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Research article

Regularity for very weak solutions to elliptic equations of p-Laplacian type

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Abstract: We study the regularity problem with non-homogeneous terms of p-Laplacian type, which is a still unsolved problem for nonlinear elliptic equations. The main results of this work are obtained by three steps. First, we use the Hodge decomposition theorem to construct a suitable test function that satisfies the solution definition. Second, by combining the solution definition with the Hodge decomposition theorem, we establish a properly formulated inverse Hölder inequality to enhance the integrability of the very weak solutions. Finally, through an iterative process, we show that the considered very weak solutions can be improved to classical weak solutions.

Keywords: regularity theory; generalized weak solutions; Hodge-type decomposition; elliptic equations; p-Laplacian class

1. Introduction

In this paper, we investigate the regularity theory of very weak solutions to the elliptic equations with p-Laplacian nonhomogeneous terms of the following type:

$$\begin{cases} \operatorname{div} S(x,u,\nabla u) = \operatorname{div}(|f|^{p-2}f), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω a Lipschitz domain in $\mathbb{R}^n$ with $n \geqslant 2$, $p \in (1,+\infty)$, u is a vector-valued function taking values in $\mathbb{R}^n$ with $N \in \mathbb{N}$, ∇u stands for the gradient matrix of u, and $f : \Omega \to \mathbb{R}^{nN}$ is a given vector field.

To introduce solutions under the very weak formulation for the elliptic system (1.1), it is necessary to impose appropriate structural conditions on the operator $S(x,u,\cdot)$. To this end, we assume that the function $S(x,u,\nabla u) : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ satisfies the following structural assumptions (H1)–(H3) for a.e. $x \in \Omega$ and every $u, u_1, u_2 \in \mathbb{R}^N$, $z_1, z_2 \in \mathbb{R}^{nN}$. Here, u, u_1 and u_2 are arbitrary functions, and z, z_1 and z_2 are the gradient matrices corresponding to u, u_1, and u_2, respectively.

H1 (Coercivity assumption) There exists a positive constant ν such that

$$S(x,u,z)z \geqslant \nu|z|^p.$$

H2 (Monotonicity assumption) There exists a positive constant β such that

$$\Big(S(x,u_1,z_1) - S(x,u_2,z_2)\Big)(z_1 - z_2) \geqslant \beta(|z_1| + |z_2|)^{p-2}|z_1 - z_2|^2.$$

H3 (Boundedness assumption) A positive constant γ can be chosen so that

$$|S(x,u,z)| \leqslant \gamma\Big(|z|^{p-1} + |u|^{\mu(p-1)} + \phi(x)\Big),$$

where $\gamma \in [\beta, +\infty), \mu \in (0, \frac{n}{n-p+1}), \phi(x) \in L^{\frac{p}{p-1}}(\Omega)$.

With these structural conditions, we are now in a position to define very weak solutions to the system (1.1).

Definition 1. *A vector-valued function $u \in W_0^{1,q}(\Omega, \mathbb{R}^N)$, max$\{1, p-1\} < q < p$ is called a very weak solution of (1.1) under the structural assumptions (H1)–(H3) if*

$$\int_\Omega S(x,u,\nabla u) \cdot \nabla\varphi dx = \int_\Omega |f|^{p-2} f \cdot \nabla\varphi dx \tag{1.2}$$

holds for all test functions $\varphi \in C_0^\infty(\Omega, \mathbb{R}^N)$.

The study of regularity theory is typically predicated on the existence of weak solutions. However, the existence of weak solutions remains unresolved for certain classes of equations within the current analytical framework. Prominent examples include elliptic equations with a p-Laplacian operator and a singular convection term.

Currently, for the p-Laplacian, the existence of solutions has been extensively studied [1–4]. However, under general structural assumptions, little is known about the existence of solutions when $f \in L^q(\Omega, \mathbb{R}^{nN})$ with $q < p$ and $p \neq 2$. Even for the simplest degenerate p-Laplacian system,

$$\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \mathrm{div}(|f|^{p-2} f)$$

subject to $u = 0$ on Ω, the existence of solutions remains an open question when $q < p$ and $p = 2$.

Iwaniec [2] observed that the integral identity for weak solutions can still hold by weakening the integrability of these solutions. This observation led to the introduction of very weak solutions with exponents below the natural exponent. He further established the existence of such solutions for homogeneous p-harmonic equations. Bulíček and Schwarzacher [5] demonstrated the existence of very weak solutions to the system of p-Laplacian type

$$\mathrm{div} S(x, \nabla u) = \mathrm{div}|f|^{p-2} f$$

in Ω, subject to the boundary condition $u = 0$ on $\partial\Omega$. Chen and Guo [6] extended this result to Eq (1.1) for $f \in L^q$ with $q \in [p - \lambda, p]$, using the method of [5].

The existence results for very weak solutions naturally lead to the problem of characterizing their relationship to weak solutions. Iwaniec and Sbordone [7] showed that very weak solutions are in fact weak solutions for A-harmonic equations

$$\mathrm{div} A(x, \nabla u) = 0.$$

Later, Kinnunen and Zhou [8] obtained the same result for the system

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$$

when the exponent p is close to two. Greco and Verde [9] extended this result to the p-Laplacian system

$$\operatorname{div}\Big((G(x)\nabla u, \nabla u)^{\frac{p-2}{2}} G(x)\nabla u\Big) = 0.$$

Subsequently, similar results have been obtained for various systems, including p-Laplacian systems [10–12], elliptic systems [13–17], and parabolic systems [18–20].

We are now interested in whether very weak solutions to Eq (1.1) can be promoted to weak solutions when $f \in L^q$ with $q < p$, given that the existence of these very weak solutions has been established in [6]. To address this issue, we aim to demonstrate the self-enhancement property of the gradients corresponding to the very weak solutions of (1.1).

The key challenge is to construct an appropriate test function, since conventional methods involving truncations and powers of u fail when the exponent is below the natural exponent p. Iwaniec and Sbordone [7] ingeniously applied the Hodge decomposition to address this challenge. Lewis [21] similarly overcame this difficulty by appealing to the Whitney extension theorem. This study adopts the method of Iwaniec and Sbordone, leveraging the stability of the Hodge-type decomposition to design an auxiliary function.

Compared to previous works, a key new aspect of our work is that an operator of A-harmonic $S(x, u, \nabla u)$ in Eq (1.1) appears to be more general, as it depends not only on the gradient ∇u and additionally on u considered in the very weak framework. In this context, we refer to the operator $S(x, u, \nabla u)$ in the system represented in (1.1) that satisfies the framework conditions (H1)–(H3) as an A-harmonic operator. Consequently, we must address the estimate issues arising from u under the structural assumption (H3). By applying regularity proof techniques, we establish a weak reverse Hölder inequality. Subsequently, we enhance the integrability exponent for very weak solutions of Eq (1.1) to the natural exponent p. These results extend the findings of [7]. And obtain the following main results.

