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Research article

On residual cumulative generalized exponential entropy and its application in human health

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Abstract: Numerous adaptations of traditional entropy concepts and their residual counterparts have emerged in statistical research. While some methodologies incorporate supplementary variables or reshape foundational assumptions, many ultimately align with conventional formulations. This study introduces a novel extension termed residual cumulative generalized exponential entropy to broaden the scope of residual cumulative entropy for continuous distributions. Key attributes of the proposed measure include non-negativity, bounds, its relationship to the continuous entropy measure, and stochastic comparisons. Practical implementations are demonstrated through case studies involving established probability models. Additionally, insights into order statistics are derived to characterize the measure's theoretical underpinnings. The residual cumulative generalized exponential entropy framework bridges concepts such as Bayesian risk assessment and excess wealth ordering. For empirical implementation, non-parametric estimation strategies are devised using data-driven approximations of residual cumulative generalized exponential entropy, with two distinct estimators of the cumulative distribution function evaluated. A practical application is showcased, using clinical diabetes data. The study further explores the role of generalized exponential entropy in identifying distributional symmetry, mainly through its application to uniform distributions to pinpoint symmetry thresholds in ordered data. Finally, the utility of generalized exponential entropy is examined in pattern analysis, with a diabetes dataset serving as a benchmark for evaluating its classification performance.

Keywords: exponential entropy; order statistics; non-parametric estimation; residual cumulative entropy; pattern recognition; stochastic order

1. Introduction

Shannon [1] first proposed the idea of entropy in 1948. It is a fundamental idea in the theories of information that measures the degree of uncertainty or knowledge contained in a given random variable. Shannon entropy is acquainted mathematically as

$$Sn(\mathbf{P}) = -\sum_{k=1}^{n} p_k \ln p_k,$$

for discrete random variables, with noting that $\mathbf{P} = (p_1, p_2, ..., p_n)$ is the vector of mass function probability, or

$$Sn(X) = -\int_{-\infty}^{\infty} f(x) \ln f(x)\, dx, \tag{1.1}$$

for a randomly generated continuous variable X, and $f(x)$ represents the probability density functional (PDF). It has proven to be a powerful tool in various fields, including communication systems, where it optimizes data transmission, and in data compression, where it measures the limits of compressibility. By capturing the average amount of information generated by random events, Shannon entropy provides a robust framework for analyzing uncertainty in diverse systems.

Building on this seminal work, Rao et al. [2] established the conception of the cumulative residual entropy model as an extension of entropy designed for survival analysis and reliability theory. Unlike traditional entropy, which measures overall uncertainty, residual cumulative entropy focuses on the uncertainty remaining in a system or process beyond a given time or threshold. Formally, given a randomly variable that is not negative X, the residual cumulative function of entropy measurement is described as:

$$RCEn(X) = \int_{0}^{\infty} \bar{F}(x) \ln \bar{F}(x)\, dx, \tag{1.2}$$

in which the survival function is encapsulated by $\bar{F}(x) = 1 - F(x)$, and the cumulative distribution function (CDF) of X is encapsulated by $F(x)$. This measure captures the tail behavior of the determined distribution, causing it to be quite helpful in applications requiring an understanding of the uncertainty associated with extreme events. Residual cumulative entropy has since been employed in numerous disciplines regarding reliability engineering, risk analysis form, and lifetime data analysis, where the quantification of remaining uncertainty is of critical importance.

Following Campbell [3], Pal and Pal [4,5] used these concerns to establish a new measure, called the exponential entropy measure, through other descriptions parallel to Shannon entropy. In the discrete situation, the formulation of the exponential entropy model is described below:

$$EXn(\mathbf{P}) = \sum_{k=1}^{n} P_k (e^{1-P_k} - 1). \tag{1.3}$$

They added the -1 term since it seems only logical that any measurement that contains data should be assigned 0 for the degenerative distribution of probabilities $(0, ..., 0, 1, 0, ..., 0)$. The authors argued that exponential entropy offers distinct benefits over Shannon's formulation. For example, they noted that exponential entropy reaches a fixed upper value in the case of a distribution that is considered to be uniform, where each probability is given by $p_k = \frac{1}{n}$ for $k = 1, 2, \ldots, n$ as demonstrated by

$$\lim_{n\to\infty} EXn\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right) = e - 1;$$

in contrast, Shannon's entropy does not exhibit this bounded behavior.

Furthermore, Panjehkeh et al. [6] examined the features and attributes of the exponential measure of entropy, along with the Shannon measure of entropy model, in both discrete and continuous situations, including the asymptotic equipartition property, invariancy under monotone transformation, subadditivity, and the chain rule. The concept of a continuous exponential measure of entropy was first presented as

$$EXn(X) = \int_{D_s} f(x)(e^{1-f(x)})dx. \tag{1.4}$$

We may observe that Panjehkeh et al. [6] eliminated the term -1 from the measurement in order to make it inherently non-negative against Shannon entropy, given that X is a randomly variable that is continuous, following the PDF f, and the support D_s. Kvalseth [7] developed a more broad variant of the exponential entropy of Eq (1.2), known as generalized entropy, given by

$$GEXn(\mathbf{P}) = \frac{1}{\beta} \sum_{k=1}^{n} p_k(e^{1-p_k^\beta} - 1), \tag{1.5}$$

where the parameter β is arbitrary and has a non-zero real value (i.e., $\beta \in \mathbb{R}\backslash\{0\}$). Moreover, the generalized exponential entropy measure has been used in many topics. Alotaibi and Elaraby [8] created a generalized exponential entropy-based method for COVID-19 disease segmentation from computed tomography scans. Fuzzy c partitioning and generalized exponential entropy were combined in their suggested method. In multi-criteria decision-making, we can see, for example, Wei et al. [9] and Dinesh and Kumar [10].

Work motivation: Since the PDF is calculated as the distribution's derivative form, the CDF appears more regularly than the density function. Furthermore, the CDF is what is relevant and/or quantifiable in practice. For instance, if the variable that is chosen randomly is the life span of a machine, the happenstance that is relevant is not whether the life span equals t, but rather whether it surpasses t. This is what prompted us to study it instead of the existing entropy based on the PDF. The question that arises is whether it can be generalized and if the traditional model can be made a special case of it. Moreover, studying the discrete case is not satisfactory without considering the continuous case. As we have mentioned that the exponential entropy in discrete case has been discussed in distinct fields; take into consider, for instance, Wei et al. [11], and Ye and Cui [12]. Therefore, it is essential to study and discuss it in the continuous case with clarification of the applications associated with the continuous side. Another important aspect to discuss is that the proposed model is based on the exponential function. Dose this function help solve some of the problems that appear in uncertainty issues?

This article aimed to present a generalization of the continuous residual cumulative entropy, known as residual cumulative generalized exponential entropy. Applications, including non-parametric estimation, are provided. On the other hand, the continuous form of the generalized exponential entropy is used to discuss the characterization of symmetry via order statistics. Moreover, a classification problem involving the pattern recognition in diabetes data based on the generalized exponential entropy model is presented. The following is the arrangement of the remaining portions of this paper: in Section 2, the concept of residual cumulative generalized exponential entropy in the continuous setting is presented. Additionally, the properties, including bounds, non-negativity, the relationship with differential entropy, and stochastic orders, are explained. In Section 3, some consequences of the residual cumulative generalized exponential entropy expansion, such as the Bayes risk and the order of excess wealth, are discussed. In Section 4, the non-parametric estimation of the empirical residual cumulative generalized exponential entropy is applied using two methods. Finally, in Section 5, the symmetry characterization using order statistics with an example based on the symmetrical uniform distribution and the classification problem using pattern recognition based on the generalized exponential entropy are illustrated.

2. Residual cumulative generalized exponential entropy measure

In this section, we will establish the concept of the residual cumulative generalized exponential entropy. Inspired by Rao et al. [2], we can depend on the function of survival $\bar{F}(x)$ to derive the residual cumulative generalized exponential entropy from the discrete case of the generalized exponential entropy in (1.5) according to the following definition.

Definition 2.1. *Consider the non-negativity continual randomly variable X following the CDF F. Then, we can realize the residual cumulative generalized exponential entropy by the following formula:*

$$
\begin{aligned}
RGEXn_\beta(X) &= \frac{1}{\beta}\int_0^\infty \bar{F}(x)\left(e^{1-\bar{F}^\beta(x)} - 1\right) dx \\
&= \frac{1}{\beta}\left[\int_0^\infty \bar{F}(x)e^{1-\bar{F}^\beta(x)}\, dx - \mu\right],
\end{aligned}
\tag{2.1}
$$

where $\beta \in \mathbb{R}\backslash\{0\}$, and the mean (expected value) $\mu = E(X) = \int_0^\infty \bar{F}(x)\, dx$.

The following proposition shows the limitation of the residual cumulative generalized exponential entropy when β tends to zero, which returns to residual cumulative entropy in (1.2).

Proposition 2.1. *Consider the non-negativity continual random variable X following the CDF F. Then, from (2.1) and (1.2), we have*

$$\lim_{\beta\to 0} RGEXn_\beta(X) = RCEn(X).$$

Proof. From (2.1), utilizing the L'Hopital's rule, we have

$$
\begin{aligned}
\lim_{\beta\to 0} RGEXn_\beta(X) &= \lim_{\beta\to 0} \frac{\int_0^\infty \bar{F}(x)\left(e^{1-\bar{F}^\beta(x)} - 1\right) dx}{\beta} \\
&= \lim_{\beta\to 0} -\int_0^\infty \bar{F}^{1+\beta}(x)e^{1-\bar{F}^\beta(x)} \ln \bar{F}(x)\, dx \\
&= \int_0^\infty \bar{F}(x) \ln \bar{F}(x)\, dx = RCEn(X).
\end{aligned}
$$

□

In the following discussions, we will discuss the non-negativity and the bounds of the residual cumulative generalized exponential entropy when $\beta > 0$.

Proposition 2.2. *Consider the non-negativity continual random variable X following the CDF F. Then, from (2.1), we can say that*

(1) The residual cumulative generalized exponential entropy is non-negative for all $\beta > 0$.

(2) To discuss the bounds of the residual cumulative generalized exponential entropy:

(a) We get

$$RGEXn_{\beta}(X) \geq (\leq) RGEXn_{1}(X),\ \forall\, 0 < \beta \leq 1 (\beta \geq 1). \tag{2.2}$$

(b) We get

$$0 \leq RGEXn_{\beta}(X) \leq \frac{\mu(e-1)}{\beta},\ \forall\, \beta \geq 0. \tag{2.3}$$

Proof. (1) Its known that $e^{1-\bar{F}^{\beta}(x)} \geq 1$, for all $\beta > 0$, and $\bar{F}(x) \in [0,1]$. Thus, $\bar{F}(x)\, e^{1-\bar{F}^{\beta}(x)} \geq \bar{F}(x)$, which implies that $\int_0^{\infty} \bar{F}(x)\, e^{1-\bar{F}^{\beta}(x)}\, dx \geq \int_0^{\infty} \bar{F}(x)\, dx = \mu$. Then, $RGEXn_{\beta}(X) \geq 0$, for all $\beta > 0$. Or, by another method, we can assume, by converse, that $\beta\, RGEXn_{\beta}(X) < 0$, then we have

$$\int_0^{\infty} \bar{F}(x)\, e^{1-\bar{F}^{\beta}(x)}\, dx - \int_0^{\infty} \bar{F}(x)\, dx < 0 \Rightarrow \int_0^{\infty} \bar{F}(x)\, e^{1-\bar{F}^{\beta}(x)}\, dx < \int_0^{\infty} \bar{F}(x) e^{0}\, dx.$$

Therefore, we deduce that $e^{1-\bar{F}^{\beta}(x)} < 1$, which implies $1 - \bar{F}^{\beta}(x) < 0$ or, equivalently, $\bar{F}^{\beta}(x) > 1$. This contradicts the fact that $\bar{F}(x) \in [0,1]$ for any x. Then, the result follows.

