obtain,

$$\left\|e^{\int_0^s a(m)dm}\phi^*(\Delta\bar{v}+e^{-\alpha\int_0^s a(m)dm}|v|^\alpha\bar{v})\right\|_{L^2(\mathbb{R}^N)}\le\|\phi\|_{L_2^{0,1}}e^{\int_0^s a(m)dm}\left(\|\nabla v\|_{L^2(\mathbb{R}^N)}+e^{-\alpha\int_0^s a(m)dm}\|v\|^{\alpha+1}_{L^{\alpha+2}(\mathbb{R}^N)}\right).$$

It follows that

$$\begin{aligned}&3\mathbb{E}\left(\left(\int_0^\tau\left\|e^{\int_0^s a(m)dm}\phi^*(\Delta\bar{v}+e^{-\alpha\int_0^s a(m)dm}|v|^\alpha\bar{v})\right\|^2_{L^2(\mathbb{R}^N)}ds\right)^{\frac{1}{2}}\right)\\&\le3\|\phi\|_{L_2^{0,1}}\mathbb{E}\left(\left(\int_0^\tau e^{2\int_0^s a(m)dm}\|\nabla v\|^2_{L^2(\mathbb{R}^N)}ds\right)^{\frac{1}{2}}\right)+3\|\phi\|_{L_2^{0,1}}\mathbb{E}\left(\left(\int_0^\tau e^{(2-2\alpha)\int_0^s a(m)dm}\|v\|^{2\alpha+2}_{L^{\alpha+2}(\mathbb{R}^N)}ds\right)^{\frac{1}{2}}\right).\end{aligned}\tag{3.7}$$

For the first term of the righthand side of (3.7), using Hölder's and Young's inequalities, we have

$$\begin{aligned}3\|\phi\|_{L_2^{0,1}}\mathbb{E}\left(\left(\int_0^\tau e^{2\int_0^s a(m)dm}\|\nabla v\|^2_{L^2(\mathbb{R}^N)}ds\right)^{\frac{1}{2}}\right)&\le3\|\phi\|_{L_2^{0,1}}\left(\int_0^{T_0}e^{2\int_0^s a(m)dm}ds\right)^{\frac{1}{2}}\mathbb{E}\left(\sup_{0\le t\le\tau}\|\nabla v\|_{L^2(\mathbb{R}^N)}\right)\\&\le\frac{1}{32}\mathbb{E}\left(\sup_{0\le t\le\tau}\|\nabla v\|^2_{L^2(\mathbb{R}^N)}\right)+C(T_0,\phi,a(t)).\end{aligned}\tag{3.8}$$

For the second term of the righthand side of (3.7), using Hölder's inequality, Young's inequality and the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}
&3\|\phi\|_{L_2^{0,1}}\mathbb{E}\left(\left(\int_0^\tau e^{(2-2\alpha)\int_0^s a(m)dm}\|v\|^{2\alpha+2}_{L^{\alpha+2}(\mathbb{R}^N)}ds\right)^{\frac{1}{2}}\right)\\
\leq&3\|\phi\|_{L_2^{0,1}}\left(\int_0^{T_0} e^{(2-2\alpha)\int_0^s a(m)dm}ds\right)^{\frac{1}{2}}\mathbb{E}\left(\sup_{0\leq t\leq\tau}\|v\|^{\alpha+1}_{L^{\alpha+2}(\mathbb{R}^N)}\right)\\
\leq&\frac{1}{4(\alpha+2)}\mathbb{E}\left(\sup_{0\leq t\leq\tau}\|v\|^{\alpha+2}_{L^{\alpha+2}(\mathbb{R}^N)}\right)+C(T_0,\phi,a(t))\\
\leq&\frac{1}{32}\mathbb{E}\left(\sup_{0\leq t\leq\tau}\|\nabla v\|^{2}_{L^{2}(\mathbb{R}^N)}\right)+C\left(T_0,\phi,a(t),\mathbb{E}\left(\|v_0\|^{2+\frac{4\alpha}{4-N\alpha}}_{L^{2}(\mathbb{R}^N)}\right)\right).
\end{aligned}\tag{3.9}$$