Theorem 1. *Let $u \in W_0^{1,q}(\Omega, \mathbb{R}^N)$ be a solution in the very weak sense for the system (1.1), subject to the structural conditions (H1)–(H3), where $f \in L^q(\Omega, \mathbb{R}^{nN})$, $q \in [p - \lambda, p]$, $1 < p < +\infty$, $0 < \lambda < 1$. This ensures the existence of an exponent q_1 that satisfies $p - \lambda \leqslant q_1 = q_1(n, N, p, \mu, \beta, \gamma) < p$ so that all very weak solutions $u \in W_0^{1,q_2}(\Omega)$ with $q_1 \leqslant q_2 < p$.*

Corollary 1 Assuming that the hypotheses of Theorem 1 are fulfilled, one can find a constant $q_1 = q_1(n, N, p, \mu, \beta, \gamma) < p$ such that for any solution $u \in W_0^{1,q}(\Omega)$ within the very weak solution framework corresponding to the elliptic model (1.1), if $q_1 < q < p$, then $u \in W_0^{1,p}(\Omega)$.

We now outline the structure of the remainder of this paper. In Section 2, we introduce the existence of very weak solutions to elliptic equations of p-Laplacian type (1.1), the theory of Hodge decomposition, a useful inequality, and a reverse Hölder inequality. In Section 3, we show that these very weak solutions to Eq (1.1) are, in fact, classical weak solutions.

2. Preliminaries

In this section, we review several basic results and inequalities that will facilitate the proof of Theorem 1.

The first result, which forms the basis of our work, establishes the existence of very weak solutions to the p-Laplacian equations (1.1) when $f \in L^q$ with $q \in [p-\lambda, p]$. This existence theorem was established through the selection of appropriately formed weight functions, in conjunction with methods such as the relative covering decomposition theorem, weighted techniques, and the divergence-curl lemma.

Lemma 1 ([6]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain; the operator $S(x, u, \nabla u)$ satisfy (H1)–(H3). Then, there exists a constant $\lambda(\nu, \beta, \gamma, n, N, p, \Omega)$ such that for all $q \in [p-\lambda, p](1 < p < \infty)$, the following results hold.*

If $f \in L^q(\Omega, \mathbb{R}^{nN})$, then there exists a very weak solution $u \in W_0^{1,q}(\Omega, \mathbb{R}^N)$ to the Eq (1.1).

Furthermore, there exist constants $C(\nu, \beta, \gamma, p, q, n, N, \Omega)$ and $\bar{C}(\nu, \beta, \gamma, p, q, n, N, \Omega)$ such that

$$\|\nabla u\|_{L^q(\Omega,\mathbb{R}^{nN})} \leqslant C\|f\|_{L^q(\Omega,\mathbb{R}^{nN})} + \bar{C}.$$

The Hodge decomposition theorem serves as the key tool in constructing a suitable test function.

Lemma 2 ([7]). *Let $\Omega \subset \mathbb{R}^n$ be a domain with a regular boundary, where $w \in W_0^{1,r}(\Omega, \mathbb{R}^N)$, $r > 1$ with λ satisfying $-1 < \lambda < r-1$. This yields the existence of a function $\varphi \in W_0^{1,\frac{r}{1+\lambda}}(\Omega, \mathbb{R}^N)$ and a matrix field with zero divergence $H \in L^{\frac{r}{1+\lambda}}(\Omega, \mathbb{R}^{nN})$ such that*

$$|\nabla w|^\lambda \nabla w = \nabla\varphi + H. \tag{2.1}$$

Moreover,

$$\|H\|_{\frac{r}{1+\lambda}} \leqslant C_r(\Omega, N)|\lambda|\|\nabla w\|_r^{1+\lambda}. \tag{2.2}$$

In Lemma 2, the most valuable case for the construction of a test function is when λ can be negative. Let $p-1 < r < p$ and $u \in W_0^{1,r}(\Omega, \mathbb{R}^N)$ is considered a solution in the very weak formulation to the p-Laplacian type equations (1.1), by letting $\lambda = r - p$, it follows that $-1 < \lambda < 0$. Hence φ in (2.1) may be employed as a test function within (1.2) since $\nabla\varphi \in L^{\frac{r}{1+r-p}}(\Omega, \mathbb{R}^{nN})$.

The following inequality will be useful for estimating the left-hand side term in the Hodge-type decomposition.

Lemma 3 ([22]). *Assume that $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^n$, where $\mathbf{X}, \mathbf{Y}$ are nonzero vectors and λ lies in $[0, 1)$, then, we have*

$$\left||\mathbf{X}|^{-\lambda}\mathbf{X} - |\mathbf{Y}|^{-\lambda}\mathbf{Y}\right| \leq 2^\lambda \frac{1+\lambda}{1-\lambda}|\mathbf{X} - \mathbf{Y}|^{1-\lambda}.$$

Finally, we introduce a reverse Hölder inequality that implies the self-improvement of the integrability exponent of $u(x)$. To clarify the notation used in the upcoming lemma, we recall that the average value of $g(x)$ over a set X is defined as

$$⨍_X g(x)dx = \frac{1}{|X|}\int_X g(x)dx,$$

where $X \subset \mathbb{R}^n$ is a Lebesgue measurable set and $|X|$ denotes the Lebesgue measure of X. In particular,

$$⨍_{M_{\mathcal{R}}(x_0)} g(x)dx = \frac{1}{\alpha_n \mathcal{R}^n}\int_{M_{\mathcal{R}}(x_0)} g(x)dx,$$

where, α_n denotes the volume of the unit ball.

Lemma 4 ([23]). *Let* $0 < \mathcal{R} < \mathcal{R}_0 \leqslant dist(x_0, \partial\Omega)$, $x_0 \in \Omega$. *Suppose that* $u(x) \in L^p(M_{\mathcal{R}}(x_0))$, $f(x) \in L^t(M_{\mathcal{R}}(x_0))$, $t > p$, $1 < p < \infty$ *satisfies the reverse Hölder inequality*

$$\fint_{M_{\mathcal{R}/2}(x_0)} |u(x)|^p dx \leqslant \theta \fint_{M_{\mathcal{R}}(x_0)} |u(x)|^p dx + C^* \Big(\fint_{M_{\mathcal{R}}(x_0)} |u(x)|^s dx \Big)^{p/s} + \fint_{M_{\mathcal{R}}(x_0)} |f(x)|^p dx$$

with $1 \leqslant s < p$, $0 \leqslant \theta < 1$. *Then there exists a constant* $p' = p'(\theta, p, n, C^*)$ *with* $t \geqslant p' > p$ *such that*

$$u \in L^{p'}_{loc}(\Omega),$$

and

$$\Big(\fint_{M_{\mathcal{R}/2}(x_0)} |u(x)|^{p'} dx \Big)^{1/p'} \leqslant C_* \Big(\fint_{M_{\mathcal{R}}(x_0)} |u(x)|^p dx \Big)^{1/p} + C_* \Big(\fint_{M_{\mathcal{R}}(x_0)} |f(x)|^{p'} dx \Big)^{1/p'}$$

where $C_* = C_*(n, C^*, p, \theta, \mathcal{R}_0)$.

