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Research article

The construction of tilting cotorsion pairs for hereditary abelian categories

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Abstract: Through tilting objects, we construct complete cotorsion pairs for specific hereditary abelian categories, such as the category of modules that are finitely generated over a finite-dimensional hereditary algebra as well as the category of coherent sheaves over weighted projective lines. We prove that a complete cotorsion pair exists in the category of coherent sheaves over a weighted projective curve $\mathbb{X}$ if and only if $\mathbb{X}$ is a weighted projective line. We also characterize the canonical tilting cotorsion pair for any weighted projective line and obtain Hovey triples in the category of vector bundles over a weighted projective line.

Keywords: cotorsion pair; tilting object; hereditary abelian category; coherent sheave; weighted projective line

1. Introduction

Cotorsion pairs were introduced by Salce [1] within the realm of abelian groups. The concept was readily extended to any abelian category, exact category, triangulated category, and even extriangulated category [2–4]. Cotorsion pairs and their connections to model structures have been intensively investigated recently. In [5], Hovey's correspondence, a significant result, reveals a bijection between abelian model structures and Hovey triples. Additionally, numerous researchers have explored techniques for constructing cotorsion pairs in various abelian categories [6–8].

Complete cotorsion pairs are plentiful: For example, in a Grothendieck category, any cotorsion pair generated by a set is complete [9]. Among these complete cotorsion pairs, the cotorsion pairs induced by tilting objects have attracted the attention of many authors [10–12]. For the module category of an ordinary ring, [11, Corollary 13.20] shows a bijection between the equivalence class of infinitely generated tilting modules and the class of hereditary complete cotorsion pairs $(\mathcal{A}, \mathcal{B})$. In these pairs, modules in $\mathcal{A}$ have projective dimension at most $n \in \mathbb{N}$, and $\mathcal{B}$ is closed under arbitrary direct sums. Moreover, when R is a hereditary, indecomposable, left pure semisimple ring, [13, Proposition 4.2] proves that every cotorsion pair $(\mathcal{A}, \mathcal{B})$ in the left R-module category Mod-R is induced by a finitely generated tilting module.

Hereditary categories serve as prototypes for many phenomena of representation theory. For an algebraically closed field k, Happel provided a characterization theorem for hereditary abelian k-categories admitting a tilting object; see [14, Theorem 3.1]. That is, if $\mathcal{H}$ is a connected hereditary abelian k-category with a tilting object, where k is a field, then $\mathcal{H}$ is derived equivalent to one of the following in the sense of Geigle-Lenzing [15]:

(i) The category mod-A of finite-dimensional modules over a finite-dimensional hereditary k-algebra A, or
(ii) The category coh-$\mathbb{X}$ of coherent sheaves on a weighted projective line $\mathbb{X}$.

The main goal of this paper is to complete cotorsion pairs for certain specific hereditary abelian categories using tilting objects. In this paper, we always assume that k is an algebraically closed field and C is a hereditary abelian k-category that is Hom-finite and Ext-finite. In this case, C is Krull–Schmidt. Here hereditary means the global dimension of C is at most 1, that is, the Yoneda $\mathrm{Ext}^2_C(-,-)$ vanishes. In particular, we will focus on the following categories in specific examples:

- Let Λ be a hereditary Artin algebra and C = mod-Λ the category of finitely generated left Λ-modules.
- Let $\mathbb{X} = (X, \omega)$ stand for a weighted projective curve, and define C = coh-$\mathbb{X}$ as the category of coherent sheaves over $\mathbb{X}$.
- Let Q be a cyclic quiver. Denote $C = \mathrm{rep}_k^{\mathrm{nil}}(Q)$ by the category of finite-dimensional nilpotent representations of Q over the field k.

We first explore the relationships between tilting objects and complete cotorsion pairs of finite type in hereditary abelian categories that do not necessarily have enough projective objects. It turns out that each complete cotorsion pair of finite type in a hereditary abelian category is induced by a tilting object. Consequently, we obtain that there are no complete cotorsion pairs of finite type in the category of coherent sheaves over a weighted projective curve whose underlying curve has genus greater than zero and no complete cotorsion pairs of finite type in the category of finite-dimensional nilpotent representations of any cyclic quiver. More precisely, we have

Theorem A. *The following statements hold.*

(i) (= **Proposition 4.3**) *In the category* coh-$\mathbb{X}$ *of coherent sheaves over a weighted projective curve* $\mathbb{X} = (X, \omega)$*, complete cotorsion pairs of finite type exist if and only if* X *has genus zero, i.e.,* $\mathbb{X}$ *is a weighted projective line.*
(ii) (= **Proposition 4.4**) *For a cyclic quiver* Q*, let* $\mathrm{rep}_k^{\mathrm{nil}}(Q)$ *denote the category of finite dimensional nilpotent* k*-representations of* Q*. Then* $\mathrm{rep}_k^{\mathrm{nil}}(Q)$ *admits no complete cotorsion pairs.*

Moreover, we offer an explicit characterization of the canonical tilting cotorsion pairs in the category of coherent sheaves on a weighted projective line. More precisely, we conduct a classification of all such cotorsion pairs in coh-$\mathbb{P}^1$, which is the category of coherent sheaves over the classical projective line $\mathbb{P}^1$. We investigate cotorsion pairs for weighted projective lines induced by canonical tilting sheaves. Moreover, we obtain Hovey triples in the category of vector bundles for any weighted projective line.

Theorem B. *The following statements hold.*

(i) (= **Proposition 4.2**) *Any hereditary complete cotorsion pair of finite type in* coh-$\mathbb{P}^1$ *has the form* $(\mathcal{A}_n, \mathcal{B}_n)$ *for some* $n \in \mathbb{Z}$*, where*

$$\mathcal{A}_n = \text{add}\,\{O(m) \mid m \leq n+1\};$$

$$\mathcal{B}_n = \text{add}\,\{O(m) \mid m \geq n\} \sqcup \text{coh}_0\text{-}\mathbb{P}^1.$$

(ii) (= **Proposition 5.4**) *Let* coh-$\mathbb{X}$ *be the category of coherent sheaves over a weighted projective line* $\mathbb{X}$*. Then there is a canonical tilting cotorsion pair* $(\mathcal{A}, \mathcal{B})$ *in* coh-$\mathbb{X}$*, where*

$$\mathcal{A} = \langle O(\vec{x}) \mid \vec{x} \leq \vec{c}\rangle;\ \mathcal{B} = \langle O(\vec{x}) \mid \vec{x} \geq 0, \text{coh}_0\text{-}\mathbb{X}\rangle.$$

(iii) (= **Proposition 6.4**) *Let* vect-$\mathbb{X}$ *be the subcategory consisting of vector bundles, where* $\mathbb{X}$ *is any weighted projective line. Suppose that the full subcategory* $\mathcal{X} \subseteq$ vect-$\mathbb{X}$ *is closed under Auslander-Reiten translations. Then under* $\mathcal{X}$*-exact structure, both* (vect-$\mathbb{X}, \mathcal{X}$) *and* ($\mathcal{X}$, vect-$\mathbb{X}$) *are complete cotorsion pairs in* vect-$\mathbb{X}$*. Consequently,* (vect-$\mathbb{X}, \mathcal{X}$, vect-$\mathbb{X}$) *is a Hovey triple, and* vect-$\mathbb{X}/\mathcal{X}$ *is a triangulated category.*

The paper is organized as follows. We will start in Section 2 with some preliminaries about cotorsion pairs, tilting objects and weighted projective lines. In Section 3, we study the relationships between tilting objects, and complete cotorsion pairs of finite type in hereditary abelian categories. As applications, in Section 4, we get the classification of complete cotorsion pairs in specific hereditary abelian categories, focusing on the category of modules that are finitely generated over a finite dimensional hereditary k-algebra and the category coh-$\mathbb{P}^1$ of coherent sheaves over the classical projective line. Regarding arbitrary weighted projective lines, in Section 5, we characterize the canonical cotorsion pairs in the category of coherent sheaves. Then, in Section 6, we successfully obtain Hovey triples in the category of vector bundles.

