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Research article

Tractability of L_2-approximation and integration over weighted Korobov spaces of increasing smoothness in the worst case setting

Weiran Ding[1], **Jiansong Li**[2,*], **Yumeng Jia**[2] **and Wanru Yang**[2]

[1] School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China
[2] School of Mathematical Sciences, Capital Normal University, Beijing 100048, China

* **Correspondence:** Email: 2210501007@cnu.edu.cn.

Abstract: This paper is devoted to the study of tractability of the L_2-approximation and integration from weighted Korobov spaces of increasing smoothness in the worst-case setting. The considered algorithms use information from the class $\Lambda^{\rm all}$, including all continuous linear functionals, and from the class $\Lambda^{\rm std}$, including function evaluations. Necessary and sufficient conditions on the weights of the function space for strong polynomial tractability, polynomial tractability, quasi-polynomial tractability, uniform weak tractability, weak tractability and (σ,τ)-weak tractability, are provided. Our results give a comprehensive picture of the weight conditions for all standard notions of algebraic tractability. It may be helpful to study the tractability of nonhomogeneous tensor product spaces.

Keywords: L_2-approximation; integration; tractability; Korobov space; increasing smoothness

1. Introduction

Solving multivariate continuous problems is a classical topic in applications, and there are thousands of papers to study this problem, usually for a fixed space of functions. Such problems can almost never be solved analytically. Since they must be solved numerically, they can only be solved approximately to within a threshold ε. To deal with these problems, we often use algorithms based on finitely many information evaluations, either from the class $\Lambda^{\rm all}$ of general linear information consisting of all continuous linear functionals, or from the class $\Lambda^{\rm std}$ of standard information consisting of function evaluations only. This motivates us to study the tractability of multivariate problems referring to $S=\{S_d : F_d \to G_d\}$, where F_d is a Banach space of functions and G_d is another Banach space. For $\varepsilon\in(0,1)$, the information complexity $n^X(\varepsilon, S_d, F_d;\Lambda)$, $X\in\{\text{ABS},\text{NOR}\}$, can be defined as the least number of linear functionals that are sufficient to obtain an ε-approximation for the information class Λ under the absolute error criterion (ABS) and normalized error criterion (NOR). Tractability describes how the information complexity $n^X(\varepsilon, S_d, F_d;\Lambda)$ behaves as a function of d and ε^{-1}, which mainly includes strong polynomial tractability (SPT), polynomial tractability (PT), quasi-polynomial tractability (QPT), uniformly weak tractability (UWT), weak tractability (WT), and (σ,τ)-weak tractability ((σ,τ)-WT).

The two most important and widely studied such problems are multivariate approximation and multivariate integration (see e.g., [1–3]). In many applications, including the solution of important computational problems such as differential and integral equations, and problems in financial mathematics, people are particularly interested in Sobolev spaces. A variant of Sobolev spaces is the Korobov space, which is often called the *Sobolev spaces of dominating mixed smoothness*. Such spaces are probably the most important spaces for the study of computational problems for periodic smooth functions. Moreover, these spaces also have many interesting applications for non-periodic functions due to interesting relations and estimates between the complexity for the periodic and non-periodic computational problems, such as multivariate integration, see e.g., [2]. This paper is devoted to discussing the tractability of multivariate L_2-approximation

$$\mathrm{APP} := \left\{\mathrm{APP}_d : \mathcal{H}(K_R) \to L_2([0,1]^d)\right\}_{d\in\mathbb{N}}$$

with $\mathrm{APP}_d(f) = f$, and multivariate integration

$$\mathrm{INT} := \left\{\mathrm{INT}_d : \mathcal{H}(K_R) \to \mathbb{R}\right\}_{d\in\mathbb{N}}$$

with $\mathrm{INT}_d(f) = \int_{[0,1]^d} f(\mathbf{x})\mathrm{d}\mathbf{x}$, for both Λ^{all} and Λ^{std} in the worst-case setting, where $\mathcal{H}(K_R)$ denotes a weighted Korobov space with increasing smoothness defined over $[0,1]^d$ with Fourier weight $R \in \{r_{d,\alpha,\gamma}, \psi_{d,\alpha,\gamma}, \omega_{d,\alpha,\gamma}\}$ (see Section 2.2 for details). Here, $\boldsymbol{\gamma} = \{(1 \geq)\gamma_1 \geq \gamma_2 \geq \dots\}$ is a positive weight sequence and $\boldsymbol{\alpha} = \{(1 <)\alpha_1 \leq \alpha_2 \leq \dots\}$ is an increasing smoothness sequence (see Section 2 for details). We remark that in our considered case the initial error is 1; therefore, the results under ABS and NOR coincide. The related problem has already been discussed in a large number papers, see e.g., [1,3–5,7]. Particularly, for $R = r_{d,\alpha,\gamma}$ with $1 < \alpha_1 = \alpha_2 = \dots$, the complete picture of multivariate L_2-approximation for $\Lambda \in \{\Lambda^{\text{all}}, \Lambda^{\text{std}}\}$ and multivariate integration was given in [8]. For $R = r_{d,\alpha,\gamma}$ with $1 < \alpha_1 \leq \alpha_2 \leq \dots$, the results on multivariate L_2-approximation for Λ^{all} were given in [9]. For a more general case, $R \in \{\psi_{d,\alpha,\gamma}, \omega_{d,\alpha,\gamma}\}$, some partial results of multivariate L_2-approximation for Λ^{all} were found in [10]. The related results will be formulated in Section 2.3. In this paper, our aim is to provide the conditions that are both sufficient and necessary for all tractability notions of multivariate L_2-approximation and multivariate integration for both information classes Λ^{all} and Λ^{std}.

The remainder of this article is organized as follows. In Section 2 we recall some basic facts on tractability and weighted Korobov spaces, and briefly formulate previous results and the main results of the current paper. In Sections 3 and 4, we discuss the tractability of L_2-approximation and integration in the worst-case setting, respectively.

2. Preliminaries and main results

In this section, we first introduce the fundamental concepts related to tractability in multivariate problems and weighted Korobov spaces, and then we recall the previous results on the tractability of L_2-approximation and integration. Finally, our results are summarized in a table.

2.1. Tractability.

Let $\{F_d\}_{d\in\mathbb{N}}$, $\{G_d\}_{d\in\mathbb{N}}$ be two sequences of Hilbert spaces, and let $\{S_d : F_d \to G_d\}_{d\in\mathbb{N}}$ be a family of continuous linear operators. We will consider two special choices of S, namely:

- L_2-approximation of functions $f \in F_d$. In this case, we have $S_d(f) = f$ and $G_d = L_2([0,1]^d)$.
- Integration of functions $f \in F_d$. In this case, we have $S_d(f) = \int_{[0,1]^d} f(\mathbf{x})\mathrm{d}\mathbf{x}$ and $G_d = \mathbb{R}$.

We approximate S_d by algorithms $A_{n,d}$ of the form

$$A_{n,d}(f) = \sum_{i=1}^{n} L_i(f) g_i, \tag{2.1}$$

where $g_i \in G_d$ and $L_i \in \Lambda$ for $i = 1, \cdots, n$. We will assume that the considered functionals L_i belong to $\Lambda^{\text{all}} = F_d^*$ or Λ^{std}. The *worst-case error for the algorithm* $A_{n,d}$ of the form (2.1) is defined as

$$e(A_{n,d}, S_d, F_d) := \sup_{\|f\|_{F_d}\le 1} \|S_d(f) - A_{n,d}(f)\|_{G_d}.$$

Then, the *n-th minimal worst-case error* is defined by

$$e(n, S_d, F_d; \Lambda) := \inf_{\substack{A_{n,d} \\ L_1,\ldots,L_n\in\Lambda}} e(A_{n,d}, S_d, F_d),$$

where the infimum is taken over all linear algorithms of the form (2.1). Particularly, for $n = 0$, the *initial error* of the problem S_d in the worst-case setting is defined by

$$e(0, S_d, F_d; \Lambda) \equiv e(0, S_d, F_d) = \sup_{\|f\|_{F_d}\le 1} \|S_d(f)\|_{G_d} = \|S_d\|_{F_d\to G_d}.$$

It is interesting to see how the worst-case errors of $A_{n,d}$ depend on the number n and on the dimension d under ABS or NOR. For this purpose, we introduce the so-called *information complexity* as

$$n^X(\varepsilon, S_d, F_d; \Lambda) := \min\left\{n \in \mathbb{N}_0 : e(n, S_d, F_d; \Lambda) \le \varepsilon\,\mathrm{CRI}_d^X\right\}$$

where

$$\mathrm{CRI}_d^X := \begin{cases} 1, & \text{if } X = \text{ ABS}, \\ e(0, S_d, F_d), & \text{if } X = \text{ NOR}. \end{cases}$$

The following notions have been frequently studied. For more about tractability we refer to [1–3] by Novak and Woźniakowski.

Consider the multivariate problem $S := \{S_d\}_{d\in\mathbb{N}}$ using the information class $\Lambda \in \{\Lambda^{\text{all}}, \Lambda^{\text{std}}\}$ under ABS or NOR. We say that S is

1) *polynomially tractable* (PT) if there are $C > 0$ and $\tau, \sigma \ge 0$ satisfying

$$n(\varepsilon, S_d, F_d; \Lambda) \le C\,\varepsilon^{-\tau} d^{\sigma} \text{ for all } d \in \mathbb{N},\ \varepsilon \in (0,1).$$

2) *strong polynomially tractable* (SPT) if there are $C > 0$ and $\tau \ge 0$ satisfying

$$n(\varepsilon, S_d, F_d; \Lambda) \le C\,\varepsilon^{-\tau} \text{ for all } d \in \mathbb{N},\ \varepsilon \in (0,1). \tag{2.2}$$

The infimum of $\tau \ge 0$ for which the inequality (2.2) is satisfied for a certain $C > 0$ is referred to as the *exponent of SPT*, denoted by $\tau^*(\Lambda)$.

3) *quasi-polynomially tractable* (QPT) if there are $t \geq 0$ and $C > 0$ satisfying

$$n(\varepsilon, S_d, F_d; \Lambda) \leq C \exp(t(1 + \ln d)(1 + \ln \varepsilon^{-1})) \text{ for all } d \in \mathbb{N},\ \varepsilon \in (0,1). \tag{2.3}$$

The infimum of $t \geq 0$ for which the inequality (2.3) is satisfied for a certain $C > 0$ is referred to as the *exponent of QPT*, denoted by $\tau^*(\Lambda)$.