3. Demonstration of the regularity theorem

In the present section, we first construct an appropriate test function using the Hodge decomposition and establish a reverse-form Hölder inequality for solutions within the very weak formulation associated with (1.1). We then explore the relationship between the aforementioned very weak solutions and standard weak solutions.

Proof of Theorem 1 By Lemma 1, there exists a very weak solution $u \in W_0^{1,q}(\Omega, \mathbb{R}^N)$ to the elliptic equations (1.1) under the structural assumptions (H1)–(H3), where $f \in L^q(\Omega, \mathbb{R}^{nN})$, $q \in [p-\lambda, p]$, $\lambda = \lambda(\nu, \beta, \gamma, n, N, p, \Omega)$. To prove Theorem 1, the key is to enhance the integrability exponent of $u \in W_0^{1,q}(\Omega, \mathbb{R}^N)$. To this end, we need to construct an appropriate test function. For convenience, we set $q = p - \lambda$ with $0 < \lambda < \frac{1}{2}$. Consequently, $u \in W_0^{1,p-\lambda}(\Omega, \mathbb{R}^N)(0 < \lambda < \frac{1}{2})$ is a very weak solution to (1.1). Let $\eta(x) \in C_0^\infty(M_{\mathcal{R}}(x_0))$ with $0 < R < min\{1, dist(x_0, \partial\Omega)\}$ be a truncation function with $\eta(x)$ taking values in [0, 1] and satisfying $|\nabla\eta(x)| \leqslant \frac{4}{\mathcal{R}}$ and identically equal to 1 on $M_{\mathcal{R}/2}(x_0)$

Based on the Hodge decomposition theorem in Lemma 2, for any $0 < \lambda < \frac{1}{2}$, there exists $\varphi \in W_0^{1,\frac{p-\lambda}{1-\lambda}}(\Omega)$ and $H \in L^{\frac{p-\lambda}{1-\lambda}}(\Omega)$ such that

$$|\nabla(\eta(x)u(x))|^{-\lambda}\nabla(\eta(x)u(x)) = \nabla\varphi + H \tag{3.1}$$

and

$$\|H\|_{\frac{p-\lambda}{1-\lambda}} \leqslant C(n,p)\lambda \|\nabla(\eta u)\|^{1-\lambda}_{p-\lambda}, \tag{3.2}$$

$$\|\nabla\varphi\|_{\frac{p-\lambda}{1-\lambda}} \leqslant C(n,p) \|\nabla(\eta u)\|^{1-\lambda}_{p-\lambda}. \tag{3.3}$$

Since $\nabla\varphi \in L^{\frac{p-\lambda}{1-\lambda}}(\Omega)$, φ is appropriate to act as a test function within the framework of very weak solutions. Consequently,

$$\int_{M_{\mathcal{R}}(x_0)} S(x, u, \nabla u) \cdot \nabla\varphi dx = \int_{M_{\mathcal{R}}(x_0)} |f|^{p-2} f \cdot \nabla\varphi dx. \tag{3.4}$$

Let

$$E(\eta, u) = |\nabla(\eta u)|^{-\lambda}\nabla(\eta u) - |\eta\nabla u|^{-\lambda}\eta\nabla u. \tag{3.5}$$

By Lemma 3, we obtain that

$$|E(\eta,u)| \leqslant 2^{\lambda}\frac{1+\lambda}{1-\lambda}|u\nabla\eta|^{1-\lambda}. \tag{3.6}$$

By combining Eq (3.1) with Eq (3.5), it follows that

$$\nabla\varphi = E(\eta,u) + |\eta\nabla u|^{-\lambda}\eta\nabla u - H.$$

Substituting this equation into Eq (3.4), we obtain

$$\begin{aligned}
&\int_{M_{\mathcal{R}}(x_0)} S(x,u,\nabla u)\cdot|\eta\nabla u|^{-\lambda}\eta\nabla u dx\\
=&-\int_{M_{\mathcal{R}}(x_0)} S(x,u,\nabla u)\cdot E(\eta,u)dx + \int_{M_{\mathcal{R}}(x_0)} S(x,u,\nabla u)\cdot H dx\\
&+\int_{M_{\mathcal{R}}(x_0)} |f|^{p-2}f\cdot\nabla\varphi dx.
\end{aligned}$$

Combining (H1), (H3), and (3.6), we can conclude that

$$\begin{aligned}
&\nu\int_{M_{\mathcal{R}}(x_0)} \eta^{1-\lambda}|\nabla u|^{p-\lambda}dx\\
\leqslant&\int_{M_{\mathcal{R}}(x_0)} S(x,u,\nabla u)|\eta\nabla u|^{-\lambda}\eta\nabla u dx\\
\leqslant&\int_{M_{\mathcal{R}}(x_0)} |S(x,u,\nabla u)||E(\eta,u)|dx + \int_{M_{\mathcal{R}}(x_0)} |S(x,u,\nabla u)||H|dx\\
&+\int_{M_{\mathcal{R}}(x_0)} |f|^{p-1}|\nabla\varphi|dx.\\
\leqslant&C_1\int_{M_{\mathcal{R}}(x_0)} |\nabla u|^{p-1}|u|^{1-\lambda}dx + C_1\int_{M_{\mathcal{R}}(x_0)} |u|^{\mu(p-1)}|u|^{1-\lambda}dx\\
&+C_1\int_{M_{\mathcal{R}}(x_0)} |\phi(x)||u|^{1-\lambda}dx + \gamma\int_{M_{\mathcal{R}}(x_0)} |\nabla u|^{p-1}|H|dx\\
&+\gamma\int_{M_{\mathcal{R}}(x_0)} |u|^{\mu(p-1)}|H|dx + \gamma\int_{M_{\mathcal{R}}(x_0)} |\phi(x)||H|dx\\
&+\int_{M_{\mathcal{R}}(x_0)} |f|^{p-1}|\nabla\varphi|dx\\
\leqslant&I_1+I_2+I_3+I_4+I_5+I_6+I_7,
\end{aligned} \tag{3.7}$$

where

$$I_1 = C_1 \int_{M_{\mathcal{R}}(x_0)} |\nabla u|^{p-1}|u|^{1-\lambda}dx;$$
$$I_2 = C_1 \int_{M_{\mathcal{R}}(x_0)} |u|^{\mu(p-1)}|u|^{1-\lambda}dx;$$
$$I_3 = C_1 \int_{M_{\mathcal{R}}(x_0)} |\phi(x)||u|^{1-\lambda}dx;$$
$$I_4 = \gamma \int_{M_{\mathcal{R}}(x_0)} |\nabla u|^{p-1}|H|dx;$$
$$I_5 = \gamma \int_{M_{\mathcal{R}}(x_0)} |u|^{\mu(p-1)}|H|dx;$$
$$I_6 = \gamma \int_{M_{\mathcal{R}}(x_0)} |\phi(x)||H|dx;$$
$$I_7 = \int_{M_{\mathcal{R}}(x_0)} |f|^{p-1}|\nabla\varphi|dx;$$

with

$$C_1 = 2^{\lambda}\frac{1+\lambda}{1-\lambda}\Big(\frac{4}{\mathcal{R}}\Big)^{1-\lambda}\gamma.$$

To derive a weak reverse Hölder inequality for $|\nabla u|^{p-\lambda}$ of the form presented in Lemma 4, we need to estimate $I_1 - I_7$ appropriately.