2. Preliminaries

Let C be a Hom-finite and Ext-finite hereditary abelian k-category. For an object M in C, we define add M as the class of all objects that are isomorphic to direct summands of finite direct sums of M. We define Gen M as the class of all M-generated objects, i.e., all objects that are epimorphic images of objects in add M. Dually, we define the symbol Cogen M.

2.1. Cotorsion pairs and Hovey's correspondence

In the category C, a *cotorsion pair* $(\mathcal{A}, \mathcal{B})$ is a pair of classes of objects such that $\mathcal{A}^{\perp_1} = \mathcal{B}$ and $\mathcal{A} = {}^{\perp_1}\mathcal{B}$, where:

$$\mathcal{A}^{\perp_1} = \{X \in C \mid \text{Ext}^1_C(A, X) = 0\ \forall A \in \mathcal{A}\};$$

$${}^{\perp_1}\mathcal{B} = \{Y \in C \mid \text{Ext}^1_C(Y, B) = 0\ \forall B \in \mathcal{B}\}.$$

By [11, Definition 5.15], the cotorsion pair *generated by* $\mathcal{S}$ is $({}^{\perp_1}(\mathcal{S}^{\perp_1}), \mathcal{S}^{\perp_1})$. The *heart* of $(\mathcal{A}, \mathcal{B})$ is $\mathcal{A} \cap \mathcal{B}$. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is of *finite type* if $\mathcal{A} \cap \mathcal{B} = \text{add}\,X$ for some $X \in C$.

We say that a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *complete* if, for any object X in C, there exist two exact sequences $0 \to B \to A \to X \to 0$ and $0 \to X \to B' \to A' \to 0$, where $B, B' \in \mathcal{B}$ and $A, A' \in \mathcal{A}$. A class

$\mathcal{A}$ of objects in C is *resolving* if it is closed under kernels of epimorphisms between its objects. In other words, for any short exact sequence $0 \to X \to Y \to Z \to 0$, $X \in \mathcal{A}$ when $Y, Z \in \mathcal{A}$. We say that class $\mathcal{B}$ is *coresolving* if $\mathcal{B}$ satisfies the dual. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *hereditary* if $\mathcal{A}$ is resolving and $\mathcal{B}$ is coresolving. According to [9, Lemma 6.17], every complete cotorsion pair in a hereditary abelian category is naturally hereditary.

Hovey's correspondence (see [5, Theorem 2.2] and [9, Theorem 6.9]) establishes a bijection between abelian model structures on an abelian category C and triples $(Q, \mathcal{W}, \mathcal{R})$ of subcategories of C, where $\mathcal{W}$ is thick (i.e., closed under retracts and satisfies the 2-out-of-3 property for short exact sequences), and both $(Q, \mathcal{W} \cap \mathcal{R})$ and $(Q \cap \mathcal{W}, \mathcal{R})$ are complete cotorsion pairs. A triple $\mathcal{M} = (Q, \mathcal{W}, \mathcal{R})$ that satisfies these criteria is known as a *Hovey triple*. A Hovey triple $\mathcal{M} = (Q, \mathcal{W}, \mathcal{R})$ is *hereditary* if the associated cotorsion pairs $(Q, \mathcal{W} \cap \mathcal{R})$ and $(Q \cap \mathcal{W}, \mathcal{R})$ are hereditary. In this case, the *homotopy category* $\mathrm{Ho}(\mathcal{M})$ is a triangulated category, which is triangle-equivalent to the stable category $(Q \cap \mathcal{R})/\omega$, where $\omega := Q \cap \mathcal{W} \cap \mathcal{R}$. For further details, we refer to [16].

2.2. *Tilting objects*

Let $C(C)$ be the category consisting of all complexes over C together with chain maps. The homotopy category of $C(C)$ is denoted by $K(C)$. We write $D(C)$ for the derived category of C, which is constructed as the localization of $K(C)$ with respect to all quasi-isomorphisms. As usual, $C^b(C)$, $K^b(C)$, and $D^b(C)$ stand for the bounded versions of these categories, respectively. Remark that both $K^b(C)$ and $D^b(C)$ are triangulated categories.

Let us recall from [14] the concept of tilting objects in a hereditary abelian category. An object T in C is said to be a *tilting* object provided that the following two assertions are valid:

(1) T has no self-extensions, that is, $\mathrm{Ext}^1_C(T, T) = 0$;

(2) T generates C homologically; that is, any object $X \in C$ with $\mathrm{Hom}_C(T, X) = 0 = \mathrm{Ext}^1_C(T, X)$ must be the zero object.

It is remarkable that T is a tilting object in C implies $\mathrm{gl.dim}\,\mathrm{End}_C(T) \leq 2$; see, for example, [17, Theorem 3.1]. Furthermore, statement (2) is equivalent to the assertion that T generates $D^b(C)$ as a triangulated category. In other words, the smallest triangulated subcategory of $D^b(C)$ that contains T is precisely $D^b(C)$ itself, as discussed in [15].

2.3. *Weighted projective lines and curves*

Recall the basic settings from [15].

In what follows, we fix a positive integer t. Let $\mathbf{p} = (p_1, p_2, \dots, p_t) \in \mathbb{Z}^t_+$. Denote by $\mathbb{L} := \mathbb{L}(\mathbf{p})$, the abelian group of rank one, which is generated by $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_t$. The generators satisfy the relations $p_1\vec{x}_1 = p_2\vec{x}_2 = \cdots = p_t\vec{x}_t$. We refer to $\vec{c} := p_i\vec{x}_i$ (for any $i \in \{1, \dots, t\}$) as the *canonical element* of $\mathbb{L}$. Every element $\vec{x} \in \mathbb{L}$ admits a *normal form* representation:

$$\vec{x} = \sum_{1 \leq i \leq t} l_i\vec{x}_i + l\vec{c}, \tag{2.1}$$

where $0 \leq l_i < p_i$ for $i = 1, \dots, t$ and $l \in \mathbb{Z}$. Furthermore, we endow $\mathbb{L}$ with an order structure by defining its positive cone as $\{\vec{x} \in \mathbb{L} \mid \vec{x} \geq 0\}$. Here, for an element $\vec{x}$, the inequality $\vec{x} \geq 0$ holds if and only if the coefficient l form representation (2.1) satisfies $l \geq 0$.

Obviously, the polynomial ring $\mathbf{k}[X_1, \ldots, X_t]$ is an $\mathbb{L}(\mathbf{p})$-graded algebra by setting $\deg X_i = \vec{x}_i$, which is denoted by $S(\mathbf{p})$.

Let $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_t\}$ be a set of distinguished closed points on the projective line $\mathbb{P}^1$, where $\lambda_1 = \infty$, $\lambda_2 = 0$, and $\lambda_3 = 1$ are fixed as the normalization. Consider the $\mathbb{L}(\mathbf{p})$ - graded ideal $I(\mathbf{p}, \boldsymbol{\lambda})$ of the algebra $S(\mathbf{p})$, which is generated by the set of polynomials

$$\{X_i^{p_i} - (X_2^{p_2} - \lambda_i X_1^{p_1}) \mid 3 \leq i \leq t\}.$$

The quotient $S(\mathbf{p}, \boldsymbol{\lambda}) := S(\mathbf{p})/I(\mathbf{p}, \boldsymbol{\lambda})$ defines an $\mathbb{L}(\mathbf{p})$ - graded algebra. For each $1 \leq i \leq t$, we denote by x_i the element in $S(\mathbf{p}, \boldsymbol{\lambda})$ obtained as the image of the variable X_i under the quotient map from $S(\mathbf{p})$ to $S(\mathbf{p}, \boldsymbol{\lambda})$.