4) *weak tractable* (WT) if

$$\lim_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, S_d, F_d; \Lambda)}{d + \varepsilon^{-1}} = 0.$$

5) (σ,τ)-*weak tractable* ((σ,τ)-WT) for $\sigma, \tau > 0$ if

$$\lim_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, S_d, F_d; \Lambda)}{d^\sigma + \varepsilon^{-\tau}} = 0.$$

6) *uniform weak tractable* (UWT) if (σ,τ)-WT holds for all $\sigma, \tau > 0$.

It is evident that the following relationship holds:

$$\text{SPT} \Rightarrow \text{PT} \Rightarrow \text{QPT} \Rightarrow \text{UWT} \Rightarrow (\sigma,\tau)\text{-WT} \quad \text{for all } (\sigma,\tau) \in (0,\infty)^2.$$

Clearly, the notions of WT and $(1,1)$-WT coincide.

2.2. *Weighted Korobov space $\mathcal{H}(K_R)$.*

Now we briefly recall some facts on weighted Korobov spaces of increasing smoothness. *Weighted Korobov spaces* are special types of the reproducing kernel Hilbert spaces $\mathcal{H}(K_R)$ with a reproducing kernel of the form

$$K_R(\mathbf{x}, \mathbf{y}) := \sum_{\mathbf{k}\in\mathbb{Z}^d} R(\mathbf{k}) \exp(2\pi i \mathbf{k} \cdot (\mathbf{x} - \mathbf{y})), \text{ for all } \mathbf{x}, \mathbf{y} \in [0,1]^d,$$

for some summable function $R : \mathbb{Z}^d \to (0, +\infty)$, i.e., $\sum_{\mathbf{k}\in\mathbb{Z}^d} R(\mathbf{k}) < +\infty$, which is often called the *Fourier weight*, and the corresponding inner product

$$\langle f, g\rangle_{\mathcal{H}(K_R)} := \sum_{\mathbf{k}\in\mathbb{Z}^d} \frac{1}{R(\mathbf{k})} \widehat{f}(\mathbf{k}) \overline{\widehat{g}(\mathbf{k})} \quad \text{and} \quad \|f\|_{\mathcal{H}(K_R)} = \sqrt{\langle f, f\rangle_{\mathcal{H}(K_R)}}.$$

Here, the Fourier coefficients $\{\widehat{f}(\mathbf{k})\}_{\mathbf{k}\in\mathbb{Z}^d}$ are given by

$$\widehat{f}(\mathbf{k}) = \int_{[0,1]^d} f(\mathbf{x}) \exp(-2\pi i \mathbf{k} \cdot \mathbf{x}) d\mathbf{x}.$$

We remark that the reproducing kernel K_R is well defined, since for all $\mathbf{x}, \mathbf{y} \in [0,1]^d$,

$$|K_R(\mathbf{x}, \mathbf{y})| \leq \sum_{\mathbf{k}\in\mathbb{Z}^d} R(\mathbf{k}) < +\infty.$$

We will consider three possible Fourier weights. Assume that $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$ is a sequence of so-called *product weights* and $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ is the smoothness parameter sequence satisfying

$$1 < \alpha_1 \leq \alpha_2 \leq \cdots. \tag{2.4}$$

1) *Weighted Korobov space.*
In this case, we consider $R(\mathbf{k})$ as

$$r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\mathbf{k}) = \prod_{j=1}^{d} r_{\alpha_j,\gamma_j}(k_j),$$

where

$$r_{\alpha,\gamma}(k) := \begin{cases} 1, & \text{for } k = 0, \\ \gamma \frac{1}{|k|^\alpha}, & \text{for } k \neq 0, \end{cases}$$

for $\gamma > 0, \alpha > 1$.
Furthermore, when $\alpha_j \in \mathbb{N}$ for all $j \in \mathbb{N}$, we define the norm

$$\|f\|^2_{r_{d,\alpha,\gamma}} := \sum_{\substack{\tau_i \in \{0,\ldots,\alpha_i\} \\ i=1,\ldots,d}} \Big(\prod_{\substack{j=1 \\ \tau_j \neq 0}}^{d} \gamma_j^{-1} \Big) \int_{[0,1]^d} (\partial_{\mathbf{x}}^{\boldsymbol{\tau}} f(\mathbf{x}))^2 \,\mathrm{d}\mathbf{x},$$

where $\partial_{\mathbf{x}}^{\boldsymbol{\tau}} := \partial_{x_1}^{\tau_1} \ldots \partial_{x_d}^{\tau_d}$ for $\boldsymbol{\tau} := (\tau_1, \ldots, \tau_d) \in \mathbb{Z}^d$. It can be checked that the norm $\|f\|_{\mathcal{H}(K_{r_{d,\alpha,\gamma}})}$ is finite if and only if the norm $\|f\|_{r_{d,\alpha,\gamma}}$ is finite with the existence of $\partial_{\mathbf{x}}^{\boldsymbol{\tau}} f$ for all $\boldsymbol{\tau} \leq \boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_d)$. This also indicates a relation between smoothness and $\boldsymbol{\alpha}$.

2) *A first variant of the weighted Korobov space.*
In this case, we consider $R(\mathbf{k})$ as

$$\psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\mathbf{k}) = \prod_{j=1}^{d} \psi_{\alpha_j,\gamma_j}(k_j),$$

where

$$\psi_{\alpha,\gamma}(k) := \begin{cases} 1, & \text{for } k = 0, \\ \gamma \frac{1}{|k|!}, & \text{for } 1 \leq |k| < \alpha, \\ \gamma \frac{(|k| - \lceil \alpha \rceil)!}{|k|!}, & \text{for } |k| \geq \alpha, \end{cases}$$

for $\gamma > 0, \alpha > 1$.

3) *A second variant of the weighted Korobov space.*
In this case, we consider $R(\mathbf{k})$ as

$$\omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\mathbf{k}) = \prod_{j=1}^{d} \omega_{\alpha_j,\gamma_j}(k_j),$$

where

$$\omega_{\alpha,\gamma}(k) := \left(1 + \frac{1}{\gamma} \sum_{s=1}^{\lceil \alpha \rceil} \beta_s(k) \right)^{-1}, \; \beta_s(k) := \begin{cases} \frac{|k|!}{(|k|-s)!}, & \text{if } |k| \geq s, \\ 0, & \text{otherwise}, \end{cases}$$

for $\gamma > 0, \alpha > 1$.

Without loss of generality, we usually assume that the weights $\boldsymbol{\gamma} = \{\gamma_j\}_{j \in \mathbb{N}}$ are ordered by

$$1 \geq \gamma_1 \geq \gamma_2 \geq \cdots > 0. \tag{2.5}$$

Remark 1 ([10]). Let $R_{\alpha_j,\gamma_j} \in \{r_{\alpha_j,\gamma_j}, \psi_{\alpha_j,\gamma_j}, \omega_{\alpha_j,\gamma_j}\}$ for all $j \in \mathbb{N}$. Then

$$R_{\alpha_j,\gamma_j}(0) = 1 \text{ and } R_{\alpha_j,\gamma_j}(1) \geq \frac{\gamma_j}{2}. \tag{2.6}$$

Furthermore, we have the following proposition from [11] (also [10, Remark 4]).

Proposition 1 ([10, 11]). *Let $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ be the smoothness parameter sequence satisfying* (2.4) *and let $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$ be the weight sequence satisfying* (2.5)*. Set $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}} = \{R_{\alpha_j,\gamma_j}\}_{j\geq 1} \in \{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}\}$. Then, for all $j \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have*

$$r_{\alpha_j,\gamma_j}(k) \leq \psi_{\alpha_j,\gamma_j}(k) \leq \lceil \alpha_j \rceil^{\lceil \alpha_j \rceil} r_{\alpha_j,\gamma_j}(k),$$

and

$$\frac{1}{3} r_{\alpha_j,\gamma_j}(k) \leq \omega_{\alpha_j,\gamma_j}(k) \leq \psi_{\alpha_j,\gamma_j}(k) \leq \lceil \alpha_j \rceil^{\lceil \alpha_j \rceil} r_{\alpha_j,\gamma_j}(k). \tag{2.7}$$

Hence, we obtain

$$\frac{1}{3} r_{\alpha_j,\gamma_j}(k) \leq R_{\alpha_j,\gamma_j}(k) \leq \lceil \alpha_1 \rceil^{\lceil \alpha_1 \rceil} r_{\alpha_1,\gamma_j}(k), \tag{2.8}$$

which implies that for all $\mathbf{k} \in \mathbb{Z}^d$,

$$r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}/3}(\mathbf{k}) \leq R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\mathbf{k}) \leq R_{d,\alpha_1,\boldsymbol{\gamma}}(\mathbf{k}) \leq r_{d,\alpha_1,\lceil \alpha_1 \rceil^{\lceil \alpha_1 \rceil}\boldsymbol{\gamma}}(\mathbf{k}), \tag{2.9}$$

and

$$r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\mathbf{k}), \omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\mathbf{k}) \leq \psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\mathbf{k}), \tag{2.10}$$

where $r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}/3} = \prod_{j=1}^d r_{\alpha_j,\gamma_j/3}$ and $r_{d,\alpha_1,\lceil \alpha_1 \rceil^{\lceil \alpha_1 \rceil}\boldsymbol{\gamma}} = \prod_{j=1}^d r_{\alpha_1,\lceil \alpha_1 \rceil^{\lceil \alpha_1 \rceil}\gamma_j}$.

Remark 2. From the definition of $\mathcal{H}(K_R)$, we know that if $R_1, R_2 : \mathbb{Z}^d \to (0, +\infty)$ are two Fourier weights for weighted Korobov spaces $\mathcal{H}(K_{R_1})$ and $\mathcal{H}(K_{R_2})$ with

$$R_1(\mathbf{k}) \leq C R_2(\mathbf{k}), \text{ for all } \mathbf{k} \in \mathbb{Z}^d,$$

then

$$C\|f\|_{\mathcal{H}(K_{R_1})} \geq \|f\|_{\mathcal{H}(K_{R_2})}, \text{ for all } f \in \mathcal{H}(K_{R_2}).$$

That is, $\mathcal{H}(K_{R_1})$ is continuously embedded into $\mathcal{H}(K_{R_2})$, denoted by $\mathcal{H}(K_{R_1}) \hookrightarrow \mathcal{H}(K_{R_2})$. Meanwhile, by (2.7), we have

$$r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\mathbf{k}) \leq \psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\mathbf{k}) \leq \left(\prod_{j=1}^d \lceil \alpha_j \rceil^{\lceil \alpha_j \rceil} \right) r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\mathbf{k}),$$

and

$$r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}/3}(\mathbf{k}) \leq \omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\mathbf{k}) \leq \left(\prod_{j=1}^d \lceil \alpha_j \rceil^{\lceil \alpha_j \rceil} \right) r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\mathbf{k}),$$

which implies that

$$\mathcal{H}(K_{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}) \hookrightarrow \mathcal{H}(K_{\psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}) \hookrightarrow \mathcal{H}(K_{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}),$$

and

$$\mathcal{H}(K_{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}/3}}) \hookrightarrow \mathcal{H}(K_{\omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}) \hookrightarrow \mathcal{H}(K_{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}).$$

These also indicate a relation between smoothness and $\boldsymbol{\alpha}$.

