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Research article

Well-posedness of linear elliptic equations with L^d-drifts under divergence-type conditions

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Abstract: We establish the well-posedness of linear elliptic equations with critical-order drifts in L^d and positive zero-order coefficients in L^1 or $L^{\frac{2d}{d+2}}$, where classical methods are often too restrictive. Our approach relies on a divergence-free transformation and a structural condition on the drift vector field, which admits a decomposition into a regular component and another whose weak divergence belongs to $L^{\tilde{q}}$ for some $\tilde{q} > \frac{d}{2}$. This condition is essential for constructing a suitable weight function ρ via the weak maximum principle and the Harnack inequality. Within this framework, we prove the existence and uniqueness of weak solutions, significantly relaxing the regularity assumptions on the zero-order coefficients in $L^{\frac{d}{2}}$.

Keywords: linear elliptic equations; well-posedness; weak solutions; divergence-free transformation; Harnack inequality; weak maximum principle

1. Introduction

This paper establishes the well-posedness (existence and uniqueness of weak solutions) (cf. Definition 2.1) of the following Dirichlet problem for a linear elliptic equation in divergence form, defined on a bounded open subset $U \subset \mathbb{R}^d$ with $d \geq 3$:

$$\begin{cases} -\operatorname{div}(A\nabla u) + \langle \mathbf{H}, \nabla u\rangle + cu = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{1.1}$$

where A is uniformly strictly elliptic and bounded on U (see (1.3)). The well-posedness of (1.1) was established in [1] not by the conventional bilinear form methods, but by employing weak convergence techniques combined with a divergence-free transformation, under the assumptions that $\mathbf{H} \in L^p(U, \mathbb{R}^d)$ with $p \in (d, \infty)$, $c \in L^1(U)$, and $f \in L^q(U)$ with $q \in (\frac{d}{2}, \infty)$. It is therefore a natural problem to investigate whether the condition $\mathbf{H} \in L^p(U, \mathbb{R}^d)$ can be relaxed to the critical case $\mathbf{H} \in L^d(U, \mathbb{R}^d)$.

However, under the sole assumption $\mathbf{H} \in L^d(U, \mathbb{R}^d)$, an extension of the result to the cases $c \in L^1(U)$ or $c \in L^{\frac{2d}{d+2}}(U)$ cannot be achieved directly by either the standard bilinear form methods or the approach in [1].

The well-posedness of (1.1) via bilinear form methods based on the Lax-Milgram theorem originates from G. Stampacchia's work [2], where it was proved that there exists a constant $\gamma > 0$, depending only on A, $\mathbf{H}$, and d, such that the problem (1.1) admits a unique weak solution whenever $c \geq \gamma$ and $c \in L^{\frac{d}{2}}(U)$ with $d \geq 3$. Similar results, under certain restrictions on the zero-order coefficients, are treated in [3]. In [4] (cf. [5, Section 8.2]), N. S. Trudinger established the well-posedness of (1.1) by developing a weak maximum principle under the assumptions $\mathbf{H} \in L^d(U, \mathbb{R}^d)$, $c \in L^{\frac{d}{2}}(U)$ with $c \geq 0$, and $f \in L^{\frac{2d}{d+2}}(U)$. In the absence of the classical coercivity property, the well-posedness of (1.1) has also been obtained via a duality argument in [6]. For another reference on non-coercive linear equations with coefficients in Lorentz spaces, [7] establishes the well-posedness of the dual problem associated with (1.1). For further results beyond the $L^2(U)$-regularity of ∇u, or for corresponding results concerning non-divergence type counterparts of (1.1), we refer to [1] and references therein.

To understand the technical challenge in the critical case of the drift coefficients, we first revisit the approach of [1] under the assumptions $\mathbf{H} \in L^p(U, \mathbb{R}^d)$ for some $p \in (d, \infty)$ and $c \in L^1(U)$ with $c \geq 0$. In [1], to apply a divergence-free transformation, one first constructs a strictly positive function $\rho \in H^{1,2}(U) \cap C(\overline{U})$ and a divergence-free vector field $\rho\mathbf{B} \in L^2(U, \mathbb{R}^d)$, which then transforms the original equation (1.1) into the form shown in (1.2) (see Theorem 4.3):

$$\begin{cases} -\operatorname{div}(\rho A \nabla u) + \langle \rho \mathbf{B}, \nabla u \rangle + \rho c u = \rho f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases} \tag{1.2}$$

This reformulation enables the derivation of a priori $H^{1,2}$ and L^∞-bounds, and through a delicate application of weak compactness methods and a duality argument, one can obtain the existence and uniqueness of a weak solution $u \in H_0^{1,2}(U)$ to (1.1).

In the critical case where $\mathbf{H} \in L^d(U, \mathbb{R}^d)$, the lack of regularity makes it difficult to construct the function $\rho \in H^{1,2}(U) \cap C(\overline{U})$ and the vector field $\rho\mathbf{B}$ as in the approach described above. The first reason is that under the condition $\mathbf{H} \in L^d(U, \mathbb{R}^d)$, obtaining the key estimate in Lemma 3.7 may not be directly derived but requires delicate computations and the use of a partition of unity. The second reason is that the method employed in [1] relies on the construction of a function ρ satisfying an elliptic Harnack inequality and Hölder continuity. However, such a construction is significantly restricted under the assumption $\mathbf{H} \in L^d(U, \mathbb{R}^d)$ (see Remark 4.2), which justifies the necessity of imposing additional conditions on $\mathbf{H}$ in $L^d(U, \mathbb{R}^d)$. Indeed, we show that the divergence-free transformation can still be successfully carried out if $\mathbf{H}$ admits a decomposition where one component has a sufficiently regular divergence in $L^{\tilde{q}}$ for some $\tilde{q} \in \left(\frac{d}{2}, \infty\right)$.

Before stating our main result, let us present the main assumption in this paper:

(T) *U is a bounded open subset of $\mathbb{R}^d$ with $d \geq 3$, and $B_r(x_0)$ is an open ball in $\mathbb{R}^d$ such that $\overline{U} \subset B_r(x_0)$. $\mathbf{H}_1 \in L^p(B_r(x_0), \mathbb{R}^d)$ for some $p \in (d, \infty)$, and $\mathbf{H}_2 \in L^d(B_r(x_0), \mathbb{R}^d)$ satisfies the following* *distributional identity (see Definition 2.2): there exists* $\tilde{h} \in L^{\tilde{q}}(B_r(x_0))$ *with* $\tilde{q} \in (\frac{d}{2}, \infty)$ *such that*

$$\int_{B_r(x_0)} \langle \mathbf{H}_2, \nabla\psi \rangle \, dx = -\int_{B_r(x_0)} \tilde{h}\psi \, dx \quad \text{for all } \psi \in C_0^\infty(B_r(x_0)),$$

i.e., $\operatorname{div} \mathbf{H}_2 = \tilde{h} \in L^{\tilde{q}}(B_r(x_0))$. $\mathbf{H} := \mathbf{H}_1 + \mathbf{H}_2 \in L^d(B_r(x_0), \mathbb{R}^d)$. $A = (a_{ij})_{1\le i,j\le d}$ *is a (possibly non-symmetric) matrix of measurable functions on* $\mathbb{R}^d$ *such that there exist constants* $M > 0$ *and* $\lambda > 0$ *satisfying*

$$\max_{1\le i,j\le d} |a_{ij}(x)| \le M, \quad \langle A(x)\xi, \xi \rangle \ge \lambda \|\xi\|^2 \quad \text{for a.e. } x \in \mathbb{R}^d \text{ and all } \xi \in \mathbb{R}^d. \tag{1.3}$$

The following is the main theorem of this paper, which shows that the conclusion of [1, Theorem 1.1] remains robust under the assumption of an L^d-drift.

Theorem 1.1. *Assume that* **(T)** *holds. Let* $c \in L^1(U)$ *with* $c \ge 0$ *in* U. *Then, the following statements hold:*

(i) Let $v \in H_0^{1,2}(U)$ *with* $cv \in L^1(U)$ *be such that*

$$\int_U \langle A\nabla v, \nabla\psi \rangle + \langle \mathbf{H}, \nabla v \rangle \psi + cv\psi \, dx = 0 \quad \text{for all } \psi \in C_0^\infty(U). \tag{1.4}$$

Then, $v = 0$.

(ii) Let $f \in L^q(U)$ *for some* $q > \frac{d}{2}$. *Then, there exists a unique weak solution* $u \in H_0^{1,2}(U) \cap L^\infty(U)$ *to* (1.1). *Moreover, u satisfies*

$$\|u\|_{H_0^{1,2}(U)} \le K_5 \|f\|_{L^{\frac{2d}{d+2}}(U)}, \tag{1.5}$$

and

$$\|u\|_{L^\infty(U)} \le K_6 \|f\|_{L^q(U)}, \tag{1.6}$$

where $K_5 = \tilde{K}_1 K_3$, $K_6 = \tilde{K}_1 K_4$, $\tilde{K}_1 > 0$ *is a constant as in Theorem 4.1, depending only on* d, λ, M, $B_r(x_0)$, p, $\tilde{q}$, $\mathbf{H}$, *the constant* $K_3 > 0$ *depends only on* d, $\frac{\lambda}{\tilde{K}_1}$, $|U|$, *and the constant* $K_4 > 0$ *depends only on* d, $\frac{\lambda}{\tilde{K}_1}$, $|U|$, q. *In particular, if* $\alpha > 0$ *is a constant,* $c \ge \alpha$ *in* U, *and* $f \in L^\theta(U) \cap L^q(U)$ *with* $\theta \in [1, \infty]$, *then u satisfies the following contraction estimate:*

$$\|u\|_{L^\theta(U)} \le \frac{\tilde{K}_1}{\alpha} \|f\|_{L^\theta(U)}. \tag{1.7}$$

(iii) Let $c \in L^{\frac{2d}{d+2}}(U)$ *and* $f \in L^{\frac{2d}{d+2}}(U)$. *Then there exists a unique weak solution* $u \in H_0^{1,2}(U)$ *to* (1.1). *Moreover, u satisfies* (1.5). *In particular, if* $\alpha > 0$ *is a constant,* $c \ge \alpha$ *in* U, *and* $f \in L^\theta(U) \cap L^{\frac{2d}{d+2}}(U)$ *with* $\theta \in [1, \infty]$, *then u satisfies* (1.7).

