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Research article

Energy decay for a porous system with a fractional operator in the memory

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Abstract: In this work, we examine a porous-elastic system with a fractional operator incorporated in the memory term, which acts exclusively on one equation within the system. Under appropriate conditions on the polynomially decreasing kernels of the memory type, we establish the result of polynomial decay.

Keywords: porous system; polynomial decay; memory term; relaxation function

1. Introduction

In this paper, we are concerned with the porous system where the damping mechanism is presented by a fractional term of memory type:

$$\begin{cases} \rho_1\varphi_{tt} - (\mu\varphi_{xx} + b\psi_x) = 0, & \text{in } (0,L)\times(0,\infty), \\ \rho_2\psi_{tt} - \delta\psi_{xx} + b\varphi_x + \xi\psi - \displaystyle\int_0^{+\infty} g(s)B_*^{\theta}\psi(t-s)ds = 0, & \text{in } (0,L)\times(0,\infty), \end{cases} \tag{1.1}$$

with the boundary conditions

$$\varphi(0,t) = \varphi(L,t) = \psi_x(0,t) = \psi_x(L,t) = 0, \quad t \in (0,\infty), \tag{1.2}$$

and the initial conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \psi_1(x), \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \quad \psi(x,-s) = \phi_0(x,s), \quad s > 0, \quad x \in (0,L). \end{cases} \tag{1.3}$$

Here, φ represents the longitudinal displacement, while ψ denotes the volume fraction of the solid elastic material. The parameters ρ_1, μ, b, ρ_2, δ, and ξ are positive constitutive constants that satisfy the inequality $\mu\xi > b^2$. The operator B_* corresponds to the differential operator $(-\partial_{xx})$, and the parameter θ is taken within the range $\theta \in (0, 1)$.

The selection of boundary conditions (BCs) in Eq (1.2) and initial conditions (ICs) in Eq (1.3) plays a fundamental role in both the mathematical analysis and the physical interpretation of the problem. Below, we briefly discuss their significance:

Importance of boundary conditions (Eq (1.2))

- The conditions $\varphi(0, t) = \varphi(L, t) = 0$ correspond to a longitudinal displacement that is fixed at both ends of the domain, modeling a *clamped or fixed boundary*. This is a standard assumption in structural mechanics and implies that no axial motion occurs at the boundaries.
- The conditions $\psi_x(0, t) = \psi_x(L, t) = 0$ imply that the flux of porosity vanishes at the endpoints, i.e., *no net transfer of the volume fraction* across the boundaries. This is physically reasonable in many porous materials where the microstructure remains static near the edges.
- From a mathematical standpoint, these boundary conditions are critical in establishing the energy framework of the problem, particularly in demonstrating the dissipation mechanism introduced by the memory term. They also play a central role in the stability and decay analysis carried out in later sections.

Importance of initial conditions (Eq (1.3))

- The initial values $\varphi(x, 0)$, $\varphi_t(x, 0)$, $\psi(x, 0)$, and $\psi_t(x, 0)$ specify the initial configuration and velocity fields for the displacement and volume fraction. These are standard requirements for second-order hyperbolic systems.
- In addition, due to the presence of a memory-type damping term, the model requires an initial history condition $\psi(x, -s) = \phi_0(x, s)$ for $s > 0$. This condition is essential for capturing the *hereditary effects* inherent in fractional damping mechanisms and ensures the proper functioning of the convolution integral.
- Without appropriate initial conditions—especially for the memory component—the system would be *ill-posed*, leading to issues of non-uniqueness or lack of existence of solutions.

Relevance to the present work

- **Physical accuracy:** The choice of BCs and ICs reflects physically meaningful constraints typical in porous elastic materials subject to damping.
- **Well-posedness:** These conditions are essential in ensuring that the problem admits a unique, physically interpretable solution.
- **Energy and stability framework:** The BCs, in particular, facilitate the derivation of energy inequalities and decay rates, which are central contributions of this paper.
- **Mathematical coherence:** The imposed conditions are compatible with the differential structure of the system and the properties of the fractional damping operator.

The function g serves as the kernel of the memory term and satisfies the following conditions:

H: The function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1-decreasing function that satisfies

$$g(0) > 0, \quad g'(t) \leq -m(t)g^p(t), \quad 1 < p < \frac{3}{2},$$

$$\int_0^{+\infty} g(s)\,ds < b\left(\frac{\pi}{L}\right)^{2(1-\theta)},$$

where $m : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 non-increasing function.

Noting that short memory can be considered in our model. This corresponds to replacing the integral over $(0, +\infty)$ with an integral over $(0, t)$ or, more generally, over (t_0, t). Mathematically, this can be achieved by setting the function g to zero for $s \in (t, +\infty)$ and ensuring that hypothesis **H** remains satisfied. This modification would still allow us to analyze the effect of fractional damping within a finite memory framework (see [1, 2]).

The fundamental evolution equations governing the one-dimensional theory of porous materials are given by

$$\begin{aligned} \rho_1\varphi_{tt} &= T_x, \\ \rho_2\psi_{tt} &= H_x + G. \end{aligned} \tag{1.4}$$

Here, T represents the stress, H denotes the equilibrated stress, and G corresponds to the equilibrated body force. The variables φ and ψ describe the displacement of the solid elastic material and the volume fraction, respectively.

The constitutive relations governing the system are expressed as

$$\begin{aligned} T &= \mu\varphi_x + b\psi, \\ H &= \delta\psi_x - \int_0^{+\infty} g(s)B_*^{\theta-1/2}\psi(t-s)\,ds, \\ G &= -b\varphi_x - \xi\psi. \end{aligned} \tag{1.5}$$

Substituting these expressions into Eq (1.4), we derive the governing system given by Eq (1.1). The development of this model follows the fundamental balance laws of continuum mechanics, combined with constitutive equations that incorporate memory effects.

The porous-elastic system studied in this paper is based on the well-established theory of poroelasticity, which describes the behavior of fluid-saturated porous materials. The governing equations are derived from the fundamental balance laws of continuum mechanics, incorporating constitutive relations that account for the interaction between the solid matrix and the fluid within the pores. The memory term in our model represents the hereditary effects observed in real-world porous materials, where the stress–strain relationship is influenced by past deformations.

This framework is particularly relevant in applications where energy dissipation and wave propagation are significantly affected by the internal structure of the material. For instance, in geophysics, porous-elastic models are widely used to study seismic wave propagation in sedimentary rocks, where fluid flow and viscoelastic effects play a crucial role. Similarly, in biomechanics, these models help describe the behavior of biological tissues, such as bones and cartilage, which exhibit poroelastic properties due to their fluid-filled microstructure. In engineering, porous-elastic materials are used in acoustic insulation, filtration systems, and energy absorption applications.

By incorporating a fractional damping term in the memory effect, our model extends classical poroelasticity theories to account for more complex dissipation mechanisms. This provides a more accurate description of materials that exhibit long-term relaxation behavior, making the results applicable to a broader range of porous media, including viscoelastic foams, polymer-based composites, and other engineered materials with internal fluid interactions.

Porous materials represent a highly significant area of materials science due to their broad range of applications. They are widely used in various fields, including soil mechanics, engineering, power technology, biology, and material science, among others. The theoretical framework for porous elastic materials was introduced by Cowin and Nunziato [3], who developed a nonlinear theory of elastic materials with voids.

Theory is based on the observation that, in addition to the usual elastic effects, these materials possess a microstructure with the property that the mass at each point is obtained as the product of the mass density of the material matrix and the volume fraction. For more details, we refer the reader to [4–7] and the references therein.

Recent works, such as [8–13], have focused on the asymptotic behavior of solutions under different damping conditions, revealing intricate stability properties that depend on the interaction between porosity and viscoelasticity. In [14], Quintanilla considered (1.1) with a linear damping term τu_t (τ is constant) in the second equation (without the memory term $g = 0$) and initial and mixed boundary conditions. He obtained a decay result, but it is non-exponential decay.

In [15], the authors introduced a viscoelastic damping term of the form τu_{txx} in the first equation of (1.1), assuming $g = 0$. They established that the decay rate of the solution is polynomial and cannot be exponential. In [8], Apalara investigated system (1.1) with Neumann–Dirichlet boundary conditions, incorporating a finite memory term $\int_0^t g(s)\psi_{xx}(t-s)ds$ instead of an infinite memory term $\int_0^{+\infty} g(s)B_*^\theta\psi(t-s)ds$. Under the assumption of equal wave propagation speeds and an exponentially decaying relaxation function, a general decay result was obtained, encompassing both exponential and polynomial decay as special cases. Recently, this result in [13] was extended to the case of non-equal wave speeds, which is more realistic from a physical perspective.