2.3. Results.

Let $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$ be a sequence of product weights and let $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ be the smoothness parameter sequence satisfying $1 < \alpha_1 \le \alpha_2 \le \cdots$. For the weight space $\mathcal{H}(K_{r_{d,\alpha,\gamma}})$ with $1 < \alpha_1 = \alpha_2 = \ldots$, the complete picture of tractability of L_2-approximation APP for both Λ^{all} and Λ^{std}, and integration INT, has been obtained in [12]. Besides, [5, 10] obtained some partial results of weighted Korobov space $\mathcal{H}(K_R)$ with $R \in \{r_{d,\alpha,\gamma}, \psi_{d,\alpha,\gamma}, \omega_{d,\alpha,\gamma}\}$. The characterization of tractability will be given in terms of decay conditions on the weight sequence $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$. Before presenting our findings, we will first outline some key notations. (We use the convention that $\inf \emptyset = \infty$.)

- The infimum of the sequence $\boldsymbol{\gamma}$ is denoted by

$$\boldsymbol{\gamma}_I := \inf_{j\ge 1} \gamma_j.$$

- The infimum of the sequence $(\ln \gamma_j^{-1} / \ln j)_{j\in\mathbb{N}}$ is denoted by

$$\delta_\gamma := \liminf_{j\to\infty} \frac{\ln \gamma_j^{-1}}{\ln j}.$$

- The sum exponent s_γ is defined as

$$s_\gamma := \inf\left\{\kappa > 0 : \sum_{j=1}^{\infty} \gamma_j^{\kappa} < \infty\right\}.$$

- The exponent t_γ is defined as

$$t_\gamma := \inf\left\{\kappa > 0 : \limsup_{d\to\infty} \frac{1}{\ln d} \sum_{j=1}^{d} \gamma_j^{\kappa} < \infty\right\}.$$

We summarize the above results in the following two tables (Tables 1 and 2).

Table 1. Tractability of L_2-approximation for $1 \ge \gamma_1 \ge \gamma_2 \ge \cdots > 0$ and $1 < \alpha = \alpha_1 = \alpha_2 = \cdots$ under ABS and NOR by the information class Λ^{all}.

	$r_{d,\alpha,\gamma}$	$\psi_{d,\alpha,\gamma}$ and $\omega_{d,\alpha,\gamma}$
SPT	$s_\gamma < \infty$	$s_\gamma < \infty$
PT	$s_\gamma < \infty$	$s_\gamma < \infty$
QPT	$\boldsymbol{\gamma}_I < 1$	suff.: $\boldsymbol{\gamma}_I < 1$
UWT	$\boldsymbol{\gamma}_I < 1$	?
(σ,τ)-WT, $\sigma \in (0,1]$	$\boldsymbol{\gamma}_I < 1$	?
WT	$\boldsymbol{\gamma}_I < 1$	?
(σ,τ)-WT, $\sigma > 1$	no extra condition on $\boldsymbol{\gamma}$	no extra condition on $\boldsymbol{\gamma}$

Table 2. Tractability of L_2-approximation for $1 \geq \gamma_1 \geq \gamma_2 \geq \cdots > 0$ and $1 < \alpha_1 \leq \alpha_2 \leq \cdots$ under ABS and NOR by the information class Λ^{all}.

	$r_{d,\alpha,\gamma}$	$\psi_{d,\alpha,\gamma}$ and $\omega_{d,\alpha,\gamma}$
SPT	$\delta_\gamma > 0$	$\delta_\gamma > 0$
PT	$\delta_\gamma > 0$	$\delta_\gamma > 0$
QPT	$\boldsymbol{\gamma}_I < 1$	?
UWT	$\boldsymbol{\gamma}_I < 1$	?
(σ, τ)-WT, $\sigma \in (0, 1]$	$\boldsymbol{\gamma}_I < 1$	?
WT	$\boldsymbol{\gamma}_I < 1$	?
(σ, τ)-WT, $\sigma > 1$	no extra condition on $\boldsymbol{\gamma}$	no extra condition on $\boldsymbol{\gamma}$

Remark 3. We remark that the condition $\delta_\gamma > 0$ is equivalent to the condition $s_\gamma < \infty$ which can be seen from the condition $1 < \alpha = \alpha_1 = \alpha_2 = \cdots$ in Tables 1 and 2.

In this paper, we outline the conditions necessary for the tractability of L_2-approximation in $\mathcal{H}(K_R)$ for both Λ^{all} and Λ^{std}, as demonstrated in Theorems 1 and 2. The results of our findings are presented in Table 3.

Table 3. Tractability of L_2-approximation for $1 \geq \gamma_1 \geq \gamma_2 \geq \cdots > 0$ and $1 < \alpha_1 \leq \alpha_2 \leq \cdots$ under ABS and NOR. (To all three weights.)

	Λ^{all}	Λ^{std}
SPT	$s_\gamma < \infty$	$\sum_{j=1}^{\infty} \gamma_j < \infty$
PT	$s_\gamma < \infty$	$\limsup_{d\to\infty} \frac{1}{\ln d} \sum_{j=1}^{d} \gamma_j < \infty$
QPT	$\boldsymbol{\gamma}_I < 1$	$\limsup_{d\to\infty} \frac{1}{\ln d} \sum_{j=1}^{d} \gamma_j < \infty$
UWT	$\boldsymbol{\gamma}_I < 1$	$\lim_{d\to\infty} \frac{1}{d^\sigma} \sum_{j=1}^{d} \gamma_j = 0 \ \forall \sigma \in (0, 1]$
(σ, τ)-WT for $\sigma \in (0, 1]$	$\boldsymbol{\gamma}_I < 1$	$\lim_{d\to\infty} \frac{1}{d^\sigma} \sum_{j=1}^{d} \gamma_j = 0$
WT	$\boldsymbol{\gamma}_I < 1$	$\lim_{d\to\infty} \frac{1}{d} \sum_{j=1}^{d} \gamma_j = 0$
(σ, τ)-WT for $\sigma > 1$	no extra condition on $\boldsymbol{\gamma}$	no extra condition on $\boldsymbol{\gamma}$

This paper also investigates the tractability of integration INT. We obtain the sufficient and necessary conditions on all tractability notions, which are the same as for the tractability of the L_2-approximation APP for the class Λ^{all}.

3. L_2-approximation in $\mathcal{H}(K_R)$.

This section focuses on the tractability of L_2-approximation on the weighted Korobov space $\mathcal{H}(K_R)$. We will study this problem for the classes Λ^{all} and Λ^{std}, respectively. More precisely, we want to approximate the embedding operators

$$\mathrm{APP}_d : \mathcal{H}(K_R) \to L_2([0, 1]^d), \ \mathrm{APP}_d(f) = f,$$

in the worst-case setting. It can be seen that APP_d is the compact embedding from the weighted space $\mathcal{H}(K_R)$ to $L_2([0,1]^d)$. To approximate APP_d with respect to the L_2-norm $\|\cdot\|_{L_2}$ over $[0,1]^d$, it follows from [1, 13] that it suffices to use linear algorithms $A^{\mathrm{app}}_{n,d}$ of the form

$$A^{\mathrm{app}}_{n,d}(f) = \sum_{i=1}^{n} L_i(f)\, g_i \tag{3.1}$$

with $g_i \in L_2([0,1]^d)$ and $L_i \in \{\Lambda^{\mathrm{all}}, \Lambda^{\mathrm{std}}\}$ for $i = 1, \ldots, n$. Remember that $\mathcal{H}(K_R)$ is a reproducing kernel Hilbert space and $\Lambda^{\mathrm{std}} \subseteq \Lambda^{\mathrm{all}}$.

The n-th minimal worst-case error with respect to the information class Λ is given by

$$e(n, \mathrm{APP}_d, \mathcal{H}(K_R); \Lambda) := \inf_{\substack{A^{\mathrm{app}}_{n,d} \\ L_1,\ldots,L_n \in \Lambda}} \sup_{\|f\|_{\mathcal{H}(K_R)} \le 1} \|f - A^{\mathrm{app}}_{n,d}(f)\|_{L_2},$$

where the infimum is extended over all algorithms of the form (3.1) with information from Λ. For $n = 0$, the initial approximation error is given by

$$e(0, \mathrm{APP}_d, \mathcal{H}(K_R)) := \sup_{\|f\|_{\mathcal{H}(K_R)} \le 1} |\mathrm{APP}_d(f)| = \|\mathrm{APP}_d\|.$$

In the following, we assume that $0 < R(\mathbf{k}) \le 1$ for all $\mathbf{k} \in \mathbb{Z}^d$, which implies that the norm of APP_d is 1, since for all $f \in \mathcal{H}(K_R)$,

$$\|\mathrm{APP}_d(f)\|^2_{L_2} = \|f\|^2_{L_2} = \sum_{\mathbf{k} \in \mathbb{Z}^d} |\widehat{f}(\mathbf{k})|^2 \le \sum_{\mathbf{k} \in \mathbb{Z}^d} \frac{1}{R(\mathbf{k})} |\widehat{f}(\mathbf{k})|^2 = \|f\|^2_{\mathcal{H}(K_R)} < +\infty,$$

and the above inequality is sharp if $f \equiv 1$. In the remaining part of this paper, we always assume that $0 < R(\mathbf{k}) \le 1$, which implies that there is no need to distinguish between ABS and NOR. For abbreviation, we write $n(\varepsilon, \mathrm{APP}_d, \mathcal{H}(K_R); \Lambda) \equiv n^X(\varepsilon, \mathrm{APP}_d, \mathcal{H}(K_R); \Lambda)$.

3.1. The information class Λ^{all}.

First we present the results on L_2-approximation in the weighted Korobov space $\mathcal{H}(K_R)$ for the information class Λ^{all}.