Although the main result of the paper concerns the existence and uniqueness of weak solutions, it is particularly noteworthy that no additional structural conditions, such as the VMO assumption on the matrix of functions A, are imposed. Furthermore, the main result Theorem 1.1 challenges the conventional belief that the optimal regularity condition for the zero-order term is $c \in L^{\frac{d}{2}}(U)$ by demonstrating that the weaker assumptions $c \in L^1(U)$ or $c \in L^{\frac{2d}{d+2}}(U)$ are sufficient. This observation suggests the possibility of further developments that could partially relax the regularity condition on the zero-order coefficient, namely $c \in L^{\frac{d}{2}}(U)$, in order to obtain the well-posedness results for non-divergence form equations as established in [8, 9].

This paper is organized as follows: Section 2 introduces the essential notations and definitions used throughout the paper. Section 3 establishes fundamental inequalities and shows Lemma 3.10, a crucial inequality for this paper. Section 4 sets up the divergence-free transformation and completes the proof of Theorem 1.1. Section 5 concludes with a discussion.

2. Notations and definitions

In this paper, we work within the d-dimensional Euclidean space $\mathbb{R}^d$, where $d \geq 1$, equipped with the standard inner product $\langle \cdot, \cdot \rangle$ and the corresponding Euclidean norm $\| \cdot \|$. For a point $x_0 \in \mathbb{R}^d$ and radius $r > 0$, we denote by $B_r(x_0)$ the open ball $\{x \in \mathbb{R}^d : \|x - x_0\| < r\}$, and we write $B_r := \{x \in \mathbb{R}^d : \|x\| < r\}$. The Lebesgue measure on $\mathbb{R}^d$ is denoted by dx, and for a measurable set $E \subset \mathbb{R}^d$, $dx(E)$ is written as $|E|$. The indicator function of a set W is denoted by 1_W. Let U be an open subset of $\mathbb{R}^d$. We denote by $C(U)$ and $C(\overline{U})$ the spaces of continuous functions on U and its closure $\overline{U}$, respectively. For $k \in \mathbb{N} \cup \{\infty\}$, the space $C^k(U)$ consists of functions that are k-times continuously differentiable on U, while $C_0^k(U)$ denotes the subspace of $C^k(U)$ consisting of functions with compact support in U. Let $s \in [1, \infty]$. We denote by $L^s(U)$ the standard Lebesgue space with norm $\| \cdot \|_{L^s(U)}$, and by $L^s(U, \mathbb{R}^d)$ the space of $\mathbb{R}^d$-valued functions whose components lie in $L^s(U)$, equipped with the norm $\|\mathbf{F}\|_{L^s(U)} := \|\|\mathbf{F}\|\|_{L^s(U)}$. For each $i \in \{1, 2, \dots d\}$, ∂_i denotes the weak partial derivative with respect to the i-th component. The weak gradient of a function u is denoted by $\nabla u := (\partial_1 u, \dots, \partial_d u)$. The Sobolev space $H^{1,s}(U)$ consists of functions in $L^s(U)$ whose weak partial derivatives also belong to $L^s(U)$. The space $H_0^{1,2}(U)$ denotes the closure of $C_0^\infty(U)$ in the $H^{1,2}(U)$-norm. By the Poincaré inequality, we write $\|u\|_{H_0^{1,2}(U)} := \|\nabla u\|_{L^2(U)}$. The dual space of $H_0^{1,2}(U)$ is denoted by $H^{-1,2}(U)$, and the duality pairing is represented by $\langle \cdot, \cdot \rangle_{H^{-1,2}(U)}$.

Definition 2.1. *Let $A = (a_{ij})_{1 \leq i,j \leq d}$ be a matrix of bounded and measurable functions on $\mathbb{R}^d$. Let $\mathbf{H} \in L^2(U, \mathbb{R}^d)$, $c \in L^1(U)$, and $f \in L^1(U)$. We say that u is a weak solution to* (1.1) *if $u \in H_0^{1,2}(U)$ and $cu \in L^1(U)$, and the following identity holds:*

$$\int_U \langle A \nabla u, \nabla \psi \rangle + \langle \mathbf{H}, \nabla u \rangle \psi + cu\psi \, dx = \int_U f\psi \, dx \quad \text{for all } \psi \in C_0^\infty(U). \tag{2.1}$$

Definition 2.2. *For a vector field $\mathbf{H} \in L^1_{\text{loc}}(U, \mathbb{R}^d)$, its divergence* $\operatorname{div} \mathbf{H}$ *is understood in the weak sense. That is, if $h \in L^1_{\text{loc}}(U)$ satisfies*

$$\int_U \langle \mathbf{H}, \nabla \varphi \rangle \, dx = -\int_U h\varphi \, dx \quad \text{for all } \varphi \in C_0^\infty(U),$$

we write $\operatorname{div} \mathbf{H} = h$ *in U. A vector field* $\mathbf{H}$ *is called divergence-free if* $\operatorname{div} \mathbf{H} = 0$.

3. Fundamental inequalities

In this section, we mainly assume the condition **(T1)** below, which is weaker than **(T)**.

(T1): *U is a bounded open subset of $\mathbb{R}^d$ with $d \geq 3$, $\mathbf{H} \in L^d(U, \mathbb{R}^d)$, and $A = (a_{ij})_{1 \leq i,j \leq d}$ is a (possibly non-symmetric) matrix of measurable functions on $\mathbb{R}^d$ such that there exist constants $M > 0$ and $\lambda > 0$ satisfying* (1.3).

Proposition 3.1. *Under the assumption* **(T1)***, define a bilinear form* $\mathcal{B} : H_0^{1,2}(U) \times H_0^{1,2}(U) \to \mathbb{R}$ *given by*

$$\mathcal{B}(f, g) := \int_U \langle A\nabla f + f\mathbf{H}, \nabla g\rangle \, dx, \quad f, g \in H_0^{1,2}(U). \tag{3.1}$$

Then, the following statements hold:

(i)

$$|\mathcal{B}(f, g)| \le K\|\nabla f\|_{L^2(U)}\|\nabla g\|_{L^2(U)} \quad \text{for all } f, g \in H_0^{1,2}(U), \tag{3.2}$$

where $K := dM + \dfrac{2(d-1)}{d-2}\|\mathbf{H}\|_{L^d(U)}$.

(ii) Let $N \ge 0$ *be a constant such that*

$$\left(\int_U 1_{\{\|\mathbf{H}\| \ge N\}}\|\mathbf{H}\|^d \, dx\right)^{\frac{2}{d}} \le \frac{\lambda^2}{16}\left(\frac{d-2}{d-1}\right)^2. \tag{3.3}$$

Then,

$$\mathcal{B}(f, f) + \frac{N^2}{\lambda}\|f\|^2_{L^2(U)} \ge \frac{\lambda}{2}\|\nabla f\|^2_{L^2(U)} \quad \text{for all } f \in H_0^{1,2}(U). \tag{3.4}$$

Proof. (i) Let $f, g \in H_0^{1,2}(U)$. By the Sobolev inequality [10, Section 5.6, Theorem 1] (cf. [11, Theorem 4.8]), we obtain

$$\|g\|_{L^{\frac{2d}{d-2}}(U)} \le \frac{2(d-1)}{d-2}\|\nabla g\|_{L^2(U)}. \tag{3.5}$$

Applying the Cauchy–Schwarz inequality and the Hölder inequality, we have

$$\begin{aligned}\left|\int_U \langle f\mathbf{H}, \nabla g\rangle \, dx\right| &\le \|\mathbf{H}\|_{L^d(U)}\|f\|_{L^{\frac{2d}{d-2}}(U)}\|\nabla g\|_{L^2(U)} \\ &\le \frac{2(d-1)}{d-2}\|\mathbf{H}\|_{L^d(U)}\|\nabla f\|_{L^2(U)}\|\nabla g\|_{L^2(U)}.\end{aligned}$$

In addition, we estimate

$$\left|\int_U \langle A\nabla f, \nabla g\rangle \, dx\right| \le dM \int_U \|\nabla f\| \, \|\nabla g\| \, dx \le dM\|\nabla f\|_{L^2(U)}\|\nabla g\|_{L^2(U)}.$$

Hence, the desired estimate (3.2) follows.