We define the *weighted projective line* $\mathbb{X} := \mathbb{X}_{\mathbf{p},\boldsymbol{\lambda}}$ as the set of all non-maximal prime homogeneous ideals of the algebra $S := S(\mathbf{p}, \boldsymbol{\lambda})$. Adopting an $\mathbb{L}$-graded adaptation of the Serre construction from [19], we can formulate the category of coherent sheaves over $\mathbb{X}$. Specifically, coh-$\mathbb{X}$ is presented as the quotient category $\text{mod}^{\mathbb{L}}\text{-}S/\text{mod}_0^{\mathbb{L}}\text{-}S$. Here, $\text{mod}^{\mathbb{L}}\text{-}S$ represents the category of finitely generated $\mathbb{L}$-graded modules over S, and $\text{mod}_0^{\mathbb{L}}\text{-}S$ is the Serre subcategory consisting of $\mathbb{L}$-graded modules of finite length.

In the quotient category $\text{mod}^{\mathbb{L}}\text{-}S/\text{mod}_0^{\mathbb{L}}\text{-}S$, the structure sheaf for the category coh-$\mathbb{X}$ is the image $\mathcal{O}$ of the algebra S. The abelian group $\mathbb{L}$ acts on the above data, including coh-$\mathbb{X}$, through grading shifts. Every line bundle in coh-$\mathbb{X}$ can be written uniquely as $\mathcal{O}(\vec{x})$, where $\vec{x}$ is an element of $\mathbb{L}$. Set the *dualizing element* of $\mathbb{L}$ as $\vec{\omega} = (t-2)\vec{c} - \sum_{1\leq i\leq t} \vec{x}_i$. Then, the category coh-$\mathbb{X}$ conforms to Serre duality, expressed by the equation $D\text{Ext}^1(X, Y) = \text{Hom}(Y, X(\vec{\omega}))$, and this duality holds functorially for any X and Y in coh-$\mathbb{X}$. Furthermore, due to Serre duality, almost split sequences exist in coh-$\mathbb{X}$, and the Auslander–Reiten translation τ is achieved by shifting with the dualizing element $\vec{\omega}$.

For an arbitrary weighted projective line $\mathbb{X}$, the category coh-$\mathbb{X}$ of coherent sheaves on $\mathbb{X}$ decomposes into two disjoint subcategories: the subcategory vect-$\mathbb{X}$ of vector bundles and the subcategory coh$_0$-$\mathbb{X}$ of torsion sheaves. Furthermore, in the category coh-$\mathbb{X}$, there exists a *canonical tilting sheaf* given by the direct sum

$$T_{\text{can}} = \bigoplus_{0\leq\vec{x}\leq\vec{c}} \mathcal{O}(\vec{x})$$

where the sum ranges over all elements $\vec{x}$ such that $0 \leq \vec{x} \leq \vec{c}$.

Recall from [20] that a *weighted projective curve* $\mathbb{X}$ is defined as a pair (X, ω). Here, X represents a smooth projective curve, and ω is a weight function that assigns integral and positive values on X. Specifically, the inequality $\omega(x) > 1$ is satisfied only at a finite number of (closed) points $x_1, \cdots, x_t$ of the curve X. We denote the category of coherent sheaves over $\mathbb{X}$ by coh-$\mathbb{X}$. This category can be decomposed into two disjoint subcategories: the subcategory vect-$\mathbb{X}$ consisting of vector bundles and the subcategory coh$_0$-$\mathbb{X}$ of torsion sheaves.

3. Cotorsion pairs in hereditary abelian categories

This section is devoted to investigating the relations between tilting objects and complete cotorsion pairs of finite type in hereditary abelian categories with not necessarily enough projectives and demonstrating that a bijective relationship exists between complete cotorsion pairs of finite type and tilting objects.

For convenience, throughout the remainder of this paper, we use the notations $\text{Ext}^1(M, N)$ and $\text{Hom}(M, N)$ to represent $\text{Ext}^1_{\mathcal{C}}(M, N)$ and $\text{Hom}_{\mathcal{C}}(M, N)$, respectively.

Suppose T is a tilting object in the category C. Define the following two classes of C:

1) The class ${}^{\perp_1}T$ consists of all objects X in C such that $\mathrm{Ext}^1(X,T)=0$;

2) The class $T^{\perp_1}$ consists of all objects Y in C such that $\mathrm{Ext}^1(T,Y)=0$.

Lemma 3.1. *Let T be a tilting object in C. Then we have ${}^{\perp_1}T = \mathrm{Cogen}\, T$, $T^{\perp_1} = \mathrm{Gen}\, T$. In addition, $({}^{\perp_1}T, T^{\perp_1})$ is a complete hereditary cotorsion pair of finite type in C.*

Proof. According to [21, Proposition 7.2.1], we have that the class ${}^{\perp_1}T$ is equal to Cogen T, the class $T^{\perp_1}$ coincides with Gen T, and the pair $({}^{\perp_1}T, T^{\perp_1})$ forms a hereditary cotorsion pair.

Next, we will prove that this cotorsion pair is complete. That is, for an arbitrary object X in C, there exist two short exact sequences:

$$0 \to B \to A \to X \to 0 \tag{3.1}$$

and

$$0 \to X \to B' \to A' \to 0 \tag{3.2}$$

with A and A' belonging to the class ${}^{\perp_1}T$, and B and B' belonging to the class $T^{\perp_1}$.

If $X \in {}^{\perp_1}T$, we take $A = X$ and $B = 0$; then (3.1) holds. Now assume $X \notin {}^{\perp_1}T$. Consider the universal add T-extension (c.f. [22, Section 3.5]) of X as follows:

$$0 \to T_1 \to Y \to X \to 0, \tag{3.3}$$

where $T_1 \cong \bigoplus_{T^{(i)}\in \mathrm{ind}\, T} (D\mathrm{Ext}^1(X,T^{(i)}) \otimes_k T^{(i)})$, and the direct sum runs over all the indecomposable direct summands ind T of T.

Therefore, whenever $\xi \in \mathrm{Ext}^1(X,T_2)$ and $T_2 \in \mathrm{add}\, T$, we can find a morphism $g \in \mathrm{Hom}(T_1,T_2)$ for which the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_1 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g} & & \downarrow & & \| & & \\ 0 & \longrightarrow & T_2 & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0. \end{array}$$

Equivalently, the morphism $\delta : \mathrm{Hom}(T_1,T_2) \to \mathrm{Ext}^1(X,T_2)$ is surjective.

Consider the long exact sequence

$$\mathrm{Hom}(T_1,T_2) \xrightarrow{\delta} \mathrm{Ext}^1(X,T_2) \longrightarrow \mathrm{Ext}^1(Y,T_2) \longrightarrow \mathrm{Ext}^1(T_1,T_2)\,.$$

Since $T_1 \in \mathrm{add}\, T$ and δ is surjective, we have $\mathrm{Ext}^1(T_1,T_2)=0$ and $\mathrm{Ext}^1(Y,T_2)=0$. Thus $Y \in {}^{\perp_1}T$. So (3.3) is the desired first short exact sequence (3.1).