Theorem 1. *Let $\boldsymbol{\alpha} = \{\alpha_j\}_{j \in \mathbb{N}}$ be the smoothness parameter sequence satisfying* (2.4) *and let $\boldsymbol{\gamma} = \{\gamma_j\}_{j \in \mathbb{N}}$ be the weight sequence satisfying* (2.5). *Consider the L_2-approximation problem* $\mathrm{APP} = \{\mathrm{APP}_d\}_{d \in \mathbb{N}}$ *for the weighted spaces $\mathcal{H}(K_R)$, $R \in \{r_{d,\alpha,\gamma}, \psi_{d,\alpha,\gamma}, \omega_{d,\alpha,\gamma}\}$ for the information class Λ^{all} under ABS and NOR. We have the following conditions:*

(i) (Cf. [10]) SPT and PT are equivalent and hold iff $s_\gamma < \infty$. In this case, the exponent of SPT is

$$\tau^*(\Lambda^{\mathrm{all}}) = 2 \max\left(\frac{1}{\alpha_1}, s_\gamma\right).$$

(ii) QPT, UWT, WT, and (σ,τ)-WT with $\sigma \in (0,1]$, are are equivalent and hold iff $\boldsymbol{\gamma}_I < 1$. In this case, the exponent of QPT satisfies

$$t^*(\Lambda^{\mathrm{all}}) \le 2 \max\left(\frac{1}{\alpha_1}, \frac{1}{\ln \boldsymbol{\gamma}_I^{-1}}\right),$$

and the equality holds for $R \in \{r_{d,\alpha,\boldsymbol{\gamma}}, \psi_{d,\alpha,\boldsymbol{\gamma}}\}$. *(In particular, if* $\boldsymbol{\gamma}_I = 0$, *then we set* $(\ln \boldsymbol{\gamma}_I^{-1})^{-1} := 0$ *and we have that* $t^*(\Lambda^{\text{all}}) = 2/\alpha_1$.*)*

(iii) (Cf. [10]) For $\sigma > 1$, (σ, τ)*-WT always holds.*

To obtain the above theorem, we recall some basic knowledge about L_2-approximation for the information class Λ^{all}. It is well known, see e.g., [1], that the n-th minimal worst-case errors $e(n, \text{APP}_d, \mathcal{H}(K_R); \Lambda^{\text{all}})$ and the information complexity $n(\varepsilon, \text{APP}_d, \mathcal{H}(K_R); \Lambda^{\text{all}})$ depend on the eigenvalues of the continuous linear operator

$$W_d = \text{APP}_d^* \text{APP}_d : \mathcal{H}(K_R) \to \mathcal{H}(K_R).$$

Let $(\lambda_{d,j}, \boldsymbol{\eta}_{d,j})_{j\in\mathbb{N}}$ be the eigenpairs of W_d, i.e.,

$$W_d \boldsymbol{\eta}_{d,j} = \lambda_{d,j} \boldsymbol{\eta}_{d,j}, \text{ for all } j \in \mathbb{N},$$

where the eigenvalues $(\lambda_{d,j})$ are ordered by $\lambda_{d,1} \geq \lambda_{d,2} \geq \cdots \geq 0$, and the eigenvectors $(\boldsymbol{\eta}_{d,j})_{j\in\mathbb{N}}$ are orthonormal,

$$\langle \boldsymbol{\eta}_{d,i}, \boldsymbol{\eta}_{d,j} \rangle_{\mathcal{H}(K_R)} = \delta_{i,j}, \text{ for all } i, j \in \mathbb{N}.$$

Then, the n-th minimal error is obtained (see [1, Corollary 4.12]) for the algorithm

$$A_{n,d}^* f = \sum_{j=1}^{n} \langle f, \boldsymbol{\eta}_{d,j} \rangle_{\mathcal{H}(K_R)} \boldsymbol{\eta}_{d,j}, \text{ for all } f \in \mathcal{H}(K_R),$$

and

$$e(n, \text{APP}_d, \mathcal{H}(K_R); \Lambda^{\text{all}}) = e(A_{n,d}^*, \mathcal{H}(K_R), \Lambda^{\text{all}}) = \lambda_{d,n+1}^{1/2}. \tag{3.2}$$

Hence, the information complexity is equal to

$$n(\varepsilon, \text{APP}_d, \mathcal{H}(K_R); \Lambda^{\text{all}}) = \left| \left\{ n \in \mathbb{N}_0 : \lambda_{d,n} > \varepsilon^2 \right\} \right|,$$

with $\varepsilon \in (0, 1)$ and $d \in \mathbb{N}$. Since it follows from [1] that the eigenpairs of the operator W_d are $(R(\mathbf{k}), e_{\mathbf{k}})$ with $\mathbf{k} \in \mathbb{Z}^d$, where

$$e_{\mathbf{k}}(\mathbf{x}) := \sqrt{R(\mathbf{k})} \exp(2\pi \mathtt{i} \mathbf{k} \cdot \mathbf{x}),$$

we have

$$n(\varepsilon, \text{APP}_d, \mathcal{H}(K_R); \Lambda^{\text{all}}) = \left| \left\{ \mathbf{k} \in \mathbb{Z}^d : R(\mathbf{k}) > \varepsilon^2 \right\} \right|. \tag{3.3}$$

Now we begin to prove Theorem 1.

Proof of Theorem 1. (i) and (iii) have been proved in [10]. We include the proof of item (i) and (ii) in a different way as a warm-up.

(i) From [1, Theorem 5.2], we know that APP is PT for Λ^{all} if and only if there exist $q \geq 0$ and $\tau > 0$ such that

$$C_{\tau,q} := \sup_{d\in\mathbb{N}} \left(\sum_{j=1}^{\infty} \lambda_{d,j}^{\tau} \right)^{1/\tau} d^{-q} < \infty, \tag{3.4}$$

and $\tau^*(\Lambda^{\text{all}}) = \inf\{2\tau : \tau \text{ satisfies (3.4)}\}$.

Since SPT implies PT, it suffices to demonstrate the sufficiency of $s_{\boldsymbol{\gamma}} < \infty$ for SPT, and the necessary of $s_{\boldsymbol{\gamma}} < \infty$ for PT.

- *Sufficiency of $s_\gamma < \infty$ for SPT.*
Now we assume that $s_\gamma < \infty$. Then, for any $\tau > \max(s_\gamma, \frac{1}{\alpha_1})$, we have $\sum_{j=1}^{\infty} \gamma_j^\tau < \infty$. Using (2.10) and (2.8), we obtain

$$\begin{aligned}
\sum_{j=1}^{\infty} \lambda_{d,j}^\tau &= \prod_{j=1}^{d} \left(1 + 2\sum_{k=1}^{\infty} (R_{\alpha_j,\gamma_j}(k))^\tau\right) \\
&\le \prod_{j=1}^{d} \left(1 + 2\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\tau} \sum_{k=1}^{\infty} (r_{\alpha_j,\gamma_j}(k))^\tau\right) \\
&\le \prod_{j=1}^{d} \left(1 + 2\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\tau} \gamma_j^\tau \zeta(\alpha_j\tau)\right) \\
&= \exp\left\{\sum_{j=1}^{d} \ln\left(1 + 2\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\tau} \gamma_j^\tau \zeta(\alpha_j\tau)\right)\right\} \\
&\le \exp\left\{2\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\tau} \sum_{j=1}^{d} \zeta(\alpha_j\tau)\gamma_j^\tau\right\} \\
&\le \exp\left\{2\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\tau} \zeta(\alpha_1\tau) \sum_{j=1}^{\infty} \gamma_j^\tau\right\} < \infty,
\end{aligned}$$

where we also used that $\ln(1+x) \le x$ for $x \ge 0$ and $\zeta(\alpha_1\tau) < \infty$ for $\tau > 1/\alpha_1$. This leads to $C_{\tau,q} < \infty$ for all $q \ge 0$, thus we have SPT, PT and

$$\tau^*(\Lambda^{\text{all}}) \le 2\max\left(s_\gamma, \tfrac{1}{\alpha_1}\right). \tag{3.5}$$

- *Necessity of $s_\gamma < \infty$ for PT.*
Now we assume that PT holds. Then there exists $\tau > 0$ such that (3.4) holds with $q = 0$. Clearly, we need $\tau > 1/\alpha_1$. Again using (2.8) and $\zeta(\alpha_j\tau) > 1$ for all j, we obtain

$$\begin{aligned}
\sum_{j=1}^{\infty} \lambda_{d,j}^\tau = \sum_{\mathbf{k}\in\mathbb{Z}^d} (R_{d,\alpha,\gamma}(\mathbf{k}))^\tau &= \prod_{j=1}^{d} \left(1 + 2\sum_{k=1}^{\infty} (R_{\alpha_j,\gamma_j}(k))^\tau\right) \\
&\ge \prod_{j=1}^{d} \left(1 + \frac{2}{3^\tau}\sum_{k=1}^{\infty} (r_{\alpha_j,\gamma_j}(k))^\tau\right) \\
&= \prod_{j=1}^{d} \left(1 + \frac{2}{3^\tau}\gamma_j^\tau \zeta(\alpha_j\tau)\right) \ge \frac{1}{3^\tau}\sum_{j=1}^{d} \gamma_j^\tau.
\end{aligned}$$

Meanwhile, (3.4) implies that $\sum_{j=1}^{\infty} \gamma_j^\tau < \infty$ and hence $s_\gamma \le \tau < \infty$. Combining both results yields that $\tau \ge \max(s_\gamma, \frac{1}{\alpha_1})$ and hence also

$$\tau^*(\Lambda^{\text{all}}) \ge 2\max\left(s_\gamma, \tfrac{1}{\alpha_1}\right). \tag{3.6}$$

Then (3.5) and (3.6) imply $\tau^*(\Lambda^{\text{all}}) = 2\max(s_\gamma, \frac{1}{\alpha_1})$.