We begin by estimating I_1.

In view of Hölder's inequality with exponents

$$p' = \frac{n(p-\lambda)}{(n+1-\lambda)(p-1)}, q' = \frac{n(p-\lambda)}{(n-p+1)(1-\lambda)},$$

where $1 < p' < \infty$, $1 < q' < \infty$, and $\frac{1}{p'} + \frac{1}{q'} = 1$, and the Sobolev embedding theorem with exponent

$$p'' = \frac{n(p-\lambda)}{n+1-\lambda},$$

such that $\frac{np''}{n-p''} = \frac{n(p-\lambda)}{n-p+1}$, we find that

$$\begin{aligned} I_1 &\leqslant C_1\Big(\int_{M_{\mathcal{R}}(x_0)} |\nabla u|^{\frac{n(p-\lambda)}{n+1-\lambda}}dx\Big)^{\frac{(n+1-\lambda)(p-1)}{n(p-\lambda)}}\Big(\int_{M_{\mathcal{R}}(x_0)} |u|^{\frac{n(p-\lambda)}{n-p+1}}dx\Big)^{\frac{(n-p+1)(1-\lambda)}{n(p-\lambda)}} \\ &\leqslant C_1C_s^{1-\lambda}\Big(\int_{M_{\mathcal{R}}(x_0)} |\nabla u|^{\frac{n(p-\lambda)}{n+1-\lambda}}dx\Big)^{\frac{(n+1-\lambda)(p-1)}{n(p-\lambda)}}\Big(\int_{M_{\mathcal{R}}(x_0)} |\nabla u|^{\frac{n(p-\lambda)}{n+1-\lambda}}dx\Big)^{\frac{(n+1-\lambda)(1-\lambda)}{n(p-\lambda)}} \\ &= C_1C_s^{1-\lambda}\Big(\int_{M_{\mathcal{R}}(x_0)} |\nabla u|^{\frac{n(p-\lambda)}{n+1-\lambda}}dx\Big)^{\frac{n+1-\lambda}{n}}. \end{aligned}$$

Using the same techniques as for I_1, we can estimate I_2.

$$
\begin{aligned}
I_2 &\leqslant C_1 \int_{M_{\mathcal{R}}(x_0)} |u|^{\mu(p-1)+1-\lambda} dx \\
&\leqslant C_1 \Big(\int_{M_{\mathcal{R}}(x_0)} |u|^{\frac{n(p-\lambda)}{n-p+1}} dx \Big)^{\frac{(n-p+1)[\mu(p-1)+1-\lambda]}{n(p-\lambda)}} \Big(\int_{M_{\mathcal{R}}(x_0)} dx \Big)^{\frac{(p-1)[n(1-\mu)+\mu(p-1)+1-\lambda]}{n(p-\lambda)}} \\
&\leqslant C_1 C(|M_{\mathcal{R}}(x_0)|, C_s) \Big(\int_{M_{\mathcal{R}}(x_0)} |\nabla u|^{\frac{n(p-\lambda)}{n+1-\lambda}} dx \Big)^{\frac{n+1-\lambda}{n(p-\lambda)}[\mu(p-1)+1-\lambda]}.
\end{aligned}
$$

We now discuss the estimate of I_2 for different values of μ.
If $0 < \mu \leqslant 1$, it is evident that

$$
I_2 \leqslant C_1 C(|M_{\mathcal{R}}(x_0)|, C_s) \Big(\int_{M_{\mathcal{R}}(x_0)} |\nabla u|^{\frac{n(p-\lambda)}{n+1-\lambda}} dx \Big)^{\frac{n+1-\lambda}{n}}.
$$

If $1 < \mu < \frac{n}{n-p+1}$, we obtain

$$
I_2 \leqslant C_1 C\Big(|M_{\mathcal{R}}(x_0)|, C_s, \|u\|_{W_0^{1,\frac{n(p-\lambda)}{n+1-\lambda}}}\Big) \Big(\int_{M_{\mathcal{R}}(x_0)} |\nabla u|^{\frac{n(p-\lambda)}{n+1-\lambda}} dx \Big)^{\frac{n+1-\lambda}{n}}.
$$

Thus, we can conclude that

$$
I_2 \leqslant C_1 C\Big(|M_{\mathcal{R}}(x_0)|, C_s, \|u\|_{W_0^{1,\frac{n(p-\lambda)}{n+1-\lambda}}}\Big) \Big(\int_{M_{\mathcal{R}}(x_0)} |\nabla u|^{\frac{n(p-\lambda)}{n+1-\lambda}} dx \Big)^{\frac{n+1-\lambda}{n}}.
$$

By applying Hölder's inequality, Poincaré's inequality, and Young's inequality, we obtain

$$
\begin{aligned}
I_3 &\leqslant C_1 \Big(\int_{M_{\mathcal{R}}(x_0)} |\phi(x)|^{\frac{p-\lambda}{p-1}} dx \Big)^{\frac{p-1}{p-\lambda}} \Big(\int_{M_{\mathcal{R}}(x_0)} |u|^{p-\lambda} dx \Big)^{\frac{1-\lambda}{p-\lambda}} \\
&\leqslant C_1 C_p^{\frac{1-\lambda}{p-\lambda}} \Big(\int_{M_{\mathcal{R}}(x_0)} |\phi(x)|^{\frac{p-\lambda}{p-1}} dx \Big)^{\frac{p-1}{p-\lambda}} \Big(\int_{M_{\mathcal{R}}(x_0)} |\nabla u|^{p-\lambda} dx \Big)^{\frac{1-\lambda}{p-\lambda}} \\
&\leqslant C_1 C_p^{\frac{1-\lambda}{p-\lambda}} \varepsilon \int_{M_{\mathcal{R}}(x_0)} |\nabla u|^{p-\lambda} dx + C_1 C_p^{\frac{1-\lambda}{p-\lambda}} C(\varepsilon) \int_{M_{\mathcal{R}}(x_0)} |\phi(x)|^{\frac{p-\lambda}{p-1}} dx.
\end{aligned}
$$

By Hölder's inequality and (3.2), we have

$$
\begin{aligned}
I_4 &\leqslant \gamma \Big(\int_{M_{\mathcal{R}}(x_0)} |\nabla u|^{p-\lambda} dx \Big)^{\frac{p-1}{p-\lambda}} \Big(\int_{M_{\mathcal{R}}(x_0)} |H|^{\frac{p-\lambda}{1-\lambda}} dx \Big)^{\frac{1-\lambda}{p-\lambda}} \\
&\leqslant C(n,p)\gamma\lambda \Big(\int_{M_{\mathcal{R}}(x_0)} |\nabla u|^{p-\lambda} dx \Big)^{\frac{p-1}{p-\lambda}} \Big(\int_{M_{\mathcal{R}}(x_0)} |\nabla(\eta u)|^{p-\lambda} dx \Big)^{\frac{1-\lambda}{p-\lambda}}.
\end{aligned}
$$