When $X \in T^{\perp_1}$, we assign $B = X$ and $A = 0$. In contrast, when $X \notin T^{\perp_1}$, there exists an object $T_1 \in \mathrm{add}\, T \subseteq \mathcal{A}$, such that $\mathrm{Ext}^1(T_1,X) \neq 0$. Consider the universal extension

$$0 \to X \to Y \to T_1 \to 0. \tag{3.4}$$

We claim $Y \in \mathcal{B}$. By applying the functor $\mathrm{Hom}(T,-)$, we get

$$\mathrm{Hom}(T,T_1) \xrightarrow{\delta} \mathrm{Ext}^1(T,X) \longrightarrow \mathrm{Ext}^1(T,Y) \longrightarrow \mathrm{Ext}^1(T,T_1) = 0.$$

Since (3.4) is a universal extension, δ is surjective, i.e., for each ξ in $\mathrm{Ext}^1(T, X)$, there is a morphism g in $\mathrm{Hom}(T, T_1)$ that gives rise to the following pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & T & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & T_1 & \longrightarrow & 0. \end{array}$$

Therefore, we have $\mathrm{Ext}^1(T, Y) = 0$. Thus $Y \in T^{\perp_1}$. So (3.4) is the desired second short exact sequence.

We conclude our proof by showing that the cotorsion pair $({}^{\perp_1}T, T^{\perp_1})$ is of finite type. Since T is a tilting object, we know that $\mathrm{add}\, T \subseteq {}^{\perp_1}T \cap T^{\perp_1}$. On the other hand, for an arbitrary indecomposable object X in ${}^{\perp_1}T \cap T^{\perp_1}$, consider a minimal right add T-approximation $f : T' \to X$. Since $T^{\perp_1} = \mathrm{Gen}\, T$, the map f is an epimorphism, which yields the exact sequence $0 \to K \to T' \to X \to 0$. After applying the $\mathrm{Hom}(T, -)$ functor to this sequence, we find that $K \in T^{\perp_1}$. Furthermore, since $X \in {}^{\perp_1}T = {}^{\perp_1}(T^{\perp_1})$, the sequence splits. As a result, X is a direct summand of T. Consequently, we conclude that ${}^{\perp_1}T \cap T^{\perp_1} = \mathrm{add}\, T$. □

Consider a bounded chain complex X in the derived category $D^b(C)$. Assume that X has non-zero terms only in the degrees ranging from a to b. In other words, $X^i \neq 0$ precisely when $a \leq i \leq b$. We define the width of X, denoted as $\mathrm{width}(X)$, as the value $b - a + 1$.

Proposition 3.2. *Assume that $(\mathcal{A}, \mathcal{B})$ is a complete cotorsion pair of finite type in the category C. Then the object T, which is defined as the direct sum $T = \oplus\{\mathrm{ind}\, \mathcal{A} \cap \mathcal{B}\}$ of all indecomposable objects in the intersection of $\mathcal{A}$ and $\mathcal{B}$, is a tilting object in C.*

Proof. Since $(\mathcal{A}, \mathcal{B})$ is of finite type, $\mathcal{A} \cap \mathcal{B} = \mathrm{add}\, Y$ for some $Y \in C$. It follows that $\mathcal{A} \cap \mathcal{B}$ is closed under direct sums. Therefore, $\mathrm{Ext}^1(M, M) = 0$ for any $M \in \mathcal{A} \cap \mathcal{B}$. In particular, we have $\mathrm{Ext}^1(T, T) = 0$. All we need to do is verify that T generates $D^b(C)$ as a triangulated category. For any object X in C, thanks to the completeness of the cotorsion pair $(\mathcal{A}, \mathcal{B})$, we can get two short exact sequences $\eta : 0 \to X \to B_1 \to A_1 \to 0$ and $0 \to B_2 \to A_2 \to X \to 0$, where $A_1, A_2 \in \mathcal{A}$, $B_1, B_2 \in \mathcal{B}$. Observe that $\mathcal{A}$ and $\mathcal{B}$ are closed under subobjects and quotients, respectively. Using this, it is easy to see that $A_1, B_2 \in \mathcal{A} \cap \mathcal{B}$. Applying the functor $\mathrm{Ext}^1(A_1, -)$ to the short exact sequence $0 \to B_2 \to A_2 \to X \to 0$, we have

$$\mathrm{Ext}^1(A_1, B_2) \to \mathrm{Ext}^1(A_1, A_2) \xrightarrow{\pi} \mathrm{Ext}^1(A_1, X) \to 0.$$

Since $\eta \in \mathrm{Ext}^1(A_1, X)$ is in the image of π, there exists an exact sequence $0 \to A_2 \to C \to A_1 \to 0$ in $\mathrm{Ext}^1(A_1, A_2)$ makes the following diagram commute.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & B_2 & = & B_2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & A_2 & \to & C & \to & A_1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & X & \to & B_1 & \to & A_1 & \to & 0. \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since $\mathcal{A}$ and $\mathcal{B}$ are closed under extensions, we have $C \in \mathcal{A} \cap \mathcal{B}$. Then, because $A_2 \in \mathcal{A} \cap \mathcal{B}$, we can conclude that $X \in \mathcal{A} \cap \mathcal{B}$. It is well known that any short exact sequence in $C(\mathcal{C})$ corresponds to an exact triangle in $D^b(\mathcal{C})$. So, as a stalk complex in $D^b(\mathcal{C})$, X belongs to the triangulated subcategory of $D^b(\mathcal{C})$ generated by T. Assume that each bounded complex $Y \in D^b(\mathcal{C})$ with width $\leq n$ (i.e., non-zero in at most n consecutive degrees) belongs to the triangulated subcategory generated by T. Let $X \in D^b(\mathcal{C})$ be a complex of width $n+1$. Without loss of generality, suppose X is non-zero in degrees $n, n-1, \ldots, 1, 0$. Consider the stupid truncation of X:

$$\sigma_{\leq n-1} X : \quad 0 \to X_{n-1} \to \cdots \to X_0 \to 0.$$

This truncation has width n, so by the inductive hypothesis, $\sigma_{\leq n-1} X$ lies in the subcategory generated by T. There is a distinguished triangle in $D^b(\mathcal{C})$:

$$X_n[n-1] \to \sigma_{\leq n-1} X \to X \to X_n[n].$$

Since $X_n[n]$ is generated by $T_n[n]$, X is also generated by T. Using induction on the width, we find that each bounded complex in $D^b(\mathcal{C})$ belongs to the triangulated subcategory generated by T. Consequently, $D^b(\mathcal{C})$ is generated by T.

□

Theorem 3.3. *There is a bijection*

$$\left\{ \text{ Complete cotorsion pairs of finite type in } \mathcal{C} \right\} \xrightarrow{1-1} \left\{ \text{ Tilting objects in } \mathcal{C} \right\}$$

induced by the assignment

$$(\mathcal{X}, \mathcal{Y}) \mapsto \oplus\{\mathrm{ind}\, \mathcal{X} \cap \mathcal{Y}\}$$

and

$$T \mapsto ({}^{\perp_1}T, T^{\perp_1}),$$

where ${}^{\perp_1}T := \{X \in \mathcal{C} \mid \mathrm{Ext}^1_{\mathcal{C}}(X, T) = 0\}$*,* $T^{\perp_1} := \{Y \in \mathcal{C} \mid \mathrm{Ext}^1_{\mathcal{C}}(T, Y) = 0\}$*, and* $\oplus\{\mathrm{ind}\, \mathcal{X} \cap \mathcal{Y}\}$ *is the direct sum of all indecomposable objects belonging to* $\mathcal{X} \cap \mathcal{Y}$*.*

Proof. According to Lemma 3.1 and Proposition 3.2, the assignments are well-defined. Now we prove that they are bijective.

Consider a complete cotorsion pair $(\mathcal{X}, \mathcal{Y})$. Set $T = \oplus\{\text{ind } \mathcal{X} \cap \mathcal{Y}\}$. Proposition 3.2 guarantees that T is a tilting object. Additionally, by Lemma 3.1, we know that the class ${}^{\perp_1}T$ is equal to Cogen T, and the class $T^{\perp_1}$ coincides with Gen T. Now we claim that $T^{\perp_1} = \mathcal{Y}$. In fact, since $\mathcal{Y}$ is closed under quotients, we have Gen $T \subseteq \mathcal{Y}$. Conversely, $\mathcal{Y} = \mathcal{X}^{\perp_1} \subseteq T^{\perp_1}$. Conversely, Lemma 3.1 yields that if $({}^{\perp_1}T, T^{\perp_1})$ is a cotorsion pair, then $({}^{\perp_1}T, T^{\perp_1}) = (\mathcal{X}, \mathcal{Y})$.