(ii) Let $f \in H_0^{1,2}(U)$. Using the Cauchy–Schwarz inequality and Young's inequality, we obtain

$$\begin{aligned}\left|\int_U \langle \mathbf{H}, \nabla f\rangle f \, dx\right| &\le \int_U \|\mathbf{H}\| \, |f| \, \|\nabla f\| \, dx \\ &\le \frac{\lambda}{4}\int_U \|\nabla f\|^2 dx + \frac{1}{\lambda}\int_U \|\mathbf{H}\|^2|f|^2 dx.\end{aligned} \tag{3.6}$$

Define the function $\phi : [0, \infty) \to \mathbb{R}$ by

$$\phi(s) := \left(\int_U 1_{\{\|\mathbf{H}\| \ge s\}}\|\mathbf{H}\|^d dx\right)^{\frac{2}{d}}, \quad s \in [0, \infty).$$

Since $\|\mathbf{H}\| \in L^d(U)$ and is finite almost everywhere, it follows from the Lebesgue dominated convergence theorem that

$$\lim_{s\to\infty} \phi(s) = 0.$$

Now choose $N \geq 0$ such that

$$\phi(N) \leq \frac{\lambda^2}{16}\left(\frac{d-2}{d-1}\right)^2. \tag{3.7}$$

Using again the Hölder inequality, (3.7), and the Sobolev inequality (3.5), we estimate

$$\begin{aligned}\int_U \|\mathbf{H}\|^2|f|^2dx &= \int_U 1_{\{\|\mathbf{H}\|\geq N\}}\|\mathbf{H}\|^2|f|^2dx + \int_U 1_{\{\|\mathbf{H}\|<N\}}\|\mathbf{H}\|^2|f|^2dx \\ &\leq \left(\int_U 1_{\{\|\mathbf{H}\|\geq N\}}\|\mathbf{H}\|^d dx\right)^{\frac{2}{d}} \|f\|^2_{L^{\frac{2d}{d-2}}(U)} + N^2\int_U |f|^2dx \\ &\leq \frac{\lambda^2}{4}\|\nabla f\|^2_{L^2(U)} + N^2\int_U |f|^2dx.\end{aligned}$$

Substituting this into (3.6), we obtain

$$\int_U \langle \mathbf{H}, \nabla f\rangle f\, dx \geq -\frac{\lambda}{2}\int_U \|\nabla f\|^2dx - \frac{N^2}{\lambda}\int_U |f|^2dx.$$

Since

$$\int_U \langle A\nabla f, \nabla f\rangle dx \geq \lambda\int_U \|\nabla f\|^2dx,$$

the desired estimate (3.4) follows. □

The following existence result is well known and can be found in [2, 5, 12]. For clarity and the reader's convenience, we state the details here.

Proposition 3.2. *Assume* **(T1)**. *Let* $\mathcal{B} : H^{1,2}_0(U) \times H^{1,2}_0(U) \to \mathbb{R}$ *denote the bilinear form defined by* (3.1). *Suppose that if* $w \in H^{1,2}_0(U)$ *satisfies*

$$\mathcal{B}(w, \varphi) = 0 \quad \text{for all } \varphi \in H^{1,2}_0(U),$$

then $w = 0$ *in* U. *Then, for each* $\psi \in H^{-1,2}(U)$, *there exists a unique* $u_\psi \in H^{1,2}_0(U)$ *such that*

$$\mathcal{B}(u_\psi, \varphi) = \langle \psi, \varphi\rangle_{H^{-1,2}(U)} \quad \text{for all } \varphi \in H^{1,2}_0(U).$$

Proof. Let $N \geq 0$ be the constant appearing in (3.3), and define $\gamma := \frac{N^2}{\lambda}$. Define a bilinear form $\mathcal{B}_\gamma : H^{1,2}_0(U) \times H^{1,2}_0(U) \to \mathbb{R}$ given by

$$\mathcal{B}_\gamma(f, g) := \mathcal{B}(f, g) + \gamma\int_U fg dx, \quad f, g \in H^{1,2}_0(U).$$

By the Lax–Milgram theorem (cf. [13, Corollary 5.8]) and Proposition 3.1, for each $\psi \in H^{-1,2}(U)$, there exists a unique $u_{\gamma,\psi} \in H^{1,2}_0(U)$ such that

$$\mathcal{B}_\gamma(u_{\gamma,\psi}, \varphi) = \langle \psi, \varphi\rangle_{H^{-1,2}(U)} \quad \text{for all } \varphi \in H^{1,2}_0(U). \tag{3.8}$$

Substituting $\varphi = u_{\gamma,\psi}$ into (3.8) and applying Proposition 3.1(ii), we obtain

$$\frac{\lambda}{2}\|\nabla u_{\gamma,\psi}\|^2_{L^2(U)} \le \mathcal{B}_\gamma(u_{\gamma,\psi}, u_{\gamma,\psi}) = \langle \psi, u_{\gamma,\psi}\rangle_{H^{-1,2}(U)} \le \|\psi\|_{H^{-1,2}(U)}\|\nabla u_{\gamma,\psi}\|_{L^2(U)},$$

and hence,

$$\|\nabla u_{\gamma,\psi}\|_{L^2(U)} \le \frac{2}{\lambda}\|\psi\|_{H^{-1,2}(U)}. \tag{3.9}$$

Define $K : H^{-1,2}(U) \to H^{1,2}_0(U)$ given by

$$K\psi := u_{\gamma,\psi}, \quad \psi \in H^{-1,2}(U). \tag{3.10}$$

Then, by (3.9), we have

$$\|K\psi\|_{H^{1,2}_0(U)} \le \frac{2}{\lambda}\|\psi\|_{H^{-1,2}(U)} \quad \text{for all } \psi \in H^{-1,2}(U).$$

Define the operator $J : H^{1,2}_0(U) \to H^{-1,2}(U)$ by, for each $u \in H^{1,2}_0(U)$,

$$\langle J(u), \varphi\rangle_{H^{-1,2}(U)} = \int_U u\varphi\, dx \quad \text{for all } \varphi \in H^{1,2}_0(U). \tag{3.11}$$

By the Rellich-Kondrachov compactness theorem, J is a compact operator, and hence so is $K \circ J$. We now state the following claim.

Claim: Let $u \in H^{1,2}_0(U)$ and $\psi \in H^{-1,2}(U)$. Then, the following statements (a)–(b) are equivalent:

$$\begin{cases} u - \gamma(K \circ J)u = K\psi \quad \text{in } H^{1,2}_0(U). & \text{(a)} \\ \mathcal{B}(u, \varphi) = \langle \psi, \varphi\rangle_{H^{-1,2}(U)} \quad \text{for all } \varphi \in H^{1,2}_0(U). & \text{(b)} \end{cases} \tag{3.12}$$

To prove the claim, first suppose that (a) holds. Then, we have

$$u = K(\psi + \gamma J(u)).$$

By the definition of K (see (3.10) and (3.8)), it follows that

$$\mathcal{B}_\gamma(u, \varphi) = \langle \psi + \gamma J(u), \varphi\rangle_{H^{-1,2}(U)} \quad \text{for all } \varphi \in H^{1,2}_0(U), \tag{3.13}$$

which is equivalent to (b) by (3.11). Conversely, assume that (b) holds. Then, (3.13) is satisfied, and hence, by the definition of K, we obtain

$$u = K(\psi + \gamma J(u)),$$

which implies (a) by (3.11). This completes the proof of the claim.

Let $I : H^{1,2}_0(U) \to H^{1,2}_0(U)$ denote the identity operator. Evaluating (a) with $\psi = 0$, it follows from the equivalence established in (3.12) that

$$u \in H^{1,2}_0(U) \text{ satisfies } (I - \gamma(K \circ J))u = 0$$

if and only if

$$\mathcal{B}(u,\varphi) = 0 \quad \text{for all } \varphi \in H_0^{1,2}(U).$$

Thus, by assumption, it follows that

$$\left\{u \in H_0^{1,2}(U) : (I - \gamma(K \circ J))u = 0\right\} = \{0\}.$$

Since $\gamma(K \circ J) : H_0^{1,2}(U) \to H_0^{1,2}(U)$ is a compact operator, the Fredholm alternative (see [13, Theorem 6.6]) implies that for each $\psi \in H^{-1,2}(U)$, there exists $u_\psi \in H_0^{1,2}(U)$ such that

$$(I - \gamma(K \circ J))u_\psi = K\psi.$$

Therefore, by the equivalence established in (3.12), the desired assertion follows. □

Proposition 3.3. (Poincaré-type inequality) *The following inequality holds:*

$$\|f\|_{L^2(U)} \leq \frac{2(d-1)}{d}|U|^{\frac{1}{d}}\|\nabla f\|_{L^2(U)} \quad \textit{for all } f \in H_0^{1,2}(U).$$

Proof. By applying the Gagliardo-Nirenberg-Sobolev inequality ([10, Section 5.6, Theorem 1]) together with the Hölder inequality, we obtain

$$\|f\|_{L^2(U)} \leq \frac{\frac{2d}{d+2}(d-1)}{d - \frac{2d}{d+2}}\|\nabla f\|_{L^{\frac{2d}{d+2}}(U)} \leq \frac{2(d-1)}{d}|U|^{\frac{1}{d}}\|\nabla f\|_{L^2(U)} \quad \text{for all } f \in H_0^{1,2}(U).$$

□

Lemma 3.4. *Let $\phi \in C^1((-\varepsilon,\infty))$ with $\varepsilon > 0$ be such that $\phi(0) = 0$ and $\phi' \in L^\infty((0,\infty))$. If $v \in H_0^{1,2}(U)$ with $v \geq 0$ in U, then $\phi(v) \in H_0^{1,2}(U)$ and $\nabla\phi(v) = \phi'(v)\nabla v$ in U.*

Proof. Extend ϕ to a function on $\mathbb{R}$, denoted again by ϕ, such that $\phi \in C^1(\mathbb{R})$ with $\phi' \in L^\infty(\mathbb{R})$. Let $(v_n)_{n\geq1} \subset C_0^\infty(U)$ be a sequence of functions such that $\lim_{n\to\infty} v_n = v$ in $H_0^{1,2}(U)$ such that

$$\lim_{n\to\infty} \|\nabla v_n - \nabla v\|_{L^2(U)} = 0 \tag{3.14}$$

and

$$\lim_{n\to\infty} v_n(x) = v(x) \quad \text{for a.e. } x \in U. \tag{3.15}$$

Then, by the chain rule, $(\phi(v_n))_{n\geq1} \subset C_0^1(U)$ and

$$\nabla\phi(v_n) = \phi'(v_n)\nabla v_n \quad \text{in } U \quad \text{for each } n \geq 1.$$

Moreover, by [11, Theorem 4.4(ii)], $\phi(v) \in H^{1,2}(U)$ satisfies $\nabla\phi(v) = \phi'(v)\nabla v$. Thus, we have

$$\begin{aligned}
&\|\nabla\phi(v) - \nabla\phi(v_n)\|_{L^2(U)} = \|\phi'(v)\nabla v - \phi'(v_n)\nabla v_n\|_{L^2(U)} \\
&\leq \|\phi'(v)\nabla v - \phi'(v_n)\nabla v\|_{L^2(U)} + \|\phi'(v_n)\nabla v - \phi'(v_n)\nabla v_n\|_{L^2(U)} \\
&\leq \|\phi'(v)\nabla v - \phi'(v_n)\nabla v\|_{L^2(U)} + \|\phi'\|_{L^\infty(\mathbb{R})}\|\nabla v - \nabla v_n\|_{L^2(U)}.
\end{aligned}$$

The first term converges to zero by the Lebesgue dominated convergence theorem and (3.15), and the second term converges to zero by (3.14). Therefore, we have $\phi(v) \in H_0^{1,2}(U)$. □

The following weak maximum principle originates from N. S. Trudinger [14], and a reformulated version is given in [12, Chapter 2] under the assumption that $\mathbf{H} \in L^p(U, \mathbb{R}^d)$ for some $p \in (d, \infty)$. However, the original result in [14] allows the critical case $\mathbf{H} \in L^d(U, \mathbb{R}^d)$. For the reader's convenience, we provide the precise statement and a detailed proof of this version below.