(2) For $0 < \beta \leq 1 (\beta \geq 1)$, we have $\bar{F}^{\beta}(x) \geq (\geq) \bar{F}(x)$, which implies that $\bar{F}(x)\, e^{1-\bar{F}^{\beta}(x)} \geq (\leq) \bar{F}(x) e^{1-\bar{F}(x)}$. Then, $\frac{1}{\beta}\left[\int_0^{\infty} \bar{F}(x) e^{1-\bar{F}^{\beta}(x)}\, dx - \mu\right] \geq (\leq) \int_0^{\infty} \bar{F}(x) e^{1-\bar{F}(x)}\, dx - \mu$, and the result follows. □

Lemma 2.1. *If $\mu = E(X) < \infty$, then $RGEXn_{\beta}(X) < \infty$, for all $\beta > 0$.*

Proof. The result is obtained directly from (2.3). □

Theorem 2.1. *If X is an absolutely continual non-negativity random variable following a PDF $f(x)$, then*

$$RGEXn_{1}(X) \geq C^{*} e^{Sn(X)},$$

with noting that

$$C^{*} = \exp\left\{\int_0^{1} \ln\left|u\left(e^{1-u} - 1\right)\right| du\right\} \simeq 0.176192,$$

and $Sn(X)$ is defined in (1.1).

Proof. Using the fact of the inequality of log-sum, it contends that

$$\int_0^{\infty} f(x) \ln \frac{f(x)}{\left|\overline{F}(x)\left(e^{1-\overline{F}(x)} - 1\right)\right|} dx \geq \ln \frac{1}{\int_0^{\infty} \left|\overline{F}(x)\left(e^{1-\overline{F}(x)} - 1\right)\right| dx} dx = -\ln RGEXn(X), \tag{2.4}$$

Moreover, the left-hand side of (2.4) can be expressed as

$$\int_0^\infty f(x)\ln\frac{f(x)}{\left|\overline{F}(x)\left(e^{1-\overline{F}(x)}-1\right)\right|}dx = -Sn(X) - \int_0^\infty f(x)\ln\left|\overline{F}(x)\left(e^{1-\overline{F}(x)}-1\right)\right|dx$$
$$= -Sn(X) - \int_0^1 \ln\left|u\left(e^{1-u}-1\right)\right|du.$$

Therefore, it follows that

$$Sn(X) + \int_0^1 \ln\left|u\left(e^{1-u}-1\right)\right|du \leq \ln RGEXn(X).$$

Applying the exponential function to both sides of the aforementioned relation, we derive

$$RGEXn(X) \geq C^* e^{Sn(X)},$$

where

$$C^* = \exp\left\{\int_0^1 \ln\left|u\left(e^{1-u}-1\right)\right|du\right\} \simeq 0.176192,$$

thereby finalizing the proof. □

Proposition 2.3. *If X is an absolutely continual non-negativity random variable with residual cumulative generalized exponential entropy* $RGEXn_\beta(X) < \infty$ *as given in (2.1),* $\forall\beta > 0$*. Then, we can obtain*

$$RGEXn_\beta(X) = \frac{1}{\beta}E(\psi_\beta(X)), \tag{2.5}$$

where

$$\psi_\beta(X) = \int_0^x \left(e^{1-\bar{F}^\beta(t)}-1\right)dt. \tag{2.6}$$

Proof. From (2.1), and utilizing the theorem of Fubini, we can express the following:

$$RGEXn_\beta(X) = \frac{1}{\beta}\int_0^\infty\left[\int_t^\infty f(x)\,dx\right]\left(e^{1-\bar{F}^\beta(t)}-1\right)dt$$
$$= \frac{1}{\beta}\int_0^\infty f(x)\left[\int_0^x\left(e^{1-\bar{F}^\beta(t)}-1\right)dt\right]dx,$$

and (2.5) is obtained by utilizing (2.6). □

In the following, we will show some examples of the residual cumulative generalized exponential entropy of well-known distributions.

Example 2.1. *Consider the non-negativity continual random variable X following the CDF F. Then,*

(1) Under the distribution of exponential (Exp(γ*)) with* $F(x) = 1 - e^{-\gamma x}$*, we get*

$$RGEXn_\beta(X) = -\frac{\left(\beta - e\,\Gamma\left[\frac{1}{\beta}\right] + e\,\Gamma\left[\frac{1}{\beta},1\right]\right)}{\gamma\beta^2},$$

where $\Gamma\left[\frac{1}{\beta}\right]$ *is the function of the Euler gamma, and* $\Gamma\left[\frac{1}{\beta}, 1\right]$ *is the function of the incomplete gamma. Moreover, with* $\beta = 1$*, we get*

$$RGEXn_1(X) = \frac{2-e}{\gamma}.$$

(2) *Under the uniform distribution* $(Ud(t_1, t_2))$ *with* $F(x) = \frac{x-t_1}{t_2-t_1}$*, and* $\beta = 1$*, we get*

$$RGEXn_1(X) = e - \frac{5}{2}.$$

Figure 1 shows the plot of the residual cumulative generalized exponential entropy and the residual cumulative entropy defined in (1.2) of the $Exp(\gamma)$ distribution. Therefore, we can see the assurance of Proposition 2.1 and Eq (2.2).

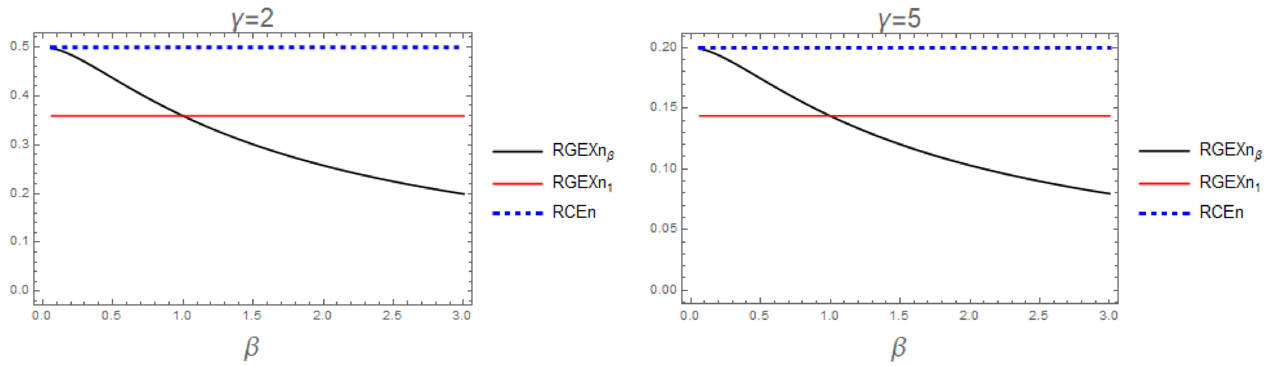

Figure 1. Plot of $RGEXn_\beta(X)$, $RGEXn_1(X)$, and $RCEn(X)$ of the $Exp(2)$ distribution (left) and the $Exp(5)$ distribution (right).

2.1. *Stochastic order*

The theorem outlined below establishes a characterization in terms of the residual cumulative generalized exponential entropy under the usual stochastic, dispersive, increasing convex, and hazard rate orders. In the usual stochastic, dispersive, increasing convex, and hazard rate orders, represented, respectively, by $X_1 \leq_{\mathrm{UstOr}} X_2$, $X_1 \leq_{\mathrm{DisOr}} X_2$, $X_1 \leq_{\mathrm{IcxOr}} X_2$, and $X_1 \leq_{\mathrm{HrOr}} X_2$, we mainly remember that the random variable X_1 is smaller than X_2 if

(1) $\bar{F}_1(x) \leq \bar{F}_2(x)$ (for the usual stochastic order),

(2) $F_2^{-1}(v) - F_1^{-1}(v)$ is increasing in $v \in (0, 1)$ (for the dispersive order),

(3) $E(\xi(X_1)) = E(\xi(X_1))$, with the existence of the expectations and for all convex increasing functions ξ (for the increasing convex order),

(4) $\frac{\bar{F}_2(x)}{\bar{F}_1(x)}$ is increasing with respect to x (for the hazard rate order).

where F_1^{-1} and F_2^{-1} are the right continually inverses of the CDF's F_1 and F_2, correspondingly; see Shaked and Shanthikumar [13].

Theorem 2.2. *Assume that two random variables that are continuously, X_1 and X_2, following distribution functions that are strictly increasing, (F_1 and F_2, respectively). If $X_1 \leq_{UstOr} X_2$, then, we obtain:*

(1) $RGEXn_\beta(X_1) \leq RGEXn_\beta(X_2)$.

(2) $RGEXn_\beta(X_1) - RGEXn_\beta(X_2) \geq \frac{1}{\beta}(\mu_1 - \mu_2)$, *where* $\mu_1 = \int_0^\infty \bar{F}_1(x)\,dx$, *and* $\mu_2 = \int_0^\infty \bar{F}_2(x)\,dx$.

Proof. 1) Let $\bar{F}_1(x) \leq \bar{F}_2(x)$, and since $e^{1-\bar{F}^\beta(x)} \geq 1$, for all $\beta > 0$. Then, the result follows.

(2) Since $\bar{F}_1(x) \leq \bar{F}_2(x)$, then $\bar{F}_1^\beta(x) \leq \bar{F}_2^\beta(x)$, for all $\beta > 0$, or equivalently $e^{1-\bar{F}_1^\beta(x)} \geq e^{1-\bar{F}_2^\beta(x)}$. Therefore, we have

$$\begin{aligned} RGEXn_\beta(X_1) - RGEXn_\beta(X_2) &= \frac{1}{\beta}\int_0^\infty \left[\bar{F}_1(x)\left(e^{1-\bar{F}_1^\beta(x)} - 1\right) - \bar{F}_2(x)\left(e^{1-\bar{F}_2^\beta(x)} - 1\right)\right] dx \\ &\geq \frac{1}{\beta}\left(\int_0^\infty \left(\bar{F}_1(x) - \bar{F}_2(x)\right) dx\right) = \frac{1}{\beta}(\mu_1 - \mu_2). \end{aligned}$$

□

Example 2.2. *Suppose that two random variables that are continuously, X_1 and X_2, following $Ud(0, t_1)$ and $Ud(0, t_2)$ distributions with the CDFs $F_1(x) = \frac{x}{t_1}$, $0 \leq x \leq t_1$, and $F_2(x) = \frac{x}{t_2}$, $0 \leq x \leq t_2$, respectively. Moreover, the means are $\mu_1 = \frac{t_1}{2}$ and $\mu_2 = \frac{t_2}{2}$. If we let $t_2 \geq t_1$, then we have $X_1 \leq_{UstOr} X_2$. With $\beta = 1$, we obtain*

$$RGEXn_1(X_1) = t_1\left(e - \frac{5}{2}\right) \leq t_2\left(e - \frac{5}{2}\right) = RGEXn_1(X_2),$$

and

$$RGEXn_1(X_1) - RGEXn_1(X_2) = \left(e - \frac{5}{2}\right)(t_1 - t_2) \geq \frac{1}{2}(t_1 - t_2) = \mu_1 - \mu_2,$$

where $(t_1 - t_2) \leq 0$. Which assures the results in Theorem 2.2.

Theorem 2.3. *Assume that two random variables that are continuously, X_1 and X_2, following distribution functions that are strictly increasing, (F_1 and F_2, respectively). If $X_1 \leq_{DisOr} X_2$, and*

$$RGEXn_\beta(X_1) = RGEXn_\beta(X_2),$$

for a fixed $\beta > 0$. Consequently, up to a location parameter, the distributions of X_1 and X_2 are identical.