(ii) From [3, Theorem 23.2] (see also [14]), we know that APP is QPT if and only if there exists a $\tau > 0$ such that

$$C_\tau := \sup_{d\in\mathbb{N}} C_{\tau,d} := \sup_{d\in\mathbb{N}} \frac{1}{d^2}\left(\sum_{j=1}^{\infty}\lambda_{d,j}^{\tau(1+\ln d)}\right)^{1/\tau} < \infty. \tag{3.7}$$

First, assume that $\boldsymbol{\gamma}_I < 1$. For $R \in \{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}\}$, we have, by (2.7) and similar to the proof of [11, Theorem 14], that

$$\begin{aligned}
\sum_{j=1}^{\infty}\lambda_{d,j}^{\tau(1+\ln d)} &= \sum_{\mathbf{k}\in\mathbb{Z}^d}(R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\mathbf{k}))^{\tau(1+\ln d)} \\
&\le \sum_{\mathbf{k}\in\mathbb{Z}^d}(\psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\mathbf{k}))^{\tau(1+\ln d)} \\
&= \prod_{j=1}^{d}\left(1 + 2\sum_{k=1}^{\infty}(\psi_{\alpha_j,\gamma_j}(k))^{\tau(1+\ln d)}\right) \\
&= \prod_{j=1}^{d}\left(1 + 2\gamma_j^{\tau(1+\ln d)}\left(\sum_{k=1}^{\lceil\alpha_j\rceil-1}\left(\frac{1}{k!}\right)^{\tau(1+\ln d)} + \sum_{k=\lceil\alpha_j\rceil}^{\infty}\left(\frac{(k-\lceil\alpha_j\rceil)!}{k!}\right)^{\tau(1+\ln d)}\right)\right) \\
&\le \prod_{j=1}^{d}\left(1 + 2\gamma_j^{\tau(1+\ln d)}\left(\sum_{k=1}^{\lceil\alpha_j\rceil-1}\left(\frac{1}{k!}\right)^{\tau(1+\ln d)} + \sum_{k=\lceil\alpha_j\rceil}^{\infty}\left(\frac{1}{(k-\lceil\alpha_j\rceil+1)^{\alpha_j}}\right)^{\tau(1+\ln d)}\right)\right) \\
&= \prod_{j=1}^{d}\left(1 + 2\gamma_j^{\tau(1+\ln d)}\left(\sum_{k=1}^{\lceil\alpha_j\rceil-1}\left(\frac{1}{k!}\right)^{\tau(1+\ln d)} + \sum_{k=1}^{\infty}\left(\frac{1}{k^{\alpha_j}}\right)^{\tau(1+\ln d)}\right)\right) \\
&= \prod_{j=1}^{d}\left(1 + 2\gamma_j^{\tau(1+\ln d)}\left(\sum_{k=1}^{\lceil\alpha_j\rceil-1}\left(\frac{1}{k!}\right)^{\tau(1+\ln d)} + \zeta(\alpha_j\tau(1+\ln d))\right)\right) \\
&\le \prod_{j=1}^{d}\left(1 + 2\gamma_j^{\tau(1+\ln d)}\left(\sum_{k=1}^{\infty}\left(\frac{1}{k!}\right)^{\tau(1+\ln d)} + \zeta(\alpha_1\tau(1+\ln d))\right)\right).
\end{aligned}$$

In order that $\zeta_d := \zeta(\alpha_1\tau(1+\ln d)) < \infty$ for all $d \in \mathbb{N}$, we need $\tau > 1/\alpha_1$. Next, we have

$$\begin{aligned}
\sum_{k=1}^{\infty}\left(\frac{1}{k!}\right)^{\tau(1+\ln d)} &= 1 + \frac{1}{2^{\tau(1+\ln d)}} + \frac{1}{6^{\tau(1+\ln d)}} + \sum_{k=4}^{\infty}\left(\frac{1}{k!}\right)^{\tau(1+\ln d)} \\
&\le 1 + \frac{1}{2^{\tau(1+\ln d)}} + \frac{1}{6^{\tau(1+\ln d)}} + \sum_{k=4}^{\infty}\left(\frac{1}{2^{\tau(1+\ln d)}}\right)^{k} \\
&= 1 + \frac{1}{2^{\tau(1+\ln d)}} + \frac{1}{6^{\tau(1+\ln d)}} + \frac{1}{2^{4\tau(1+\ln d)}}\frac{1}{1-1/2^{\tau(1+\ln d)}} \\
&= 1 + \frac{1}{2^{\tau(1+\ln d)}} + \frac{1}{6^{\tau(1+\ln d)}} + \frac{1}{2^{4\tau(1+\ln d)} - 2^{3\tau(1+\ln d)}} \\
&\le 1 + \frac{1}{2^{\tau(1+\ln d)}} + \frac{1}{6^{\tau(1+\ln d)}} + \frac{1}{2^{3\tau\ln d}}\ \frac{1}{2^{4\tau} - 2^{3\tau}} \\
&\le 1 + \frac{1}{2^{\tau\ln d}}\left(2 + \frac{1}{2^{4\tau} - 2^{3\tau}}\right) = 1 + \frac{c_\tau}{d^{\tau\ln 2}},
\end{aligned}$$

where $c_\tau := 2 + 1/(2^{4\tau} - 2^{3\tau})$. This gives

$$C_{\tau,d} \le \frac{1}{d^2}\left\{\prod_{j=1}^{d}\left(1 + 2\gamma_j^{\tau(1+\ln d)}\left(\frac{c_\tau}{d^{\tau\ln 2}} + \zeta_d\right)\right)\right\}^{1/\tau}$$
$$= \exp\left\{\frac{1}{\tau}\sum_{j=1}^{d}\ln\left(1 + 2\gamma_j^{\tau(1+\ln d)}\left(\zeta_d + \frac{c_\tau}{d^{\tau\ln 2}}\right)\right) - 2\ln d\right\}$$
$$\le \exp\left\{\frac{2}{\tau}\left(\zeta_d + \frac{c_\tau}{d^{\tau\ln 2}}\right)\sum_{j=1}^{d}\gamma_j^{\tau(1+\ln d)} - 2\ln d\right\},$$

where we used that $\ln(1+x) \le x$ for all $x \ge 0$. Using the inequality

$$\zeta(x) \le 1 + \frac{1}{x-1}, \text{ for all } x > 1,$$

we have

$$\zeta_d \le 1 + \frac{1}{(\alpha_1\tau - 1) + \alpha_1\tau\ln d}.$$

Then we obtain

$$C_{\tau,d} \le \exp\left\{\frac{2}{\tau}\left(1 + \frac{1}{(\alpha_1\tau - 1) + \alpha_1\tau\ln d} + \frac{c_\tau}{d^{\tau\ln 2}}\right)\sum_{j=1}^{d}\gamma_j^{\tau(1+\ln d)} - 2\ln d\right\}.$$

Now we distinguish two cases:

- Case 1: $\boldsymbol{\gamma}_I = 0$. In this case, we have $\lim_{j\to\infty}\gamma_j = 0$ and hence, there is a $J \in \mathbb{N}$ such that $\gamma_j \le e^{-1/\tau}$ for all $j \ge J$. Then

$$\sum_{j=1}^{d}\gamma_j^{\tau(1+\ln d)} \le \sum_{j=1}^{J-1}1 + \sum_{j=J}^{d}e^{-\ln d} = J.$$

  Note that J depends on τ, and it is finite for any fixed τ. Thus, if $\tau > 1/\alpha_1$ and $\boldsymbol{\gamma}_I = 0$ then

$$C_{\tau,d} \quad \le \exp\left\{\frac{2}{\tau}\left(1 + \frac{1}{(\alpha_1\tau - 1) + \alpha_1\tau\ln d} + \frac{c_\tau}{d^{\tau\ln 2}}\right)J - 2\ln d\right\} \to 0 \text{ if } d \to \infty.$$

  By the characterization in (3.7), this implies QPT.
- Case 2: $\boldsymbol{\gamma}_I \in (0,1)$. In this case, for any $\gamma_* \in (\boldsymbol{\gamma}_I, 1)$, there exists a $j_0 \in \mathbb{N}$ such that $\gamma_j \le \gamma_* < 1$ for all $j > j_0$. Then for every $d \in \mathbb{N}$,

$$\sum_{j=1}^{d}\gamma_j^{\tau(1+\ln d)} \le j_0 + \gamma_*^{\tau(1+\ln d)}\max(d - j_0, 0)$$
$$= j_0 + \frac{\gamma_*^\tau \max(d - j_0, 0)}{d^{\tau\ln\gamma_*^{-1}}}$$
$$\le j_0 + 1,$$

whenever $\tau \geq (\ln \gamma_*^{-1})^{-1}$. Thus, if $\tau > 1/\alpha_1$ and $\tau \geq (\ln \gamma_*^{-1})^{-1}$, then

$$C_{\tau,d} \leq \exp\left\{\frac{2}{\tau}\left(1 + \frac{1}{(\alpha_1\tau - 1) + \alpha_1\tau \ln d} + \frac{c_\tau}{d^{\tau \ln 2}}\right)(j_0 + 1) - 2\ln d\right\}$$
$$\to 0, \quad \text{if } d \to \infty.$$

Again, by the characterization in (3.7), this implies QPT.

Clearly, QPT implies UWT, WT, and (σ, τ)-WT with $\sigma \in (0, 1]$. So it remains to prove that WT implies $\boldsymbol{\gamma}_I < 1$. To this end, we let $\boldsymbol{\gamma}_I = 1$, i.e., $\gamma_j \equiv 1$. Then we derive from (2.6) that for $\mathbf{k} \in \{0, 1\}^d$,

$$R_{d,\alpha,\boldsymbol{\gamma}}(\mathbf{k}) = \prod_{j=1}^{d} R_{\alpha_j,\gamma_j}(k_j) \geq \prod_{j=1}^{d}\left(\frac{\gamma_j}{2}\right) =: \eta,$$

which, by (3.3), implies that for any $\varepsilon \in (0, \sqrt{\eta})$,

$$n(\varepsilon, \mathrm{APP}_d, \mathcal{H}(K_R); \Lambda^{\mathrm{all}}) \geq \left|\left\{\mathbf{k} \in \{0, 1\}^d : R_{d,\alpha,\boldsymbol{\gamma}}(\mathbf{k}) > \varepsilon^2\right\}\right| = 2^d.$$

This means that APP suffers from the curse of dimensionality, and thus, we cannot have (σ, τ)-WT for any $\sigma \in (0, 1]$. This deduces that QPT, UWT, WT, and (σ, τ)-WT with $\sigma \in (0, 1]$ are equivalent, and hold iff $\boldsymbol{\gamma}_I < 1$.