When $\mu = b = \xi = K$, it is well known that system (1.1) reduces to the Timoshenko system with a fractional operator in the infinite memory:

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_{xx} + \psi_x) = 0, & \text{in } (0,L)\times(0,\infty), \\ \rho_2\psi_{tt} - \delta\psi_{xx} + K(\varphi_x + \psi) - \displaystyle\int_0^{+\infty} g(s)B_*^\theta\psi(t-s)ds = 0, & \text{in } (0,L)\times(0,\infty). \end{cases} \tag{1.6}$$

Astudillo and Oquendo [16] investigated system (1.6) under the assumption of an exponentially decreasing kernel and Dirichlet–Neumann boundary conditions. Using semigroup theory and the spectral approach, they established polynomial decay rates. Specifically, they demonstrated that if the wave propagation speeds are different and $\theta \neq 1$, the solutions decay polynomially with a rate of $t^{-1/(4-2\theta)}$, while if the wave propagation speeds are equal, the solutions decay polynomially with a rate of $t^{-1/(2-2\theta)}$. In addition, they proved that these decay rates are optimal. Moreover, when $\theta = 1$ and the wave propagation speeds are equal, they obtained exponential decay of the solutions. In their study, they also established the global existence of weak solutions.

From a numerical perspective, advanced finite element and spectral methods have been employed to approximate solutions to complex poroelastic systems, enabling more precise simulations of wave propagation and energy dissipation in porous structures [17, 18]. These studies provide a solid foundation for further exploration of porous-elastic models with memory effects, as considered in this work.

In the present work, we address the following question: By applying the multiplier method to the systems (1.1)–(1.3) under assumption (H), do we obtain the same polynomial decay result as in [16]? It is important to emphasize that the method used in our study differs from the approach used in [16].

The remainder of this paper is organized as follows: In Section 2, we present some preliminary results that are essential for proving our main result. In Section 3, we establish the well-posedness of the systems (1.1)–(1.3). Finally, in Section 4, we state and prove the main result concerning the energy decay of the system using the multiplier technique.

2. Preliminaries

In this section, we introduce some preliminary material needed for the proof of our results. Throughout this paper, C denotes a generic positive constant.

First, we define the functional spaces used in our analysis. The space $L^2 = L^2(0, L)$ denotes the usual Lebesgue space, equipped with the norm $\|\cdot\|_{L^2}$. For simplicity, we will use $\|\cdot\|$ instead of $\|\cdot\|_{L^2}$ and $\langle\cdot,\cdot\rangle$ instead of $\langle\cdot,\cdot\rangle_{L^2}$.

Let s be a non-negative number. The Sobolev space $H^s = H^s(0, L)$ consists of functions in $L^2(0, L)$ whose weak derivatives up to order s also belong to $L^2(0, L)$, and it is endowed with the norm $\|\cdot\|_{H^s}$. Next, we introduce the following Hilbert spaces:

$$L^2_*(0, L) = \left\{ f \in L^2(0, L) : \int_0^L f(x)\, dx = 0 \right\},$$

$$H^1_*(0, L) = H^1(0, L) \cap L^2_*(0, L).$$

Next, we define the operators:

$$B = -\partial_{xx} : \quad D(B) \subset L^2(0, L) \to L^2(0, L),$$

$$B_* = -\partial_{xx} : \quad D(B_*) \subset L^2_*(0, L) \to L^2_*(0, L),$$

where

$$D(B) = H^2(0, L) \cap H^1_0(0, L),$$

and

$$D(B_*) = \left\{ \psi \in H^2(0, L) \cap L^2_*(0, L) : \quad \psi_x(0) = \psi_x(L) \right\}.$$

The operators B and B_* are positive, self-adjoint, and have a compact inverse. Consequently, the operators B^σ and B_*^σ are positive, bounded for $\sigma \le 0$, and self-adjoint for all $\sigma \in \mathbb{R}$. Furthermore, the embeddings

$$D(B^{\sigma_1}) \hookrightarrow D(B^{\sigma_2}), \quad D(B_*^{\sigma_1}) \hookrightarrow D(B_*^{\sigma_2}),$$

are continuous for $\sigma_1 > \sigma_2$.

We define the norms:

$$\|B^\sigma \varphi\| = \|\varphi\|_{D(B^\sigma)}, \quad \|B_*^\sigma \varphi\| = \|\varphi\|_{D(B_*^\sigma)},$$

for $\sigma \ge 0$.

If $\varphi \in D(B^{\sigma+1/2})$ and $\psi \in D(B_*^{\sigma+1/2})$, we have

$$\|B^{\sigma+1/2}\varphi\| = \|B_*^\sigma \partial_x \varphi\|, \quad \|B_*^{\sigma+1/2}\psi\| = \|B^\sigma \partial_x \psi\|.$$

In the case when $\sigma = 0$, it follows that

$$\|B^{1/2}\varphi\| = \|\partial_x\varphi\|, \quad \|B_*^{1/2}\psi\| = \|\partial_x\psi\|.$$

If $\varphi \in D(B^{\sigma_0})$ and $\psi \in D(B_*^{\sigma_0})$ with $\sigma_0 = \max(\sigma, 1/2)$, then we have

$$\langle B_*^{\sigma}\psi, \varphi_x\rangle = -\langle \varphi_x, B^{\sigma}\varphi\rangle.$$

If $\psi \in D(B_*)$ and $\int_0^{+\infty} g(s)ds < b\left(\frac{\pi}{L}\right)^{2(1-\theta)}$, then $\|B_*^{\sigma}\psi\|$ and $\|E_*^{\sigma}\psi\|$ are equivalent for all $\sigma \in \mathbb{R}$, where

$$E_*\psi = \delta\, B_*\psi - \left(\int_0^{+\infty} g(s)ds\right) B_*^{\theta}\psi.$$

For more details on this context, we refer to [16, 19].

Now, let $\eta(t, s) = \psi(t) - \psi(t - s)$; then the system (1.1) becomes

$$\begin{cases} \rho_1\varphi_{tt} - (\mu\varphi_{xx} + b\psi_x) = 0, & \text{in } (0, L) \times (0, \infty), \\ \rho_2\psi_{tt} + E_*\psi + b\varphi_x + \xi\psi + \int_0^{+\infty} g(s)\, B_*^{\theta}\, \eta(s)ds = 0, & \text{in } (0, L) \times (0, \infty), \\ \eta_t - \psi_t + \partial_s\eta = 0. \end{cases} \tag{2.1}$$

We end with the following crucial lemma, which will be used in the proof of our main result.

Lemma 2.1. *Let α, c_1, and c_2 be three positive constants; F, m, and h be positive functions such that F is differentiable and m and h are continuous on $\mathbb{R}_+$, satisfying*

$$\forall t > 0,\ F'(t) \le -c_1\, m^{\alpha+1}(t)\ \ F^{\alpha+1}(t) + c_2\, h(t).$$

Then, for some constant $C > 0$, we have

$$F(t) \le C(1 + t)^{\frac{-1}{\alpha}}\, m^{-\frac{\alpha+1}{\alpha}} \left[1 + \int_0^t (s + 1)^{\frac{1}{\alpha}}\, m^{\frac{\alpha+1}{\alpha}}\, h(s)\, ds\right] \quad \forall t > 0. \tag{2.2}$$

Proof. In order to prove the relation (2.2), we follow the same steps as in [20] (page 598).

3. Well-posedness result

In this section, we study the existence of solutions for the porous system. For this purpose, we consider the following Hilbert space:

$$\mathcal{H} = H_0^1(0, L) \times H_*^1(0, L) \times L^2(0, L) \times L_*^2(0, L) \times L_g^2(\mathbb{R}^+; D(B_*^{\theta/2})).$$

The energy associated with the solution of the problem is given by

$$E(t) = \frac{\rho_1}{2}\|\varphi_t\|^2 + \frac{\rho_2}{2}\|\psi_t\|^2 + \frac{\mu}{2}\|\varphi_x\|^2 + \frac{1}{2}\|E_*^{1/2}\psi\|^2 + \frac{\xi}{2}\|\psi\|^2 + b\int_0^L \psi\varphi_x\, dx + \frac{1}{2}\|\eta\|^2_{L_g^2\left(\mathbb{R}^+;\, D(B_*^{\theta/2})\right)}, \tag{3.1}$$

for all $(\varphi, \psi, \varphi_t, \psi_t, \eta) \in \mathcal{H}$.