Proposition 3.5. (Weak maximum principle) *Assume* **(T1)**. *Let* $\mathcal{B} : H_0^{1,2}(U) \times H_0^{1,2}(U) \to \mathbb{R}$ *denote the bilinear form defined by* (3.1). *Let* $u \in H_0^{1,2}(U)$ *satisfy*

$$\mathcal{B}(u, \varphi) \leq 0 \quad \text{for all } \varphi \in H_0^{1,2}(U) \text{ with } \varphi \geq 0 \text{ in } U. \tag{3.16}$$

Then, $u \leq 0$ *in* U.

Proof. Let $\phi \in C^1((-\varepsilon, \infty))$ with $\varepsilon > 0$ be such that $\phi \geq 0$ on $[0, \infty)$, $\phi(0) = 0$, and $\phi' \in L^\infty((0, \infty))$. By [11, Theorem 4.4] and Lemma 3.4, we have $u^+ \in H_0^{1,2}(U)$ and $\phi(u^+) \in H_0^{1,2}(U)$, with

$$\nabla \phi(u^+) = \phi'(u^+) \nabla u^+ = \phi'(u^+) 1_{\{u>0\}} \nabla u \in L^2(U, \mathbb{R}^d).$$

Substituting $\varphi = \phi(u^+)$ into (3.16), we obtain

$$\int_U \langle A \nabla u,\, \phi'(u^+) 1_{\{u>0\}} \nabla u \rangle \, dx + \int_U \langle u\mathbf{H},\, \phi'(u^+) 1_{\{u>0\}} \nabla u \rangle \, dx \leq 0.$$

Since $1_{\{u>0\}} = (1_{\{u>0\}})^2$ and $1_{\{u>0\}} \nabla u = \nabla u^+$, it follows that

$$\int_U \langle A \nabla u^+,\, \phi'(u^+) \nabla u^+ \rangle \, dx \leq \int_U \langle -u^+ \mathbf{H},\, \phi'(u^+) \nabla u^+ \rangle \, dx. \tag{3.17}$$

Given $\varepsilon > 0$, define $\phi_\varepsilon \in C^1((-\varepsilon, \infty))$ by

$$\phi_\varepsilon(t) := \frac{t}{t + \varepsilon}, \quad t \in [0, \infty).$$

Then, clearly $\phi_\varepsilon \geq 0$ on $[0, \infty)$, $\phi_\varepsilon(0) = 0$, and $\phi'_\varepsilon \in L^\infty((0, \infty))$, where

$$\phi'_\varepsilon(t) = \frac{\varepsilon}{(t + \varepsilon)^2} \quad \text{for all } t \in [0, \infty).$$

Thus, substituting $\phi = \phi_\varepsilon$ into (3.17) yields

$$\int_U \frac{1}{(u^+ + \varepsilon)^2} \langle A \nabla u^+,\, \nabla u^+ \rangle \, dx \leq \int_U \left\langle -u^+ \mathbf{H},\, \frac{1}{(u^+ + \varepsilon)^2} \nabla u^+ \right\rangle dx. \tag{3.18}$$

For each $\varepsilon > 0$, define $\psi_\varepsilon \in C^1((-\varepsilon, \infty))$ by

$$\psi_\varepsilon(t) := \ln\left(1 + \frac{t}{\varepsilon}\right), \quad t \in [0, \infty).$$

Then, $\psi'_\varepsilon \in L^\infty((0, \infty))$ satisfies

$$\psi'_\varepsilon(t) = \frac{1}{t + \varepsilon} \quad \text{for all } t \in [0, \infty).$$

Again, by Lemma 3.4, $\psi_\varepsilon(u^+) \in H_0^{1,2}(U)$, and inequality (3.18) implies that

$$\begin{aligned}\lambda\|\nabla\psi_\varepsilon(u^+)\|^2_{L^2(U)} &\le \int_U \langle A\nabla\psi_\varepsilon(u^+),\ \nabla\psi_\varepsilon(u^+)\rangle\, dx\\ &\le \int_U \left\langle \frac{-u^+}{u^+ + \varepsilon}\mathbf{H},\ \nabla\psi_\varepsilon(u^+)\right\rangle dx\\ &\le \|\mathbf{H}\|_{L^2(U)}\|\nabla\psi_\varepsilon(u^+)\|_{L^2(U)}.\end{aligned}$$

Hence,

$$\|\nabla\psi_\varepsilon(u^+)\|_{L^2(U)} \le \frac{1}{\lambda}\|\mathbf{H}\|_{L^2(U)}.$$

By Proposition 3.3, it follows that

$$\|\psi_\varepsilon(u^+)\|^2_{L^2(U)} \le \frac{4(d-1)^2}{\lambda^2 d^2}|U|^{\frac{2}{d}}\|\mathbf{H}\|^2_{L^2(U)}.$$

Now, suppose there exists a measurable subset $V \subset U$ with $|V| > 0$ such that $u^+(x) > 0$ for all $x \in V$. Then, by Fatou's lemma,

$$\infty = \int_V \liminf_{\varepsilon\to 0^+} |\psi_\varepsilon(u^+)|^2\, dx \le \liminf_{\varepsilon\to 0^+} \int_V |\psi_\varepsilon(u^+)|^2\, dx \le \frac{4(d-1)^2}{\lambda^2 d^2}|U|^{\frac{2}{d}}\|\mathbf{H}\|^2_{L^2(U)} < \infty,$$

which is a contradiction. Therefore, $u^+ = 0$ a.e. in U, as desired. □

Corollary 3.6. *Assume* **(T1)**. *Let $\mathcal{B} : H_0^{1,2}(U)\times H_0^{1,2}(U) \to \mathbb{R}$ denote the bilinear form defined by* (3.1). *Let $g \in H^{-1,2}(U)$. Then, there exists a unique function $u_g \in H_0^{1,2}(U)$ such that*

$$\mathcal{B}(u_g, \varphi) = \langle g, \varphi\rangle_{H^{-1,2}(U)} \quad \textit{for all } \varphi \in H_0^{1,2}(U).$$

Proof. Let $w \in H_0^{1,2}(U)$ satisfy

$$\mathcal{B}(w, \varphi) = 0 \quad \text{for all } \varphi \in H_0^{1,2}(U).$$

Then, by Proposition 3.5, $w \le 0$ and $-w \le 0$ in U, and hence $w = 0$ in U. Thus, the assertion follows from Proposition 3.2. □

The following lemma provides a standard energy estimate, but due to the assumption that $\mathbf{H} \in L^d(U, \mathbb{R}^d)$, a delicate use of partition of unity and compactness arguments is required. In contrast, if one assumes $\mathbf{H} \in L^p(U, \mathbb{R}^d)$ for some $p > d$, the estimate could likely be derived more easily using an interpolation inequality. We refer to [15] for the derivation of the $H^{1,q}$-estimate under appropriate regularity assumptions on the coefficient matrix A and the domain U.

Lemma 3.7. *Assume* **(T1)**. *For each $n \ge 1$, let $A_n = (a_{n,ij})_{1\le i,j\le d}$ be a matrix of measurable functions on $\mathbb{R}^d$, satisfying*

$$\max_{1\le i,j\le d} |a_{n,ij}(x)| \le M, \quad \langle A_n(x)\xi, \xi\rangle \ge \lambda\|\xi\|^2 \quad \textit{for a.e. } x \in \mathbb{R}^d \textit{ and all } \xi \in \mathbb{R}^d. \tag{3.19}$$

Assume also that $\lim_{n\to\infty} a_{n,ij} = a_{ij}$ *in* $L^2(U)$ *for all* $1 \le i, j \le d$. *Let* η *be a standard mollifier on* $\mathbb{R}^d$, *and for each* $n \ge 1$, *define* $\eta_n \in C_0^\infty(B_{1/n})$ *given by* $\eta_n(x) := n^d \eta(nx)$, $x \in \mathbb{R}^d$. *Define*

$$\mathbf{H}_n := \mathbf{H} * \eta_n, \quad n \ge 1,$$

where $\mathbf{H}$ *is the zero extension of* $\mathbf{H} \in L^d(U, \mathbb{R}^d)$ *to* $\mathbb{R}^d$. *Given* $g \in H^{-1,2}(U)$, *let* $u_{n,g} \in H_0^{1,2}(U)$ *be the unique function satisfying*

$$\int_U \langle A_n \nabla u_{n,g} + u_{n,g}\, \mathbf{H}_n, \nabla \varphi \rangle \, dx = \langle g, \varphi \rangle_{H^{-1,2}(U)} \quad \text{for every } \varphi \in H_0^{1,2}(U), \tag{3.20}$$

as in Corollary 3.6. Then, there exist constants $c_1, c_2 > 0$ *which only depend on* d, λ, M, $\mathbf{H}$ *and* U ($c_1, c_2 > 0$ *are independent of* n *and* g) *such that*

$$\|\nabla u_{n,g}\|_{L^2(U)} \le c_1 \|u_{n,g}\|_{L^2(U)} + c_2 \|g\|_{H^{-1,2}(U)}.$$