Now we turn to calculate the exponent of QPT. Again from [3, Theorem 23.2], we know that the exponent of QPT is

$$t^*(\Lambda^{\mathrm{all}}) = \inf\{2\tau : \tau \text{ satisfies (3.7)}\}.$$

From the above part of the proof, it follows that τ satisfies (3.7) as long as $\tau > 1/\alpha_1$ and $\tau > (\ln \boldsymbol{\gamma}_I^{-1})^{-1}$, where we put $(\ln \boldsymbol{\gamma}_I^{-1})^{-1} := 0$ whenever $\boldsymbol{\gamma}_I = 0$. Therefore,

$$t^*(\Lambda^{\mathrm{all}}) \leq 2\max\left(\frac{1}{\alpha_1}, \frac{1}{\ln \boldsymbol{\gamma}_I^{-1}}\right).$$

Assume now that we have QPT for $R \in \{r_{d,\alpha,\boldsymbol{\gamma}}, \psi_{d,\alpha,\boldsymbol{\gamma}}\}$. Then, (3.7) holds true for some $\tau > 0$. Consideration of the special case $d = 1$ together with (3.7) yields that

$$\begin{aligned} C_\tau \geq \left(\sum_{j=1}^{\infty} \lambda_{1,j}^\tau\right)^{1/\tau} &= \left(1 + 2\sum_{k=1}^{\infty} R_{\alpha_1,\gamma_1}^\tau(k)\right)^{1/\tau} \\ &\geq \left(1 + 2\sum_{k=1}^{\infty} r_{\alpha_1,\gamma_1}^\tau(k)\right)^{1/\tau} \\ &= (1 + 2\zeta(\alpha_1\tau)\gamma_1^\tau)^{1/\tau}, \end{aligned}$$

where we used (2.10), and hence we must have $\tau > 1/\alpha_1$. This already implies the result $t^*(\Lambda^{\mathrm{all}}) = 2/\alpha_1$ whenever $\boldsymbol{\gamma}_I = 0$.

It remains to study the case $\boldsymbol{\gamma}_I > 0$. Now, again according to (3.7) and (2.10), there exists a $\tau > 1/\alpha_1$ such that for all $d \in \mathbb{N}$ we have

$$C_\tau \geq \frac{1}{d^2}\left\{\prod_{j=1}^{d}\left(1 + \sum_{k=1}^{\infty}(R_{\alpha_j,\gamma_j}(k))^{\tau(1+\ln d)}\right)\right\}^{1/\tau}$$

$$
\begin{aligned}
&\geq \frac{1}{d^2}\left\{\prod_{j=1}^{d}\left(1+\sum_{k=1}^{\infty}(r_{\alpha_j,\gamma_j}(k))^{\tau(1+\ln d)}\right)\right\}^{1/\tau} \\
&= \frac{1}{d^2}\left\{\prod_{j=1}^{d}\left(1+\gamma_j^{\tau(1+\ln d)}\zeta(\alpha_j\tau(1+\ln d))\right)\right\}^{1/\tau} \\
&\geq \frac{1}{d^2}\left\{\prod_{j=1}^{d}\left(1+\gamma_j^{\tau(1+\ln d)}\right)\right\}^{1/\tau} \\
&= \exp\left\{\frac{1}{\tau}\sum_{j=1}^{d}\ln\left(1+\gamma_j^{\tau(1+\ln d)}\right)-2\ln d\right\}.
\end{aligned}
$$

Taking the logarithm leads to

$$
\begin{aligned}
\ln C_\tau &\geq \frac{1}{\tau}\sum_{j=1}^{d}\ln\left(1+\gamma_j^{\tau(1+\ln d)}\right)-2\ln d \\
&\geq \frac{d}{\tau}\ln\left(1+\boldsymbol{\gamma}_I^{\tau(1+\ln d)}\right)-2\ln d
\end{aligned}
$$

for all $d \in \mathbb{N}$. Since $\boldsymbol{\gamma}_I \in (0,1)$ and since $\ln(1+x) \geq x\ln 2$ for all $x \in [0,1]$, it follows that for all $d \in \mathbb{N}$ we have

$$
\begin{aligned}
\ln C_\tau &\geq \frac{d\ln 2}{\tau}\boldsymbol{\gamma}_I^{\tau(1+\ln d)}-2\ln d \\
&= \frac{\boldsymbol{\gamma}_I^{\tau}\ln 2}{\tau}d^{1-\tau\ln\boldsymbol{\gamma}_I^{-1}}-2\ln d,
\end{aligned}
$$

which implies that $\tau \geq (\ln\boldsymbol{\gamma}_I^{-1})^{-1}$. Therefore, we will get

$$
t^*(\Lambda^{\mathrm{all}}) \geq 2\max\left(\frac{1}{\alpha_1},\frac{1}{\ln\boldsymbol{\gamma}_I^{-1}}\right)
$$

and the claimed result follows.

(iii) The result for (σ,τ)-weak tractability for $\sigma > 1$ for the class Λ^{all} follows from the corresponding result for the class Λ^{std} from Theorem 2.

The proof is completed. □

3.2. The information class Λ^{std}.

In the next theorem, we present the respective conditions for tractability of L_2-approximation in the weighted Korobov space $\mathcal{H}(K_R)$ for the information class Λ^{std}.

Theorem 2. *Let* $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ *be the smoothness parameter sequence satisfying* (2.4) *and let* $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$ *be the weight sequence satisfying* (2.5)*. Consider the* L_2*-approximation problem* $\mathrm{APP} = \{\mathrm{APP}_d\}_{d\in\mathbb{N}}$ *for the weighted spaces* $\mathcal{H}(K_R)$*,* $R \in \{r_{d,\alpha,\gamma}, \psi_{d,\alpha,\gamma}, \omega_{d,\alpha,\gamma}\}$ *for the information class* Λ^{std} *under ABS and NOR. Then we have the following conditions:*

(i) SPT holds iff

$$\sum_{j=1}^{\infty} \gamma_j < \infty$$

(which implies $s_\gamma \le 1$). In this case, the exponent of SPT is

$$\tau^*(\Lambda^{\mathrm{std}}) = 2 \max\left(\frac{1}{\alpha_1}, s_\gamma\right).$$

(ii) PT, QPT are equivalent and hold iff

$$\limsup_{d\to\infty} \frac{1}{\ln d} \sum_{j=1}^{d} \gamma_j < \infty. \tag{3.8}$$

(iii) WT holds iff

$$\lim_{d\to\infty} \frac{1}{d} \sum_{j=1}^{d} \gamma_j = 0. \tag{3.9}$$

(iv) UWT holds iff

$$\lim_{d\to\infty} \frac{1}{d^\sigma} \sum_{j=1}^{d} \gamma_j = 0, \text{ for all } \sigma \in (0,1]. \tag{3.10}$$

(v) For $\sigma \in (0,1]$, (σ,τ)-WT holds iff

$$\lim_{d\to\infty} \frac{1}{d^\sigma} \sum_{j=1}^{d} \gamma_j = 0. \tag{3.11}$$

For $\sigma > 1$, (σ,τ)-WT always holds.

The proof of Theorem 2 is based on relations between the minimal errors of Λ^{std} and Λ^{all}, in particular on [17, Theorem 1] and on [6, Theorem 1] (see also [3, Theorem 26.10]). These results provide that the trace of the operator $W_d = \mathrm{APP}_d^*\mathrm{APP}_d$ is finite. Recall that the trace of W_d is given by the sum of its eigenvalues, i.e.,

$$\mathrm{trace}(W_d) = \sum_{j=1}^{\infty} \lambda_{d,j} = \sum_{\mathbf{k}\in\mathbb{Z}^d} R_{d,\alpha,\gamma}(\mathbf{k}) = \prod_{j=1}^{d}\left(1 + 2\sum_{k=1}^{\infty} R_{\alpha_j,\gamma_j}(k)\right).$$

Using (2.10), we obtain

$$\begin{aligned}\mathrm{trace}(W_d) &\le \prod_{j=1}^{d}\left(1 + 2\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\sum_{k=1}^{\infty} r_{\alpha_j,\gamma_j}(k)\right)\\ &= \prod_{j=1}^{d}\left(1 + 2\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\zeta(\alpha_j)\gamma_j\right)\\ &\le \prod_{j=1}^{d}\left(1 + 2\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\zeta(\alpha_1)\gamma_j\right) \qquad (3.12)\end{aligned}$$

and

$$\operatorname{trace}(W_d) \geq \prod_{j=1}^{d}\left(1+\frac{2}{3}\sum_{k=1}^{\infty} r_{\alpha_j,\gamma_j}(k)\right) = \prod_{j=1}^{d}\left(1+\frac{2\gamma_j}{3}\zeta(\alpha_j)\right).$$

Hence trace(W_d) is infinite if and only if $\alpha_1 = \alpha_2 = \cdots = 1$. Note that in general there is no relation between the power of Λ^{all} and Λ^{std} whenever the trace of W_d is infinite. For a discussion of this issue we refer to [3, Section 26.3].

In order to prove the sufficiency of PT for Λ^{std}, we use a criterion from [3] as stated below.

Lemma 2. ([3, Theorem 26.13]). *Consider multivariate approximation* APP $= \{\text{APP}_d\}_{d\in\mathbb{N}}$ *in the worst-case setting for ABS or NOR, where*

$$\text{APP}_d : F_d \to G_d \text{ with } \text{APP}_d(f) = f,$$

and F_d is a reproducing kernel Hilbert space continuously embedded in G_d. Assume that the trace of W_d is finite for all d, and there are two constants $C, q \geq 0$ such that

$$\frac{\operatorname{trace}(W_d)}{(\text{CRI}_d^X)^2} \leq Cd^q, \text{ for all } d \in \mathbb{N}.$$

Then polynomial tractabilities of APP *for* Λ^{all} *and* Λ^{std} *are equivalent. Particularly, if $q = 0$, then strong polynomial tractabilities of* APP *for* Λ^{all} *and* Λ^{std} *are equivalent. More precisely,*

$$n^X(\varepsilon, \text{APP}_d, F_d; \Lambda^{\text{all}}) \leq C^{\text{all}}\varepsilon^{-p^{\text{all}}}d^{q^{\text{all}}} \text{ for all } \varepsilon \in (0,1), d \in \mathbb{N},$$

implies that there exists a constant $C^{\text{std}} \geq 0$ such that

$$n^X(\varepsilon, \text{APP}_d, F_d; \Lambda^{\text{std}}) \leq C^{\text{std}}\varepsilon^{-p^{\text{std}}}d^{q^{\text{std}}} \text{ for all } \varepsilon \in (0,1), d \in \mathbb{N},$$

where

$$p^{\text{std}} = p^{\text{all}} + 2 \;\; \text{and} \;\; q^{\text{std}} = q^{\text{all}} + q.$$

Now we prove Theorem 2.

Proof of Theorem 2. It follows from (4.2) in Section 4 that necessary conditions of tractability for integration INT are also necessary for L_2 approximation APP. Consequently, the necessity of the conditions outlined in items (i)–(v) is derived from Theorem 3. It is enough to examine whether the conditions in items (i)–(v) are sufficient.

(i) Clearly, $\alpha_1 > 1$ implies that trace(W_d) is finite for all $d \in \mathbb{N}$. Then according to [17, Theorem 1], there is a positive integer c satisfying for all $n \in \mathbb{N}$

$$e(c\,n, \text{APP}_d, \mathcal{H}(K_R); \Lambda^{\text{std}})^2 \leq \frac{1}{n}\sum_{k=n}^{\infty} e(k, \text{APP}_d, \mathcal{H}(K_R); \Lambda^{\text{all}})^2. \tag{3.13}$$

Now let $\sum_{j=1}^{\infty}\gamma_j < \infty$. Then, the sum exponent $s_\gamma \leq 1$.