Let T be a tilting object. Clearly, add $T \subseteq {}^{\perp_1}T \cap T^{\perp_1}$. Conversely, for any $X \in \text{ind } {}^{\perp_1}T \cap T^{\perp_1}$, take a minimal right add T-approximation $f : T' \to X$. According to Lemma 3.1, the morphism f is surjective. As a result, there exists an exact sequence $0 \to K \to T' \to X \to 0$. After applying the Hom$(T, -)$ functor to this exact sequence, we conclude that $K \in T^{\perp_1}$. Since $X \in {}^{\perp_1}T = {}^{\perp_1}(T^{\perp_1})$, this exact sequence splits, making X a direct summand of T. Therefore, $\mathcal{X} \cap \mathcal{Y} = {}^{\perp_1}T \cap T^{\perp_1} = \text{add } T$. □

Remark 3.4. Given a ring R, there exists an alternative definition for a cotorsion pair to be of finite type [10, Section 1]. That is, a cotorsion pair $(\mathcal{X}, \mathcal{Y})$ in Mod-R is called *of finite type* if it is generated by a class of modules possessing a finite projective resolution consisting of finitely generated projective modules. In this case, its heart equals Add K for some module K; see [18, Lemma 5.4]. Moreover, each cotorsion pair generated by tilting modules is of finite type in the sense of [10, Section 1].

Let $\mathcal{X}$ be a subcategory of mod-R. For an R-module homomorphism $\varphi : X \to M$ with $X \in \mathcal{X}$, φ is a right $\mathcal{X}$-approximation of M when $\text{Hom}_R(X_0, \varphi) : \text{Hom}_R(X_0, X) \to \text{Hom}_R(X_0, M)$ is surjective for all $X_0 \in \mathcal{X}$. We call $\mathcal{X}$ contravariantly finite if each R-module admits a right $\mathcal{X}$-approximation.

It was found out by Auslander and Reiten [23] that the notion of a contravariantly finite resolving subcategory is closely related to tilting theory. In particular, let Λ be a hereditary Artin algebra. The mapping $T \to T^{\perp_1}$ establishes a bijection between the isomorphism classes of basic tilting modules and the covariantly finite coresolving subcategories in mod-Λ. By Theorem 3.3, we get the following result.

Corollary 3.5. *For a hereditary Artin algebra Λ, the assignment $\mathcal{B} \to (\mathcal{A}, \mathcal{B})$ (with $\mathcal{A} = {}^{\perp_1}\mathcal{B}$) defines a bijection between the set of covariantly finite coresolving subcategories of* mod-Λ *and the set of complete cotorsion pairs of finite type in* mod-Λ.

Let Λ be a Henselian Gorenstein local ring. Takahashi [24, Theorem 1.2] classified the contravariantly finite resolving subcategories of mod-Λ, showing that there are only three: the subcategory proj(Λ) of projective modules, the subcategory MCM(Λ) of maximal Cohen-Macaulay modules, and mod-Λ itself. Since there is a bijection between resolving, contravariantly finite subcategories and coresolving, covariantly finite subcategories of mod-Λ, Theorem 3.3 and the preceding corollary yield the following classification result. When Λ is hereditary, since proj(Λ) = MCM(Λ), there are only two complete cotorsion pairs in mod-Λ.

Corollary 3.6. *Let Λ be a hereditary commutative Henselian (e.g., complete) local artin ring. Then all the complete cotorsion pairs of finite type in* mod-Λ *are the following:*

- (proj(Λ), mod-Λ)
- (mod-Λ, inj(Λ))

where inj(Λ) *denotes the full subcategory consisting of all finitely generated injective Λ-modules.*

4. Classification of cotorsion pairs for certain hereditary abelian categories

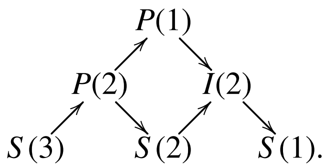
In this section, we obtain the classification results of complete cotorsion pairs for certain hereditary abelian categories, based on the main Theorem 3.3.

Recall that there are two kinds of connected hereditary abelian k-categories with tilting objects up to derived equivalences, namely, the module categories for finite-dimensional hereditary k-algebras and the categories of coherent sheaves over weighted projective lines. In this section, using Theorem 3.3, we classify all complete cotorsion pairs of finite type in two specific categories: the category of quiver representations and the category of coherent sheaves.

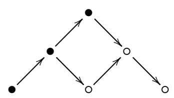
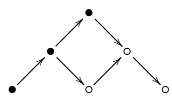
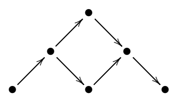
Example 4.1. *Let A be the path algebra of the quiver of type $\mathbb{A}_3$:*

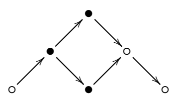
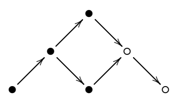
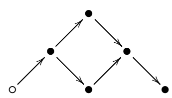
$$\overset{1}{\circ} \longrightarrow \overset{2}{\circ} \longrightarrow \overset{3}{\circ}.$$

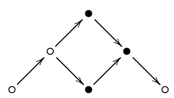
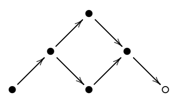
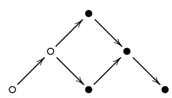
The Auslander–Reiten quiver $\Gamma(\text{mod-}A)$ *of the module category* mod-A *appears as*

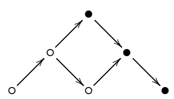
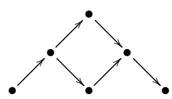
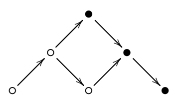


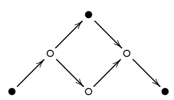
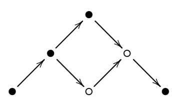
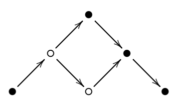
It is well-known that there are only 5 tilting A-modules; hence, there are 5 complete cotorsion pairs in mod-A *according to Theorem 3.3. They are indicated as follows, where we mark by* $\bullet$ *in* $\Gamma(\text{mod-}A)$ *to indicate the indecomposable direct summands in the tilting module T and in the induced cotorsion pairs $(\mathcal{A}, \mathcal{B})$ of finite type, respectively.*

(1) *If* $T =$  , *then* $\mathcal{A} =$  $\mathcal{B} =$  .

(2) *If* $T =$  , *then* $\mathcal{A} =$  $\mathcal{B} =$  .

(3) *If* $T =$  , *then* $\mathcal{A} =$  $\mathcal{B} =$  .

(4) *If* $T =$  , *then* $\mathcal{A} =$  $\mathcal{B} =$  .

(5) *If* $T =$  , *then* $\mathcal{A} =$  $\mathcal{B} =$  .

In general, we can give the number of complete cotorsion pairs over Dynkin algebras in terms of that of tilting modules (c.f. [25, Theorem 1]). Dynkin algebras are the connected hereditary Artin algebras that are representation-finite; thus, their valued quivers are of Dynkin type $\Delta_n = \mathbb{A}_n, \mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{F}_4, \mathbb{G}_2$. Here we list the number of complete cotorsion pairs over

Dynkin algebras as follows:

$\mathbb{A}_n$	$\mathbb{B}_n$	$\mathbb{C}_n$	$\mathbb{D}_n$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$	$\mathbb{F}_4$	$\mathbb{G}_2$
$\frac{1}{n+1}\binom{2n}{n}$	$\binom{2n-1}{n-1}$	$\binom{2n-1}{n-1}$	$\frac{3n-4}{2n-2}\binom{2n-2}{n-2}$	418	2431	17342	66	5

Next, we will classify all the complete cotorsion pairs of finite type in the category coh-$\mathbb{P}^1$ of coherent sheaves over the projective line $\mathbb{P}^1$.