Proof. First, note that for any open sets V, W with $\overline{V} \subset W$,

$$\|\mathbf{H}_n\|_{L^d(V)} \le \|\mathbf{H}\|_{L^d(W)} \tag{3.21}$$

(see the proof of [10, Theorem 7, Appendices]). Let $x \in \overline{U}$ and $r_x > 0$ be such that

$$\frac{2(d-1)}{d-2} \|\mathbf{H}\|_{L^d(B_{2r_x}(x))} \le \frac{\lambda}{4}. \tag{3.22}$$

Let $\zeta \in C_0^\infty(B_{r_x}(x))$. Given $g \in H^{-1,2}(U)$, substituting $\varphi = \zeta^2 u_{n,g} \in H_0^{1,2}(U)$ in (3.20) and using (3.5), we have

$$\begin{aligned}
&\lambda \|\zeta \nabla u_{n,g}\|^2_{L^2(U)} \le \int_U \langle A_n \nabla u_{n,g}, \zeta^2 \nabla u_{n,g} \rangle \, dx \\
&= -\int_U \langle A_n \nabla u_{n,g}, 2u_{n,g} \zeta \nabla \zeta \rangle \, dx - \int_U \langle u_{n,g} \mathbf{H}_n, 2\zeta u_{n,g} \nabla \zeta \rangle \, dx \\
&\qquad - \int_U \langle u_{n,g} \mathbf{H}_n, \zeta^2 \nabla u_{n,g} \rangle \, dx + \langle g, \zeta^2 u_{n,g} \rangle_{H^{-1,2}(U)} \\
&\le 2dM \|\zeta \nabla u_{n,g}\|_{L^2(U)} \|u_{n,g} \nabla \zeta\|_{L^2(U)} + 2\|\mathbf{H}_n\|_{L^d(B_{r_x}(x))} \|\zeta u_{n,g}\|_{L^{\frac{2d}{d-2}}(U)} \|u_{n,g} \nabla \zeta\|_{L^2(U)} \\
&\qquad + \|\mathbf{H}_n\|_{L^d(B_{r_x}(x))} \|\zeta u_{n,g}\|_{L^{\frac{2d}{d-2}}(U)} \|\zeta \nabla u_{n,g}\|_{L^2(U)} + \|g\|_{H^{-1,2}(U)} \left(\|\zeta^2 \nabla u_{n,g}\|_{L^2(U)} + \|2\zeta u_{n,g} \nabla \zeta\|_{L^2(U)} \right) \\
&\le 2dM \|\zeta \nabla u_{n,g}\|_{L^2(U)} \|u_{n,g} \nabla \zeta\|_{L^2(U)} + \frac{4(d-1)}{d-2} \|\mathbf{H}_n\|_{L^d(B_{r_x}(x))} \|\nabla(\zeta u_{n,g})\|_{L^2(U)} \|u_{n,g} \nabla \zeta\|_{L^2(U)} \\
&\qquad + \frac{2(d-1)}{d-2} \|\mathbf{H}_n\|_{L^d(B_{r_x}(x))} \|\nabla(\zeta u_{n,g})\|_{L^2(U)} \|\zeta \nabla u_{n,g}\|_{L^2(U)} \\
&\qquad + \|g\|_{H^{-1,2}(U)} \|\zeta\|_{L^\infty(U)} \|\zeta \nabla u_{n,g}\|_{L^2(U)} + \|g\|_{H^{-1,2}(U)} \|2\zeta\|_{L^\infty(U)} \|u_{n,g} \nabla \zeta\|_{L^2(U)}.
\end{aligned}$$

Using Young's inequality, we obtain that

$$\frac{\lambda}{2} \|\zeta \nabla u_{n,g}\|^2_{L^2(U)}$$

$$\leq \left(\frac{8d^2M^2}{\lambda} + \frac{40(d-1)^2}{\lambda(d-2)^2}\|\mathbf{H}_n\|^2_{L^d(B_{r_x}(x))} + \frac{4(d-1)}{d-2}\|\mathbf{H}_n\|_{L^d(B_{r_x}(x))} + 2\|\zeta\|^2_{L^\infty(U)}\right)\|u_{n,g}\nabla\zeta\|^2_{L^2(U)}$$
$$+ \left(\frac{2}{\lambda}\|\zeta\|^2_{L^\infty(U)} + \frac{1}{2}\right)\|g\|^2_{H^{-1,2}(U)} + \frac{2(d-1)}{d-2}\|\mathbf{H}_n\|_{L^d(B_{r_x}(x))}\|\zeta\nabla u_{n,g}\|^2_{L^2(U)}. \tag{3.23}$$

Applying (3.21) and (3.22) to (3.23),

$$\|\zeta\nabla u_{n,g}\|^2_{L^2(U)}$$
$$\leq \frac{4}{\lambda}\left(\frac{8d^2M^2}{\lambda} + \frac{40(d-1)^2}{\lambda(d-2)^2}\|\mathbf{H}\|^2_{L^d(U)} + \frac{4(d-1)}{d-2}\|\mathbf{H}\|_{L^d(U)} + 2\|\zeta\|^2_{L^\infty(U)}\right)\|\nabla\zeta\|^2_{L^\infty(U)}\|u_{n,g}\|^2_{L^2(U)}$$
$$+ \frac{4}{\lambda}\left(\frac{2}{\lambda} + \frac{1}{2}\right)\|g\|^2_{H^{-1,2}(U)}.$$

Since $\overline{U}$ is compact and $\{B_{r_x}(x) : x \in \overline{U}\}$ is an open cover of $\overline{U}$, there exists $x_1, \dots, x_N \in \overline{U}$ such that

$$\overline{U} \subset \bigcup_{i=1}^N B_{r_{x_i}}(x_i).$$

Let $(\zeta_i)_{i=1}^N$ be the smooth partition of unity with $\text{supp}(\zeta_i) \subset B_{r_{x_i}}(x_i)$ such that

$$\sum_{i=1}^N \zeta_i = 1 \quad \text{on } \overline{U}.$$

Therefore,

$$\|\nabla u_{n,g}\|_{L^2(U)} = \left\|\sum_{i=1}^N \zeta_i\nabla u_{n,g}\right\|_{L^2(U)} \leq \sum_{i=1}^N \left\|\zeta_i\nabla u_{n,g}\right\|_{L^2(U)}$$
$$\leq c_1\|u_{n,g}\|_{L^2(U)} + c_2\|g\|_{H^{-1,2}(U)},$$

where

$$c_1 = \sum_{i=1}^N \frac{2}{\sqrt{\lambda}}\left(\frac{8d^2M^2}{\lambda} + \frac{40(d-1)^2}{\lambda(d-2)^2}\|\mathbf{H}\|^2_{L^d(U)} + \frac{4(d-1)}{d-2}\|\mathbf{H}\|_{L^d(U)} + 2\|\zeta_i\|^2_{L^\infty(U)}\right)^{\frac{1}{2}}\|\nabla\zeta_i\|_{L^\infty(U)}$$

and

$$c_2 = \frac{2N}{\sqrt{\lambda}}\left(\frac{2}{\lambda} + \frac{1}{2}\right)^{\frac{1}{2}}.$$

□

The following lemma is inspired by the compactness arguments in [10, Section 6.2, Theorem 6], and its key feature is that the constant $C > 0$ remains independent of both the index n and the external data $g \in H^{-1,2}(U)$, even though the coefficients are given as a sequence rather than a single function. Different from [16, Lemma 3.3], the main feature here is that uniform estimates are obtained for the mollifications of $\mathbf{H}$, assuming $\mathbf{H} \in L^d(U, \mathbb{R}^d)$.

Lemma 3.8. *Assume* **(T1)**. *For each $n \geq 1$, let $A_n = (a_{n,ij})_{1\leq i,j\leq d}$ be a matrix of measurable functions on $\mathbb{R}^d$ satisfying* (3.19). *Assume also that $\lim_{n\to\infty} a_{n,ij} = a_{ij}$ in $L^2(U)$ for all $1 \leq i, j \leq d$. Let η be a standard mollifier on $\mathbb{R}^d$, and for each $n \geq 1$, define $\eta_n \in C_0^\infty(B_{1/n})$ given by $\eta_n(x) := n^d\eta(nx)$, $x \in \mathbb{R}^d$. Define*

$$\mathbf{H}_n := \mathbf{H} * \eta_n,$$

where $\mathbf{H}$ is the zero extension of $\mathbf{H} \in L^d(U, \mathbb{R}^d)$ to $\mathbb{R}^d$. Given $g \in H^{-1,2}(U)$, let $u_{n,g} \in H_0^{1,2}(U)$ be the unique function satisfying

$$\int_U \langle A_n\nabla u_{n,g} + u_{n,g}\,\mathbf{H}_n, \nabla\varphi\rangle\, dx = \langle g, \varphi\rangle_{H^{-1,2}(U)} \quad \text{for every } \varphi \in H_0^{1,2}(U),$$

as in Corollary 3.6. Then, the following statements hold:

(i) There exists a constant $C > 0$ independent of $n \geq 1$ and $g \in H^{-1,2}(U)$ such that

$$\|u_{n,g}\|_{L^2(U)} \leq C\|g\|_{H^{-1,2}(U)} \quad \text{for all } n \geq 1 \text{ and } g \in H^{-1,2}(U). \tag{3.24}$$

Moreover,

$$\|\nabla u_{n,g}\|_{L^2(U)} \leq (c_1C + c_2)\|g\|_{H^{-1,2}(U)}, \tag{3.25}$$

where $c_1, c_2 > 0$ are constants as in Lemma 3.7.