- First we consider $s_\gamma < 1$. Then, using Theorem 1, we have SPT for Λ^{all} with exponent

$$\tau^*(\Lambda^{\mathrm{all}}) = 2\max\left(\frac{1}{\alpha_1}, s_\gamma\right) < 2.$$

Hence, for every $\tau > \tau^*(\Lambda^{\mathrm{all}})$ there is $C > 0$ such that

$$n(\varepsilon, \mathrm{APP}_d, \mathcal{H}(K_R); \Lambda^{\mathrm{all}}) \le C\varepsilon^{-\tau},$$

which yields that

$$e(k, \mathrm{APP}_d, \mathcal{H}(K_R); \Lambda^{\mathrm{all}}) \le \frac{C}{k^{1/\tau}}.$$

Inserting into (3.13) gives

$$\begin{aligned} e(c\,n, \mathrm{APP}_d, \mathcal{H}(K_R); \Lambda^{\mathrm{std}})^2 &\le \frac{C}{n}\sum_{k=n}^{\infty}\frac{1}{k^{2/\tau}} \le \frac{C}{n}\int_n^{\infty}\frac{1}{x^{2/\tau}}\mathrm{d}x \\ &= \frac{C}{n}\frac{\tau}{2-\tau}\frac{1}{n^{2/\tau-1}} \le \frac{C\,\tau}{2-\tau}\frac{1}{n^{2/\tau}}. \end{aligned}$$

Hence, there exists a number $a_\tau > 0$ such that

$$e(c\,n, \mathrm{APP}_d, \mathcal{H}(K_R); \Lambda^{\mathrm{std}}) \le \frac{a_\tau}{n^{1/\tau}},$$

which implies that

$$n(\varepsilon, \mathrm{APP}_d, \mathcal{H}(K_R); \Lambda^{\mathrm{std}}) \le \lceil c\, a_\tau^\tau\, \varepsilon^{-\tau}\rceil.$$

Hence, since $\tau > \tau^*(\Lambda^{\mathrm{std}})$ is arbitrary, we have SPT with

$$\tau^*(\Lambda^{\mathrm{std}}) = 2\max\left(\frac{1}{\alpha_1}, s_\gamma\right).$$

(Note that trivially $\tau^*(\Lambda^{\mathrm{std}}) \ge \tau^*(\Lambda^{\mathrm{all}}) = 2\max(1/\alpha_1, s_\gamma)$.)

- Next we consider $s_\gamma = 1$. From (3.13) and (3.2) we obtain

$$e(c\,n, \mathrm{APP}_d, \mathcal{H}(K_R); \Lambda^{\mathrm{std}})^2 \le \frac{1}{n}\sum_{k=n}^{\infty}\lambda_{d,k+1} \le \frac{1}{n}\sum_{k=1}^{\infty}\lambda_{d,k} = \frac{\mathrm{trace}(W_d)}{n}. \tag{3.14}$$

Since $\sum_{j=1}^{\infty}\gamma_j < \infty$ and $\alpha_1 > 1$, we derive from (3.12) that

$$\begin{aligned} \mathrm{trace}(W_d) &\le \exp\left\{\sum_{j=1}^{d}\ln\left(1 + 2\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\zeta(\alpha_1)\gamma_j\right)\right\} \\ &\le \exp\left\{2\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\zeta(\alpha_1)\sum_{j=1}^{\infty}\gamma_j\right\} \\ &=: \Gamma < \infty. \end{aligned}$$

Hence, insertion into (3.14) gives

$$e(c\,n, \mathrm{APP}_d, \mathcal{H}(K_R); \Lambda^{\mathrm{std}})^2 \le \frac{\Gamma}{n}.$$

From this, we obtain in the same way as above SPT with

$$\tau^*(\Lambda^{\mathrm{std}}) = 2 = 2\max\left(\frac{1}{\alpha_1}, s_\gamma\right).$$

(ii) Assume that (3.8) holds. This implies that there exists a constant $M > 0$ such that

$$\frac{1}{\ln d}\sum_{j=1}^{d}\gamma_j < M, \quad \text{for all } d \in \mathbb{N}.$$

Then, by (3.12) we have

$$\begin{aligned}\mathrm{trace}(W_d) &\le \exp\left\{2\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\zeta(\alpha_1)\sum_{j=1}^{d}\gamma_j\right\}\\ &\le \exp\left(2\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\zeta(\alpha_1)\ln d^M\right) = d^{2\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\zeta(\alpha_1)M}.\end{aligned}$$

Meanwhile, since the sequence $(\gamma_j)_{j\in\mathbb{N}}$ is non-increasing, we have

$$\frac{d\,\gamma_d}{\ln d} \le \frac{1}{\ln d}\sum_{j=1}^{d}\gamma_j < M, \text{ for all } d \in \mathbb{N},$$

which implies that* $\gamma_j = O(j^{-1}\ln j)$ and $s_\gamma = 1$. By Theorem 1 (this implies that APP is SPT for Λ^{all}, i.e., there exist two constants $C^{\mathrm{all}}, p^{\mathrm{all}} \ge 0$ such that

$$n(\varepsilon, \mathrm{APP}_d, \mathcal{H}(K_R); \Lambda^{\mathrm{all}}) \le C^{\mathrm{all}}\varepsilon^{-p^{\mathrm{all}}} \ \text{ for all } \varepsilon \in (0,1) \text{ and } d \in \mathbb{N}.$$

Now Lemma 2 implies that there is a constant $C^{\mathrm{std}} > 0$ such that

$$n(\varepsilon, \mathrm{APP}_d, \mathcal{H}(K_R); \Lambda^{\mathrm{std}}) \le C^{\mathrm{std}}\varepsilon^{-p^{\mathrm{std}}}d^{q^{\mathrm{std}}} \ \text{ for all } \varepsilon \in (0,1) \text{ and } d \in \mathbb{N},$$

where

$$p^{\mathrm{std}} = p^{\mathrm{all}} + 2 \ \text{ and } \ q^{\mathrm{std}} = \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\zeta(\alpha_1)M.$$

Hence, we have PT and QPT for Λ^{std}.

(iii)–(v) If any of the three conditions (3.9), (3.10) or (3.11) holds, then this leads to $\boldsymbol{\gamma}_I < 1$, since otherwise, for every $\sigma \in (0,1]$,

$$\lim_{d\to\infty}\frac{1}{d^\sigma}\sum_{j=1}^{d}\gamma_j = \lim_{d\to\infty}\frac{d}{d^\sigma} = \lim_{d\to\infty}d^{1-\sigma} \ge 1.$$

*Here, $A_j = O(B_j)$ means that there exists a constant $C > 0$ independent of j such that $A_j \le CB_j$.

Therefore, we know from Theorem 1 that WT, UWT, and (σ,τ)-WT with $\sigma \in (0,1]$ hold for $\Lambda^{\rm all}$.

We only prove the sufficiency of (3.11) for (σ,τ)-WT with $\sigma \in (0,1]$ since the other two weak tractabiliies can be deduced similarly. Thus, it suffices to show that (3.11) implies (σ,τ)-WT with $\sigma \in (0,1]$. Indeed, we observe from (3.12) that

$$\begin{aligned}\frac{\ln(\mathrm{trace}(W_d))}{d^\sigma} &\le \frac{1}{d^\sigma}\ln\left(\prod_{j=1}^{d}(1+2\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\zeta(\alpha_1)\gamma_j)\right)\\ &= \frac{1}{d^\sigma}\sum_{j=1}^{d}\ln(1+2\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\zeta(\alpha_1)\gamma_j)\\ &\le \frac{2\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}\zeta(\alpha_1)}{d^\sigma}\sum_{j=1}^{d}\gamma_j,\end{aligned}$$

where we used that $\ln(1+x) \le x$ for all $x \ge 0$. Thus (3.11) implies

$$\lim_{d\to\infty}\frac{\ln(\mathrm{trace}(W_d))}{d^\sigma} = 0.$$

Following the proof of [3, Theorem 26.11], we can obtain (σ,τ)-WT with $\sigma \in (0,1]$ for $\Lambda^{\rm std}$.

The proof is completed. □

4. Integration in $\mathcal{H}(K_R)$.

In this section, we investigate the tractability of integration operators defined on the weighted Korobov space $\mathcal{H}(K_R)$. More precisely, we want to approximate the integral operators

$$\mathrm{INT}_d : \mathcal{H}(K_R) \to \mathbb{R},\ \mathrm{INT}_d(f) = \int_{[0,1]^d} f(\mathbf{x})\mathrm{d}\mathbf{x} = \widehat{f}(\mathbf{0}),$$

in the worst-case setting. We approximate INT_d by means of linear algorithms $A^{\rm int}_{n,d}$ of the form

$$A^{\rm int}_{n,d}(f) := \sum_{i=1}^{n}\lambda_i f(\mathbf{x}_i), \tag{4.1}$$

with nodes $\mathbf{x}_1,\dots,\mathbf{x}_n \in [0,1]^d$ and integration weights $\lambda_1,\dots,\lambda_n \in \mathbb{R}$.

The n-th minimal error for integration in $\mathcal{H}(K_R)$ is defined as

$$e(n,\mathrm{INT}_d,\mathcal{H}(K_R)) := \inf_{A^{\rm int}_{n,d}}\sup_{\|f\|_{\mathcal{H}(K_R)}\le 1}\left|\mathrm{INT}_d(f) - A^{\rm int}_{n,d}(f)\right|$$

where the infimum is taken over all algorithms of the form (4.1). For $n = 0$, the initial integration error is given by

$$e(0,\mathrm{INT}_d,\mathcal{H}(K_R)) := \sup_{\|f\|_{\mathcal{H}(K_R)}\le 1}|\mathrm{INT}_d(f)| = \|\mathrm{INT}_d\|.$$

In the following, we assume that $R(\mathbf{0}) = 1$, which leads to the norm of INT_d is 1, since

$$\|\mathrm{INT}_d\| = \sup_{0\ne f\in\mathcal{H}(K_R)}\frac{|\widehat{f}(\mathbf{0})|}{\|f\|_{\mathcal{H}(K_R)}}$$

$$= \sup_{0\neq f\in\mathcal{H}(K_R)} \frac{|\widehat{f}(\mathbf{0})|}{\sqrt{\sum_{\mathbf{k}\in\mathbb{Z}^d} R^{-1}(\mathbf{k})\,|\widehat{f}(\mathbf{k})|^2}}$$
$$\leq \sup_{0\neq f\in\mathcal{H}(K_R)} \frac{|\widehat{f}(\mathbf{0})|}{\sqrt{|\widehat{f}(\mathbf{0})|^2}} = 1$$

and the upper bound can be attained by choosing $f = 1$. Therefore, there is no need to distinguish between ABS and NOR. For abbreviation, we write $n(\varepsilon, \mathrm{INT}_d, \mathcal{H}(K_R)) \equiv n^X(\varepsilon, \mathrm{INT}_d, \mathcal{H}(K_R))$.