Proposition 4.2. *Any hereditary complete cotorsion pair of finite type in* coh-$\mathbb{P}^1$ *has the form* $(\mathcal{A}_n, \mathcal{B}_n)$ *for some* $n \in \mathbb{Z}$*, where*

$$\mathcal{A}_n = \text{add}\,\{O(m) \mid m \le n+1\};$$

$$\mathcal{B}_n = \text{add}\,\{O(m) \mid m \ge n\} \sqcup \text{coh}_0\text{-}\mathbb{P}^1.$$

Proof. Recall that any tilting sheaf in coh-$\mathbb{P}^1$ has the form $T_n := O(n) \oplus O(n+1)$ for some $n \in \mathbb{Z}$. By Theorem 3.3, every complete cotorsion pair of finite type, induced by a tilting sheaf, has the form $(\mathcal{A}_n, \mathcal{B}_n)$, where

$$\mathcal{A}_n = {}^{\perp_1}T_n = \text{add}\,\{O(m) \mid m \le n+1\}$$

and

$$\mathcal{B}_n = T_n^{\perp_1} = \text{add}\,\{O(m) \mid m \ge n\} \sqcup \text{coh}_0\text{-}\mathbb{P}^1.$$

This finishes the proof. □

Proposition 4.3. *In the category* coh-$\mathbb{X}$ *of coherent sheaves over a weighted projective curve* $\mathbb{X} = (X, \omega)$*, complete cotorsion pairs of finite type exist if and only if* X *has genus zero, i.e.,* $\mathbb{X}$ *is a weighted projective line.*

Proof. Via the p-cycle construction from [26], the category coh-$\mathbb{X}$ of coherent sheaves on the weighted projective curve $\mathbb{X} = (X, w)$ is built from coh$X$, the coherent sheaf category of the base curve X. Theorem 4.3 in [26] states that coh-$\mathbb{X}$ is a hereditary abelian category, and it has a tilting object exactly when coh$X$ does. Further, Corollary A.7 in [20] shows that coh$X$ admits a tilting object if and only if X has genus zero, meaning $\mathbb{X}$ is the classical projective line. Applying the bijection from Theorem 3.3 finalizes the proof. □

At the end of this section, we will prove that the category of finite-dimensional nilpotent representations of a cyclic quiver has no complete cotorsion pairs.

Proposition 4.4. *For a cyclic quiver* Q*, let* $\text{rep}_k^{\text{nil}}(Q)$ *denote the category of finite-dimensional nilpotent* k*-representations of* Q*. Then* $\text{rep}_k^{\text{nil}}(Q)$ *admits no complete cotorsion pairs.*

Proof. Recall that the Auslander–Reiten quiver of $\text{rep}_k^{\text{nil}}(Q)$ is a non-homogeneous tube. Hence there are no tilting objects in $\text{rep}_k^{\text{nil}}(Q)$ (see [27, Section 5]). Then the bijection in Theorem 3.3 implies the result. □

5. Canonical cotorsion pairs for weighted projective lines

This section focuses on the study of cotorsion pairs for weighted projective lines induced by canonical tilting sheaves.

It should be noted that for every weighted projective line $\mathbb{X}$, there exists a specific canonical tilting sheaf $T_{\text{can}} = \bigoplus_{0 \le \vec{x} \le \vec{c}} \mathcal{O}(\vec{x})$ in coh-$\mathbb{X}$. In the following, we fix the tilting sheaf $T = T_{\text{can}}$, and describe in more details on the cotorsion pair $(\mathcal{A}, \mathcal{B})$ induced by T, i.e.,

$$\mathcal{A} = {}^{\perp_1}T := \{X \in C \mid \text{Ext}^1(X, T) = 0\}; \quad \mathcal{B} = T^{\perp_1} := \{Y \in C \mid \text{Ext}^1(T, Y) = 0\}.$$

Lemma 5.1. *$\mathcal{A} \subseteq$ vect-$\mathbb{X}$ and $\mathcal{B} \supseteq$ coh$_0$-$\mathbb{X}$.*

Proof. Note that there are no extensions from vector bundles to torsion sheaves in coh-$\mathbb{X}$, so $\text{Ext}^1(\text{vect-}\mathbb{X}, \text{coh}_0\text{-}\mathbb{X}) = 0$. Given $T \in$ vect-$\mathbb{X}$, we get $\text{Ext}^1(T, \text{coh}_0\text{-}\mathbb{X}) = 0$, which implies coh$_0$-$\mathbb{X} \subseteq \mathcal{B}$. Since $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair with $\text{Ext}^1(\mathcal{A}, \mathcal{B}) = 0$, we have $\text{Ext}^1(\mathcal{A}, \text{coh}_0\mathbb{X}) = 0$. Thus, $\mathcal{A} \subseteq$ vect-$\mathbb{X}$. □

Now we give a more detailed description for $\mathcal{A}$ and $\mathcal{B}$.

Proposition 5.2. *$\mathcal{A} = {}^{\perp_1}\mathcal{O}$ and $\mathcal{B} = \mathcal{O}(\vec{c})^{\perp_1}$.*

Proof. First we prove $\mathcal{A} = {}^{\perp_1}\mathcal{O}$. To do so, it is enough to show that for any $X \in$ vect-$\mathbb{X}$, the condition $\text{Ext}^1(X, \mathcal{O}) = 0$ is equivalent to $\text{Ext}^1(X, T) = 0$.

The 'if' part is obvious since $\mathcal{O}$ is a direct summand of T. In the following, we only consider the 'only if' part. Whenever $0 \le \vec{x} \le \vec{c}$, the following exact sequence exists:

$$0 \to \mathcal{O} \to \mathcal{O}(\vec{x}) \to S \to 0,$$

where $S \in$ coh$_0$-$\mathbb{X}$. Applying the functor $\text{Ext}^1(X, -)$, we get a right exact sequence

$$\text{Ext}^1(X, \mathcal{O}) \to \text{Ext}^1(X, \mathcal{O}(\vec{x})) \to \text{Ext}^1(X, S) \to 0.$$

Since $X \in$ vect-$\mathbb{X}, S \in$ coh$_0$-$\mathbb{X}$. Hence $\text{Ext}^1(X, S) = 0$. Therefore, $\text{Ext}^1(X, \mathcal{O}) = 0$ implies that $\text{Ext}^1(X, \mathcal{O}(\vec{x})) = 0$. It follows that $\text{Ext}^1(X, T) = 0$.