Combining (3.7)–(3.9), we get

$$\begin{aligned}
&3\mathbb{E}\left(\left(\int_0^\tau \left\|e^{\int_0^s a(m)dm}\phi^*(\Delta\bar{v}+e^{-\alpha\int_0^s a(m)dm}|v|^\alpha\bar{v})\right\|^2_{L^2(\mathbb{R}^N)}ds\right)^{\frac{1}{2}}\right)\\
\leq&\frac{1}{16}\mathbb{E}\left(\sup_{0\leq t\leq\tau}\|\nabla v\|^{2}_{L^{2}(\mathbb{R}^N)}\right)+C\left(T_0,\phi,a(t),\mathbb{E}\left(\|v_0\|^{2+\frac{4\alpha}{4-N\alpha}}_{L^{2}(\mathbb{R}^N)}\right)\right).
\end{aligned}\tag{3.10}$$

Therefore, together with Lemma 3.1, we finally obtain

$$\mathbb{E}\left(\sup_{0\leq t\leq\tau}H(v,t)\right)\leq\mathbb{E}(H(v_0))+C\left(T_0,\phi,a(t),\mathbb{E}\left(\|v_0\|^{2+\frac{4\alpha}{4-N\alpha}}_{L^{2}(\mathbb{R}^N)}\right)\right)+\frac{1}{2}\mathbb{E}\left(\sup_{0\leq t\leq\tau}H(v,t)\right).\tag{3.11}$$

Then, case (1) holds.

Case (2): If $\lambda=-1$, taking the expectation and using a martingale inequality to (2.8), we have

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\leq t\leq\tau}H(v,t)\right)\leq&\mathbb{E}(H(v_0))+\frac{\alpha}{\alpha+2}\mathbb{E}\left(\int_0^\tau\int_{\mathbb{R}^N}|a(s)|e^{-\alpha\int_0^s a(m)dm}|v|^{\alpha+2}dxds\right)\\
&+3\mathbb{E}\left(\left(\int_0^\tau \left\|e^{\int_0^s a(m)dm}\phi^*(\Delta\bar{v}+e^{-\alpha\int_0^s a(m)dm}|v|^\alpha\bar{v})\right\|^2_{L^2(\mathbb{R}^N)}ds\right)^{\frac{1}{2}}\right)\\
&+\frac{1}{2}\|\phi\|^2_{L_2^{0,1}}\int_0^\tau e^{2\int_0^s a(m)dm}ds\\
&+\frac{1}{2}\sum_{k\in\mathbb{N}}\mathbb{E}\left(\int_0^\tau\int_{\mathbb{R}^N}e^{(2-\alpha)\int_0^s a(m)dm}\left(|v|^\alpha|\phi e_k|^2+\alpha|v|^{\alpha-2}(\mathrm{Im}(\bar{v}\phi e_k))^2\right)dxds\right).
\end{aligned}\tag{3.12}$$

From (2.4), we obtain that

$$\frac{1}{\alpha+2}e^{-\alpha\int_0^t a(s)ds}\|v\|^{\alpha+2}_{L^{\alpha+2}(\mathbb{R}^N)}\leq H(v,t).\tag{3.13}$$

Note that the condition $a(t)\in L^1_{loc}(0,\infty)$ ensures that the term $e^{-\alpha\int_0^t a(s)ds}$ in (3.13) remains nonnegative and bounded regardless of the sign of $a(t)$ (positive or negative). It follows, for the second term of the righthand side of (3.12), that we have

$$\frac{\alpha}{\alpha+2}\mathbb{E}\left(\int_0^\tau\int_{\mathbb{R}^N}|a(s)|e^{-\alpha\int_0^s a(m)dm}|v|^{\alpha+2}dxds\right)\leq\alpha\mathbb{E}\left(\int_0^\tau|a(s)|H(v,s)ds\right)\leq\alpha\mathbb{E}\left(\int_0^\tau|a(s)|\sup_{0\leq s\leq\tau}H(v,s)ds\right). \tag{3.14}$$