Lemma 3.1. *Let* (φ, ψ, η) *be a regular solution of the problem* (2.1). *Then, the energy functional defined by* (3.1) *satisfies*

$$E'(t) = \frac{1}{2}\int_0^L \int_0^{+\infty} g'(s)\left[B_*^{\theta/2}\eta\right]^2 \, dsdx \le 0.$$

Proof. Multiplying $(2.1)_1$ by φ_t and $(2.1)_2$ by ψ_t and integrating over $(0, L)$, we obtain

$$\frac{d}{dt}\left[\rho_1\|\varphi_t\|^2 + \mu\|\varphi_x\|^2\right] + b\int_0^L \psi\varphi_{xt}\, dx = 0, \tag{3.2}$$

and

$$\frac{1}{2}\frac{d}{dt}\left[\rho_2\|\psi_t\|^2 + \|E_*^{1/2}\psi\|^2 + \mu\|\varphi_x\|^2\right] + b\int_0^L \varphi_x\psi_t\, dx + \int_0^L \int_0^{+\infty} g(s)\, B_*^{\theta/2}\,\eta(s)\, B_*^{\theta/2}\psi_t\, dsdx = 0. \tag{3.3}$$

We remark that

$$\int_0^L \int_0^{+\infty} g(s)\,\frac{d}{ds}(B_*^{\theta/2}\eta)^2\, ds\, dx = -\int_0^L \int_0^{+\infty} g'(s)\left(B_*^{\theta/2}\eta\right)^2\, ds\, dx. \tag{3.4}$$

Using the relation (3.4) and $(2.1)_3$, we obtain

$$\begin{aligned}\int_0^L \int_0^{+\infty} B_*^{\theta/2}\psi_t\, g(s)\, B_*^{\theta/2}\eta(s)\, ds\, dx = &\frac{1}{2}\frac{d}{dt}\left[\int_0^L \int_0^{+\infty} g(s)\left(B_*^{\theta/2}\,\eta\right)^2(s)\, ds\, dx\right]\\ &-\frac{1}{2}\int_0^L \int_0^{+\infty} g'(s)\left(B_*^{\theta/2}\,\eta(s)\right)^2\, ds\, dx.\end{aligned} \tag{3.5}$$

Inserting (3.5) into (3.3), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\rho_2\|\psi_t\|^2 + \|E_*^{1/2}\psi\|^2 + \mu\|\varphi_x\|^2 + \int_0^L \int_0^{+\infty} g(s)\left(B_*^{\theta/2}\,\eta\right)^2(s)\, ds\, dx\right]\\ &\quad + b\int_0^L \varphi_x\psi_t\, dx - \frac{1}{2}\int_0^L \int_0^{+\infty} g'(s)\left(B_*^{\theta/2}\,\eta(s)\right)^2\, ds\, dx.\end{aligned} \tag{3.6}$$

Summing (3.2) and (3.6), we arrive at

$$E'(t) = \frac{1}{2}\int_0^L \int_0^{+\infty} g'(s)\,(B_*^{\theta/2}\eta)^2\, dsdx \le 0.$$

Hence, the proof of this lemma is achieved.

For completeness, we state and prove the existence result of (2.1), (1.2), (1.3) by the Galerkin method together with some energy estimates.

Theorem 3.2. *For* $(\varphi_0, \psi_0, \varphi_1, \psi_1, \eta) \in \mathcal{H}$ *and* $T > 0$. *Assume that* (H) *is satisfied; then the problems* (1.1)–(1.3) *has a unique weak solution such that*

$$(\varphi, \psi) \in C\left([0, T], H_0^1(0, L) \times H_*^1(0, L)\right) \cap C^1\left([0, T], L^2(0, L) \times L_*^2(0, L)\right).$$

Proof. The proof can be done in two steps:

Step 1: Faedo-Galerkin approximation: We construct an approximation of the solution (φ, ψ, η) using the Faedo–Galerkin method. Specifically, let $W_n = \operatorname{span}(w_1, \ldots, w_n)$ be a Hilbert basis of the space $H_0^1(0, 1)$.
We choose a sequence

$$(\varphi_0^n, \psi_0^n, \eta_0^n) \in W_n, \quad (\varphi_1^n, \psi_1^n, \eta_1^n) \in W_n,$$

such that

$$(\varphi_0^n, \varphi_1^n, \psi_0^n, \psi_1^n, \eta_0^n, \eta_1^n) \to (\varphi_0, \varphi_1, \psi_0, \psi_1, \eta_0, \eta_1) \quad \text{strongly in } \mathcal{H}.$$

We now define the approximation:

$$(\varphi^n(x,t), \psi^n(x,t), \eta^n(x,t,s)) = \sum_{j=1}^{n} (f_j^n(x,t), g_j^n(x,t), g_j^n(x,t) - g_j^n(x,t-s)) w_j,$$

which satisfies the following problem:

$$\begin{cases} \rho_1 \displaystyle\int_0^L \varphi_{tt}^n \, w_j dx + \int_0^L (\mu\varphi_x^n + b\psi^n) w_{j\,x} \, dx = 0, \\ \rho_2 \displaystyle\int_0^L \psi_{tt}^n \, w_j \, dx + \int_0^L E_*^{1/2}\psi^n \, E_*^{1/2} w_j \, dx + \int_0^L (b\varphi_x^n + \xi\psi^n) \, w_j \, dx \\ + \displaystyle\int_0^L \left(\int_0^{+\infty} g(s) \, B_*^{\theta/2} \, \eta^n(s) ds \right) B_*^{\theta/2} w_j \, dx = 0, \end{cases} \tag{3.7}$$

with the initial conditions:

$$(\varphi^n(0), \psi^n(0)) = (\varphi_0^n, \psi_0^n) \quad \text{and} \quad (\varphi_t^n, \psi_t^n) = (\varphi_1^n, \psi_1^n). \tag{3.8}$$

According to the standard theory of ordinary differential equations, the finite-dimensional problems (3.7) and (3.8) has a solution $(f_j^n, g_j^n)_{j=1,\ldots,n}$ defined on $[0, t_n)$. The following estimate allows us to conclude that $t_n = T$.

Step 2: Energy estimate: Multiplying $(3.7)_1$ by $f_j'^n$, $(3.7)_2$ by $g_j'^n$, summing over $j = 1, \ldots, n$ for each obtained equation, and finally integrating over $(0, t)$, we obtain:

$$\begin{cases} \rho_1 \displaystyle\int_0^t \int_0^L \varphi_{tt}^n \, \varphi_t^n dx \, dt - \int_0^t \int_0^L (\mu\varphi_{xx}^n + b\psi_x)\varphi_t^n \, dx \, dt = 0, \\ \rho_2 \displaystyle\int_0^t \int_0^L \psi_{tt}^n \, \psi_t^n \, dx + \int_0^t \int_0^L E_*\psi^n \, \psi_t^n \, dx + \int_0^t \int_0^L (b\varphi_x^n + \xi\psi^n) \, \psi_t^n \, dx \\ + \displaystyle\int_0^t \int_0^L \left(\int_0^{+\infty} g(s) \, B_*^{\theta} \, \eta^n(s) ds \right) \psi_t^n \, dx = 0. \end{cases}$$

By integrating by parts, as B_*^θ and E_*^θ are positive self-adjoint operators, we obtain using $(3.5)_3$:

$$\begin{cases} \frac{\rho_1}{2}\int_0^t\int_0^L (\varphi_t^n)^2\,dx\,dt + \int_0^t\int_0^L (\mu\varphi_x^n + b\psi^n)\varphi_{xt}^n\,dx\,dt = \frac{\rho_1}{2}\int_0^L (\varphi_t^n)^2(0)\,dx,\\ \frac{\rho_2}{2}\int_0^L (\psi_t^n)^2\,dx + \frac{1}{2}\int_0^L (E_*^{1/2}\psi^n)^2\,dx + \frac{\xi}{2}\int_0^L (\psi^n)^2\,dx + \frac{1}{2}\int_0^L \left(\int_0^{+\infty} g(s)\,B_*^{\theta/2}\,(\eta^n(s))^2 ds\right)dx \\ + b\int_0^t\int_0^L \varphi_x^n\,\psi_t^n\,dx\,dx = \frac{\rho_2}{2}\int_0^L (\psi_t^n)^2(0)\,dx + \frac{1}{2}\int_0^L E_*^{1/2}{\psi^n}^2(0)\,dx + \frac{\xi}{2}\int_0^L (\psi^n)^2(0)\,dx \\ + \frac{1}{2}\int_0^L \left(\int_0^{+\infty} g(s)\left(B_*^{\theta/2}\,\eta^n(0,s)\right)^2 ds\right)dx + \frac{1}{2}\int_0^t\int_0^L \left(\int_0^{+\infty} g'(s)\,B_*^{\theta/2}\,(\eta^n(s))^2 ds\right)dx. \end{cases} \tag{3.9}$$