(ii) Given $g \in H^{-1,2}(U)$, let $u_g \in H_0^{1,2}(U)$ be the unique function satisfying

$$\int_U \langle A\nabla u_g + u_g\,\mathbf{H}, \nabla\varphi\rangle\, dx = \langle g, \varphi\rangle_{H^{-1,2}(U)} \quad \text{for every } \varphi \in H_0^{1,2}(U),$$

as in Corollary 3.6. Then, there exists a subsequence of $(u_{n,g})_{n\geq1}$, say again $(u_{n,g})_{n\geq1}$ such that

$$\lim_{n\to\infty} u_{n,g} = u_g \quad \text{weakly in } H_0^{1,2}(U) \quad \text{and} \quad \lim_{n\to\infty} u_{n,g} = u_g \quad \text{in } L^2(U). \tag{3.26}$$

In particular,

$$\|u_g\|_{L^2(U)} \leq C\|g\|_{H^{-1,2}(U)} \quad \text{for all } n \geq 1 \text{ and } g \in H^{-1,2}(U).$$

and

$$\|\nabla u_g\|_{L^2(U)} \leq (c_1C + c_2)\|g\|_{H^{-1,2}(U)},$$

where $C > 0$ is the constant as in (i) and $c_1, c_2 > 0$ are constants as in Lemma 3.7.

Proof. (i) Suppose, by contradiction, that (3.24) does not hold. Then, for each $k \in \mathbb{N}$, there exist $\tilde{g}_k \in H^{-1,2}(U)$ and $n_k \in \mathbb{N}$ such that

$$\|u_{n_k,\tilde{g}_k}\|_{L^2(U)} > k\|\tilde{g}_k\|_{H^{-1,2}(U)}.$$

Define

$$g_k := \frac{\tilde{g}_k}{\|u_{n_k,\tilde{g}_k}\|_{L^2(U)}} \in H^{-1,2}(U).$$

By Corollary 3.6, it follows that

$$u_{n_k,g_k} = \frac{u_{n_k,\tilde{g}_k}}{\|u_{n_k,\tilde{g}_k}\|_{L^2(U)}}.$$

Therefore,

$$\|u_{n_k,g_k}\|_{L^2(U)} = 1 \tag{3.27}$$

and

$$\|g_k\|_{H^{-1,2}(U)} < \frac{1}{k}. \tag{3.28}$$

Meanwhile, we have

$$\int_U \langle A_{n_k} \nabla u_{n_k,g_k} + u_{n_k,g_k} \mathbf{H}_{n_k}, \nabla\varphi\rangle\, dx = \langle g_k, \varphi\rangle_{H^{-1,2}(U)} \quad \text{for every } \varphi \in H_0^{1,2}(U).$$

Using Lemma 3.7, (3.27) and (3.28), it follows that

$$\begin{aligned} \|\nabla u_{n_k,g_k}\|_{L^2(U)} &\le c_1 \|u_{n,g}\|_{L^2(U)} + c_2 \|g\|_{H^{-1,2}(U)} \\ &\le c_1 + c_2, \end{aligned} \tag{3.29}$$

where $c_1, c_2 > 0$ are constants which only depend on d, λ, M, $\mathbf{H}$ and U (c_1, c_2 are independent of n and g).

Case 1) Suppose that the set $\{n_k : k \ge 1\}$ is bounded. Then, there exists $N \in \mathbb{N}$ and a subsequence $(k_j)_{j\ge1} \subset (k)_{k\ge1}$ such that $n_{k_j} = N$ for all $j \ge 1$. In this case, from (3.29), we deduce that

$$\|\nabla u_{N,g_{k_j}}\|_{L^2(U)} \le c_1 + c_2 \quad \text{for all } j \ge 1.$$

Moreover,

$$\int_U \langle A_N \nabla u_{N,g_{k_j}} + u_{N,g_{k_j}} \mathbf{H}_N, \nabla\varphi\rangle\, dx = \langle g_{k_j}, \varphi\rangle_{H^{-1,2}(U)} \quad \text{for all } \varphi \in H_0^{1,2}(U) \text{ and } j \ge 1. \tag{3.30}$$

By the weak compactness of bounded sets in $H_0^{1,2}(U)$ and the Rellich-Kondrachov compactness theorem, there exists a subsequence of $(u_{N,g_{k_j}})_{j\ge1}$, which we denote again by $(u_{N,g_{k_j}})_{j\ge1}$, and a function $u \in H_0^{1,2}(U)$ such that

$$\lim_{j\to\infty} u_{N,g_{k_j}} = u \quad \text{weakly in } H_0^{1,2}(U), \qquad \lim_{j\to\infty} u_{N,g_{k_j}} = u \quad \text{in } L^2(U). \tag{3.31}$$

Passing to the limit in (3.30) along this subsequence and using the fact that $g_{k_j} \to 0$ in $H^{-1,2}(U)$ as $j \to \infty$ (see (3.28)), we obtain

$$\int_U \langle A_N \nabla u + u\,\mathbf{H}_N, \nabla\varphi\rangle\, dx = 0 \quad \text{for all } \varphi \in H_0^{1,2}(U).$$

By the uniqueness in Corollary 3.6, it follows that $u = 0$ in U. On the other hand, by (3.27) and (3.31),

$$1 = \lim_{j\to\infty} \|u_{N,g_{k_j}}\|_{L^2(U)} = \|u\|_{L^2(U)} = 0,$$

which is a contradiction.

Case 2) Suppose now that the set $\{n_k : k \ge 1\}$ is unbounded. Then, there exists a subsequence $(k_j)_{j\ge1} \subset (k)_{k\ge1}$ such that

$$\lim_{j\to\infty} n_{k_j} = \infty$$

and

$$\int_U \langle A_{n_{k_j}} \nabla u_{n_{k_j},g_{k_j}} + u_{n_{k_j},g_{k_j}} \mathbf{H}_{n_{k_j}}, \nabla\varphi \rangle \, dx = \langle g_{k_j}, \varphi \rangle_{H^{-1,2}(U)} \quad \text{for all } \varphi \in H_0^{1,2}(U). \tag{3.32}$$

By (3.29), we have

$$\|\nabla u_{n_{k_j},g_{k_j}}\|_{L^2(U)} \le c_1 + c_2.$$

Consequently, there exists a subsequence of $(u_{n_{k_j},g_{k_j}})_{j\ge1}$, say again $(u_{n_{k_j},g_{k_j}})_{j\ge1}$, and a function $u \in H_0^{1,2}(U)$ such that

$$\lim_{j\to\infty} u_{n_{k_j},g_{k_j}} = u \quad \text{weakly in } H_0^{1,2}(U), \qquad \lim_{j\to\infty} u_{n_{k_j},g_{k_j}} = u \quad \text{in } L^2(U). \tag{3.33}$$

Using (3.33), we now pass to the limit in the weak formulation in (3.32), and hence we get

$$\int_U \langle A\nabla u + u\,\mathbf{H}, \nabla\varphi \rangle \, dx = 0 \quad \text{for all } \varphi \in H_0^{1,2}(U).$$

By the uniqueness in Corollary 3.6, it follows that $u = 0$ in U. On the other hand, by (3.27) and (3.33),

$$1 = \lim_{j\to\infty} \|u_{n_{k_j},g_{k_j}}\|_{L^2(U)} = \|u\|_{L^2(U)} = 0.$$

Since both cases result in a contradiction, the initial assumption must be false. Thus, (3.24) does hold. Therefore, (3.25) directly follows from Lemma 3.7.

(ii) By the weak compactness of bounded subsets in $H_0^{1,2}(U)$ and the Rellich-Kondrachov compactness theorem applied to (3.25), there exists a subsequence of $(u_{n,g})_{n\ge1}$, which we still denote by $(u_{n,g})_{n\ge1}$, such that (3.26) holds. The rest follows from the lower semi-continuity of the norm used in the estimates (3.24) and (3.25). □

Remark 3.9. *Assume* **(T1)***, where $d \ge 3$ is replaced by $d = 2$, and suppose that $\mathbf{H} \in L^p(U,\mathbb{R}^2)$ for some $p \in (2,\infty)$. In analogy with the proofs of Propositions 3.1, 3.2, 3.5 and Corollary 3.6, we obtain that for each $g \in H^{-1,2}(U)$, there exists a unique function $u_g \in H_0^{1,2}(U)$ satisfying*

$$\int_U \langle A\nabla u_g + u_g\,\mathbf{H}, \nabla\varphi \rangle \, dx = \langle g, \varphi \rangle_{H^{-1,2}(U)} \quad \textit{for all } \varphi \in H_0^{1,2}(U).$$

For each $n \ge 1$, let $A_n = (a_{n,ij})_{1\le i,j\le d}$ be a matrix-valued function satisfying (3.19)*, and assume that* $\lim_{n\to\infty} a_{n,ij} = a_{ij}$ *in $L^2(U)$ for all $1 \le i, j \le d$. Let $\mathbf{H}_n \in L^p(U,\mathbb{R}^2)$ be the mollification of the zero extension of $\mathbf{H}$ to $\mathbb{R}^d$ as in Lemma 3.8. For each $n \ge 1$, let $u_{n,g} \in H_0^{1,2}(U)$ be the unique function satisfying*

$$\int_U \langle A_n\nabla u_{n,g} + u_{n,g}\,\mathbf{H}_n, \nabla\varphi \rangle \, dx = \langle g, \varphi \rangle_{H^{-1,2}(U)} \quad \textit{for all } \varphi \in H_0^{1,2}(U).$$

Then, by a similar argument to that in the proof of Lemma 3.8, we obtain the results in Lemma 3.8.