Proof. Suppose that $X_1 \leq_{\text{DisOr}} X_2$ (i.e., the function $F_2^{-1}(v) - F_1^{-1}(v)$ is decreasing in v), and $RGEXn_\beta(X_1) = RGEXn_\beta(X_2)$. Then, by a change of variable $v = F(x)$, we observe that (according to the equality given)

$$RGEXn_\beta(X_2) - RGEXn_\beta(X_1) = \frac{1}{\beta}\int_0^1 ((1-v)\left(e^{1-(1-v)^\beta} - 1\right))d[F_2^{-1}(v) - F_1^{-1}(v)] = 0,$$

for a fixed $\beta > 0$. Since $X_1 \leq_{\text{DisOr}} X_2$, we are aware of that $F_2^{-1}(v) - F_1^{-1}(v)$ is a function considered to be decreasing of v. We now assert that for every $0 \leq v \leq 1$, $F_2^{-1}(v) - F_1^{-1}(v) = c$ (constant). Suppose, to the contrary, that there exists a subinterval (θ_1, θ_2) within $[0, 1]$ where $F_2^{-1}(v) - F_1^{-1}(v)$ fails to remain constant across (θ_1, θ_2). In this case,

$$\begin{aligned} 0 &= \frac{1}{\beta}\int_0^1 ((1-v)\left(e^{1-(1-v)^\beta} - 1\right) d[F_2^{-1}(v) - F_1^{-1}(v)] \\ &\geq \frac{1}{\beta}\int_{\theta_1}^{\theta_2} ((1-v)\left(e^{1-(1-v)^\beta} - 1\right) d[F_2^{-1}(v) - F_1^{-1}(v)] > 0, \end{aligned}$$

which is a contradiction. Consequently, for all $0 \leq v \leq 1$, $F_2^{-1}(v) - F_1^{-1}(v) = c$ (constant), indicating that X_1 and X_2 have an equal distribution according to the location parameter. Alternatively, in more detail, the proof can be argued by contradiction. Suppose that $F_2^{-1}(v) - F_1^{-1}(v)$ is not constant on some subinterval (θ_1, θ_2). Then, because $F_2^{-1}(v) - F_1^{-1}(v)$ is decreasing, the integral over (θ_1, θ_2) will be positive:

$$\frac{1}{\beta}\int_{\theta_1}^{\theta_2}(1-v)\left(e^{1-(1-v)^\beta} - 1\right) d\left[F_2^{-1}(v) - F_1^{-1}(v)\right] > 0.$$

This contradicts the equality $RGEXn_\beta(X_1) = RGEXn_\beta(X_2)$, implying that $F_2^{-1}(v) - F_1^{-1}(v)$ must be constant for all $v \in [0, 1]$. □

Remark 2.1. *As a deeper explanation of Theorem 2.3, the phrase "Consequently, up to a location parameter, the distributions of X_1 and X_2 are identical" means that the distributions of X_1 and X_2 are the same except for a shift or translation along the real number line. In other words, X_1 and X_2 have the same shape and structure in their distributions, but one is a shifted version of the other. For example, let X_1 and X_2 be $Exp(\gamma)$ distributed random variables with the rate parameter $\gamma > 0$. If X_1 and X_2 both follow the same exponential distribution with the rate parameter γ, then their CDFs are:*

$$F_1(x) = 1 - e^{-\gamma x}, \quad F_2(x) = 1 - e^{-\gamma x}.$$

Here, X_1 and X_2 are identically distributed, and there is no shift ($c = 0$). Suppose that X_2 is a shifted version of X_1 by a constant $c > 0$. In this case, the CDF of X_2 is:

$$F_2(x) = F_1(x - c) = 1 - e^{-\gamma(x-c)}, \quad x \geq c.$$

Here, X_2 is the same as X_1 but shifted to the right by c. The distributions are identical up to the location parameter c.

The following assertion gives an alternate formula for the residual cumulative generalized exponential entropy of X. The sequel uses this formulation, which is in respect of an expanding convex function, to derive a number of findings.

Lemma 2.2. *The following is true for $\beta > 0$ if X represents an entirely constantly non-negativity random variable with the limiting mean $\mu = \mathbb{E}(X)$:*

$$RGEXn_\beta(X) \geq \frac{\psi_\beta(\mu)}{\beta},$$

with noting that the function $\psi_\beta(\cdot)$ is given in (2.6).

Proof. Given that $\psi_\beta(\cdot)$ is a function to be convex, Jensen's inequality may be used as $E(\psi_\beta(X)) \geq \psi_\beta(E(X))$, which obtains the proof. □

The following characteristics of stochastic order are useful for comparing risk measurements and are also obtained by using Proposition 2.3.

Proposition 2.4. *For $\beta > 0$, it is true that if X_1 and X_2 are completely continuously non-negative random variables that correspond to $X_1 \leq_{IcxOr} X_2$, then*

$$\psi_\beta(X_1) \leq_{IcxOr} \psi_\beta(X_2),$$

where (2.6) defines the function $\psi_\beta(\cdot)$. $X_1 \leq_{IcxOr} X_2$ specifically suggests

$$RGEXn_\beta(X_1) \leq RGEXn_\beta(X_2).$$

Proof. According to Theorem 4.A.8 of Shaked and Shanthikumar [13], $\psi_\beta(X_1) \leq_{\mathrm{IcxOr}} \psi_\beta(X_2)$ is an function which considered to be increasing convex for $\beta > 0$ in the function $\psi_\beta(\cdot)$. Specifically, using Eq (2.5) and the concept of rising convex order, we obtain $RGEXn_\beta(X_1) \leq RGEXn_\beta(X_2)$. □

Proposition 2.5. *Assume that $X_1, \ldots, X_m$ be m independent non-negativity absolutely continuous random variables with the collective CDF F, and $Z_1, \ldots, Z_m$ are another set of m independent non-negative continuous random variables with the collective CDF F^*. If $X_i \leq_{IcxOr} Z_i$ for $i = 1, 2, \ldots, m$, then for all $\beta > 0$, we obtain*

$$RGEXn_\beta(\max\{X_1, X_2, \ldots, X_m\}) \leq RGEXn_\beta(\max\{Z_1, Z_2, \ldots, Z_m\}).$$

Proof. Given that $X_i \leq_{\mathrm{IcxOr}} Z_i$ for $i = 1, 2, \ldots, m$, using Shaked and Shanthikumar's [13] Corollary 4.A.16 , we get

$$\max\{X_1, X_2, \ldots, X_m\} \leq_{\mathrm{IcxOr}} \max\{Z_1, Z_2, \ldots, Z_m\}.$$

The result then follows directly from Proposition 2.3. □

Proposition 2.6. *For $\beta > 0$, the following is true if X_1 and X_2 are non-negative randomized variables that match $X_1 \leq_{HrOr} X_2$:*

$$\frac{RGEXn_\beta(X_1)}{E(X_1)} \leq \frac{RGEXn_\beta(X_1)}{E(X_2)}.$$

Proof. Shaked and Shanthikumar [13] determined that, assuming $X_1 \leq_{\mathrm{HrOr}} X_2$, the function $\psi_\beta(\cdot)$, described in (2.6), is an increasing function to be convex such that $\psi_\beta(0) = 0$:

$$\frac{E(\psi_\beta(X_1))}{E(X_1)} \leq \frac{E(\psi_\beta(X_2))}{E(X_2)}.$$

Therefore, Proposition 2.4 completes the evidence. □

2.2. Order statistics-based characterization

In this subsection, we will examine the characterization of the residual cumulative generalized exponential entropy based on order statistics. Using the well-known Müntz-Szász theorem, we first review the idea of a full sequence of functions and a lemma; for further information, see [14–16].

Lemma 2.3. *(Müntz-Szász theorem; see Higgins [16], pp, 95–96). On a limited interval (θ_1, θ_2), for a function which considered integrable $\Theta(z)$, if*

$$\int_{\theta_1}^{\theta_2} z^{t_i}\Theta(z)\,dz = 0, \quad i \geq 1,$$

for nearly all $z \in (\theta_1, \theta_2)$, $\Theta(z) = 0$, where $\{t_i, i \geq 1\}$ is a sequence of positive integers that increasing strictly fulfilling

$$\sum_{j=1}^{\infty} \frac{1}{t_j} = +\infty.$$

In functional analysis, the well-known Lemma 2.3 states that the set of values $\{z^{t_1}, z^{t_2}, \cdots : 1 \leq t_1 < t_2 < \dots\}$ constitutes a complete sequence. It is important to note that Galambos [17] presents a straightforward version of the Müntz-Szász theorem along with a proof (see Theorem AII.3). Furthermore, the Müntz-Szász theory for $\{\psi_{t_i}(z), t_i \geq 1\}$ was extended by Hwang and Lin [14], where $\psi(z)$ is monotone and absolutely continuous on a range (θ_1, θ_2). We examine characterization using the first-order statistics in the following theorem. Let $Z_1, \dots, Z_t$ be completely continuously non-negativity random variables with the routine PDF f and CDF F, and let t be independent and distributed in an identical manner. The function that is considered to be the survival function of the first-order statistics is thus expressed as follows: $\bar{F}_{1;t}(z) = [\bar{F}(z)]^t$, $z \geq 0$.

Theorem 2.4. *Assume that X and Z are two completely continual, non-negativity random variables, each with a PDF of f and h and a CDF of F and H. Then, if and only if F and H are members of the same distribution family, albeit, with a different scale and location, we have*

$$\frac{RGEXn_\beta(X_{1;t})}{E(X_{1;t})} = \frac{RGEXn_\beta(Z_{1;t})}{E(Z_{1;t})},$$

for every $t = t_k$, $k \geq 1$, and for a given $\beta > 0$, such that

$$\sum_{k=1}^{\infty} t_k^{-1} = \infty.$$

Proof. The necessary condition is inessential. For the sufficiency condition, after letting $v = \bar{F}(x)$, we realize that

$$E(X_{1;t}) = \int_0^\infty \bar{F}_{1;t}(x)\,dx = \int_0^1 \frac{v^t}{f\left(\bar{F}^{-1}(v)\right)}\,dv,$$

and that,

$$\beta\, RGEXn_\beta(X_{1;t}) = \int_0^1 \frac{v^t\left(e^{1-v^{t\beta}} - 1\right)}{f\left(\bar{F}^{-1}(v)\right)}\,dv.$$

Consequently,

$$\frac{RGEXn_\beta(X_{1;t})}{E(X_{1;t})} = \frac{RGEXn_\beta(Z_{1;t})}{E(Y_{1;t})},$$

is equivalent to

$$\frac{\int_0^1 \frac{v^t\left(e^{1-v^{t\beta}}-1\right)}{f\left(\bar{F}^{-1}(v)\right)}\,dv}{\int_0^1 \frac{v^t}{f\left(\bar{F}^{-1}(v)\right)}\,dv} = \frac{\int_0^1 \frac{v^t\left(e^{1-v^{t\beta}}-1\right)}{h\left(\bar{H}^{-1}(v)\right)}\,dv}{\int_0^1 \frac{v^t}{h\left(\bar{H}^{-1}(v)\right)}\,dv}.$$

Therefore,

$$\int_0^1 v^t\left(e^{1-v^{t\beta}}-1\right)\left[\frac{1}{f\left(\bar{F}^{-1}(v)\right)} - \frac{1}{\Psi h\left(\bar{H}^{-1}(v)\right)}\right]dv, = \int_0^1 v^t\Theta(v)\,dv,$$

where

$$\Theta(v) = \left(e^{1-v^{t\beta}}-1\right)\left[\frac{1}{f\left(\bar{F}^{-1}(v)\right)} - \frac{1}{\Psi\, h\left(\bar{H}^{-1}(v)\right)}\right], \tag{2.7}$$

$$\Psi = \frac{\int_0^1 \frac{v^t}{f\left(\bar{F}^{-1}(v)\right)}\,dv}{\int_0^1 \frac{v^t}{h\left(\bar{H}^{-1}(v)\right)}\,dv}.$$

For every $t = t_k,\ k \geq 1$, the latter relation is hypothesized to exist, so that $\sum_{k=1}^{\infty} t_k^{-1} = \infty$. Applying the Müntz-Szász theorem to the whole sequence $v^t, t \geq 1$ and Lemma 2.3, the relation (2.7) provides $\Theta(v) = 0$, or equivalently, $f\left(\bar{F}^{-1}(v)\right) = \Psi\, h\left(\bar{H}^{-1}(v)\right)$, for every $0 < v < 1$. We may observe that $F^{-1}(v) = \Psi\, H^{-1}(v) + c$, for any $0 < v < 1$ and a real constant c, if we remember that $\frac{d}{dv}\overline{F}^{-1}(v) = \frac{1}{f\left(\bar{F}^{-1}(v)\right)}$. By the same manner, in Psarrakos and Toomaj [18], the CDFs F and H are members of the same distribution family, with a different scale and location. □

Theorem 2.5. *Consider two completely continuous, non-negativity random variables, X and Z, each with PDFs of f and h and CDFs of F and H. For a change in location, F and H are members of the same distribution family, if and only if*

$$RGEXn_\beta(X_{1;t}) = RGEXn_\beta(Z_{1;t}),$$

for a fixed $\beta > 0$ *and for every* $t = t_j,\ j \geq 1$, *where* $\sum_{k=1}^{\infty} \frac{1}{t_k} = \infty$.