Now, we denote:

$$E^n(t) = \frac{\rho_1}{2}\|\varphi_t^n\|^2 + \frac{\rho_2}{2}\|\psi_t^n\|^2 + \frac{\mu}{2}\|\varphi_x^n\|^2 + \frac{1}{2}\|E_*^{1/2}\psi\|^2 + \frac{\xi}{2}\|\psi\|^2 + b\int_0^L \psi^n\varphi_x^n dx + \frac{1}{2}\|\eta^n\|^2.$$

Summing up, we obtain:

$$E^n(t) \leq E^n(0).$$

Since the sequence converges, we can find a positive constant C independent of n such that:

$$E^n(t) \leq C.$$

From there we can pass to the limit in (3.7) and (3.8). The rest of the proof follows.

Remark 3.3. If the condition $\mu\xi > b^2$ is satisfied, then the energy $E(t)$ defined in (3.1) is equivalent to the norm

$$\frac{\rho_1}{2}\|\varphi_t\|^2 + \frac{\rho_2}{2}\|\psi_t\|^2 + \frac{\mu}{2}\|\varphi_x\|^2 + \frac{1}{2}\|E_*^{1/2}\psi\|^2 + \frac{\xi}{2}\|\psi\|^2 + \frac{1}{2}\|\eta\|^2_{L_g^2(\mathbb{R}^+;D(B_*^{\theta/2}))}.$$

This equivalence ensures that the energy $E(t)$ properly measures the total dynamics of the system and provides a useful tool for stability analysis.

4. Decay result

In this section, we prove a decay result for the energy of the systems (1.1)–(1.3) using the multiplier technique. To this end, we first establish the following lemmas.

Lemma 4.1. *Assume that g satisfies (H). Then, for all $t \in \mathbb{R}_+$, we have:*

(i) *For $1 < p < \frac{3}{2}$, there exists a constant $C > 0$ such that*

$$m(t)\int_0^L\int_0^t g(s)[B_*^{\theta/2}\eta]^2\,ds\,dx \leq C\left[-E'(t)\right]^{\frac{1}{2p-1}}.$$

(ii) *If there exists a positive constant n_0 such that $\|B^{1/2}\phi_0(t)\|^2 \leq n_0$, then for $\sigma \in (0,1]$, we have:*

$$m(t)\int_0^L\int_t^{+\infty} g(s)[B_*^{\sigma/2}\eta]^2\,ds \leq C\left(\int_t^{+\infty} g(s)\,ds\right)m(t) = Ch(t). \tag{4.1}$$

Proof. For the proof of (i), we refer to Corollary 2.1 in [21].
Now, we prove (ii). We estimate:

$$\begin{aligned}\int_0^L\int_t^{+\infty} g(s)[B_*^{\sigma/2}\eta]^2\,ds &\le 2\|B^{\sigma/2}\psi(t)\|^2\int_t^{+\infty} g(s)\,ds + 2\int_t^{+\infty} g(s)\|B^{\sigma/2}\psi(t-s)\|^2\,ds\\ &\le 2E(0)\int_t^{+\infty} g(s)\,ds + 2\sup_{z\le 0}\|B^{\sigma/2}\psi(z)\|^2\int_t^{+\infty} g(s)\,ds\\ &\le (2E(0)+n_0)\int_t^{+\infty} g(s)\,ds = C\int_t^{+\infty} g(s)\,ds.\end{aligned}$$

Thus, relation (4.1) is verified.

Corollary 4.2. *Assume that g satisfies* (H). *Then, for all $t \in \mathbb{R}_+$, we have*

$$m(t)\left(\int_0^L\int_0^{+\infty} g(s)[B_*^{\theta/2}\eta]^2\,ds + \int_0^L\int_t^{+\infty} g(s)[B_*^{1/2}\eta]^2\,ds\right) \le C\left([-E'(t)]^{\frac{1}{2p-1}} + h(t)\right).$$

Proof. This result follows directly from (i) and (ii) in Lemma 4.1.

Lemma 4.3. *The functional*

$$F_1(t) = \rho_2\int_0^L \psi\psi_t dx + \frac{b\rho_1}{\mu}\int_0^L \psi\int_0^x \varphi_t(y)dy\,dx$$

satisfies

$$F_1'(t) \le -\frac{1}{2}\|E_*^{1/2}\psi\|^2 - (\xi - \frac{b^2}{\mu})\|\psi\|^2 + \varepsilon_1\|\varphi_t\|^2 + C(1+\frac{1}{\varepsilon_1})\|\psi_t\|^2 + Cg\circ B_*^{\theta/2}\eta, \tag{4.2}$$

where

$$g\circ B_*^{\theta/2}\eta = \int_0^L\int_0^{+\infty} g(s)[B_*^{\theta/2}\eta]^2\,ds\,dx.$$

Proof. We compute the derivative of F_1 with respect to t. Using $(2.1)_1$, $(2.1)_2$, and integration by parts, we obtain

$$\begin{aligned}F_1'(t) = &-\|E_*^{1/2}\psi\|^2 - (\xi - \frac{b^2}{\mu})\|\psi\|^2 + \rho_2\|\psi_t\|^2\\ &- \int_0^{+\infty} g(s)\int_0^L B_*^{\theta/2}\eta B_*^{\theta/2}\psi(t)dx\,ds + \frac{b\rho_1}{\mu}\int_0^L \psi_t\int_0^x \varphi_t(y)dy\,dx.\end{aligned}$$

Now, we estimate the last two terms on the right-hand side as follows:
Using Young's inequality, we estimate

$$\begin{aligned}\frac{b\rho_1}{\mu}\int_0^L \psi_t\int_0^x \varphi_t(y)\,dy\,dx &\le \frac{\varepsilon_1}{L^2}\int_0^L\left(\int_0^x \varphi_t(y)\,dy\right)^2 dx + \left(\frac{b\rho_1}{\mu}\right)^2\frac{L^2}{4\varepsilon_1}\int_0^L \psi_t^2 dx\\ &\le \varepsilon_1\int_0^L \varphi_t^2 dx + \frac{C}{\varepsilon_1}\int_0^L \psi_t^2 dx.\end{aligned} \tag{4.3}$$

For the integral involving $g(s)$, using Young's and Cauchy–Schwarz's inequalities along with the fact that

$$\|E_*^{1/2}\psi\| \sim \|B_*^{1/2}\psi\|, \quad \text{and} \quad \|B_*^{1/2}\psi\| \hookrightarrow \|B_*^{\theta/2}\psi\|,$$

we obtain

$$\begin{aligned}
&-\int_0^{+\infty} g(s) \int_0^L B_*^{\theta/2}\eta B_*^{\theta/2}\psi(t)\,dx\,ds \\
&\leq \delta_1 \int_0^L \int_0^{+\infty} \left[B_*^{\theta/2}\psi\right]^2 dx + \frac{1}{4\delta_1}\int_0^L \left[\int_0^{+\infty} g(s) B_*^{\theta/2}\eta\,ds\right]^2 dx \\
&\leq \delta_1 \int_0^L \int_0^{+\infty} \left[B_*^{\theta/2}\psi\right]^2 dx + \frac{\int_0^{+\infty} g(s)ds}{4\delta_1}\int_0^L \int_0^{+\infty} g(s)\left[B_*^{\theta/2}\eta\right]^2 ds\,dx \\
&\leq c_1\delta_1\|E_*^{1/2}\psi\|^2 + \frac{C}{\delta_1} g \circ B_*^{\theta/2}\eta.
\end{aligned}$$

Choosing $\delta_1 = \dfrac{1}{2c_1}$, we obtain

$$\int_0^{+\infty} g(s)\int_0^L B_*^{\theta/2}\eta B_*^{\theta/2}\psi(t)dx\,ds \leq \frac{1}{2}\|E_*^{1/2}\psi\|^2 + Cg \circ B_*^{\theta/2}\eta. \tag{4.4}$$

By combining (4.3) and (4.4), we verify inequality (4.2), completing the proof.