By employing Minkowski's inequality and Poincaré's inequality, we derive

$$\begin{aligned}
&\Big(\int_{M_{\mathcal{R}}(x_0)}|\nabla(\eta u)|^{p-\lambda}dx\Big)^{\frac{1}{p-\lambda}}\\
=&\Big(\int_{M_{\mathcal{R}}(x_0)}|u\nabla\eta+\eta\nabla u|^{p-\lambda}dx\Big)^{\frac{1}{p-\lambda}}\\
\leqslant&\Big(\int_{M_{\mathcal{R}}(x_0)}|u\nabla\eta|^{p-\lambda}dx\Big)^{\frac{1}{p-\lambda}}+\Big(\int_{M_{\mathcal{R}}(x_0)}|\eta\nabla u|^{p-\lambda}dx\Big)^{\frac{1}{p-\lambda}}\\
\leqslant&\frac{4}{\mathcal{R}}C_p^{\frac{1}{p-\lambda}}\Big(\int_{M_{\mathcal{R}}(x_0)}|\nabla u|^{p-\lambda}dx\Big)^{\frac{1}{p-\lambda}}+\Big(\int_{M_{\mathcal{R}}(x_0)}|\nabla u|^{p-\lambda}dx\Big)^{\frac{1}{p-\lambda}}\\
=&\Big(\frac{4}{\mathcal{R}}C_p^{\frac{1}{p-\lambda}}+1\Big)\Big(\int_{M_{\mathcal{R}}(x_0)}|\nabla u|^{p-\lambda}dx\Big)^{\frac{1}{p-\lambda}}.
\end{aligned}\tag{3.8}$$

Substituting (3.8) into the preceding estimate of I_4, we further obtain

$$\begin{aligned}
I_4&\leqslant C(n,p)\Big(\frac{4}{\mathcal{R}}C_p^{\frac{1}{p-\lambda}}+1\Big)^{1-\lambda}\gamma\lambda\Big(\int_{M_{\mathcal{R}}(x_0)}|\nabla u|^{p-\lambda}dx\Big)^{\frac{p-1}{p-\lambda}}\Big(\int_{M_{\mathcal{R}}(x_0)}|\nabla u|^{p-\lambda}dx\Big)^{\frac{1-\lambda}{p-\lambda}}\\
&\leqslant C(n,p)\Big(\frac{4}{\mathcal{R}}C_p^{\frac{1}{p-\lambda}}+1\Big)^{1-\lambda}\gamma\lambda\int_{M_{\mathcal{R}}(x_0)}|\nabla u|^{p-\lambda}dx.
\end{aligned}$$

By Hölder's inequality, (3.2) and (3.8), we can estimate

$$\begin{aligned}
I_5&\leqslant\gamma\Big(\int_{M_{\mathcal{R}}(x_0)}|u|^{\mu(p-\lambda)}dx\Big)^{\frac{p-1}{p-\lambda}}\Big(\int_{M_{\mathcal{R}}(x_0)}|H|^{\frac{p-\lambda}{1-\lambda}}dx\Big)^{\frac{1-\lambda}{p-\lambda}}\\
&\leqslant C(n,p)\gamma\lambda\Big(\int_{M_{\mathcal{R}}(x_0)}|u|^{\mu(p-\lambda)}dx\Big)^{\frac{p-1}{p-\lambda}}\Big(\int_{M_{\mathcal{R}}(x_0)}|\nabla(\eta u)|^{p-\lambda}dx\Big)^{\frac{1-\lambda}{p-\lambda}}\\
&\leqslant C(n,p)\Big(\frac{4}{\mathcal{R}}C_p^{\frac{1}{p-\lambda}}+1\Big)^{1-\lambda}\gamma\lambda\Big(\int_{M_{\mathcal{R}}(x_0)}|u|^{\mu(p-\lambda)}dx\Big)^{\frac{p-1}{p-\lambda}}\Big(\int_{M_{\mathcal{R}}(x_0)}|\nabla u|^{p-\lambda}dx\Big)^{\frac{1-\lambda}{p-\lambda}}.
\end{aligned}$$

As before, we consider μ in two cases.

Assuming $0<\mu\leqslant 1$, and utilizing Hölder's, Young's, and Poincaré's inequalities, the following equation is obtained.

$$\begin{aligned}
&\Big(\int_{M_{\mathcal{R}}(x_0)}|u|^{\mu(p-\lambda)}dx\Big)^{\frac{p-1}{p-\lambda}}\\
\leqslant&\Big(\int_{M_{\mathcal{R}}(x_0)}|u|^{p-\lambda}dx\Big)^{\frac{\mu(p-1)}{p-\lambda}}\Big(\int_{M_{\mathcal{R}}(x_0)}dx\Big)^{\frac{(1-\mu)(p-1)}{p-\lambda}}\\
\leqslant&\Big(\int_{M_{\mathcal{R}}(x_0)}|u|^{p-\lambda}dx\Big)^{\frac{p-1}{p-\lambda}}+\Big(\int_{M_{\mathcal{R}}(x_0)}dx\Big)^{\frac{p-1}{p-\lambda}}\\
\leqslant&C_p^{\frac{p-1}{p-\lambda}}\Big(\int_{M_{\mathcal{R}}(x_0)}|\nabla u|^{p-\lambda}dx\Big)^{\frac{p-1}{p-\lambda}}+\Big(\int_{M_{\mathcal{R}}(x_0)}dx\Big)^{\frac{p-1}{p-\lambda}}.
\end{aligned}$$

If $1 < \mu < \frac{n}{n-p+1}$, using the Sobolev embedding theorem, we can find

$$
\begin{aligned}
&\Big(\int_{M_{\mathcal{R}}(x_0)} |u|^{\mu(p-\lambda)}dx\Big)^{\frac{p-1}{p-\lambda}} \\
\leqslant& C_s^{\mu(p-1)}\Big(\int_{M_{\mathcal{R}}(x_0)} |\nabla u|^{p-\lambda}dx\Big)^{\frac{\mu(p-1)}{p-\lambda}} \\
\leqslant& C\Big(C_s, \|u\|_{W_0^{1,p-\lambda}}\Big)\Big(\int_{M_{\mathcal{R}}(x_0)} |\nabla u|^{p-\lambda}dx\Big)^{\frac{p-1}{p-\lambda}}.
\end{aligned}
$$

Combining the above two inequalities, we can deduce that

$$
\Big(\int_{M_{\mathcal{R}}(x_0)} |u|^{\mu(p-\lambda)}dx\Big)^{\frac{p-1}{p-\lambda}} \leqslant C\Big(C_p, C_s, \|u\|_{W_0^{1,p-\lambda}}\Big)\Big(\int_{M_{\mathcal{R}}(x_0)} |\nabla u|^{p-\lambda}dx\Big)^{\frac{p-1}{p-\lambda}} + \Big(\int_{M_{\mathcal{R}}(x_0)} dx\Big)^{\frac{p-1}{p-\lambda}}.
$$