Proof. For $\Psi = 1$, the proof is comparable to Theorem 2.4. □

3. Results on the expansion of the residual cumulative generalized exponential entropy measure

In this section, we will examine the expansion of the residual cumulative generalized exponential entropy and obtain some results. The definition of the residual cumulative generalized exponential entropy is:

$$RGEXn_\beta(X) = \frac{1}{\beta}\int_0^\infty \bar{F}(x)\left(e^{1-\bar{F}^\beta(x)}-1\right)dx.$$

Using the Taylor series expansion for $e^{1-\bar{F}^{\beta}(x)}-1$, we have:

$$\begin{aligned} e^{1-\bar{F}^{\beta}(x)}-1 &= (1-\bar{F}^{\beta}(x)) + \frac{(1-\bar{F}^{\beta}(x))^2}{2!} + \frac{(1-\bar{F}^{\beta}(x))^3}{3!} + \cdots \\ &= \sum_{k=1}^{\infty} \frac{(1-\bar{F}^{\beta}(x))^k}{k!}. \end{aligned}$$

For $\beta > 0$, $\bar{F}^{\beta}(x) \geq 0$ for all x, so $1-\bar{F}^{\beta}(x)$ is finite and the series converges. Substituting into the integral:

$$RGEXn_{\beta}(X) = \frac{1}{\beta}\sum_{k=1}^{\infty}\frac{1}{k!}\int_0^{\infty}\bar{F}(x)\left(1-\bar{F}^{\beta}(x)\right)^k dx.$$

For $(1-\bar{F}^{\beta}(x))^k$, we use the binomial theorem to obtain

$$(1-\bar{F}^{\beta}(x))^k = \sum_{j=0}^{k}\binom{k}{j}(-1)^j\bar{F}^{\beta j}(x).$$

Substituting this back yields

$$RGEXn_{\beta}(X) = \frac{1}{\beta}\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\int_0^{\infty}\bar{F}^{1+\beta j}(x)dx. \tag{3.1}$$

For $\beta > 0$, the survival function $\bar{F}(x)$ typically decreases to 0 as $x \to \infty$, ensuring the convergence of $\bar{F}^{1+\beta j}(x)$.

3.1. The mean residual life function's risk Bayes

The residual or excess of Z, assuming that it surpasses a threshold t, is represented by $Z_t = [Z-t \mid Z > t]$ if the random variable Z represents the lifespan of a component or a system. In contrast, $[Z \mid W]$ often indicates a random variable with a similar distribution as Z conditional on W. The PDF of Z_t is obtained as follows:

$$f(z \mid t) = \frac{f(z)}{\bar{F}(t)}, \quad z > t.$$

The equation for the function of the average residual life of Z with a finite mean μ can potentially be calculated as

$$M(t) = M(Z;t) = E_{Z>t}[Z-t \mid Z > t], \quad t \geq 0, \tag{3.2}$$

in this case, $E_{Z>t}$ denotes the expectation of the residual PDF $f(z \mid t)$. With Z_{β} in place of Z, the function of the mean residual life of Z_{β}, represented as $M_{\beta}(t)$, may be found using (3.2). We note that the best choice under the quadratic loss function is the function of the mean residual life of Z_{β}:

$$Qls(\delta, Z_{\beta} \mid t) = (Z_{\beta} - t - \delta)^2, \quad Z_{\beta} > t,$$

for excess of prediction, i.e.,

$$\delta^*(t) = \arg\min_{\delta} E_{Z_{\beta}>t}[Qls(\delta, Z_{\beta} \mid t)] = M_{\beta}(t), \quad \beta > 0.$$

Asadi et al. [19] noted that $M_\beta(t)$ is a risk local measure that depends on the threshold t. The Bayes risk is its global risk, as

$$Br(M_\beta) = E_\Pi[M_\beta(t)],$$

where $\Pi(t)$ is the distribution of prior for the threshold t.

Theorem 3.1. *Let Z have the function of the mean residual life M, and assume the baseline prior $\Pi(t) = f(t)$. Then, the residual cumulative generalized exponential entropy can be expressed by the Bayes risk of $M_\beta(t)$ as follows:*

$$\begin{aligned} RGEXn_\beta(Z) &= \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j\, j\, Br(M_{\beta j+1}) \\ &= \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j\, j\, E[M_{\beta j+1}(Z)]. \end{aligned} \tag{3.3}$$

Proof. Under the substitution $\Pi(t) = f(t)$, and utilizing the theorem of Fubini, we can express the Bayes risk as

$$\begin{aligned} Br(M_\alpha) &= \int_0^\infty M_\alpha(t)\Pi(t)\, dt = \int_0^\infty M_\alpha(t) f(t)\, dt \\ &= \int_0^\infty \left(\frac{\int_t^\infty \bar{F}^\alpha(z)\, dz}{\bar{F}^\alpha(t)} \right) f(t)\, dt \\ &= \int_0^\infty \bar{F}^\alpha(z) \left(\int_0^z \frac{f(t)\, dt}{\bar{F}^\alpha(t)}\, dt \right) dz \\ &= \int_0^\infty \bar{F}^\alpha(z) \left(\frac{1}{\alpha - 1} \left(\bar{F}^{-\alpha+1}(z) - 1 \right) \right) dz \\ &= \frac{1}{\alpha - 1} \int_0^\infty \left(\bar{F}(z) - \bar{F}^\alpha(z) \right) dz \\ &= \frac{1}{\alpha - 1} \left(\mu - \int_0^\infty \bar{F}^\alpha(z)\, dz \right), \end{aligned}$$

where $1 \neq \alpha > 0$. Therefore, we can see that

$$\int_0^\infty \bar{F}^\alpha(z)\, dz = \mu - (\alpha - 1) Br(M_\alpha). \tag{3.4}$$

Substituting from (3.4) in (3.1), we obtain

$$\begin{aligned} RGEXn_\beta(Z) &= \frac{1}{\beta} \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \left[\mu - (\beta j) Br(M_{\beta j+1}) \right] \\ &= -\sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j\, j\, Br(M_{\beta j+1}), \end{aligned}$$

where the last line is obtained from noting that the inner summation:

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^j = (1-1)^k = 0, \quad \text{for } k \geq 1.$$

□

Theorem 3.2. *Assume that Z is a non-negativity, exactly continually random variable following the PDF $f(z)$. In this case*

$$RGEXn_\beta(Z) = -\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\, j\left(E(Z_{\beta j+1}) + E(g_{\beta j+1}(Z))\right),$$

where $E(Z_\beta) = \int_0^\infty \bar{F}^\beta(z)\,dz$, $g_\beta(v) = \int_0^v M'(u)\bar{F}^{\beta-1}(u)\,du$, and $v > 0$.

Proof. From (3.1), we can rewrite it as

$$RGEXn_\beta(Z) = -\frac{1}{\beta}\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\, j\left[\frac{1}{\beta j}\left(\mu - \int_0^\infty \bar{F}^{1+\beta j}(z)dz\right)\right]. \tag{3.5}$$

Let $\eta(z) = \frac{f(z)}{\bar{F}(z)}$ be the function of the hazard rate. We then have the integration

$$\begin{aligned}\int_0^\infty M(z)\eta(z)\bar{F}^\beta(z)\,dz &= \int_0^\infty \left(\frac{\int_z^\infty \bar{F}(t)\,dt}{\bar{F}^{2-\beta}(z)}\right) f(z)\,dz \\ &= \int_0^\infty \bar{F}(z)\left(\int_0^z f(t)\bar{F}^{\beta-2}(t)\,dt\right)dz \\ &= \frac{1}{\beta-1}\left(\mu - \int_0^\infty \bar{F}^\beta(z)dz\right).\end{aligned} \tag{3.6}$$

By substituting (3.6) in (3.5), and using the result $M(u)\eta(u) = 1 + M'(u)$, we get

$$\begin{aligned}RGEXn_\beta(Z) &= -\frac{1}{\beta}\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\, j\left[\int_0^\infty M(z)\eta(z)\bar{F}^{\beta j+1}(z)\,dz\right] \\ &= -\frac{1}{\beta}\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\, j\left[E(Z_{\beta j+1}) + \int_0^\infty M'(z)\bar{F}^{\beta j+1}(z)\,dz\right] \\ &= -\frac{1}{\beta}\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\, j\left[E(Z_{\beta j+1}) + \int_0^\infty M'(z)\int_x^\infty f(u)[\bar{F}(z)]^{\beta j}\,du\,dz\right] \\ &= -\frac{1}{\beta}\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\, j\left[E(Z_{\beta j+1}) + \int_0^\infty f(u)\int_0^u M'(z)[\bar{F}(z)]^{\beta j}\,dz\,du\right],\end{aligned}$$

then the result follows. □

3.2. Relation to the order of excess wealth

This part examines the connection between the residual cumulative generalized exponential entropy and the wealth order excess, sometimes referred to as the spread right order. Examining the standard deviations of two distribution functions is usually the simplest method to compare their variability. However, Shaked and Shanthikumar [13] developed and thoroughly analyzed stochastic ordering and different transformations for comparing the variability, since numerical measurements alone may not always give adequate information. One of them is the order of excess wealth, which is used to evaluate spread. The wealth excess convert for a non-negativity random variable Z follows a CDF H and PDF h is given by (cf. Fernandez-Ponce et al. [20])

$$\Delta_Z(v) = \int_{H^{-1}(v)}^{\infty} \overline{H}(z)dz = \int_{v}^{1} (1-q) \cdot \frac{1}{h(H^{-1}(q))}\, dq$$
$$= \int_{v}^{1} \left(H^{-1}(q) - H^{-1}(v)\right) dq,$$

with noting that $H^{-1}(v) = \inf\{z : F(z) \geq v\}$, $v \in (0,1)$, is the quantile function of H, and $dz = \frac{d}{dq}H^{-1}(q)\, dq = \frac{1}{h(H^{-1}(q))}\, dq$. Therefore, the difference $H^{-1}(q) - H^{-1}(v)$ measures the excess above the threshold $F^{-1}(v)$ at a level q.

This function and the function of the mean residual life are also connected in this manner by the following connection:

$$M_Z(F^{-1}(u)) = \frac{\Delta_Z(v)}{1-v}, 0 < v < 1. \tag{3.7}$$

Equation (3.7) is used to prove the following theorem.

Theorem 3.3. *If Z has a CDF of H and is a completely continuous, non-negative random variable, then*

$$RGEXn_\beta(Z) = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j\, j \left[\int_0^1 M_Z(H^{-1}(v))(1-v)^{\beta j} dv\right]. \tag{3.8}$$

for all $\beta > 0$.

Proof. The residual cumulative generalized exponential entropy given in (3.3), can be rewritten as

$$RGEXn_\beta(Z) = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j\, j\, E[M_{\beta j+1}(Z)]$$
$$= \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \frac{j}{\beta j+1}\, E[M_Z(Z_{\beta j+1})],$$

and the result follows. □

Example 3.1. *(1) Suppose that Z has a distribution of $Ud(0, d)$. It is clear to see that*

$$M_Z(F^{-1}(v)) = \frac{d(1-v)}{2}.$$

Consequently, using (3.8), we obtain

$$\begin{aligned} RGEXn_\beta(Z) &= \sum_{k=1}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\,\frac{j\,d}{2}\left[\int_0^1 (1-v)^{\beta j+1}dv\right] \\ &= \sum_{k=1}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\,\frac{j\,d}{2(\beta j+2)}, \end{aligned}$$

$\beta > 0$.

(2) Let us examine the Pareto distribution, characterized by a scale parameter $p > 0$ and a shape parameter $s > 0$, where the function of survival is provided by $\overline{H}(z) = \frac{p^s}{(z+p)^s}$ for $z \geq 0$. It is straightforward to observe that

$$M_Z(F^{-1}(v)) = \frac{p(1-v)^{\frac{-1}{s}}}{(s-1)}.$$

Consequently, using (3.8), we obtain

$$\begin{aligned} RGEXn_\beta(Z) &= \sum_{k=1}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\,\frac{j\,p}{(s-1)}\left[\int_0^1 (1-v)^{\beta j-\frac{1}{s}}dv\right] \\ &= \sum_{k=1}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\,\frac{j\,s\,p}{(s(\beta j+1)-1)(s-1)}, \end{aligned}$$

$\beta > 0$.