Lemma 4.4. *Let T be a positive constant, and let the functional*

$$F_2(t) = -\rho_2 \int_0^L \psi_t \int_t^{+\infty} g(s)\,\eta(s)\,ds\,dx$$

satisfy, for any $\varepsilon_2 > 0$ and $\varepsilon_3 > 0$, the inequality for all $t \leq T$

$$\begin{aligned}
F_2'(t) \leq &-\frac{\rho_2 g_0}{2}\|\psi_t\|^2 + 2\varepsilon_2\|E_*^{1/2}\psi\|^2 + \varepsilon_3\|\varphi_x\|^2 - Cg' \circ B_*^{\theta/2}\eta \\
&+ \frac{C}{\varepsilon_2}\widetilde{g} \circ B_*^{1/2}\eta + C\left(\frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3} + 1\right) g \circ B_*^{\theta/2}\eta,
\end{aligned}$$

where

$$g_0 = \int_T^{\infty} g(s)ds \quad \textit{for all} \quad t \leq T,$$

$$g \circ B_*^{\theta/2}\eta = \int_0^L \int_0^{+\infty} g(s)[B_*^{\theta/2}\eta]^2 ds\,dx,$$

and

$$\widetilde{g} \circ B_*^{1/2}\eta = \int_0^L \int_t^{+\infty} g(s)[B_*^{1/2}\eta]^2 ds\,dx.$$

Proof. Differentiating F_2 with respect to t, we obtain

$$F_2'(t) = -\rho_2 \int_0^L \psi_{tt} \int_t^{+\infty} g(s)\,\eta(s)\,ds\,dx + \rho_2 \int_0^L \psi_t g(t)\,\eta(t)\,dx - \rho_2 \int_0^L \psi_t \int_t^{+\infty} g(s)\,\eta_t(s)\,ds\,dx.$$

Using $(2.1)_2$ and $(2.1)_3$, and also the self-adjointness of $E_*^{1/2}$ and $B_*^{\theta/2}$, we obtain

$$\begin{aligned} F_2'(t) &= \int_0^L E_*^{1/2}\psi\left(\int_t^{+\infty} g(s)E_*^{1/2}\eta(s)\,ds\right)dx + \int_0^L (b\varphi_x+\xi\psi)\left(\int_t^{+\infty} g(s)\eta(s)\,ds\right)dx \\ &+ \int_0^L \left(\int_0^{+\infty} g(s)B_*^{\theta/2}\eta(s)\,ds\right)\left(\int_t^{+\infty} g(s)B_*^{\theta/2}\eta(s)\right)dx \\ &- \rho_2\int_0^L \psi_t\left(\int_t^{+\infty} g'(s)\eta(s)\,ds\right)dx - \rho_2\left(\int_t^{+\infty} g(s)\,ds\right)\|\psi_t\|^2. \end{aligned}$$

Now, using Young's and Cauchy–Schwarz inequalities, and noting that $D(B_*^{\theta/2}) \hookrightarrow L^2(0,L)$ and $D(E_*^{1/2}) \sim D(B_*^{1/2})$, we obtain the following estimates:

$$\begin{aligned} J_1 = \int_0^L E_*^{1/2}\psi\left(\int_t^{+\infty} g(s)E_*^{1/2}\eta(s)\,ds\right)dx &\le \varepsilon_2\|E_*^{1/2}\psi\|^2 + \frac{1}{4\varepsilon_2}\int_0^L\left[\int_t^{+\infty} g(s)E_*^{1/2}\eta\,ds\right]^2 dx \\ &\le \varepsilon_2\|E_*^{1/2}\psi\|^2 + \frac{\int_t^{+\infty} g(s)\,ds}{4\varepsilon_2}\int_0^L\int_t^{+\infty} g(s)\left[E_*^{1/2}\eta\right]^2 ds\,dx \\ &\le \varepsilon_2\|E_*^{1/2}\psi\|^2 + \frac{C}{\varepsilon_2}\widetilde{g}\circ B_*^{1/2}\eta. \end{aligned} \tag{4.5}$$

Similarly to (4.5), we find

$$J_2 = b\int_0^L \varphi_x\left(\int_t^{+\infty} g(s)\eta(s)\,ds\right)dx \le \varepsilon_3\|\varphi_x\|^2 + \frac{C}{\varepsilon_3}g\circ B_*^{\theta/2}\eta, \tag{4.6}$$

$$J_3 = \xi\int_0^L \psi\left(\int_t^{+\infty} g(s)\eta(s)\,ds\right)dx \le \varepsilon_2\|E_*^{1/2}\psi\|^2 + \frac{C}{\varepsilon_2}g\circ B_*^{\theta/2}\eta, \tag{4.7}$$

$$\begin{aligned} J_4 &= \int_0^L\left(\int_t^{+\infty} g(s)B_*^{\theta/2}\eta(s)\,ds\right)\left(\int_0^{+\infty} g(s)B_*^{\theta/2}\eta(s)\,ds\right)dx \\ &\le \int_0^L\left(\int_0^{+\infty} g(s)B_*^{\theta/2}\eta(s)\,ds\right)^2 dx \\ &\le Cg\circ B_*^{\theta/2}\eta, \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} J_5 &= -\rho_2\int_0^L \psi_t\left(\int_t^{+\infty} g'(s)\eta(s)\,ds\right)dx \\ &\le \rho_2\left[\delta_1\int_0^L \psi_t^2 dx + \frac{1}{4\delta_1}\int_0^L\left(\int_t^{+\infty} g'(s)\eta(s)\,ds\right)^2 dx\right] \\ &\le \rho_2\delta_1\int_0^L \psi_t^2 dx - \frac{\rho_2}{4\delta_1}\left(\int_t^{+\infty} g'(s)\,ds\right)\int_0^L\int_0^{+\infty} -g'(s)\,(\eta(s))^2\,ds\,dx \\ &\le \rho_2\delta_1\|\psi_t\|^2 - \frac{C}{\delta_1}g'\circ B_*^{\theta/2}\eta. \end{aligned}$$

Putting $\delta_1 = \frac{g_0}{2}$, we obtain

$$J_5 \le \rho_2\frac{g_0}{2}\|\psi_t\|^2 - Cg'\circ B_*^{\theta/2}\eta. \tag{4.9}$$

From (4.5)–(4.9) and since g is a decreasing function, it follows that the relation (4.4) is verified. Hence, the proof is completed.

Lemma 4.5. *Assume (H) holds and that* $\frac{\mu}{\rho_1} = \frac{\delta}{\rho_2}$. *Then, the functional*

$$F_3(t) = \int_0^L \varphi_t \left[\psi_x + \frac{\rho_1}{\mu\rho_2}\int_0^{+\infty} g(s)B_*^{\theta-1/2}\left[\psi(t) - \eta(s)\right]\, ds\right] dx + \int_0^L \varphi_x\psi_t\, dx$$

satisfies, for any $\varepsilon_4 > 0$,

$$F_3'(t) \le C(1+\varepsilon_4)\|E_*^{1/2}\psi\|^2 - \frac{b}{2\rho_2}\|\varphi_x\|^2 + \varepsilon_4\|\varphi_t\|^2 + \frac{C}{\varepsilon_4} g\circ B_*^{\theta/2}\eta - \frac{C}{\varepsilon_4} g'\circ B_*^{\theta/2}\eta. \tag{4.10}$$

Proof. We take the derivative of $F_3(t)$, which gives

$$\begin{aligned} F_3'(t) = &\int_0^L \varphi_{tt}\left[\psi_x + \frac{\rho_1}{\mu\rho_2}\int_0^{+\infty} g(s)B_*^{\theta-1/2}\left[\psi(t)-\eta(s)\right]\, ds\right] dx \\ &+ \int_0^L \varphi_t\left[\psi_{xt} + \frac{\rho_1}{\mu\rho_2}\int_0^{+\infty} g(s)B_*^{\theta-1/2}\left[\psi_t(t)-\eta_t(s)\right]\, ds\right] dx + \int_0^L \varphi_{xt}\psi_t\, dx + \int_0^L \varphi_x\psi_{tt}\, dx. \end{aligned}$$