Substituting the above inequality into the estimate for I_5, we obtain the following bound:

$$
I_5 \leqslant C\Big(n, p, \gamma, C_P, C_s, \|u\|_{W_0^{1,p-\lambda}}\Big)\lambda\left[\int_{M_{\mathcal{R}}(x_0)} |\nabla u|^{p-\lambda}dx + \int_{M_{\mathcal{R}}(x_0)} dx\right].
$$

To estimate I_6, we apply Hölder's inequality, (3.2), (3.8), and Young's inequality to obtain

$$
\begin{aligned}
I_6 \leqslant& \gamma\Big(\int_{M_{\mathcal{R}}(x_0)} |\phi(x)|^{\frac{p-\lambda}{p-1}}dx\Big)^{\frac{p-1}{p-\lambda}}\Big(\int_{M_{\mathcal{R}}(x_0)} |H|^{\frac{p-\lambda}{1-\lambda}}dx\Big)^{\frac{1-\lambda}{p-\lambda}} \\
\leqslant& C(n,p)\gamma\lambda\Big(\int_{M_{\mathcal{R}}(x_0)} |\phi(x)|^{\frac{p-\lambda}{p-1}}dx\Big)^{\frac{p-1}{p-\lambda}}\Big(\int_{M_{\mathcal{R}}(x_0)} |\nabla(\eta u)|^{p-\lambda}dx\Big)^{\frac{1-\lambda}{p-\lambda}} \\
\leqslant& C(n,p)\Big(\frac{4}{\mathcal{R}}C_P^{\frac{1}{p-\lambda}} + 1\Big)^{1-\lambda}\gamma\lambda\Big(\int_{M_{\mathcal{R}}(x_0)} |\phi(x)|^{\frac{p-\lambda}{p-1}}dx\Big)^{\frac{p-1}{p-\lambda}}\Big(\int_{M_{\mathcal{R}}(x_0)} |\nabla u|^{p-\lambda}dx\Big)^{\frac{1-\lambda}{p-\lambda}} \\
\leqslant& C(n,p)\Big(\frac{4}{\mathcal{R}}C_P^{\frac{1}{p-\lambda}} + 1\Big)^{1-\lambda}\gamma\lambda\Big(\int_{M_{\mathcal{R}}(x_0)} |\nabla u|^{p-\lambda}dx + \int_{M_{\mathcal{R}}(x_0)} |\phi(x)|^{\frac{p-\lambda}{p-1}}dx\Big).
\end{aligned}
$$

Finally, by applying Hölder's inequality, (3.3), (3.8), and Young's inequality, we arrive at

$$
\begin{aligned}
I_7 \leqslant& \Big(\int_{M_{\mathcal{R}}(x_0)} |f|^{p-\lambda}dx\Big)^{\frac{p-1}{p-\lambda}}\Big(\int_{M_{\mathcal{R}}(x_0)} |\nabla\varphi|^{\frac{p-\lambda}{1-\lambda}}dx\Big)^{\frac{1-\lambda}{p-\lambda}} \\
\leqslant& C(n,p)\Big(\int_{M_{\mathcal{R}}(x_0)} |f|^{p-\lambda}dx\Big)^{\frac{p-1}{p-\lambda}}\Big(\int_{M_{\mathcal{R}}(x_0)} |\nabla(\eta u)|^{p-\lambda}dx\Big)^{\frac{1-\lambda}{p-\lambda}} \\
\leqslant& C(n,p)\Big(\frac{4}{\mathcal{R}}C_P^{\frac{1}{p-\lambda}} + 1\Big)^{1-\lambda}\Big(\int_{M_{\mathcal{R}}(x_0)} |f|^{p-\lambda}dx\Big)^{\frac{p-1}{p-\lambda}}\Big(\int_{M_{\mathcal{R}}(x_0)} |\nabla u|^{p-\lambda}dx\Big)^{\frac{1-\lambda}{p-\lambda}} \\
\leqslant& C(n,p)\Big(\frac{4}{\mathcal{R}}C_P^{\frac{1}{p-\lambda}} + 1\Big)^{1-\lambda}\Big(\varepsilon\int_{M_{\mathcal{R}}(x_0)} |\nabla u|^{p-\lambda}dx + C(\varepsilon)\int_{M_{\mathcal{R}}(x_0)} |f|^{p-\lambda}dx\Big).
\end{aligned}
$$

Substituting the estimates for I_1 through I_7 into (3.7) and using the definition of η, the final expression is derived.

$$
\begin{aligned}
&\nu\int_{M_{\mathcal{R}/2}(x_0)}|\nabla u|^{p-\lambda}dx\\
\leqslant\ & C_1C_s^{1-\lambda}\Big(\int_{M_{\mathcal{R}}(x_0)}|\nabla u|^{\frac{n(p-\lambda)}{n+1-\lambda}}dx\Big)^{\frac{n+1-\lambda}{n}}\\
&+C_1C\Big(|M_{\mathcal{R}}(x_0)|,C_s,\|u\|_{W_0^{1,\frac{n(p-\lambda)}{n+1-\lambda}}}\Big)\Big(\int_{M_{\mathcal{R}}(x_0)}|\nabla u|^{\frac{n(p-\lambda)}{n+1-\lambda}}dx\Big)^{\frac{n+1-\lambda}{n}}\\
&+C_1C_P^{\frac{1-\lambda}{p-\lambda}}\varepsilon\int_{M_{\mathcal{R}}(x_0)}|\nabla u|^{p-\lambda}dx+C_1C_P^{\frac{1-\lambda}{p-\lambda}}C(\varepsilon)\int_{M_{\mathcal{R}}(x_0)}|\phi(x)|^{\frac{p-\lambda}{p-1}}dx\\
&+C(n,p)\Big(\frac{4}{\mathcal{R}}C_P^{\frac{1}{p-\lambda}}+1\Big)^{1-\lambda}\gamma\lambda\int_{M_{\mathcal{R}}(x_0)}|\nabla u|^{p-\lambda}dx\\
&+C\Big(n,p,\gamma,C_P,C_s,\|u\|_{W_0^{1,p-\lambda}}\Big)\lambda\left(\int_{M_{\mathcal{R}}(x_0)}|\nabla u|^{p-\lambda}dx+\int_{M_{\mathcal{R}}(x_0)}dx\right)\\
&+C(n,p)\Big(\frac{4}{\mathcal{R}}C_P^{\frac{1}{p-\lambda}}+1\Big)^{1-\lambda}\gamma\lambda\int_{M_{\mathcal{R}}(x_0)}|\nabla u|^{p-\lambda}dx\\
&+C(n,p)\Big(\frac{4}{\mathcal{R}}C_P^{\frac{1}{p-\lambda}}+1\Big)^{1-\lambda}\gamma\lambda\int_{M_{\mathcal{R}}(x_0)}|\phi(x)|^{\frac{p-\lambda}{p-1}}dx\\
&+C(n,p)\Big(\frac{4}{\mathcal{R}}C_P^{\frac{1}{p-\lambda}}+1\Big)^{1-\lambda}\varepsilon\int_{M_{\mathcal{R}}(x_0)}|\nabla u|^{p-\lambda}dx\\
&+C(n,p)\Big(\frac{4}{\mathcal{R}}C_P^{\frac{1}{p-\lambda}}+1\Big)^{1-\lambda}C(\varepsilon)\int_{M_{\mathcal{R}}(x_0)}|f|^{p-\lambda}dx.
\end{aligned}
$$