4. Empirical residual cumulative generalized exponential entropy

The residual cumulative generalized exponential entropy is estimated in this section using the empirical residual cumulative entropy. For any $\beta > 0$, and the random sample $X_1, X_2, \ldots, X_n$, the empirical estimation of the residual cumulative generalized exponential entropy is expressed as

$$\begin{aligned} RGEXn_\beta(\widehat{F}_n) &= \frac{1}{\beta}\int_0^{\infty}\widehat{F}_n(x)\left[e^{1-\widehat{F}_n^{\beta}(x)}-1\right]dx \\ &= \frac{1}{\beta}\sum_{j=1}^{n-1}\int_{X_{j;n}}^{X_{j+1;n}}\left(1-\frac{j}{n}\right)\left[e^{1-\left(1-\frac{j}{n}\right)^{\beta}}-1\right]dx \\ &= \frac{1}{\beta}\sum_{j=1}^{n-1}\Omega_{j+1}\left(1-\frac{j}{n}\right)\left[e^{1-\left(1-\frac{j}{n}\right)^{\beta}}-1\right], \end{aligned} \tag{4.1}$$

with noting that $X_{1;n} \leq X_{2;n} \leq \ldots \leq X_{n;n}$ are the associated order statistics of the random sample, and the sample spacings are $\Omega_{j+1} = X_{j+1;n} - X_{j;n}$, $j = 1, 2, ..., n-1$. For the sample that corresponds to F, the empirical distribution function is described by $\widehat{F}_n(x) = \sum_{j=1}^{n-1} \frac{j}{n} \mathbb{A}_{[x_j, x_{j+1}]}(x), x \geq 0$, with the indicator function, $\mathbb{A}_{\varpi}(x) = 1, x \in \varpi$. Moreover, with $\beta > 0$, we can use the expansion form of the residual cumulative generalized exponential entropy in (3.1) to present its empirical expression as follows:

$$\begin{aligned} RGEXn_\beta(\widehat{F}_n) &= \frac{1}{\beta} \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \int_0^\infty \bar{F}^{1+\beta j}(x) dx \\ &= \frac{1}{\beta} \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \sum_{l=1}^{n-1} \Omega_{l+1} \left(1 - \frac{l}{n}\right)^{1+\beta j}. \end{aligned} \tag{4.2}$$

Utilizing the expansion form of the empirical residual cumulative generalized exponential entropy in (4.2), we can now provide a central limit theorem for this measure, which is derived from a random sample with an exponential distribution.

Theorem 4.1. *A sample selected at random $X_1, X_2, \ldots, X_n$ drawn from a common $Exp(\gamma)$ distribution is considered. In this case,*

$$\frac{RGEXn_\beta(\widehat{F}_n) - E\left[RGEXn_\beta(\widehat{F}_n)\right]}{\sqrt{Var\left[RGEXn_\beta(\widehat{F}_n)\right]}} \longrightarrow \textit{standard normal distribution},$$

where $\beta > 0$.

Proof. The empirically residual cumulative generalized exponential entropy measure can be written as a total of the independent exponential random variables X_l, $l = 1, 2, ..., n$, using the expansion (4.2), where the variance and expected value are given by (noting that the spacing Ω_{l+1} are independent and distributed by $Exp(\gamma(n - l))$)

$$E[X_l] = \frac{1}{n\gamma\beta} \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \left(1 - \frac{l}{n}\right)^{\beta j}, \tag{4.3}$$

and

$$Var[X_l] = \frac{1}{n^2\gamma^2\beta^2} \left[\sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \left(1 - \frac{l}{n}\right)^{\beta j}\right]^2. \tag{4.4}$$

Consider the following: $\Phi_{l,q} = E[|X_l - E(X_l)|^q]$, $q = 2, 3$. From (4.4), one can derive the following estimates for n considered to be large, as follows

$$\begin{aligned} \sum_{l=1}^{n} \Phi_{l,2} = \sum_{l=1}^{n} E[|X_l - E(X_l)|^2] = \sum_{l=1}^{n} Var[X_l] &= \frac{1}{n^2\gamma^2\beta^2} \sum_{l=1}^{n} \left[\sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \left(1 - \frac{l}{n}\right)^{\beta j}\right]^2 \\ &\approx \frac{1}{n^2\gamma^2\beta^2} \int_0^1 (e^{1-x^\beta} - 1)^2 \, dx = \frac{1}{n^2\gamma^2\beta^2} \, g_2. \end{aligned}$$

Additionally, given whatever random variable X_l with an exponential distribution, the following result may be reached. From (4.3), we obtain (observing that $E[|X_l - E(X_l)|^3] = \frac{2(6-e)[E(X_l)]^3}{e}$, see, [21, 22])

$$\sum_{l=1}^{n} \Phi_{l,3} = \frac{2(6-e)}{e} \sum_{l=1}^{n} [E(X_l)]^3 = \frac{2(6-e)}{e(n\gamma\beta)^3} \sum_{l=1}^{n} \left[\sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \left(1 - \frac{l}{n}\right)^{\beta j} \right]^3$$
$$\approx \frac{2(6-e)}{en^3\gamma^3\beta^3} \int_0^1 (e^{1-x^\beta} - 1)^3\, dx = \frac{2(6-e)}{en^3\gamma^3\beta^3}\, g_3.$$

Taking note of that

$$g_q = \int_0^1 (e^{1-x^\beta} - 1)^q\, dx,\ q = 2, 3,$$

and the integrand $(e^{1-x^\beta} - 1)^q$ is bounded and continuous over the interval $[0, 1]$. Therefore, it is true for large n, given an adequate function Gn, that

$$\frac{\sum_{l=1}^{n} \Phi_{l,3}}{\sum_{l=1}^{n} \Phi_{l,2}} \approx \frac{Gn}{n} \underset{n\to\infty}{\longrightarrow} 0.$$

Thus, the central limit theorem's Lyapunov condition is satisfied (see, for example, [23]), thus completing the proof. □

4.1. Procedure of the second estimator

In this subsection, a different non-parametric estimator can be developed as follows. The residual cumulative generalized exponential entropy given in (3.1) can be rewritten as

$$RGEXn_\beta(X) = \frac{1}{\beta} \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \int_0^\infty \bar{F}^{1+\beta j}(x) dx$$
$$= \frac{1}{\beta} \left(\sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j (1 + \beta j) \left(\int_0^\infty x f(x)\, \bar{F}^{\beta j}(x)\, dx \right) \right). \tag{4.5}$$

The residual cumulative generalized exponential entropy was introduced as an L-functional by Zardasht [24]. Similarly, $RGEXn_\beta(X)$ in (4.5) can be represented as

$$RGEXn_\beta(X) = \frac{1}{\beta} \left(\sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j (1 + \beta j) \left(\int_0^\infty x\bar{F}^{\beta j}(x)\, dF(x) \right) \right)$$
$$= \frac{1}{\beta} \left(\sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j (1 + \beta j) \left(\int_0^\infty x Ln_{\beta j}(F(x)) dF(x) \right) \right), \tag{4.6}$$

where $Ln_{\beta j}(v) = (1 - u)^{\beta j}$, $0 \le u \le 1$. In the follow-up, we can produce an estimate for $RGEXn_\beta(X)$ using the following L-statistic by replacing F in (4.6) with $\widehat{F}_n$:

$$RGEXn_{\beta}*(\widehat{F}_n) = \frac{1}{\beta} \left(\sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j (1 + \beta j) \left(\int_0^\infty x Ln_{\beta j}\left(\widehat{F}_n\right) d\widehat{F}_n \right) \right)$$
$$\approx \frac{1}{\beta} \left(\sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \frac{(1 + \beta j)}{n} \left(\sum_{l=1}^{n} X_{l;n} Ln_{\beta j}\left(\frac{l}{n}\right) \right) \right). \tag{4.7}$$

4.2. Application

Example 4.1. *A random sample $X_1, X_2, \ldots, X_n$ selected from the $EXP(1)$ distribution is considered. $\Omega_{l+1} = X_{(l+1)} - X_{(l)}$ provides the sample spacing for $l = 1, 2, \ldots, n-1$, which are independent. Each Ω_{j+1} has an exponential distribution with a parameter of $(n-l)$. Thus, from (4.1), we obtain the following:*

(1) The mean of $RGEXn_\beta(\widehat{F}_n)$ is

$$E[RGEXn_\beta(\widehat{F}_n)] = \frac{1}{n\beta}\sum_{l=1}^{n-1}\left(e^{1-\left(1-\frac{l}{n}\right)^\beta} - 1\right).$$

(2) The variance of $RGEXn_\beta(\widehat{F}_n)$ is

$$Var[RGEXn_\beta(\widehat{F}_n)] = \frac{1}{n^2\beta^2}\sum_{l=1}^{n-1}\left(e^{1-\left(1-\frac{l}{n}\right)^\beta} - 1\right)^2.$$

It is possible to infer, from Eq (4.7) and the relations (4.6.6)–(4.6.8) provided by Arnold et al. [25], that

(1) The mean of $RGEXn^*_\beta(\widehat{F}_n)$ is

$$E[RGEXn^*_\beta(\widehat{F}_n)] = \frac{1}{\beta}\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\frac{(1+\beta j)}{n}\sum_{l=1}^{n}Ln_{\beta j}\left(\frac{l}{n}\right)\sum_{p=1}^{l}\frac{1}{n-p+1}.$$

(2) The variance of $RGEXn^*_\beta(\widehat{F}_n)$ is

$$\begin{aligned}Var[RGEXn^*_\beta(\widehat{F}_n)] =& \frac{1}{\beta^2}\left(\sum_{l=1}^{n}\left(\left(\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\frac{(1+\beta j)}{n}Ln_{\beta j}\left(\frac{l}{n}\right)\right)^2\sum_{p=1}^{l}\frac{1}{(n-p+1)^2}\right)\right.\\ &+ 2\sum_{l_1=1}^{n}\sum_{l_2=1}^{n}\left(\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\frac{(1+\beta j)}{n}Ln_{\beta j}\left(\frac{l_1}{n}\right)\right)\left(\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\right.\\ &\left.\left.\times\ (-1)^j\frac{(1+\beta j)}{n}Ln_{\beta j}\left(\frac{l_2}{n}\right)\right)\sum_{p=1}^{l_1}\frac{1}{(n-p+1)^2}\right).\end{aligned}$$

Remark 4.1. *The intervals of confidence for $RGEXn_\beta(X)$ can possibly be computed using the results of Theorem 4.1 if the random variables have an exponential distribution. In particular, from (4.1) and (4.7), it is true for any specific $\beta > 0$ that*

$$RGEXn_\beta(\widehat{F}_n) \pm \chi_{\frac{\delta}{2}}\sqrt{Var[RGEXn_\beta(\widehat{F}_n)]},$$

$$RGEXn^*_\beta(\widehat{F}_n) \pm \chi_{\frac{\delta}{2}}\sqrt{Var[RGEXn^*_\beta(\widehat{F}_n)]},$$

where the critical point of the standard normal distribution at $\frac{\delta}{2}$ is shown by $\chi_{\frac{\delta}{2}}$.

The mean and variance of the empirical residual cumulative generalized exponential function of entropy for the aforementioned relationships are shown in Table 1. Sample sizes of 10, 20, 30, 40, and 50 were chosen, with varying values of the order β. The residual cumulative generalized exponential entropy's precise values are computed as follows: $RGEXn_1(X) = 0.718282$, $RGEXn_2(X) = 0.515039$, and $RGEXn_3(X) = 0.398348$. On the basis of the results in Table 1, we can conclude the following

(1) It is evident that the mean converges to the true value, and the variability of the empirical measurement approaches zero as the sample size increases.

(2) For any fixed n and increasing β, the variance decreases.

(3) For any large n, the second estimator provides a more accurate result (by decreasing the variance) compared with the first.

Table 1. Empirical residual cumulative generalized exponential entropy's expected value and variance for the exponential distribution with a unit mean for $\beta = 1, 2$, and 3 and $n = 10, 20, 30, 40$, and 50 sample sizes.