Dually, one can prove $\mathcal{B} = \mathcal{O}(\vec{c})^{\perp_1}$, which is omitted here. □

Corollary 5.3. *The following statements hold.*

(1) *$\mathcal{O}(\vec{x}) \in \mathcal{A}$ if and only if $\vec{x} \le \vec{c}$;*
(2) *$\mathcal{O}(\vec{x}) \in \mathcal{B}$ if and only if $\vec{x} \ge 0$.*

Proof. For all $\vec{x}, \vec{y}$ in the lattice $\mathbb{L}$, applying Serre duality yields

$$\text{Ext}^1(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{y})) \cong D\text{Hom}(\mathcal{O}(\vec{y}), \mathcal{O}(\vec{x} + \vec{\omega})),$$

which vanishes if and only if $\vec{x} + \vec{\omega} - \vec{y} \not\ge 0$, or equivalently, $\vec{x} + \vec{\omega} - \vec{y} \le \omega + \vec{c}$, i.e., $\vec{x} - \vec{y} \le \vec{c}$. Therefore, by Proposition 5.2,

$$\mathcal{O}(\vec{x}) \in \mathcal{A} \Leftrightarrow \text{Ext}^1(\mathcal{O}(\vec{x}), \mathcal{O}) = 0 \Leftrightarrow \vec{x} \le \vec{c};$$
$$\mathcal{O}(\vec{x}) \in \mathcal{B} \Leftrightarrow \text{Ext}^1(\mathcal{O}(\vec{c}), \mathcal{O}(\vec{x})) = 0 \Leftrightarrow \vec{x} \ge 0.$$

□

For an object $M \in \mathcal{C}$, we denote by $\langle M\rangle$ the full subcategory of $\mathcal{C}$ generated by add M (under isomorphisms and extensions). For a subcategory $\mathcal{S}$ of $\mathcal{C}$, denote by $\langle \mathcal{S}\rangle = \langle M \mid M \in \mathcal{S}\rangle$. Next, we obtain the following conclusion.

Proposition 5.4. *Let* coh-$\mathbb{X}$ *be the category of coherent sheaves over a weighted projective line* $\mathbb{X}$. *Then there is a canonical tilting cotorsion pair* $(\mathcal{A}, \mathcal{B})$ *in* coh-$\mathbb{X}$, *where*

$$\mathcal{A} = \langle O(\vec{x}) \mid \vec{x} \leq \vec{c}\rangle;\ \mathcal{B} = \langle O(\vec{x}) \mid \vec{x} \geq 0, \mathrm{coh}_0\text{-}\mathbb{X}\rangle.$$

Proof. To prove $\mathcal{A} = \langle O(\vec{x}) \mid \vec{x} \leq \vec{c}\rangle$, Corollary 5.3 implies $\langle O(\vec{x}) \mid \vec{x} \leq \vec{c}\rangle \subseteq \mathcal{A}$. Thus, to prove the equality, it suffices to show that any indecomposable object X in $\mathcal{A}$ belongs to $\langle O(\vec{x}) \mid \vec{x} \leq \vec{c}\rangle$. We use induction on $r = r(X)$. Since Lemma 5.1 indicates $\mathcal{A} \subseteq$ vect-$\mathbb{X}$, we have $r \geq 1$.

If $r = 1$, then X is a line bundle, say $X = O(\vec{x})$. By Corollary 5.3, we obtain $\vec{x} \leq \vec{c}$. Assume that $Y \in \langle O(\vec{x}) \mid \vec{x} \leq \vec{c}\rangle$ for any $Y \in \mathcal{A}$ with $r(Y) < r$. Now consider $X \in \mathrm{ind}\,\mathcal{A}$ with $r(X) = r$. Since $\mathcal{A} = \mathrm{Cogen}(T)$, X is cogenerated by add T. Therefore, there exists an injection

$$f : X \longrightarrow \bigoplus_{0 \leq \vec{x} \leq \vec{c}} O(\vec{x})^{\oplus m(\vec{x})},$$

where $m(\vec{x}) \geq 0$. Let $f_{\vec{x}} : X \to O(\vec{x})$ be a non-zero summand of f. Then $\mathrm{im} f_{\vec{x}} = O(\vec{y})$ for some $\vec{y} \leq \vec{x} \leq \vec{c}$. Since $O(\vec{y})$ is in $\langle O(\vec{x}) \mid \vec{x} \leq \vec{c}\rangle$, there exists an exact sequence $0 \to Y \to X \to O(\vec{y}) \to 0$. As $\mathcal{A}$ is closed under subobjects, $Y \in \mathcal{A}$ and $r(Y) < r(X)$. By the induction hypothesis, $Y \in \langle O(\vec{x}) \mid \vec{x} \leq \vec{c}\rangle$, which implies $X \in \langle O(\vec{x}) \mid \vec{x} \leq \vec{c}\rangle$.

To prove $\mathcal{B} = \langle O(\vec{x}) \mid \vec{x} \geq 0, \mathrm{coh}_0\text{-}\mathbb{X}\rangle$, Lemma 5.1 and Corollary 5.3 imply $\langle O(\vec{x}) \mid \vec{x} \geq 0, \mathrm{coh}_0\text{-}\mathbb{X}\rangle \subseteq \mathcal{B}$. So, it suffices to prove that any indecomposable vector bundle X in $\mathcal{B}$ is in $\langle O(\vec{x}) \mid \vec{x} \geq 0, \mathrm{coh}_0\text{-}\mathbb{X}\rangle$. Similar to the above, we'll use induction on $r = r(X)$.

When $r = 1$, Corollary 5.3(2) concludes the proof. Suppose that for any $Y \in \mathcal{B}$ with $r(Y) < r$, we have $Y \in \langle O(\vec{x}) \mid \vec{x} \geq 0, \mathrm{coh}_0\text{-}\mathbb{X}\rangle$. Now, let $X \in \mathrm{ind}\,\mathcal{B}$ with $r(X) = r$. Since $\mathcal{B} = \mathrm{Gen}(T)$, there exist a surjection

$$g : \bigoplus_{0 \leq \vec{x} \leq \vec{c}} O(\vec{x})^{\oplus m_{\vec{x}}} \longrightarrow X.$$

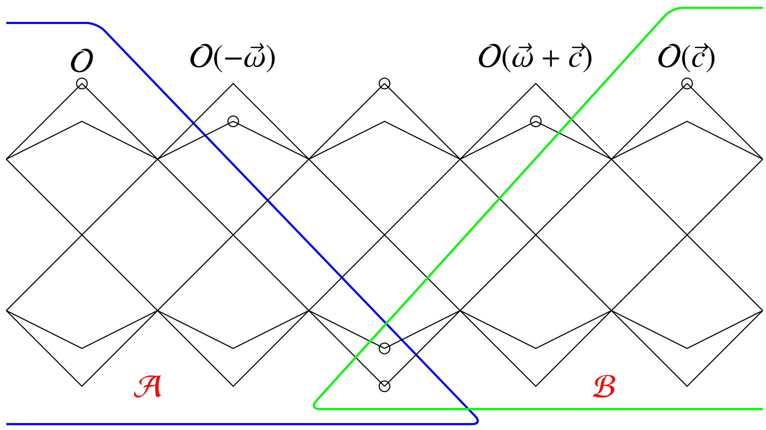
Let $g_{\vec{x}} : O(\vec{x}) \to X$ be a non-zero summand of g. Then $g_{\vec{x}}$ is injective, yielding a short exact sequence $0 \to O(\vec{x}) \to X \to Y \to 0$. Note that $\mathcal{B}$ is closed under quotients. Hence $Y \in \mathcal{B}$ with $r(Y) < r$. Then by induction, $Y \in \langle O(\vec{x}) \mid \vec{x} \geq 0, \mathrm{coh}_0\text{-}\mathbb{X}\rangle$. It follows that $X \in \langle O(\vec{x}) \mid \vec{x} \geq 0, \mathrm{coh}_0\text{-}\mathbb{X}\rangle$. We are done. □

We now present a concrete example to illustrate the canonical torsion pair $(\mathcal{A}, \mathcal{B})$ in the category of coherent sheaves. Since the torsion sheaves are all included in $\mathcal{B}$, we will only focus on vector bundles.