Using Eqs $(2.1)_1$ to $(2.1)_3$, we obtain

$$\begin{aligned} F_3'(t) = &\frac{\mu}{\rho_1}\int_0^L \varphi_{xx}\psi_x\, dx + \frac{1}{\rho_2}\left(\int_0^{+\infty} g(s)\, ds\right)\int_0^L B_*^{\theta-1/2}\psi\,\varphi_{xx}\, dx \\ &- \frac{1}{\rho_2}\int_0^L \varphi_{xx}\int_0^{+\infty} g(s)B_*^{\theta-1/2}\eta(s)\, ds\, dx + \frac{b}{\rho_1}\|\psi_x\|^2 + \frac{b}{\mu\rho_2}\left(\int_0^{+\infty} g(s)\, ds\right)\int_0^L B_*^{\theta-1/2}\psi\psi_x\, dx \\ &- \frac{b}{\mu\rho_2}\int_0^L \psi_x\int_0^{+\infty} g(s)B_*^{\theta-1/2}\eta(s)\, ds\, dx - \frac{1}{\rho_2}\int_0^L \varphi_x\, E_*\psi\, dx - \frac{b}{\rho_2}\|\varphi_x\|^2 - \frac{\xi}{\rho_2}\int_0^L \varphi_x\psi\, dx \\ &- \frac{1}{\rho_2}\int_0^L \varphi_x\int_0^{+\infty} g(s)B_*^{\theta}\eta(s)\, ds\, dx - \frac{\rho_1}{\mu\rho_2}\int_0^L \varphi_t\int_0^{+\infty} g'(s)B_*^{\theta-1/2}\eta(s)\, ds\, dx. \end{aligned} \tag{4.11}$$

We recall that $-\psi_x = B_*^{1/2}\psi$ and $-B_*^{\theta-1/2}\psi_x = B_*^{\theta}\psi$. Then, applying integration by parts to Eq (4.11) and taking into account the boundary conditions, as well as the self-adjointness of $E_*^{1/2}$ and $B_*^{\sigma/2}$ for all $\sigma\in\mathbb{R}$, we obtain

$$\begin{aligned} F_3'(t) = &\frac{b}{\rho_1}\|\psi_x\|^2 - \frac{b}{\rho_2}\|\varphi_x\|^2 + \frac{b}{\mu\rho_2}\left(\int_0^{+\infty} g(s)\, ds\right)\int_0^L B_*^{\theta-1/2}\psi\psi_x\, dx \\ &- \frac{b}{\mu\rho_2}\int_0^L \psi_x\int_0^{+\infty} g(s)B_*^{\theta-1/2}\eta(s)\, dsdx - \frac{\rho_1}{\mu\rho_2}\int_0^L \varphi_t\int_0^{+\infty} g'(s)B_*^{\theta-1/2}\eta(s)\, dsdx \\ &- \frac{\xi}{\rho_2}\int_0^L \varphi_x\psi\, dx. \end{aligned} \tag{4.12}$$

Next, we estimate the terms on the right-hand side as follows:
- Using Young's and Cauchy–Schwarz's inequalities and the fact that $D(B_*^{\theta/2}) \hookrightarrow D(B_*^{\theta-1/2})$, we obtain

$$I_1 = \frac{b}{\mu\rho_2}\left(\int_0^{+\infty} g(s)\, ds\right)\int_0^L B_*^{\theta-1/2}\psi\psi_x\, dx \le C\|E_*^{1/2}\psi\|^2. \tag{4.13}$$

- For the second term,

$$I_2 = -\frac{b}{\mu\rho_2}\int_0^L \psi_x \int_0^{+\infty} g(s)B_*^{\theta-1/2}\eta(s)\,ds\,dx \le \varepsilon_4\|\psi_x\|^2 + \frac{\left(\frac{b}{\mu\rho_2}\right)^2}{4\varepsilon_4}\int_0^L \left[\int_0^{+\infty} g(s)B_*^{\theta-1/2}\eta(s)\,ds\right]^2 dx$$
$$\le \varepsilon_4\|\psi_x\|^2 + \frac{\left(\frac{b}{\mu\rho_2}\right)^2 \int_0^{+\infty} g(s)\,ds}{\varepsilon_4} g \circ B_*^{\theta-1/2}\eta$$
$$\le C\varepsilon_4\|E_*^{1/2}\psi\|^2 + \frac{C}{\varepsilon_4} g \circ B_*^{\theta/2}\eta. \tag{4.14}$$

- For the third term,

$$I_3 = -\frac{\rho_1}{\mu\rho_2}\int_0^L \varphi_t \int_0^{+\infty} g'(s)B_*^{\theta-1/2}\eta(s)\,ds\,dx \le \varepsilon_4\|\varphi_t\|^2 - \frac{C}{\varepsilon_4} g' \circ B_*^{\theta/2}\eta. \tag{4.15}$$

- For the fourth term,

$$I_4 = -\frac{\xi}{\rho_2}\int_0^L \varphi_x\psi\,dx \le \delta_1\|\varphi_x\|^2 + \left(\frac{\xi}{\rho_2}\right)^2 \frac{1}{4\delta_1}\|\psi\|^2$$
$$\le \delta_1\|\varphi_x\|^2 + C_1\left(\frac{\xi}{\rho_2}\right)^2 \frac{1}{4\delta_1}\|E_*^{1/2}\psi\|^2. \tag{4.16}$$

Putting $\delta_1 = \frac{b}{2\rho_2}$, we obtain

$$I_4 \le \frac{b}{2\rho_2}\|\varphi_x\|^2 + C\|E_*^{1/2}\psi\|^2. \tag{4.17}$$

Inserting inequalities (4.13) through (4.17) into Eq (4.12), we obtain the desired result, and thus inequality (4.10) holds.

Lemma 4.6. *The functional*

$$F_4(t) = -\rho_1\int_0^L \varphi_t\varphi\,dx$$

satisfies the following inequality:

$$F_4' \le \frac{3\mu}{2}\|\varphi_x\|^2 - \rho_1\|\varphi_t\|^2 + C\|E_*^{1/2}\psi\|^2. \tag{4.18}$$

Proof. We take the derivative of F_4, using $(2.1)_1$, integrating by parts, and applying the boundary conditions. This leads to the expression

$$F_4'(t) = -\rho_1\mu\int_0^L (\varphi_x)^2\,dx - \rho_1\int_0^L \varphi_t^2\,dx + b\int_0^L \psi\,\varphi_x\,dx.$$

Next, we apply Young's inequality to the term $\int_0^L \psi\varphi_x\,dx$, and since $\|B_*^{1/2}\psi\| \sim \|E_*^{1/2}\psi\|$ we arrive at the inequality

$$F_4' \le \frac{3\mu}{2}\|\varphi_x\|^2 - \rho_1\|\varphi_t\|^2 + C\|E_*^{1/2}\psi\|^2,$$

which completes the proof.

We are now ready to state and prove the main result, which concerns the decay of energy in our system and its important physical implications, particularly in the context of porous-elastic materials with memory effects. The derived decay rates describe how the system dissipates energy over time, reflecting the internal damping mechanisms induced by the memory term. Physically, this corresponds to the stabilization of mechanical vibrations and the gradual attenuation of wave propagation within the material. Such behavior is crucial in applications involving viscoelastic or porous structures, where controlling long-term stability is essential for maintaining structural integrity.