Sample size	β	$E[RGEXn_\beta(\widehat{F_n})]$	$Var[RGEXn_\beta(\widehat{F_n})]$	$E[RGEXn^*_\beta(\widehat{F_n})]$	$Var[RGEXn^*_\beta(\widehat{F_n})]$
	1	0.633799	0.0618118	0.636254	0.086091
$n = 10$	2	0.471249	0.0302985	0.57848	0.0651343
	3	0.368875	0.0174016	0.530634	0.056318
	1	0.675683	0.0343049	0.027132	0.617761
$n = 20$	2	0.493352	0.0160719	0.55787	0.0203291
	3	0.38382	0.00911087	0.507678	0.017622
	1	0.689803	0.023654	0.611674	0.0135907
$n = 30$	2	0.500628	0.0109196	0.550958	0.0101643
	3	0.388709	0.00616504	0.499906	0.0061073
	1	0.696893	0.0180386	0.608645	0.00826977
$n = 40$	2	0.504248	0.0082666	0.547493	0.00618269
	3	0.391136	0.00465795	0.495995	0.00438061
	1	0.701156	0.014575	0.606833	0.00560691
$n = 50$	2	0.506414	0.00665019	0.545411	0.00419222
	3	0.392587	0.00374276	0.493641	0.00365316

Example 4.2. *Reaven and Miller [26] investigated the connection between insulin and blood chemistry indicators of glucose tolerances in 145 non-fat individuals. They visualized the data in three dimensions using the Stanford Accelerator Linear Center's PRIM9 technology and found an odd pattern that resembled a big blob with two wings pointing in separate directions. Three categories were created from the 145 observations: Overt diabetes, Chemical diabetics, and Normals. Additionally, the 145 observations were divided into three groups: 33 for overt diabetes, 36 for chemical diabetes, and 76 for normal diabetes. Five factors for every single individual were examined as follows:*

(1) Relative of weight (Λ_1),

(2) Test plasma glucose level (Λ_2),

(3) Fasting plasma glucose level (Λ_3),

(4) Plasma insulin during the test (Λ_4), and

(5) Steady state plasma glucose (Λ_5).

For the Λ_5 data set, we fit it to the exponential distribution with the parameter $\gamma = 0.00433$. Figure 2 shows the histogram of the Λ_5 data set with the exponential curve, along with the empirical and theoretical CDFs. To statistically validate this fit, we performed a Kolmogorov-Smirnov test, which yielded a p-value of 0.051032. This result supports the suitability of the $EXP(0.00433)$ distribution for modeling these data.

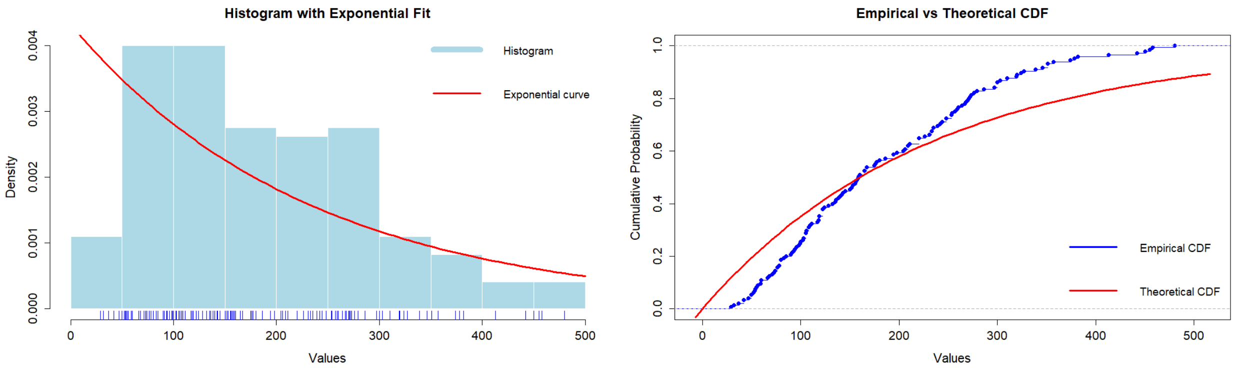

Figure 2. Histogram of the Λ_5 data set along with the exponential curve (left) and the empirical and theoretical CDFs (right).

Figure 3 shows the theoretical residual cumulative generalized exponential entropy, which was computed using this exponential parameter distribution and the confidence interval (at $\delta = 0.05$) obtained from Eq (4.1). It is clear that the estimators' confidence intervals contain the theoretical value.

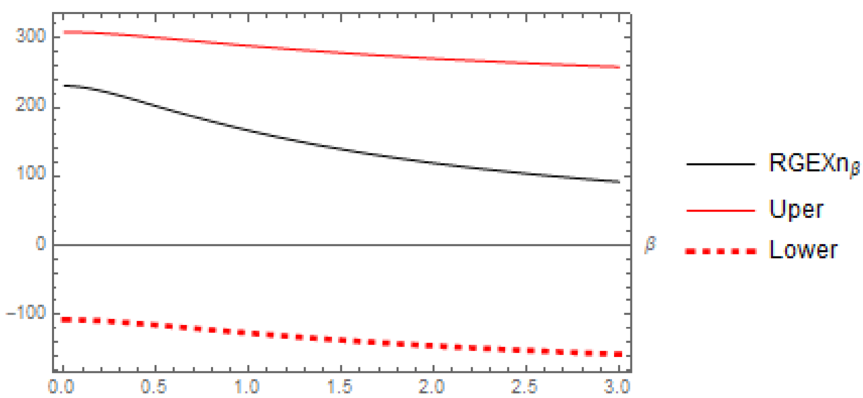

Figure 3. Theoretical measure of the residual cumulative generalized exponential entropy with 95% confidence intervals in Example 4.2.

5. Characterization of symmetry and pattern of recognition using generalized exponential entropy

In this section, we will apply some properties and applications with the generalized exponential entropy including symmetry characterization and classification problem with pattern of recognition.

5.1. Characterization of symmetry

Using (1.5), we can define the continuous case of the generalized exponential entropy of the continual random variable Z with the PDF f as follows:

$$GEXn(Z) = \frac{1}{\beta}\int_{-\infty}^{\infty} f(z)(e^{1-f^{\beta}(z)} - 1)\, dz, \tag{5.1}$$

where $\beta \in \mathbb{R}\backslash\{0\}$. Several interesting features of the extended exponential entropy of order statistics appear when the PDF of the underlying identical besides the independent of random variables is symmetric. With an underlying distribution Z containing the lth-order statistic $Z_{l;t}$, $1 \leq l \leq t$, the PDF of a sample of size t is derived by

$$f_{l;t}(z) = \frac{1}{Beta_f(l, t-l+1)} F^{l-1}(z)\overline{F}^{t-l}(z)f(z), \tag{5.2}$$

with nothing that $Beta_f(l, t-l+1) = \frac{\Gamma(l)\Gamma(t-l+1)}{\Gamma(t+1)}$. We begin with two lemmas discussed by Fashandi and Ahmadi [27] and Balakrishnan and Selvitella [28], respectively, the definition of $f_{r;t}$ in (5.2) and the symmetry assumption serve as the immediate foundation for the proof.

Lemma 5.1. *[27] The following result is supported by G_Z, PDF f, and CDF F, and Z is a continuous random variable as*

$$f\left(F^{-1}(v)\right) = f\left(F^{-1}(1-v)\right) \quad \text{for all } v \in (0,1),$$

which suggests the symmetry of $F(z)$ with respect to a constant $g_n \in G_Z$.

Lemma 5.2. *[28] Assume that the parent distribution of the order statistic $Z_{l;t}$, $l = 1, ..., t$, has a PDF f with noting $f(\mu + z) = f(\mu - z)$, $z \geq 0$. We proceed with the following analysis:*

$$F(\mu + z) = \overline{F}(\mu - z), \qquad f_{l;t}(\mu + z) = f_{t-l+1;t}(\mu - z).$$

Theorem 5.1. *Let $Z_1, ..., Z_t$ be identical, including independent distributed observations over Z whose PDF is regarded as symmetric around its mean μ. Consequently, we have*

(1) In the event that t is deemed to be odd, $GEXn_{\beta}(Z_{l;t}) = GEXn_{\beta}(Y_{t-l+1;t})$, $l = 1, ..., t$.

(2) If and only if $GEXn_{\beta}(Z_{1;t}) = GEXn_{\beta}(Z_{t;t})$, $\forall t \geq 1$, then Z has a symmetric PDF.

Proof. (1) Lemma 5.2 and Eq (5.1) provide us with

$$\begin{aligned} GEXn_{\beta}(Z_{l;t}) &= \frac{1}{\beta}\int_{-\infty}^{\infty} f_{l;t}(z)(e^{1-f_{l;t}^{\beta}(z)} - 1)\, dz \frac{1}{\beta}\int_{-\infty}^{\infty} f_{l;t}(\mu + z)(e^{1-f_{l;t}^{\beta}(\mu+z)} - 1)\, dz \\ &= \frac{1}{\beta}\int_{-\infty}^{\infty} f_{t-l+1;t}(\mu - z)(e^{1-f_{t-l+1;t}^{\beta}(\mu-z)} - 1)\, dz \\ &= \frac{1}{\beta}\int_{-\infty}^{\infty} f_{t-l+1;t}(z)(e^{1-f_{t-l+1;t}^{\beta}(z)} - 1)\, dz = GEXn_{\beta}(Z_{t-l+1;t}). \end{aligned}$$

(2) The first part of this theorem establishes the necessity. We now turn to the sufficiency. Suppose that $GEXn_\beta(Z_{l;t}) = GEXn_\beta(Z_{t;t})$ holds for all $t \geq 1$. Using Lemma 5.1, we derive the following, for every $v \in (0, 1)$,

$$f(F^{-1}(1-v)) = f(F^{-1}(v)),$$

which leads to $-\frac{d}{du}F^{-1}(1-v) = \frac{d}{dv}F^{-1}(v)$. Consequently, $-F^{-1}(1-v) = F^{-1}(v) + g_n$, and thus $f(-F^{-1}(v) - g_n) = f(F^{-1}(v))$, where g_n is a constant, which is valid for all $v \in (0, 1)$. Substituting $F^{-1}(v) = -\frac{g_n}{2} + z$, we obtain $f(-\frac{g_n}{2} - z) = f(-\frac{g_n}{2} + z)$ for all $z \in \mathbb{R}$, thereby completing the proof of the theorem.

□

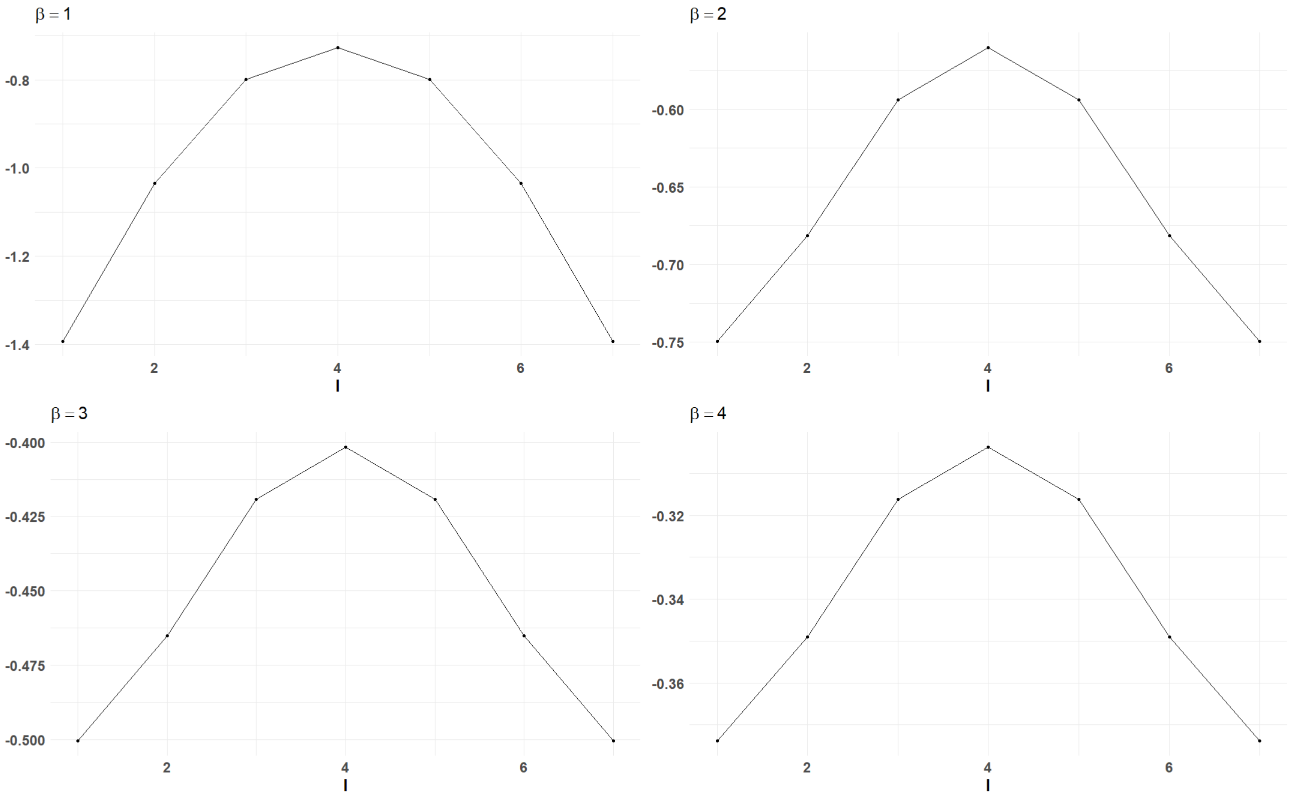

Figure 4. Generalized exponential entropy of the lth-order statistics of $Ud(-1, 1)$ distribution.