Obviously, we have

$$e(n, \mathrm{INT}_d, \mathcal{H}(K_R)) \leq e(n, \mathrm{APP}_d, \mathcal{H}(K_R); \Lambda^{\mathrm{std}}),$$

which implies

$$n(\varepsilon, \mathrm{INT}_d, \mathcal{H}(K_R)) \leq n(\varepsilon, \mathrm{APP}_d, \mathcal{H}(K_R); \Lambda^{\mathrm{std}}). \tag{4.2}$$

Now we formulate our result of the tractability of integration INT in the weighted Korobov space $\mathcal{H}(K_R)$ for $R \in \{r_{d,\alpha,\gamma}, \psi_{d,\alpha,\gamma}, \omega_{d,\alpha,\gamma}\}$.

Theorem 3. *Let $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ be the smoothness parameter sequence satisfying* (2.4)*, and let $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$ be the weight sequence satisfying* (2.5). *Consider the integration problem* INT $= \{\mathrm{INT}_d\}_{d\in\mathbb{N}}$ *for the weighted spaces $\mathcal{H}(K_R)$, $R \in \{r_{d,\alpha,\gamma}, \psi_{d,\alpha,\gamma}, \omega_{d,\alpha,\gamma}\}$. Then we have the following conditions:*

(i) SPT holds iff

$$\sum_{j=1}^{\infty} \gamma_j < \infty$$

(which is equivalent to $s_\gamma \leq 1$). In this case the exponent of SPT satisfies

$$\tau^*(\Lambda^{\mathrm{std}}) = 2\max\left(\frac{1}{\alpha_1}, s_\gamma\right).$$

(ii) PT, QPT are equivalent and hold iff

$$\limsup_{d\to\infty} \frac{1}{\ln d} \sum_{j=1}^{d} \gamma_j < \infty.$$

(iii) WT holds iff

$$\lim_{d\to\infty} \frac{1}{d} \sum_{j=1}^{d} \gamma_j = 0.$$

(iv) UWT holds iff

$$\lim_{d\to\infty} \frac{1}{d^\sigma} \sum_{j=1}^{d} \gamma_j = 0, \textit{ for all } \sigma \in (0,1].$$

(v) For $\sigma \in (0,1]$ (σ,τ)-WT holds iff

$$\lim_{d\to\infty} \frac{1}{d^\sigma} \sum_{j=1}^{d} \gamma_j = 0.$$

For $\sigma > 1$, (σ,τ)-WT always holds.

To verify Theorem 3, we shall give the lower bound of the n-th minimal error for INT in $\mathcal{H}(K_R)$. we will employ the subsequent proposition and lemma.

Proposition 3 ([15]). *For $j = 1, \dots, d$, let $\beta_j \in (0, 1]$ and let H_j be a reproducing kernel space on $[0, 1]$ such that $(1, \beta_j \cos(2\pi x), \beta_j \sin(2\pi x))$ are orthonormal in H_j. Consider the integration problem INT for $f \in \mathbf{H}_d = H_1 \otimes \cdots \otimes H_d$. We have*

$$e(n, \mathrm{INT}_d, \mathbf{H}_d)^2 \geq 1 - n \prod_{j=1}^{d} (1 + \beta_j^2)^{-1}.$$

Lemma 4. *Let $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ be the smoothness parameter sequence satisfying (2.4), and let $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$ be the weight sequence satisfying (2.5). Set $R \in \{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}\}$. Then, for all $n \in \mathbb{N}$,*

$$e(n, \mathrm{INT}_d, \mathcal{H}(K_R)) \geq e(n, \mathrm{INT}_d, \mathcal{H}(K_{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}/3}})).$$

Proof. This can be deduced from Remark 2, (2.9), and the definition of the n-th minimal error for integration. □

Applying Proposition 3 and Lemma 4 to the weighted space $\mathcal{H}(K_{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}/3}})$, we obtain the lower bound of $e(n, \mathrm{INT}_d, \mathcal{H}(K_R))$ for $R \in \{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}\}$.

Lemma 5. *Let $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ be the smoothness parameter sequence satisfying (2.4), and let $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$ be the weight sequence. Set $R \in \{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}\}$. Then we have*

$$e(n, \mathrm{INT}_d, \mathcal{H}(K_R))^2 \geq 1 - n \prod_{j=1}^{d} \left(1 + \frac{1}{3}\gamma_j\right)^{-1}. \tag{4.3}$$

This implies that for all $\varepsilon \in (0, 1)$,

$$n(\varepsilon, \mathrm{INT}_d, \mathcal{H}(K_R)) \geq (1 - \varepsilon^2) \prod_{j=1}^{d} \left(1 + \frac{1}{3}\gamma_j\right). \tag{4.4}$$

Proof. Clearly, the weighted Korobov space $\mathcal{H}(K_{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}/3}})$ satisfies the conditions of Proposition 3 with $\beta_j = \sqrt{\gamma_j/3}$, $j = 1, \dots, d$, which, by Proposition 3 and Lemma 4, implies that for $R \in \{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}\}$,

$$e(n, \mathrm{INT}_d, \mathcal{H}(K_R))^2 \geq e(n, \mathrm{INT}_d, \mathcal{H}(K_{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}/3}}))^2 \geq 1 - n \prod_{j=1}^{d} \left(1 + \frac{1}{3}\gamma_j\right)^{-1},$$

which gives (4.3). For all $\varepsilon \in (0, 1)$, letting $e(n, \mathrm{INT}_d, \mathcal{H}(K_R)) \leq \varepsilon$, we will obtain (4.4). This completes the proof. □

Remark 4. We remark that for $R \in \{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}\}$, the constant $1/3$ in (4.3) and (4.4) can be replaced by 1. That is,

$$e(n, \mathrm{INT}_d, \mathcal{H}(K_R))^2 \geq 1 - n \prod_{j=1}^{d} \left(1 + \gamma_j\right)^{-1},$$

and for all $\varepsilon \in (0, 1)$,

$$n(\varepsilon, \mathrm{INT}_d, \mathcal{H}(K_R)) \geq (1 - \varepsilon^2) \prod_{j=1}^{d} \left(1 + \gamma_j\right).$$

Now we turn to give the proof of Theorem 3.

Proof of Theorem 3. According to (4.2), we know that the sufficient condition on the tractability of the L_2-approximation problem for $\Lambda^{\rm std}$ also works for the integration problem in $\mathcal{H}(K_R)$. Therefore, we only need to verify the necessary conditions.

First, we assert that $\lim_{j\to\infty}\gamma_j = 0$. If it does not hold, there would exist a constant $\gamma_* > 0$ such that $\gamma_j \ge \gamma_* > 0$ for every $j \in \mathbb{N}$. According to Lemma 5, it can be derived that

$$n(\varepsilon, {\rm INT}_d, \mathcal{H}(K_R)) \ge (1-\varepsilon^2)\left(1+\frac{1}{3}\gamma_*\right)^d.$$

Thus $n(\varepsilon, {\rm INT}_d, \mathcal{H}(K_R))$ grows exponentially fast in d, which implies that INT suffers from the curse of dimensionality. This proves our claim.

Now we assume that $\sum_{j=1}^\infty \gamma_j = \infty$. Then we can derive from $\lim_{j\to\infty}\gamma_j = 0$ and the inequality $x\ln 2 \le \ln(1+x) \le x, x\in[0,1]$ that[†]

$$\begin{aligned}\prod_{j=1}^d\left(1+\frac{1}{3}\gamma_j\right) &= \exp\left\{\sum_{j=1}^d \ln\left(1+\frac{1}{3}\gamma_j\right)\right\}\\ &= \Theta\left(\exp\left(\frac{1}{3}\sum_{j=1}^d\gamma_j\right)\right). \end{aligned} \tag{4.5}$$

Then it follows from (4.4) and (4.5) that

$$\lim_{d\to\infty} n(\varepsilon, {\rm INT}_d, \mathcal{H}(K_R)) = \infty,$$

which implies that INT cannot be SPT. Thus $\sum_{j=1}^\infty \gamma_j < \infty$ is necessary for SPT.

Next assume that $\limsup_{d\to\infty}\frac{1}{\ln d}\sum_{j=1}^d\gamma_j = \infty$. Then we can derive from $\lim_{j\to\infty}\gamma_j = 0$ that

$$\prod_{j=1}^d\left(1+\frac{1}{3}\gamma_j\right) = \Theta\left(d^{\frac{1}{3\ln d}\sum_{j=1}^d\gamma_j}\right).$$

Combining with (4.4), we obtain that $n(\varepsilon, {\rm INT}_d, \mathcal{H}(K_R))$ goes to infinity faster than any power of d, and INT cannot be PT. Thus, $\limsup_{d\to\infty}\frac{1}{\ln d}\sum_{j=1}^d\gamma_j < \infty$ is necessary for PT.

Finally, assume that

$$\lim_{d+\varepsilon^{-1}\to\infty}\frac{\ln n(\varepsilon, {\rm INT}_d, \mathcal{H}(K_R))}{d^\sigma + \varepsilon^{-\tau}} = 0,\ \sigma\in(0,1].$$

Then we can derive from (4.4) and (4.5) that

$$\lim_{d\to\infty}\frac{1}{d^\sigma}\sum_{j=1}^d\gamma_j = 0.$$

This implies the condition is necessary for the three WT notions.

The proof is completed. □

[†]Here, $A_d = \Theta(B_d)$ means that there exsits a constant $C > 0$ independent of d such that $C^{-1}B_d \le A_d \le CB_d$.

5. Concluding remarks

This paper gives a complete picture of the tractability of multivariate L_2-approximation for both Λ^{all} and Λ^{std}, and multivariate integration from weighted Korobov spaces of increasing smoothness in the worst-case setting. According to the results in [16], the corresponding tractability conditions of L_2-approximation in the randomized case setting are the same as in the worst-case setting. Moreover, our results may be helpful to study the tractability of nonhomogeneous tensor product spaces.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

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