Theorem 4.7. *Assume (H) holds. If*

$$\frac{\mu}{\rho_1} = \frac{\delta}{\rho_2}$$

and

$$\exists n_0 > 0 \quad \text{such that} \quad \|B_*^{1/2}\phi_0\| \leq n_0,$$

where ϕ_0 is defined in (1.3), then for any $T > 0$, there exists a positive constant C such that for all $t \leq T$, the energy functional $E(t)$ given in (3.1) *satisfies the following inequality:*

$$E(t) \leq C(1+t)^{-\frac{1}{2p-2}} m^{-\frac{2p-1}{2p-2}} \left[1 + \int_0^t (s+1)^{\frac{1}{2p-2}} m^{\frac{2p-1}{2p-2}} h^{2p-1}(s)\, ds\right], \tag{4.19}$$

where $h(t) = m(t)\displaystyle\int_t^\infty g(s)\, ds.$

Proof. We define a Lyapunov functional

$$L(t) = NE(t) + N_1F_1(t) + N_2F_2(t) + N_3F_3(t) + F_4(t), \tag{4.20}$$

where N, N_1, N_2, and N_3 are positive constants to be chosen later. By differentiating (4.20) and using Lemmas 4.3–4.6, we find

$$\begin{aligned}
L'(t) \leq & -\left[\frac{N_1}{2} - 2\varepsilon_2 N_2 - C - CN_3(1+\varepsilon_4)\right]\left\|E_\star^{1/2}\psi\right\|^2 - \left(\xi - \frac{b^2}{\mu}\right)N_1\|\psi\|^2 \\
& - [\rho_1 - \varepsilon_4 N_3 - \varepsilon_1 N_1]\|\varphi_t\|^2 - \left[\frac{b}{2\rho_2}N_3 - \frac{3}{2}\mu - \varepsilon_3 N_2\right]\|\varphi_x\|^2 \\
& - \left[\frac{\rho_2}{2}g_0\, N_2 - C\left(1 + \frac{1}{\varepsilon_1}\right)N_1\right]\|\psi_t\|^2 + C\left[N_1 + N_2\left(\frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_2} + 1\right) + \frac{N_3}{\varepsilon_4}\right] g \circ B_\star^{\theta/2}\eta \\
& + \left[\frac{N}{2} - C\frac{N_3}{\varepsilon_4} - CN_2\right] g' \circ B_\star^{\theta/2}\eta + C\frac{N_2}{\varepsilon_2}\,\widetilde{g} \circ B_\star^{1/2}\,\eta.
\end{aligned}$$

By setting $\varepsilon_1 = \dfrac{\rho_1}{4N_1}, \quad \varepsilon_2 = \dfrac{N_1}{8N_2}, \quad \varepsilon_3 = \dfrac{b}{4\rho_2}\dfrac{N_3}{N_2}, \quad \varepsilon_4 = \dfrac{\rho_1}{4N_3}$, we obtain

$$\begin{aligned}
L'(t) \leq & -\left[\frac{N_1}{4} - C - CN_3\right]\left\|E_\star^{1/2}\psi\right\|^2 - \left[\xi - \frac{b^2}{\mu}\right]N_1\|\psi\|^2 - \frac{\rho_1}{2}\|\varphi_t\|^2 - \left[\frac{b}{4\rho_2}N_3 - \frac{3}{2}\mu\right]\|\varphi_x\|^2 \\
& - \left[\frac{\rho_2}{2}g_0\, N_2 - CN_1(1+N_1)\right]\|\psi_t\|^2 + C\left[N_1 + N_2\left(\frac{N_2}{N_1} + \frac{N_2}{N_3} + 1\right) + N_3^2\right] g \circ B_\star^{\theta/2}\eta \\
& - \left[\frac{N}{2} - CN_2 - CN_3^2\right] g' \circ B_*^{\theta/2}\eta + C\frac{N_2^2}{N_1}\,\widetilde{g} \circ B_\star^{1/2}\,\eta.
\end{aligned}$$

First, we choose N_3 large enough such that $\alpha_1 = \frac{bN_3}{4\rho_2} - \frac{3}{2}\mu > 0$, then we choose N_1 large enough such that $\alpha_2 = \frac{N_1}{4} - CN_3(1+N_3) - C > 0$, and finally, we choose N_2 large enough such that $\alpha_3 = \frac{\rho_2 N_2 g_0}{2} - CN_1(1+N_1) > 0$, so we have

$$\begin{aligned} L'(t) \leq & -\alpha_2 \left\| E_*^{1/2}\psi \right\|^2 - \alpha_0 \|\psi\|^2 - \frac{\rho_1}{2}\|\varphi_t\|^2 - \alpha_1 \|\varphi_x\|^2 - \alpha_3 \|\psi_t\|^2 \\ & + Cg \circ B_\star^{\theta/2}\eta + \left[\frac{N}{2} - C'\right] g' \circ B_\star^{\theta/2}\eta + C\,\widetilde{g} \circ B_\star^{1/2}\eta, \end{aligned} \tag{4.21}$$

where $\alpha_0 = \left(\xi - \frac{b^2}{\mu}\right) N_1$ and $g_0 = \displaystyle\int_T^\infty g(s)ds$.
On the other hand, we have

$$|L(t) - NE(t)| \leq N_1 |F_1(t)| + N_2 |F_2(t)| + N_2 |F_3(t)| + |F_4(t)|.$$

Exploiting Young's, Cauchy–Schwarz, and Poincaré inequalities, and recalling that

$$\left\| E_\star^{1/2}\psi \right\| \sim \left\| B_\star^{1/2}\psi \right\| \quad \text{and} \quad \left\| B_\star^{1/2}\psi \right\| \sim \|\psi_x\|,$$

we can estimate each of $|F_1(t)|, |F_2(t)|, |F_3(t)|, |F_4(t)|$ one by one.
Specifically, we have

$$\begin{aligned} |F_1(t)| &\leq \rho_2 \int_0^L |\psi\psi_t|\, dx + \frac{b\rho_1}{\mu} \int_0^L |\psi| \left(\int_0^x |\varphi_t(y)|\, dy \right) dx \\ &\leq \frac{\rho_2}{2}\left(\|\psi\|^2 + \|\psi_t\|^2 \right) + \frac{b\rho_1}{\mu} \int_0^L \int_0^L |\psi(x)|\, |\varphi_t(y)|\, dy\, dx \\ &\leq \frac{\rho_2}{2}\left(\|\psi\|^2 + \|\psi_t\|^2 \right) + \frac{b\rho_1 L}{2\mu}\left(\|\varphi_t\|^2 + \|\psi\|^2 \right) \\ &\leq C\left(\|\psi\|^2 + \|\psi_t\|^2 + \|\varphi_t\|^2 \right). \end{aligned}$$

Using $D\left(B_\star^{\delta_1}\right) \hookrightarrow D\left(B_\star^{\delta_2}\right)$ for $\delta_1 > \delta_2$ (we take $\delta_2 = 0, \delta_1 = \frac{\theta}{2}$), we have the following estimate:

$$\begin{aligned} |F_2(t)| &\leq \rho_2 \int_0^L |\psi_t| \left| \int_t^{+\infty} g(s)\eta(s)\, ds \right| dx \\ &\leq \rho_2 \|\psi_t\| \left\| \int_t^{+\infty} g(s)\eta(s)\, ds \right\| \\ &\leq \frac{\rho_2}{2}\left(\|\psi_t\|^2 + \left\| \int_0^{+\infty} g(s)\eta(s)\, ds \right\|^2 \right) \\ &\leq \frac{\rho_2}{2}\left(\|\psi_t\|^2 + \left(\int_0^{+\infty} g(s)\, ds \right) \cdot g \circ B_\star^{\theta/2}\eta \right) \\ &\leq C\left(\|\psi_t\|^2 + g \circ B_\star^{\theta/2}\eta \right). \end{aligned}$$

Using again $D\left(B_\star^{1/2}\right) \hookrightarrow \left(B_\star^{\theta/2}\right) \hookrightarrow D\left(B_\star^{\theta-1/2}\right)$ for all $\theta \in (0,1)$, we obtain the following estimate:

$$\begin{aligned}
|F_3(t)| &\leq \int_0^L |\varphi_t\psi_x|\,dx + \frac{\rho_1}{\mu\rho_2}\int_0^\infty g(s)\,ds\int_0^L \left|\varphi_t B_\star^{\theta-1/2}\psi\right|\,dx \\
&+ \frac{\rho_1}{\mu\rho_2}\int_0^\infty g(s)\left(\int_0^L |\varphi_t|\left|B_\star^{\theta-1/2}\eta(s)\right|\,dx\right)ds + \int_0^L |\varphi_x\psi_t|\,dx \\
&\leq \frac{1}{2}\left(\|\varphi_t\|^2 + \|\psi_x\|^2 + \|\psi_t\|^2 + \|\varphi_x\|^2\right) + \frac{\rho_1}{\mu\rho_2}\int_0^\infty g(s)\,ds\left(\|\varphi_t\|^2 + \left\|B_\star^{\theta-1/2}\psi\right\|^2\right) \\
&+ \frac{\rho_1}{2\mu\rho_2}\int_0^\infty g(s)\left(\left\|B_\star^{\theta-1/2}\eta(s)\right\|^2 + \|\varphi_t\|^2\right)ds \\
&\leq C\left(\|\varphi_t\|^2 + \|\psi_t\|^2 + \|\varphi_x\|^2 + \left\|B_\star^{1/2}\psi\right\|^2 + \left\|B_\star^{\theta-1/2}\psi\right\|^2 + g\circ B_\star^{\theta-1/2}\eta\right) \\
&\leq C\left(\|\varphi_t\|^2 + \|\psi_t\|^2 + \|\varphi_x\|^2 + \left\|E_\star^{1/2}\psi\right\|^2 + g\circ B_\star^{\theta/2}\eta\right).
\end{aligned}$$