Corollary 5.1. *In alignment with Theorem 5.1, let* $\Delta GEXn_\beta(Z_{p;t}) = GEXn_\beta(Z_{p+1;t}) - GEXn_\beta(Z_{p;t})$ *represent the forward difference operator with respect to p, where* $1 \leq p \leq t-1$*. It follows that* $\Delta GEXn_\beta(Z_{l;t}) = -\Delta GEXn_\beta(Z_{t-l;t})$*, for* $l = 1, ..., t$.

Remark 5.1. *Define* Ξ_t *as* $GEXn_\beta(Z_{1;t}) - GEXn_\beta(Z_{t;t})$*. The condition* $\Xi_t = 0$*, for* $t = 1, 2, ...,$ *holds if and only if Z exhibits symmetry. Consequently,* Ξ_t *can serve as a fundamental measure of symmetry and as a statistic for testing symmetry.*

From the assumptions outlined in Corollary 5.1, it can be deduced that the fractional generalized entropy $GEXn_\beta(Z_{l;n})$ attains either a local maximum at the median. This property is illustrated using the $Ud(-1, 1)$ distribution. Specifically, for the generalized exponential entropy of the median ($l = 4$) with $t = 7$, the maximum values are -0.7269227 for $\beta = 1$, -0.56026 for $\beta = 2$, -0.4016726 for $\beta = 3$, and -0.3036904 for $\beta = 4$ (refer to Figure 4).

5.2. *Pattern of recognition*

The probabilistic measures of information in the classification problems using pattern recognition will be covered in this section. For classification problems, uncertainty measurements are a helpful tool; for instance, one might refer to [29] and [30] in this context. We will use the data presented in Example 4.2 to discuss its relevant pattern of recognition using the generalized exponential entropy given in (1.5). The variables and their correlation are presented in Figure 5.

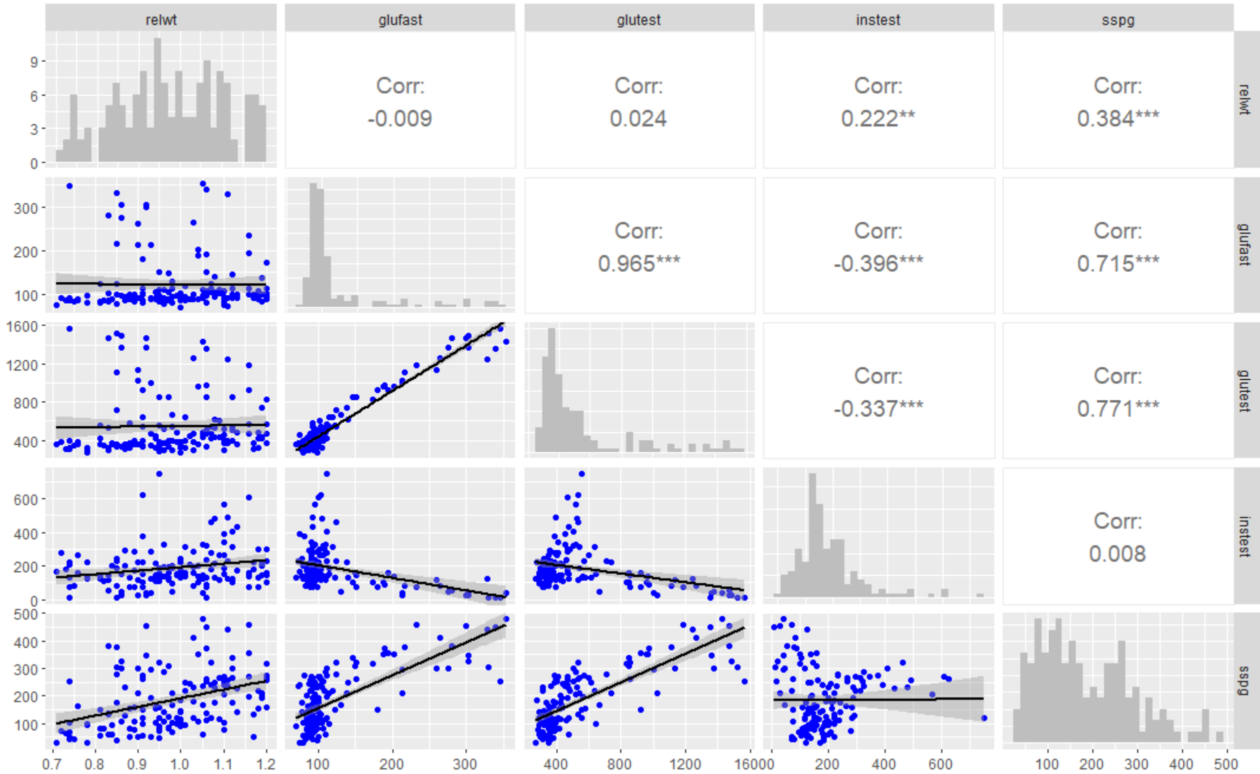

Figure 5. The variables and their correlation of Example 4.2 data (*** : p-value < 0.001; ** : p-value < 0.01).

As indicated in Table 2(I), we select 30 samples for each data categorization in order to build an interval number model. We next identify a sample that has both the greatest and lowest values. An anonymous test case of every possible occurrence in the dataset is displayed. Assuming that the selected singleton data sample (1.18, 108, 486, 297, 220) originates from the chemical diabetic group (a singleton is a quantity that occurred only once in a dataset). To identify the singletons in a sample, we must first select a sample from a sequence of data and then note the occurrences of the values in that sample.

Table 2. (I) The three specified groups' interval numbers. (II) Distributions of probabilities according to the interval values.

(I) Item	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5
Normal	[0.74, 1.2]	[74, 112]	[269, 418]	[81, 267]	[29, 273]
Chemical diabetic	[0.83, 1.2]	[75, 114]	[413, 643]	[109, 748]	[60, 300]
Overt diabetic	[0.74, 1.2]	[120, 353]	[538, 1520]	[10, 460]	[150, 458]
(II) Item	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5
$\mathbb{P}$(normal)	0.319397	0.45749	0.318917	0.395996	0.329123
$\mathbb{P}$(chemical diabetic)	0.361206	0.483345	0.603506	0.234745	0.405461
$\mathbb{P}$(overt diabetic)	0.319397	0.0591652	0.0775768	0.369259	0.265417

Subsequently, we create five distinct probability distributions using Kang et al.'s [31] technique, which is based on the closeness between interval numbers. $R_1 = [\lambda_1, \lambda_2]$ and $R_2 = [\lambda_1^*, \lambda_2^*]$ are the two ranges taken into account. The distance between the ranges R_1 and R_2 is then calculated by

$$I(R_1, R_2) = \left[\left(\frac{\lambda_1 + \lambda_2}{2}\right) - \left(\frac{\lambda_1^* + \lambda_2^*}{2}\right)\right]^2 + \frac{1}{3}\left[\left(\frac{\lambda_1 - \lambda_2}{2}\right)^2 + \left(\frac{\lambda_1^* - \lambda_2^*}{2}\right)^2\right].$$

Furthermore, their similarity $\rho(R_1, R_2)$ is explained as

$$\rho(R_1, R_2) = \frac{1}{1 + \zeta\, I(R_1, R_2)},$$

in which ζ is the supporting coefficient; one example of its application is to set ζ to 5. For range the R_1, we use the ranges listed in Table 2(I); for the range R_2, we use individual values from the selected sample to generate the given probability distributions (for example, the range of for the value 1.18 in the Λ_1 attribute is $R_2 = [1.18, 1.18]$). Table 2(II) shows that each of the five evaluated criteria produces three similarity values. A probability distribution is then created using the normalized representation of this data. These probability distributions are then evaluated using our generalized exponential entropy measure (with $\beta = 1$, $\beta = 2$, and $\beta = 3$) and are shown in Table 3(I). We also use $W^*(T) = e^{-T}$ as the weighting foundation due to the monotonicity of the function that is considered to be exponential. The weights are then obtained by normalizing them. For example, when the generalized exponential entropy's Λ_3 characteristic is used, the procedure produces

$$W^*(\Lambda_3) = \frac{e^{-GEXn(\Lambda_3)}}{e^{-GEXn(\Lambda_1)} + e^{-GEXn(\Lambda_2)} + e^{-GEXn(\Lambda_3)} + e^{-GEXn(\Lambda_4)} + e^{-GEXn(\Lambda_5)}},$$

The weighted values $W^*(\Lambda_t)$, $t = 1, 2, 3, 4, 5$, corresponding to the five characteristics, are presented in Table 3(II). Therefore, the final probability distribution of the generalized exponential entropy measure is listed as:

(1) Under $\beta = 1$, we obtain

$$\mathbb{P}(\text{normal}) = 0.36586,\ \mathbb{P}(\text{chemical diabetic}) = 0.427895,\ \mathbb{P}(\text{overt diabetic}) = 0.206244.$$

(2) Under $\beta = 2$, we obtain

$$\mathbb{P}(\text{normal}) = 0.364894,\ \mathbb{P}(\text{chemical diabetic}) = 0.424175,\ \mathbb{P}(\text{overt diabetic}) = 0.210931.$$

(3) Under $\beta = 3$, we obtain

$$\mathbb{P}(\text{normal}) = 0.364377,\ \mathbb{P}(\text{chemical diabetic}) = 0.421197,\ \mathbb{P}(\text{overt diabetic}) = 0.214426.$$

Following that, it was concluded that the chosen sample is most likely to be in the chemical diabetic category. As a result, in this case, the conclusion was correct.

Table 3. (I) Generalized exponential entropy measurements (II) $W^*(\Lambda_t)$, $t = 1,2,3,4,5$, where the weights represent the five variables.

(I) Item	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5
$\beta = 1$	0.945856	0.748902	0.722495	0.922879	0.931814
$\beta = 2$	0.71489	0.604572	0.5662	0.699949	0.705139
$\beta = 3$	0.539139	0.488135	0.4556	0.531995	0.534171
(II) Item	$W^*(\Lambda_1)$	$W^*(\Lambda_2)$	$W^*(\Lambda_3)$	$W^*(\Lambda_4)$	$W^*(\Lambda_5)$
$\beta = 1$	0.181642	0.221183	0.227102	0.185864	0.18421
$\beta = 2$	0.188615	0.210614	0.218853	0.191454	0.190463
$\beta = 3$	0.194114	0.204272	0.211027	0.195506	0.195081

Table 4. The recognition rates of entropy and generalized exponential entropy approaches.