Rearranging this inequality yields

$$
\begin{aligned}
&\nu\int_{M_{\mathcal{R}/2}(x_0)}|\nabla u|^{p-\lambda}dx\\
\leqslant\ & \bigg[\Big(2C(n,p)\Big(\frac{4}{\mathcal{R}}C_P^{\frac{1}{p-\lambda}}+1\Big)^{1-\lambda}\gamma+C\Big(n,p,\gamma,C_P,C_s,\|u\|_{W_0^{1,p-\lambda}}\Big)\Big)\lambda\\
&+\Big(C_1C_P^{\frac{1-\lambda}{p-\lambda}}+C(n,p)\Big(\frac{4}{\mathcal{R}}C_P^{\frac{1}{p-\lambda}}+1\Big)^{1-\lambda}\Big)\varepsilon\bigg]\int_{M_{\mathcal{R}}(x_0)}|\nabla u|^{p-\lambda}dx\\
&+\Big[C_1C_s^{1-\lambda}+C_1C\Big(|M_{\mathcal{R}}(x_0)|,C_s,\|u\|_{W_0^{1,\frac{n(p-\lambda)}{n+1-\lambda}}}\Big)\Big]\Big(\int_{M_{\mathcal{R}}(x_0)}|\nabla u|^{\frac{n(p-\lambda)}{n+1-\lambda}}dx\Big)^{\frac{n+1-\lambda}{n}}\\
&+\Big[C_1C_P^{\frac{1-\lambda}{p-\lambda}}C(\varepsilon)+C(n,p)\Big(\frac{4}{\mathcal{R}}C_P^{\frac{1}{p-\lambda}}+1\Big)^{1-\lambda}\gamma\lambda\Big]\int_{M_{\mathcal{R}}(x_0)}|\phi(x)|^{\frac{p-\lambda}{p-1}}dx\\
&+C(n,p)\Big(\frac{4}{\mathcal{R}}C_P^{\frac{1}{p-\lambda}}+1\Big)^{1-\lambda}C(\varepsilon)\int_{M_{\mathcal{R}}(x_0)}|f|^{p-\lambda}dx\\
&+C\Big(n,p,\gamma,C_P,C_s,\|u\|_{W_0^{1,p-\lambda}}\Big)\lambda\int_{M_{\mathcal{R}}(x_0)}dx.
\end{aligned}
$$

Dividing both sides of the above inequality by $|M_{\mathcal{R}}(x_0)| = \alpha_n\mathcal{R}^n$, where α_n denotes the volume of the unit ball, we obtain

$$
\begin{aligned}
&\nu \fint_{M_{\mathcal{R}/2}(x_0)} |\nabla u|^{p-\lambda} dx \\
&\leqslant 2^n\Bigg[\Big(2C(n,p)\Big(\frac{4}{\mathcal{R}}C_P^{\frac{1}{p-\lambda}}+1\Big)^{1-\lambda}\gamma + C\Big(n,p,\gamma,C_P,C_s,\|u\|_{W_0^{1,p-\lambda}}\Big)\Big)\lambda \\
&\quad + \Big(C_1 C_P^{\frac{1-\lambda}{p-\lambda}} + C(n,p)\Big(\frac{4}{\mathcal{R}}C_P^{\frac{1}{p-\lambda}}+1\Big)^{1-\lambda}\Big)\varepsilon\Bigg]\fint_{M_{\mathcal{R}}(x_0)} |\nabla u|^{p-\lambda} dx \\
&\quad + 2^n\Big[C_1 C_s^{1-\lambda} + C_1 C\Big(|M_{\mathcal{R}}(x_0)|, C_s, \|u\|_{W_0^{1,\frac{n(p-\lambda)}{n+1-\lambda}}}\Big)\Big]\Big(\alpha_n\mathcal{R}^n\Big)^{\frac{1-\lambda}{n}}\Big(\fint_{M_{\mathcal{R}}(x_0)} |\nabla u|^{\frac{n(p-\lambda)}{n+1-\lambda}} dx\Big)^{\frac{n+1-\lambda}{n}} \\
&\quad + 2^n\Big[C_1 C_P^{\frac{1-\lambda}{p-\lambda}} C(\varepsilon) + C(n,p)\Big(\frac{4}{\mathcal{R}}C_P^{\frac{1}{p-\lambda}}+1\Big)^{1-\lambda}\gamma\lambda\Big]\fint_{M_{\mathcal{R}}(x_0)} |\phi(x)|^{\frac{p-\lambda}{p-1}} dx \\
&\quad + 2^n C(n,p)\Big(\frac{4}{\mathcal{R}}C_P^{\frac{1}{p-\lambda}}+1\Big)^{1-\lambda} C(\varepsilon)\fint_{M_{\mathcal{R}}(x_0)} |f|^{p-\lambda} dx \\
&\quad + C\Big(n,p,\gamma,C_P,C_s,\|u\|_{W_0^{1,p-\lambda}}\Big)\lambda.
\end{aligned}
$$

Choosing λ, ε small enough such that

$$
\begin{aligned}
&2^n\Big[\Big(2C(n,p)(\frac{4}{\mathcal{R}}C_P^{\frac{1}{p-\lambda}}+1)^{1-\lambda}\gamma + C(n,p,\gamma,C_P,C_s,\|u\|_{W_0^{1,p-\lambda}})\Big)\lambda \\
&\quad + \Big(C_1 C_P^{\frac{1-\lambda}{p-\lambda}} + C(n,p)(\frac{4}{\mathcal{R}}C_P^{\frac{1}{p-\lambda}}+1)^{1-\lambda}\Big)\varepsilon\Big] < \nu,
\end{aligned}
$$

and setting $\tau = \frac{n(p-\lambda)}{n+1-\lambda}$, then we can deduce that

$$
\fint_{M_{\mathcal{R}/2}(x_0)} |\nabla u|^{p-\lambda} dx \leqslant \theta \fint_{M_{\mathcal{R}}(x_0)} |\nabla u|^{p-\lambda} dx + M\Big(\fint_{M_{\mathcal{R}}(x_0)} |\nabla u|^{\tau} dx\Big)^{\frac{p-\lambda}{\tau}} + \fint_{M_{\mathcal{R}}(x_0)} |F|^{p-\lambda} dx,
$$

with $0 < \theta < 1$, $1 < \tau < p-\lambda$ and here

$$
M = \frac{2^n}{\nu}\Big[C_1 C_s^{1-\lambda} + C_1 C\Big(|M_{\mathcal{R}}(x_0)|, C_s, \|u\|_{W_0^{1,\frac{n(p-\lambda)}{n+1-\lambda}}}\Big)\Big]\Big(\alpha_n\mathcal{R}^n\Big)^{\frac{1-\lambda}{n}},
$$

and

$$
\begin{aligned}
F = \Bigg\{&\frac{2^n}{\nu}\Big[C_1 C_P^{\frac{1-\lambda}{p-\lambda}} C(\varepsilon) + C(n,p)\Big(\frac{4}{\mathcal{R}}C_P^{\frac{1}{p-\lambda}}+1\Big)^{1-\lambda}\gamma\lambda\Big]|\phi(x)|^{\frac{p-\lambda}{p-1}} \\
&+ \frac{2^n}{\nu}C(n,p)\Big(\frac{4}{\mathcal{R}}C_P^{\frac{1}{p-\lambda}}+1\Big)^{1-\lambda} C(\varepsilon)|f|^{p-\lambda} + \frac{1}{\nu}C\Big(n,p,\gamma,C_P,C_s,\|u\|_{W_0^{1,p-\lambda}}\Big)\lambda\Bigg\}^{\frac{1}{p-\lambda}}.
\end{aligned}
$$