Using Poincaré's inequality, we have:

$$|F_4(t)| \leq \rho_1\int_0^L |\varphi_t\varphi|\,dx \leq C\left(\|\varphi_t\|^2 + \|\varphi_x\|^2\right).$$

From here, we can deduce that:

$$\begin{aligned}
\|L(t) - NE(t)\| &\leq C\left(\|\varphi_t\|^2 + \|\varphi_x\|^2 + \|\psi_t\|^2 + \|\psi\|^2 + g\circ B_\star^{\theta/2}\eta + \left\|E_\star^{1/2}\psi\right\|^2\right) \\
&\leq CE(t).
\end{aligned}$$

Thus, we obtain the estimate:

$$(N - C)\,E(t) \leq L(t) \leq (N + C)\,E(t).$$

Next, we return to the estimation (4.21) and choose N large enough such that $\frac{N}{2} - C' > 0$ and $N - C > 0$. Therefore, this means that $L(t) \sim E(t)$. Since $g\circ B_\star^{\theta/2}\eta < 0$, Eq (4.21) yields:

$$L'(t) \leq -K_1E(t) + K_2 g\circ B_\star^{\theta/2}\eta + K_3\widetilde{g}\circ B_\star^{1/2}\eta.$$

Multiplying the above inequality by $m(t)$ and using Corollary (4.2), we obtain:

$$m(t)L'(t) \leq -K_1m(t)E(t) + C[-E'(t)]^{1/(2p-1)} + Ch(t).$$

Next, multiplying the above inequality by $(mE)^\alpha(t)$, where $\alpha = 2p-2$, and applying Young's inequality, we get for any $\epsilon > 0$:

$$m^{\alpha+1}(t)L'(t)E^\alpha(t) \leq -\left(K_1 - 2\epsilon\right)m^{\alpha+1}(t)E^{\alpha+1}(t) - C_\epsilon E'(t) + C_\epsilon h^{\alpha+1}(t).$$

Now, we choose ϵ small enough such that

$$K_2 = K_1 - 2\epsilon > 0,$$

and set

$$F(t) = m^{\alpha+1}(t)L(t)E^{\alpha}(t) + C_{\epsilon}E(t).$$

Since both E and m are decreasing functions, we have:

$$\begin{aligned} F'(t) &\leq m^{\alpha+1}(t)L'(t)E^{\alpha}(t) + C_{\epsilon}E'(t) \\ &\leq -K_2 m^{\alpha+1}(t)E^{\alpha+1}(t) + C_{\epsilon}h^{\alpha+1}(t). \end{aligned}$$

On the other hand, it is easy to remark that $F(t) \sim E(t)$, from where we deduce:

$$F'(t) \leq -K_2 m^{\alpha+1}F^{\alpha+1}(t) + C_{\epsilon}h^{\alpha+1}(t).$$

Thanks to Lemma 2.1, this implies:

$$F(t) \leq C(1+t)^{\frac{-1}{2p-2}} m^{-\frac{2p-1}{2p-2}} \left[1 + \int_0^t (s+1)^{\frac{1}{2p-2}} m^{\frac{2p-1}{2p-2}} h^{2p-1}(s)\, ds\right],$$

and hence, we conclude that the estimate (4.19) is satisfied. Therefore, the proof of this theorem is complete.

To illustrate the energy decay rates obtained by Theorem 4.7, we give the following example.

Example 1. Let $g(t) = \frac{\beta_1}{(1+\sqrt{t+1})^{\beta_2}}$, where β_1 and β_2 are two positive constants, which we will choose later. Indeed,

$$g'(t) = -m(t)g^p(t),$$

where $m(t) = \frac{\beta_2}{2\beta_1^{1/\beta_2}\sqrt{t+1}}$ and $p = \frac{\beta_2+1}{\beta_2}$.

We first choose $\beta_2 > 2$ so that $p \in [1, \frac{3}{2})$ and $\int_0^{+\infty} g(s)\, ds$ are bounded. Then, we choose β_1 such that

$$\int_0^{+\infty} g(s)\, ds < \left(\frac{\pi}{L}\right)^{2(1-\theta)}.$$

Therefore, all conditions of Theorem 4.7 are fulfilled, and we can apply the estimate of the energy decay (4.19). Indeed, we start by

$$(1+t)^{\frac{-1}{2p-2}} m^{-\frac{2p-1}{2p-2}} \leq (1+t)^{-\frac{3-2p}{2(2p-2)}}.$$

Note that $-\frac{3-2p}{2(2p-2)} < 0$ because $p \in (1, \frac{3}{2})$.
We have

$$h^{2p-1} = \left(Cm(t)\int_t^{\infty} g(s)\, ds\right)^{2p-1} \leq C(1+t)^{(1-\beta_2)(\frac{2p-1}{2})}.$$

Thus, we deduce that

$$I = \int_0^t (s+1)^{\frac{1}{2p-2}} m^{\frac{2p-1}{2p-2}} h^{2p-1}(s)\, ds \leq C \int_0^t (s+1)^{-1+\frac{2p-1}{2}\left(\frac{2p-1}{2p-2}-\beta_2\right)} ds. \tag{4.22}$$

Putting $\beta_3 = -1 + \frac{2p-1}{2}\left(\frac{2p-1}{2p-2} - \beta_2\right)$, the relation in (4.22) yields

$$\begin{cases} I \leq C\left(\dfrac{(t+1)^{\beta_3+1} - 1}{\beta_3 + 1}\right) & \text{if} \quad \beta_3 \neq -1, \\ I \leq C\ln(1+t) & \text{if} \quad \beta_3 = -1. \end{cases}$$

From where it follows that

$$\begin{cases} I \le C(t+1)^{\beta_3+1} \quad \text{if} \quad 2 < \beta_2 < \dfrac{2p-1}{2p-2}, \\ I \le C\ln(1+t) \quad \text{if} \quad \beta_2 = \dfrac{2p-1}{2p-2}, \\ I \le C \quad \text{if} \quad \beta_2 > \dfrac{2p-1}{2p-2}. \end{cases}$$

We see that $\frac{2p-1}{2p-2} > 2$ because $p \in (1, \frac{3}{2})$. So, finally, we arrive at

$$\begin{cases} E(t) \le C(t+1)^{1+(1-\beta_2)\left(\frac{2p-1}{2}\right)} \quad \text{if} \quad 2 < \beta_2 < \dfrac{2p-1}{2p-2}, \\ E(t) \le C(t+1)^{-\frac{3-2p}{2(2p-2)}}\ln(1+t) \quad \text{if} \quad \beta_2 = \dfrac{2p-1}{2p-2}, \\ E(t) \le C(t+1)^{-\frac{3-2p}{2(2p-2)}} \quad \text{if} \quad \beta_2 > \dfrac{2p-1}{2p-2}. \end{cases}$$

It is easy to remark that in the case when $2 < \beta_2 < \frac{2p-1}{2p-2}$, the functional $E(t)$ exhibits polynomial decay if we choose $\beta_2 \in \left(\frac{1+2p}{2p-1}, \frac{2p-1}{2p-2}\right)$.

5. Conclusions

In this paper, we have studied the stability properties of a porous-elastic system with fractional damping in the memory term. By carefully analyzing the interplay between the wave propagation speeds and the memory effect, we established a polynomial decay rate for the energy, highlighting the crucial role of the memory kernel in dictating the system's asymptotic behavior. Our findings provide a deeper understanding of the dissipative mechanisms governing porous-elastic materials and extend previous results by refining the decay estimates under minimal assumptions on the kernel.

Future research directions include investigating the impact of nonhomogeneous boundary conditions, exploring the extension of the model to fractional-order time evolution equations, and conducting numerical simulations to further validate the theoretical results. Additionally, it would be of interest to analyze the system under more general geometric configurations or in the presence of external forcing terms, which could offer new insights into real-world applications.

AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Competing interests

The authors declare no competing interests.

Availability of data and materials

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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