For the third term of the righthand side of (3.12), using Hölder's inequality and Young's inequality, we have

$$\begin{aligned}
&3\mathbb{E}\left(\left(\int_0^\tau\left\|e^{\int_0^s a(m)dm}\phi^*(\Delta\bar{v}+e^{-\alpha\int_0^s a(m)dm}|v|^\alpha\bar{v})\right\|^2_{L^2(\mathbb{R}^N)}ds\right)^{\frac{1}{2}}\right)\\
\leq&3\|\phi\|_{L_2^{0,1}}\mathbb{E}\left(\left(\int_0^\tau e^{2\int_0^s a(m)dm}\|\nabla v\|^2_{L^2(\mathbb{R}^N)}ds\right)^{\frac{1}{2}}\right)+3\|\phi\|_{L_2^{0,1}}\mathbb{E}\left(\left(\int_0^\tau e^{(2-2\alpha)\int_0^s a(m)dm}\|v\|^{2\alpha+2}_{L^{\alpha+2}(\mathbb{R}^N)}ds\right)^{\frac{1}{2}}\right)\\
\leq&\frac{1}{32}\mathbb{E}\left(\sup_{0\leq t\leq\tau}\|\nabla v\|^2_{L^2(\mathbb{R}^N)}\right)+C(T_0,\phi,a(t))+\frac{1}{16(\alpha+2)}\mathbb{E}\left(\sup_{0\leq t\leq\tau}e^{-\alpha\int_0^t a(m)dm}\|v\|^{\alpha+2}_{L^{\alpha+2}(\mathbb{R}^N)}\right)\\
\leq&\frac{1}{8}\mathbb{E}\left(\sup_{0\leq t\leq\tau}H(v,t)\right)+C(T_0,\phi,a(t)).
\end{aligned} \tag{3.15}$$

For the last term in (3.12), using Hölder's inequality, we have

$$\begin{aligned}
&\frac{1}{2}\sum_{k\in N}\mathbb{E}\left(\int_0^\tau\int_{\mathbb{R}^N}e^{(2-\alpha)\int_0^s a(m)dm}\left(|v|^\alpha|\phi e_k|^2+\alpha|v|^{\alpha-2}\left(\mathrm{Im}\,(\bar{v}\phi e_k)\right)^2\right)dxds\right)\\
\leq&\frac{1+\alpha}{2}\sum_{k\in\mathbb{N}}\mathbb{E}\left(\int_0^\tau\int_{\mathbb{R}^N}e^{(2-\alpha)\int_0^s a(m)dm}|v|^\alpha|\phi e_k|^2dxds\right)\\
\leq&\frac{1+\alpha}{2}\sum_{k\in\mathbb{N}}\mathbb{E}\left(\int_0^\tau e^{(2-\alpha)\int_0^s a(m)dm}\|v\|^\alpha_{L^{\alpha+2}(\mathbb{R}^N)}\|\phi e_k\|^2_{L^{\alpha+2}(\mathbb{R}^N)}ds\right)\\
\leq&\frac{1}{8(\alpha+2)}\mathbb{E}\left(\sup_{0\leq t\leq\tau}e^{-\alpha\int_0^t a(m)dm}\|v\|^{\alpha+2}_{L^{\alpha+2}(\mathbb{R}^N)}\right)+C(T_0,\phi,a(t))\\
\leq&\frac{1}{8}\mathbb{E}\left(\sup_{0\leq t\leq\tau}H(v,t)\right)+C(T_0,\phi,a(t)).
\end{aligned} \tag{3.16}$$

Combining (3.14)–(3.16), and Gronwall's inequality, we finally have

$$\mathbb{E}\left(\sup_{0\leq t\leq\tau}H(v,t)\right)\leq C\left(T_0,\phi,a(t),\mathbb{E}(H(v_0))\right). \tag{3.17}$$

Then, case (2) holds. In conclusion, we finish the proof of Lemma 3.2. □

Remark 3.3. Suppose that $u(t,x)=e^{-\int_0^t a(s)ds}v(t,x)$, and we obtain the global existence for Eq (1.1).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is supported by National Natural Science Foundation of China (No.12001256) and by the Fundamental Research Funds for the Universities of the Educational Department of Liaoning Provincial of China (No.JYTMS20231046).

Conflict of interest

The authors declare there is no conflicts of interest.

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