Method	Normal	Chemical diabetic	Overt diabetic	In general
Entropy method	100%	55.5%	60.6%	72.03%
Generalized exponential entropy method	100%	44.4%	33.3%	59.23%

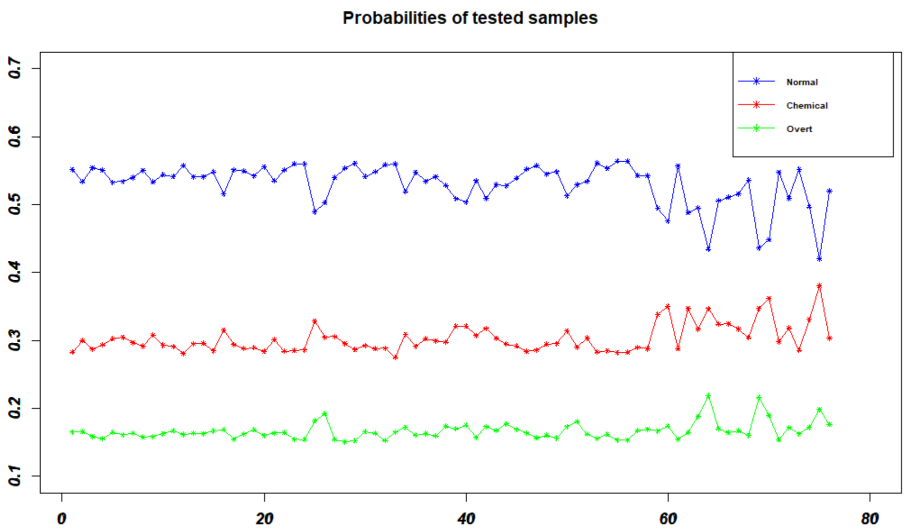

Figure 6. Probabilities of the 76 normal categories tested samples.

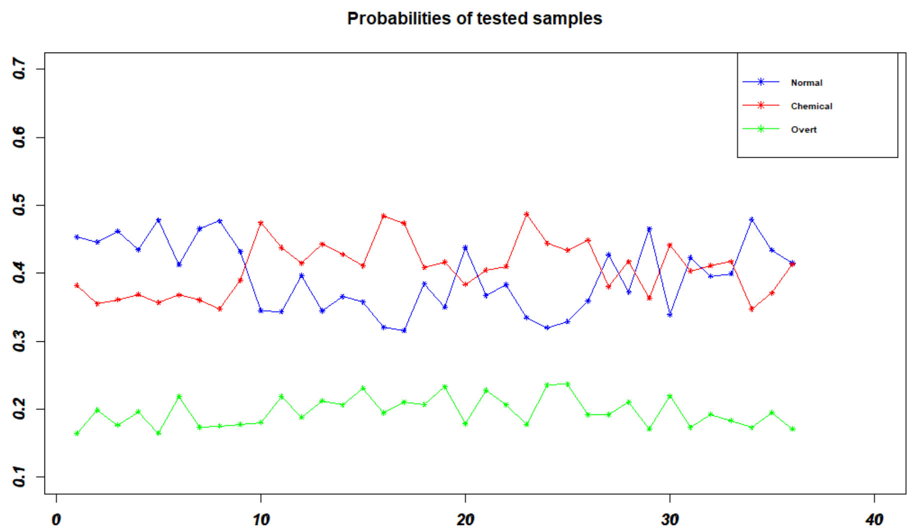

Figure 7. Probabilities of the 36 chemical diabetic categories tested samples.

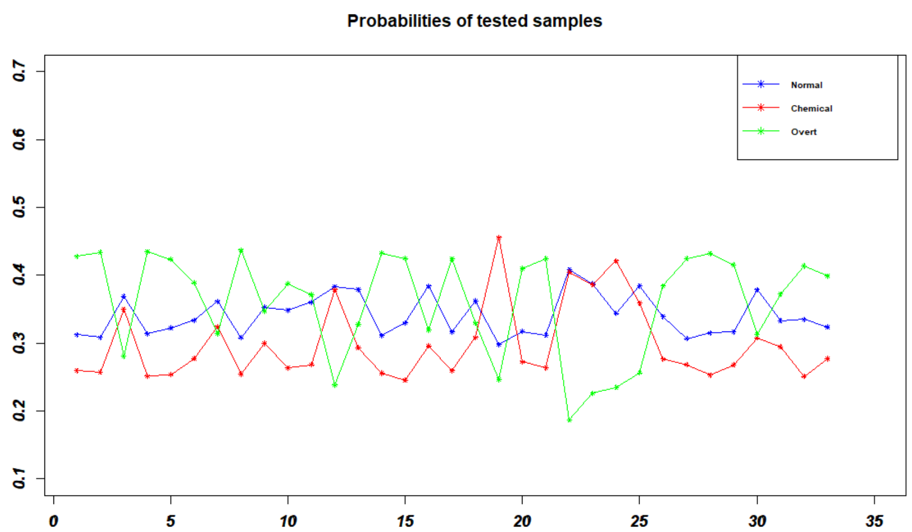

Figure 8. Probabilities of the 33 overt diabetic categories tested samples.

Using this strategy, we examined all 145 samples via the entropy and generalized exponential entropy approaches over a range of β values, including 33 in the overt diabetic, 36 in the chemical diabetic, and 76 in the normal categories. The recognition rates are displayed in Table 4, where the entropy technique yields 72.03%. In the meanwhile, 59.23 is obtained using the extended exponential entropy technique. Moreover, when $\beta = 1$, we can see the probabilities when we choose the 76 normal categories tested samples in Figure 6, the 36 chemical diabetic categories tested samples in Figure 7, and the 33 overt diabetic categories tested samples in Figure 8 (the higher probability indicates the right choice).

It must be emphasized that the results obtained in this section are based on the specific dataset used in our analysis and should not be interpreted as a general statement of preference. The performance of the proposed model may vary with different datasets, where its advantages could become more evident.

6. Conclusions and future work

In this study, we emphasized that the study of the continuous case must be carried out and not just the discrete case, as its importance appeared in many of the applications used, such as dealing with the order statistics, as dealing with them in the discrete case is not flexible and limited. In addition, it is important to study the model based on the distribution function because of its many advantages. Besides, we have seen that the residual cumulative generalized exponential entropy measure is a generalization of the original model presented by Rao et al. [2]; this is one of the reasons why we do not rely on the original model. Moreover, proving that this measure is bounded contributed to solving some problems that have appeared in some theories and made it flexible in dealing with them. In addition, the measure's reliance on the exponential function made us use its expansion in studying some topics, like the order of excess wealth, Bayes risk, and estimations. Moreover, the real data were selected to see the application of this measure in terms of non-parametric estimators and the extent of suitability of these estimators with the proposed measure. On the other hand, this data was used to solve the problem of classification, as is clear at the end in Table 4, where the extent of the efficiency of using this measure to solve the problem at hand was shown.

Overall, the implications of the residual cumulative generalized exponential entropy as an extension of the residual cumulative entropy, which tends to it when $\beta \to 0$, has been presented. Numerous findings have been examined, including non-negativity, limits, relationship to the measure of classical differential measure of entropy, and preservation features, with a few well-known and familiar stochastic comparisons. Besides, some of those features are verified with some well-known distributions. In addition, some characterization of our model based on the first-order statistics has been obtained. Moreover, under the Taylor series expansion, some results on the expansion of the residual cumulative generalized exponential entropy, such as Bayes risk and the connection with the transform of excess wealth, have been explained. On top of that, we have examined the issue of using its empirical CDF to estimate the residual cumulative generalized exponential entropy. In this context, we estimate this measure using two distinct empirical estimators of the CDF. A theorem of the centralized limitation for the empirical measurement of randomly samples drawn from a distribution that is considered to be exponential is developed for the first estimator. A theorem of the central limit solution for empirical measuring constructed from a randomly sample using an unknown distribution is provided; however, it is also provided for the second estimator. Both methods were used on the data, and we found that they are close in their results, as the average values give results close to the true values, but there is some slight advantage for the second estimator. Moreover, the continuous case of generalized exponential entropy is discussed to illustrate the symmetry characterization of order statistics, using an example of a symmetric uniform distribution that shows that the median is the point of symmetry. On the other hand, generalized exponential entropy has been discussed in many areas, such as multi-criteria decision-making. Therefore, we have applied this model to the classification issue by utilizing the pattern recognition of a diabetes dataset and comparing it with the classical entropy, which shows superiority to the classical entropy measure.

In future work, we can implement the obtained measure to different topics like the concomitants of order statistics and tests of hypotheses and compare it against other existing entropy-based models in terms of interpretability, efficiency, or computational feasibility. Moreover, we aim to extend our analysis to additional common distributions, such as the normal and gamma distributions, which currently require numerical solutions. Investigating these distributions in greater detail will help illustrate the broader applicability of the measure and provide further insights into its practical utility. Furthermore, while the current study relies on closed-form expressions for the variance and mean (which eliminate the need for simulation-based data), future work will incorporate simulation studies to compute the mean square error. This will allow us to further validate the estimator's accuracy and assess the convergence of bias as the sample sizes increase. Moreover, we plan to explore the integration of the stochastic precedence order to evaluate its potential contributions and impact on our theoretical framework, thereby broadening the scope of stochastic ordering relations analyzed in this study.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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Appendix

The following Mathematica codes are provided to allow readers to reproduce the results presented in the paper.

A.1. Code of calculations in Table 1 using Wolfram Mathematica

```
Beta= 2; Gamma = 1;
fx= 1 - Exp[-Gamma x];
RGEXn= N[1/Beta Integrate[(1 - fx) (Exp[1 - (1 - fx)^Beta] - 1),
          {x, 0, Infinity}]];
n = 10;
mean= N[1/(n Beta) Sum[(Exp[1 - (1 - 1/n)^Beta] - 1), {1, 1, n - 1}]];
var= N[1/(n Beta)^2 Sum[(Exp[1 - (1 - 1/n)^Beta] - 1)^2,
       {1, 1, n - 1}]]
```

A.2. Code of Figure 3 using Wolfram Mathematica

```
y= Sort[{data}];
n= Length[y];
Beta= Beta;
```

```
Gamma= 0.00433;
fx= 1 − Exp[−Gamma x];
RGEXn= N[1/Beta Integrate[(1 − fx) (Exp[1 − (1 − fx)^Beta] − 1),
       {x, 0, Infinity}]];
s1= N[1/Beta Sum[(y[[i + 1]] − y[[i]]) (1 − i/n)
   (Exp[1 − (1 − i/n)^Beta] − 1), {i, 1, n − 1}]];
var1= N[Variance[y]];
c1= s1 + 1.96 Sqrt[var1];
c2= s1 − 1.96 Sqrt[var1];
Plot[{STMn, c1, c2}, {Beta, 0.001, 3},
 PlotLegends −> {"RGEXn", "Upper", "Lower"},
 AxesLabel −> Automatic,
 PlotStyle −> {Black, Red, {Dashed, Red, AbsoluteThickness[3]}},
 Frame −> True]
```

A.3. Code of Figure 4 using R software

```
rm(list=ls())
# Load necessary libraries
library(ggplot2)

# Define parameters
Theta <− 4
n <− 7
a <− −1
b <− 1

# Define the function for fyos1
fyos1 <− function(y, r) {
  ffy <− (y − a) / (b − a)
  fy <− 1
  return((gamma(n + 1) ∗ ffy^(r − 1) ∗ (1 − ffy)^(n − r) ∗ fy) /
          (gamma(r) ∗ gamma(n − r + 1)))
}

# Define the generalized function GF
GF <− function(r) {
  integrand <− function(y) {
    fyos1_val <− fyos1(y, r)
    return(fyos1_val∗(exp(1−(fyos1_val)^(Theta))−1))
  }

  result <− integrate(integrand, a, b)$value
```

```
  return((1 / Theta) * result)
}

# Create the plot data
r_values <- 1:n
GF_values <- sapply(r_values, GF)

# Create a data frame for ggplot
df <- data.frame(r = r_values, GF = GF_values)

# Plot using ggplot2
ggplot(df, aes(x = r, y = GF)) +
  geom_point() +
  geom_line() +
  labs(x = "l", y = "", title = expression(paste(beta, "= 3"))) +
  theme_minimal(base_size = 15)
```

A.4. Code of Figure 5 using R software

```
# Clear the environment
rm(list = ls())
library(heplots)
library(GGally)

data("Diabetes")

# Select only the numerical variables
df <- Diabetes[, c("relwt", "glufast", "glutest", "instest", "sspg")]

# Create a scatterplot matrix
ggpairs(df, lower = list(continuous = "smooth"))
```