The following lemma is a generalization of [16, Lemma 3.4], and in particular, it remains valid even under the assumption $\mathbf{H} \in L^d(U,\mathbb{R}^d)$. Its proof requires a highly delicate approximation argument, in which Lemma 3.8 plays a central role.

Lemma 3.10. *Assume* (**T1**). *Assume that $u \in H^{1,2}(U)$ satisfies*

$$\int_U \langle A\nabla u + u\mathbf{H}, \nabla\varphi\rangle dx \le 0 \quad \textit{for all } \varphi \in C_0^\infty(U),\ \varphi \ge 0.$$

Then, we have

$$\int_U \langle A\nabla u^+ + u^+\mathbf{H}, \nabla\varphi\rangle dx \le 0 \quad \textit{for all } \varphi \in C_0^\infty(U),\ \varphi \ge 0.$$

Proof. Let V be an arbitrary open subset of U with $\overline{V} \subset U$. To show the assertion, it is enough to show that

$$\int_U \langle A\nabla u^+ + u^+\mathbf{H}, \nabla\varphi\rangle dx \le 0 \quad \text{for all } \varphi \in C_0^\infty(V),\ \varphi \ge 0.$$

Let W be an open set with a smooth boundary such that

$$\overline{V} \subset W \subset \overline{W} \subset U.$$

Let B be an open ball such that $\overline{U} \subset B$. By [11, Theorem 4.7], $u \in H^{1,2}(W)$ can be extended to a function $\hat{u} \in H_0^{1,2}(B)$. Moreover, by [11, Theorem 4.4(iii)], we have $\hat{u}^+ \in H_0^{1,2}(B)$ with

$$\nabla\hat{u}^+ = \begin{cases} \nabla\hat{u} & \text{a.e. on } \{\hat{u} > 0\}, \\ 0 & \text{a.e. on } \{\hat{u} \le 0\}. \end{cases}$$

Extend $\mathbf{H} \in L^d(U, \mathbb{R}^d)$ to $\mathbb{R}^d$ by zero extension. Define

$$\mathbf{F} := A\nabla\hat{u} + \hat{u}\mathbf{H} \in L^2(\mathbb{R}^d, \mathbb{R}^d).$$

Let η be a standard mollifier on $\mathbb{R}^d$, and for each $n \ge 1$, define $\eta_n \in C_0^\infty(B_{1/n})$ by $\eta_n(x) := n^d\eta(nx)$, $x \in \mathbb{R}^d$. For each $n \in \mathbb{N}$ and $1 \le i, j \le d$, define

$$a_{n,ij} := a_{ij} * \eta_n, \quad A_n := (a_{n,ij})_{1\le i,j\le d}, \quad \mathbf{H}_n := \mathbf{H} * \eta_n, \quad \mathbf{F}_n := \mathbf{F} * \eta_n \quad \text{on } \mathbb{R}^d.$$

Then, $a_{n,ij} \in C^\infty(\mathbb{R}^d)$, and $\mathbf{H}_n, \mathbf{F}_n \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ for all $n \ge 1$ and $1 \le i, j \le d$, and it holds that

$$\lim_{n\to\infty} a_{n,ij} = a_{ij} \text{ in } L^2(B, \mathbb{R}^d), \quad \lim_{n\to\infty} \mathbf{H}_n = \mathbf{H} \text{ in } L^d(B, \mathbb{R}^d), \quad \lim_{n\to\infty} \mathbf{F}_n = \mathbf{F} \text{ in } L^2(B, \mathbb{R}^d).$$

Furthermore, (3.19) holds. Choose $\delta > 0$ such that $\overline{B}_\delta(x) \subset W$ for all $x \in \overline{V}$. Pick $N \in \mathbb{N}$ with $\frac{1}{N} < \delta$. Then, for any $n \ge N$ and $\varphi \in C_0^\infty(V)$ with $\varphi \ge 0$, we have $\varphi * \eta_n \in C_0^\infty(W)$, $\varphi * \eta_n \ge 0$, and

$$\begin{aligned} \int_U \langle \mathbf{F}_n, \nabla\varphi\rangle dx &= \int_{\mathbb{R}^d} \langle \mathbf{F}_n, \nabla\varphi\rangle dx = \int_{\mathbb{R}^d} \langle \mathbf{F}, \nabla(\varphi * \eta_n)\rangle dx \\ &= \int_{\mathbb{R}^d} \langle A\nabla\hat{u} + \hat{u}\mathbf{H}, \nabla(\varphi * \eta_n)\rangle dx = \int_U \langle A\nabla u + u\mathbf{H}, \nabla(\varphi * \eta_n)\rangle dx \le 0. \end{aligned} \tag{3.34}$$

According to Corollary 3.6, there exists a unique function $u_n \in H_0^{1,2}(B)$ such that

$$\int_B \langle A_n\nabla u_n + u_n\mathbf{H}_n, \nabla\tilde{\varphi}\rangle dx = \int_B \langle \mathbf{F}_n, \nabla\tilde{\varphi}\rangle dx \quad \text{for all } \tilde{\varphi} \in C_0^\infty(B). \tag{3.35}$$

By Lemma 3.8(i), we obtain

$$\|u_n\|_{H_0^{1,2}(B)} \leq (c_1 C + c_2)\|\mathbf{F}_n\|_{L^2(B,\mathbb{R}^d)} \leq (c_1 C + c_2)\|\mathbf{F}\|_{L^2(B,\mathbb{R}^d)},$$

where $c_1, c_2 > 0$ are constants as in Lemma 3.7 and $C > 0$ is the constant as in Lemma 3.8(i). By the weak compactness of bounded subsets in $H_0^{1,2}(B)$ and using [11, Theorem 4.4(iii)], there exist $\tilde{u} \in H_0^{1,2}(B)$ and a subsequence (still denoted by u_n) such that

$$\lim_{n\to\infty} u_n = \tilde{u} \quad \text{and} \quad \lim_{n\to\infty} u_n^+ = \tilde{u}^+ \quad \text{weakly in } H_0^{1,2}(B). \tag{3.36}$$

Hence letting $n \to \infty$, we get

$$\int_B \langle A\nabla\tilde{u} + \tilde{u}\mathbf{H}, \nabla\tilde{\varphi}\rangle dx = \int_B \langle \mathbf{F}, \nabla\tilde{\varphi}\rangle dx = \int_B \langle A\nabla\hat{u} + \hat{u}\mathbf{H}, \nabla\tilde{\varphi}\rangle dx \quad \text{for all } \tilde{\varphi} \in C_0^\infty(B).$$

By the uniqueness in Proposition 3.5, we conclude that $\tilde{u} = \hat{u}$ in $H_0^{1,2}(B)$. Thus, by (3.36), we have

$$\lim_{n\to\infty} u_n = \hat{u} \quad \text{and} \quad \lim_{n\to\infty} u_n^+ = \hat{u}^+ \quad \text{weakly in } H_0^{1,2}(B). \tag{3.37}$$

Define the operator

$$\mathcal{L}_n u_n := \sum_{i,j=1}^d a_{n,ij}\partial_i\partial_j u_n + \langle \mathbf{H}_n + \operatorname{div} A_n, \nabla u_n\rangle + (\operatorname{div}\mathbf{H}_n)u_n.$$

Then from (3.34) and (3.35), we deduce that for all $n \geq N$ and $\varphi \in C_0^\infty(V)$ with $\varphi \geq 0$,

$$-\int_V \mathcal{L}_n u_n \cdot \varphi dx = \int_V \langle A_n\nabla u_n + u_n\mathbf{H}_n, \nabla\varphi\rangle dx = \int_U \langle \mathbf{F}_n, \nabla\varphi\rangle dx \leq 0,$$

which implies

$$\mathcal{L}_n u_n \geq 0 \quad \text{in } V \quad \text{for all } n \geq N. \tag{3.38}$$

Let ϕ be a standard mollifier on $\mathbb{R}$, and for each $n \geq 1$, define $\phi_n \in C_0^\infty(-1/n, 1/n)$ by $\phi_n(t) := n\phi(nt)$ for $t \in \mathbb{R}$. For each $\varepsilon > 0$, define

$$f_\varepsilon(z) := \begin{cases} \sqrt{z^2+\varepsilon^2} - \varepsilon & \text{if } z \geq 0, \\ 0 & \text{if } z < 0. \end{cases}$$

Then, $f_\varepsilon \in C^1(\mathbb{R})$ and its derivative f_ε' belongs to $H^{1,\infty}(\mathbb{R}) \cap C(\mathbb{R})$. In particular, we have

$$f_\varepsilon'(z) = \begin{cases} \frac{z}{\sqrt{z^2+\varepsilon^2}} & \text{if } z \geq 0, \\ 0 & \text{if } z < 0, \end{cases} \quad \text{and} \quad f_\varepsilon''(z) = \begin{cases} \frac{\varepsilon^2}{(z^2+\varepsilon^2)^{3/2}} & \text{if } z > 0, \\ 0 & \text{if } z < 0. \end{cases}$$

Observe that

$$\lim_{\varepsilon\to 0+} f_\varepsilon(z) = z^+, \quad \text{and} \quad \lim_{\varepsilon\to 0+} f_\varepsilon'(z) = 1_{(0,\infty)}(z) \quad \text{for all } z \in \mathbb{R}. \tag{3.39}$$

Let $f_{\varepsilon,k} := f_\varepsilon * \phi_k$. Then $f_{\varepsilon,k}' \geq 0$ and $f_{\varepsilon,k}'' \geq 0$ on $\mathbb{R}$. Moreover,

$$\lim_{k\to\infty} f_{\varepsilon,k}(u_n) = f_\varepsilon(u_n), \quad \text{and} \quad \lim_{k\to\infty} f_{\varepsilon,k}'(u_n) = f_\varepsilon'(u_n) \quad \text{uniformly on } U. \tag{3.40}$$