Thus, by Lemma 4, one can find an exponent $q' > q = p-\lambda$ for which $u \in W_0^{1,q'}(\Omega)$ holds. Note that $f \in L^q(\Omega, \mathbb{R}^{nN})$ for $q \in [p-\lambda, p]$. By similar reasoning, we derive an alternative estimate comparable to the reverse Hölder inequality for $|\nabla u|^{p-\lambda}$, where the exponents q' and τ' replace $q = p-\lambda$ and τ, respectively. Specifically,

$$\fint_{M_{\mathcal{R}/2}(x_0)} |\nabla u|^{q'} dx \leqslant \theta \fint_{M_{\mathcal{R}}(x_0)} |\nabla u|^{q'} dx + M\left(\fint_{M_{\mathcal{R}}(x_0)} |\nabla u|^{\tau'} dx\right)^{\frac{q'}{\tau'}} + \fint_{M_{\mathcal{R}}(x_0)} |F|^{q'} dx.$$

We then obtain $u \in W_0^{1,q''}(\Omega)$ for some $q'' > q'$. Moreover, the reverse Hölder inequality remains valid with the new exponents q'' and τ'' in place of q' and τ', respectively. Therefore, by repeating the above process, we can continuously improve the integrability of ∇u. Consequently, we infer that $u \in W_0^{1,p}(\Omega)$ for any $u \in W_0^{1,q}(\Omega)$ with $q \in [p - \lambda, p)$.

The proof of Theorem 1 is completed.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

References

1. L. Diening, P. Kaplický, S. Schwarzacher, BMO estimates for the p-laplacian, *Nonlinear Anal. Theory Methods Appl.*, **75** (2012), 637–650. https://doi.org/10.1016/j.na.2011.08.065

2. T. Iwaniec, p-harmonic tensors and quasiregular mappings, *Ann. Math.*, **136** (1992), 589–624. https://doi.org/10.2307/2946602

3. G. Mingione, Nonlinear aspects of calderón-zygmund theory, *Jahresber. Dtsch. Math. Ver.* , **112** (2010), 159–191. https://doi.org/10.1365/s13291-010-0004-5

4. J. Serrin, Isolated singularities of solutions of quasi-linear equations, *Acta Math.*, **113** (1965), 219–240. https://doi.org/10.1007/BF02391778

5. M. Bulíček, S. Schwarzacher, Existence of very weak solutions to elliptic systems of p-laplacian type, *Calc. Var. Partial Differ. Equations*, **55** (2016), 52. https://doi.org/10.1007/s00526-016-0986-7

6. S. Chen, L. Guo, Existence of very weak solutions for a class of p-laplacian elliptic equations, *J. Guangxi Univ. (Nat. Sci. Ed.)*, **48** (2023), 236–245. https://doi.org/10.13624/j.cnki.issn.1001-7445.2023.0236

7. T. Iwaniec, C. Sbordone, Weak minima of variational integrals, *J. Reine Angew. Math.*, **454** (1994), 143–161. https://doi.org/10.1515/crll.1994.454.143

8. J. Kinnunen, S. Zhou, A note on very weak p-harmonic mappings, *Electron. J. Differ. Equations*, **1997** (1997), 1–4.
9. L. Greco, A. Verde, A regularity property of p-harmonic functions, *Ann. Acad. Sci. Fenn. Math.*, **25** (2000), 317–323.
10. J. Kinnunen, J. L. Lewis, Very weak solutions of parabolic systems of p-laplacian type, *Ark. Mat.*, **40** (2002), 105–132. https://doi.org/10.1007/BF02384505
11. B. Stroffolini, A stability result for p-harmonic systems with discontinuous coefficients, *Electron. J. Differ. Equations*, **2001** (2001), 1–7.
12. S. Zhou, Very weak solutions of p-laplacian type equations with VMO coefficients, *J. Partial Differ. Equations*, **14** (2001), 12–20.
13. S. Baasandorj, S. Byun, W. Kim, Self-improving properties of very weak solutions to double phase systems, *Trans. Am. Math. Soc.*, **376** (2023), 8733–8768. https://doi.org/10.1090/tran/9039
14. S. Byun, M. Lim, Gradient estimates of very weak solutions to general quasilinear elliptic equations, *J. Funct. Anal.*, **283** (2012), 109668 https://doi.org/10.1016/j.jfa.2022.109668
15. M. Carozza, A. P. di Napoli, On very weak solutions of a class of nonlinear elliptic systems, *Comment. Math. Univ. Carolin.*, **41** (2000), 493–508.
16. G. Du, J. Han, Global higher integrability for very weak solutions to nonlinear subelliptic equations, *Boundary Value Probl.*, **2017** (2017), 93. https://doi.org/10.1186/s13661-017-0825-6
17. J. Merker, J. Rakotoson, Very weak solutions of poisson's equation with singular data under neumann boundary conditions, *Calc. Var. Partial Differ. Equations*, **52** (2015), 705–726. https://doi.org/10.1007/s00526-014-0730-0
18. K. Adimurthi, S. Byun, J. Oh, Interior and boundary higher integrability of very weak solutions for quasilinear parabolic equations with variable exponents, *Nonlinear Anal.*, **194** (2020), 111370. https://doi.org/10.1016/j.na.2018.10.014
19. V. Bögelein, Q. Li, Very weak solutions of degenerate parabolic systems with non-standard $p(x,t)$-growth, *Nonlinear Anal. Theory Methods Appl.*, **98** (2014), 190–225. https://doi.org/10.1016/j.na.2013.12.009
20. J. Rakotoson, Regularity of a very weak solution for parabolic equations and applications, *Adv. Differ. Equations*, **16** (2011), 867–894. https://doi.org/10.57262/ade/1355703179
21. J. L. Lewis, On the very weak solutions of certain elliptic systems, *Commun. Partial Differ. Equations*, **18** (1993), 1515–1537. https://doi.org/10.1080/03605309308820984
22. T. Iwaniec, L. Migliaccio, L. Nania, C. Sbordone, Integrability and removability results for quasiregular mappings in high dimensions, *Math. Scand.*, **75** (1994), 263–279.
23. N. G. Meyers, A. Elcrat, Some results on regularity for solutions of non-linear elliptic systems and quasi-regular functions, *Duke Math. J.*, **42** (1975), 121–136. https://doi.org/10.1215/S0012-7094-75-04211-8