Example 5.5. *Consider a weighted projective line* $\mathbb{X}$ *of weight type* $(2, 2, 4)$. *Then*

$$T_{\mathrm{can}} = \bigoplus_{0 \leq \vec{x} \leq \vec{c}} O(\vec{x}) = O \oplus O(\vec{x}_1) \oplus O(\vec{x}_2) \oplus O(\vec{x}_3) \oplus O(2\vec{x}_3) \oplus O(3\vec{x}_3) \oplus O(\vec{c}).$$

The Auslander–Reiten quiver Γ(vect-$\mathbb{X}$) *of the subcategory of vector bundles* vect-$\mathbb{X}$ *has the following shape:*



Each indecomposable direct summand of T has been marked by $\circ$ in $\Gamma(\text{vect-}\mathbb{X})$. Note that the canonical cotorsion pair $(\mathcal{A}, \mathcal{B})$ can be written as

$$\mathcal{A} = \text{Cogen } T = \{X \mid \text{Ext}^1(X, T) = 0\}; \quad \mathcal{B} = \text{Gen } T = \{X \mid \text{Ext}^1(T, X) = 0\}.$$

According to Proposition 5.4, we obtain that

$$\mathcal{A} \cap \text{vect-}\mathbb{X} = \mathcal{A} = \langle \mathcal{O}(\vec{x}) \mid \vec{x} \leq \vec{c} \rangle; \quad \mathcal{B} \cap \text{vect-}\mathbb{X} = \langle \mathcal{O}(\vec{x}) \mid \vec{x} \geq 0 \rangle.$$

Observe that $\mathcal{A} \cap \mathcal{B} = \text{add } T$. Besides, the remaining indecomposable vector bundles belonging to $\mathcal{A}$ (resp., *$\mathcal{B}$) are sitting in the area bounded by the blue* (resp., *green) curve.*

6. Complete cotorsion pairs in vect-$\mathbb{X}$

For any weighted projective line $\mathbb{X}$, the subcategory vect-$\mathbb{X} \subseteq$ coh-$\mathbb{X}$ of vector bundles is an exact category. The properties of cotorsion pairs in vect-$\mathbb{X}$ are quite different from those of coh-$\mathbb{X}$. In fact, vect-$\mathbb{X}$ carries various Frobenius exact structures, c.f. [28], which make it possible to construct compatible cotorsion pairs and then Hovey triples in vect-$\mathbb{X}$.

In this section, we focus on cotorsion pairs in vect-$\mathbb{X}$. Let $\mathcal{X}$ be a full subcategory of vect-$\mathbb{X}$, closed under Auslander–Reiten translations. That is, for any $X \in \mathcal{X}$, we have $\tau X = X(\vec{\omega}) \in \mathcal{X}$.

Definition 6.1. In vect-$\mathbb{X}$, a sequence

$$\eta : 0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0 \tag{6.1}$$

is $\mathcal{X}$-*exact* if $\text{Hom}(X, \eta)$ is exact for all $X \in \mathcal{X}$. In this case, u and v are called an $\mathcal{X}$-*monomorphism* and an $\mathcal{X}$-*epimorphism*, respectively.

Lemma 6.2. *The sequence η in* (6.1) *is $\mathcal{X}$-exact if and only if* $\text{Hom}(\eta, X)$ *is exact for any $X \in \mathcal{X}$.*

Proof. According to the definition, the $\mathcal{X}$-exactness of η is equivalent to the exactness of $\text{Hom}(X, \eta)$ for every $X \in \mathcal{X}$. Thanks to coh-$\mathbb{X}$ being a hereditary abelian category, this condition is in turn equivalent to the exactness of $\text{Ext}^1(X, \eta)$. Serre duality further reveals that $\text{Ext}^1(X, \eta)$ is exact precisely when $\text{Hom}(\eta, \tau X)$ is exact. From the fact that $\tau\mathcal{X} = \mathcal{X}$, we can thus conclude the proof. □

For any almost-split sequence $0 \to X \to Y \to Z \to 0$ in vect-$\mathbb{X}$, the sequence is $\mathcal{X}$-exact if and only if Z (or equivalently, $X = \tau Z$) does not belong to the subcategory $\mathcal{X}$.

The following result is well-known; for the proof, we refer to [29]; see also [30, Theorem B.2] and [31, Proposition 2.16].

Proposition 6.3. *The class of $\mathcal{X}$-exact sequences induces an exact structure on* vect-$\mathbb{X}$*, which is Frobenius. In this structure, the indecomposable projective (resp., injective) objects are precisely the vector bundles in $\mathcal{X}$.*

Proposition 6.4. *Under $\mathcal{X}$-exact structure, both* (vect-$\mathbb{X}, \mathcal{X}$) *and* ($\mathcal{X}$, vect-$\mathbb{X}$) *are complete cotorsion pairs in* vect-$\mathbb{X}$*. Consequently,* (vect-$\mathbb{X}, \mathcal{X}$, vect-$\mathbb{X}$) *is a Hovey triple, and* vect-$\mathbb{X}/\mathcal{X}$ *is a triangulated category.*

Proof. This is a consequence of the fact that the objects in $\mathcal{X}$ are both projective and injective in the category vect-$\mathbb{X}$. □

At the end of this section, we investigate when the triangulated category vect-$\mathbb{X}/\mathcal{X}$ has Auslander–Reiten triangles. Denote by $\underline{\mathrm{Hom}}_{\mathcal{X}}(X, Y)$ the homomorphism space between X and Y in the triangulated category vect-$\mathbb{X}/\mathcal{X}$. In vect-$\mathbb{X}$, an $\mathcal{X}$-monomorphism $u : X \to I$ is a *injective envelope* of X if I is injective and for any composition $X \xrightarrow{u} I \xrightarrow{v} Y$ that is an $\mathcal{X}$-monomorphism, v must also be an $\mathcal{X}$-monomorphism. We write $X \xrightarrow{j_X} I(X)$ for the injective envelope of X.

Proposition 6.5. *The triangulated category* vect-$\mathbb{X}/\mathcal{X}$ *is both Hom-finite and Krull–Schmidt. Its Serre duality is characterized by the isomorphism*

$$\underline{\mathrm{Hom}}_{\mathcal{X}}(X, Y[1]) = D\underline{\mathrm{Hom}}_{\mathcal{X}}(Y, X(\vec{\omega})).$$

Moreover, vect-$\mathbb{X}/\mathcal{X}$ *admits Auslander–Reiten triangles, and the $\vec{\omega}$-shift acts as its AR-translation.*

Proof. Since vect-$\mathbb{X}$ is Hom-finite, the category vect-$\mathbb{X}/\mathcal{X}$ inherits both Hom-finiteness and the Krull–Schmidt property. We now discuss its Serre duality.

Applying $\mathrm{Hom}(X, -)$ and $\mathrm{Hom}(-, X(\vec{\omega}))$ to the exact sequence $\mu : 0 \to Y \to I(Y) \to Y[1] \to 0$, we get the exact sequences:

$$\mathrm{Hom}(X, I(Y)) \to \mathrm{Hom}(X, Y[1]) \to \underline{\mathrm{Hom}}_{\mathcal{X}}(X, Y[1]) \to 0, \tag{6.2}$$

$$\mathrm{Hom}(I(Y), X(\vec{\omega})) \to \mathrm{Hom}(Y, X(\vec{\omega})) \to \underline{\mathrm{Hom}}_{\mathcal{X}}(Y, X(\vec{\omega})) \to 0. \tag{6.3}$$

Using Serre duality in coh-$\mathbb{X}$, dualizing (6.3) gives the exact sequence:

$$0 \to D\underline{\mathrm{Hom}}_{\mathcal{X}}(Y, X(\vec{\omega})) \to \mathrm{Ext}^1(X, Y) \to \mathrm{Ext}^1(X, I(Y)). \tag{6.4}$$

From the long exact Hom-Ext sequence $\mathrm{Hom}(X, \mu)$ and the sequences (6.2) and (6.4), we obtain the natural isomorphism $\underline{\mathrm{Hom}}_{\mathcal{X}}(X, Y[1]) \cong D\underline{\mathrm{Hom}}_{\mathcal{X}}(Y, X(\vec{\omega}))$ as asserted. □

Use of AI tools declaration

The authors declare that Artificial Intelligence (AI) tools played no part in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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