Thus, for any $\varphi \in C_0^\infty(V)$ with $\varphi \geq 0$, it follows from (3.40) and (3.38) that

$$\begin{aligned}
&\int_U \langle A_n \nabla f_\varepsilon(u_n) + f_\varepsilon(u_n)\mathbf{H}_n, \nabla\varphi\rangle dx = \lim_{k\to\infty} \int_U \langle A_n \nabla f_{\varepsilon,k}(u_n) + f_{\varepsilon,k}(u_n)\mathbf{H}_n, \nabla\varphi\rangle dx \\
&= \lim_{k\to\infty} \Big(- \int_U f'_{\varepsilon,k}(u_n)\mathcal{L}_n u_n \varphi\, dx - \int_U f''_{\varepsilon,k}(u_n)\langle A_n \nabla u_n, \nabla u_n\rangle \varphi\, dx \\
&\qquad\qquad - \int_U \operatorname{div} \mathbf{H}_n (f_{\varepsilon,k}(u_n) - u_n f'_{\varepsilon,k}(u_n))\varphi\, dx \Big) \\
&\leq - \int_U \operatorname{div} \mathbf{H}_n (f_\varepsilon(u_n) - u_n f'_\varepsilon(u_n))\varphi\, dx.
\end{aligned}$$

Since the right-hand side tends to zero as $\varepsilon \to 0^+$ by (3.39), we conclude from [11, Theorem 4.4(iii)] and (3.39) that for all $n \geq N$,

$$\int_U \langle A_n \nabla u_n^+ + u_n^+ \mathbf{H}_n, \nabla\varphi\rangle dx \leq 0 \quad \text{ for all } \varphi \in C_0^\infty(V),\ \varphi \geq 0.$$

Finally, by taking the weak limit of u_n^+ in (3.37) as $n \to \infty$, we obtain

$$\int_U \langle A \nabla u^+ + u^+ \mathbf{H}, \nabla\varphi\rangle dx \leq 0 \quad \text{ for all } \varphi \in C_0^\infty(V),\ \varphi \geq 0,$$

which completes the proof. □

4. Proof of main result

The following theorem is a key result of this paper, which corresponds to [1, Theorem 3.1] but weakens the assumption on $\mathbf{H}$ from $L^p(U, \mathbb{R}^d)$ to $L^d(U, \mathbb{R}^d)$ by taking advantage of the additional structure on its divergence. This additional structure allows us to apply Hölder regularity and the Harnack inequality. The idea of the proof originates from [17, Theorem 1] (cf. [12, Chapter 2]), where the coefficient matrix A is assumed to lie in VMO.

Theorem 4.1. *Assume that* **(T)** *holds. Then the following statements hold:*

(i) Let $x_1 \in U$. Then, there exists $\rho \in H^{1,2}(B_r(x_0)) \cap C(B_r(x_0))$ with $\rho(x) > 0$ for all $x \in B_r(x_0)$ and $\rho(x_1) = 1$ such that

$$\int_{B_r(x_0)} \langle A^T \nabla\rho + \rho\mathbf{H}, \nabla\varphi\rangle dx = 0, \quad \textit{for all } \varphi \in C_0^\infty(B_r(x_0)). \tag{4.1}$$

(ii) Let ρ be as in Theorem 4.1(i). Then, there exists a constant $\tilde{K}_1 \geq 1$ which only depends on d, λ, M, $B_r(x_0)$, p, $\tilde{q}$ and $\mathbf{H}$ such that

$$1 \leq \max_{\overline{U}} \rho \leq \tilde{K}_1 \min_{\overline{U}} \rho \leq \tilde{K}_1.$$

Proof. (i) By Corollary 3.6, there exists $v \in H_0^{1,2}(B_r(x_0))$ such that

$$\int_{B_r(x_0)} \langle A^T \nabla v + v\mathbf{H}, \nabla\varphi\rangle dx = - \int_{B_r(x_0)} \langle \mathbf{H}, \nabla\varphi\rangle\, dx \quad \text{ for all } \varphi \in H_0^{1,2}(B_r(x_0)). \tag{4.2}$$

Let $w = v + 1 \in H^{1,2}(B_r(x_0))$. Let $\mathcal{T} : H^{1,2}(B_r(x_0)) \to L^2(\partial B_r(x_0))$ be the trace operator as in [11, Theorem 4.6]. Then,

$$\mathcal{T}(w) = \mathcal{T}(v) + 1 = 1 \quad \text{in } L^2(\partial B_r(x_0)). \tag{4.3}$$

Observe that from (4.2)

$$\int_{B_r(x_0)} \langle A^T \nabla w + w\mathbf{H}, \nabla\varphi\rangle dx = 0 \quad \text{for all } \varphi \in H_0^{1,2}(B_r(x_0)). \tag{4.4}$$

Meanwhile, $-w = -v - 1 \le -v$ in $B_r(x_0)$, and hence $0 \le (-w)^+ \le (-v)^+$ in $B_r(x_0)$. Since $(-v)^+ \in H_0^{1,2}(B_r(x_0))$, it follows by [1, Proposition A.9] that $(-w)^+ \in H_0^{1,2}(B_r(x_0))$. Therefore, applying Lemma 3.10 to (4.4) where w is replaced by $-w$, we have

$$\int_{B_r(x_0)} \langle A^T \nabla(-w)^+ + (-w)^+\mathbf{H}, \nabla\varphi\rangle dx \le 0 \quad \text{for all } \varphi \in H_0^{1,2}(B_r(x_0)) \text{ with } \varphi \ge 0.$$

By Proposition 3.5, $(-w)^+ \le 0$ in $B_r(x_0)$, which implies

$$w \ge 0 \quad \text{in } B_r(x_0). \tag{4.5}$$

Let $w_n \in C_0^\infty(B_r(x_0))$ be such that $\lim_{n\to\infty} w_n = w$ in $H_0^{1,2}(B_r(x_0))$. For each $\varphi \in C_0^\infty(B_r(x_0))$, we have

$$\begin{aligned}
\int_{B_r(x_0)} \langle w\mathbf{H}, \nabla\varphi\rangle\, dx &= \int_{B_r(x_0)} \langle w\mathbf{H}_1, \nabla\varphi\rangle\, dx + \int_{B_r(x_0)} \langle w\mathbf{H}_2, \nabla\varphi\rangle\, dx \\
&= \int_{B_r(x_0)} \langle w\mathbf{H}_1, \nabla\varphi\rangle\, dx + \lim_{n\to\infty}\left(\int_{B_r(x_0)} \langle \mathbf{H}_2, \nabla(w_n\varphi)\rangle\, dx - \int_{B_r(x_0)} \langle \mathbf{H}_2, \varphi\nabla w_n\rangle\, dx\right) \\
&= \int_{B_r(x_0)} \langle w\mathbf{H}_1, \nabla\varphi\rangle\, dx + \lim_{n\to\infty}\left(-\int_{B_r(x_0)} \tilde{h} w_n\varphi\, dx - \int_{B_r(x_0)} \langle \mathbf{H}_2, \varphi\nabla w_n\rangle\, dx\right) \\
&= \int_{B_r(x_0)} \langle w\mathbf{H}_1, \nabla\varphi\rangle\, dx - \int_{B_r(x_0)} \langle \mathbf{H}_2, \nabla w\rangle\varphi\, dx - \int_{B_r(x_0)} \tilde{h} w\varphi\, dx.
\end{aligned}$$

Thus, (4.4) implies that

$$\int_{B_r(x_0)} \langle A^T \nabla w + w\mathbf{H}_1, \nabla\varphi\rangle dx - \int_{B_r(x_0)} \Big(\langle \mathbf{H}_2, \nabla w\rangle + \tilde{h} w\Big)\varphi\, dx = 0 \quad \text{for all } \varphi \in H_0^{1,2}(B_r(x_0)). \tag{4.6}$$

Since $\mathbf{H}_1 \in L^p(B_r(x_0), \mathbb{R}^d)$, $\mathbf{H}_2 \in L^d(B_r(x_0), \mathbb{R}^d)$ and $\tilde{h} \in L^{\tilde{q}}(B_r(x_0))$ with $\tilde{q} \in (\frac{d}{2}, \infty)$, it follows by [2, Théorème 7.2] that w has a continuous version in $B_r(x_0)$, say again $w \in H^{1,2}(B_r(x_0)) \cap C(B_r(x_0))$ (indeed, w has a locally Hölder continuous version in $B_r(x_0)$). Moreover, it follows from (4.5) that $w(x) \ge 0$ for all $x \in B_r(x_0)$.

Claim: $w(x) > 0$ for every $x \in B_r(x_0)$.

To show the claim, we proceed by contradiction. Suppose there exists $y_0 \in B_r(x_0)$ such that $w(y_0) = 0$. Then, applying the Harnack inequality (see [2, Théorème 8.1]) to (4.6), we deduce that w must vanish identically on $B_R(x_0)$ for all $R \in (\|y_0 - x_0\|, r)$. Given that R is arbitrary, it follows that $w = 0$ on $B_r(x_0)$, which implies $\mathcal{T}(w) = 0$ on $L^2(\partial B_r(x_0))$. This, however, contradicts (4.3). Therefore, we conclude that our claim holds.

Let $x_1 \in U$. Since $w(x_1) > 0$, we define the normalized function $\rho \in H^{1,2}(B_r(x_0)) \cap C(B_r(x_0))$ by

$$\rho(x) := \frac{1}{w(x_1)} w(x), \quad x \in B_r(x_0).$$

Thus, (4.1) is fulfilled by (4.4).

(ii) Observe that by (4.6),

$$\int_{B_r(x_0)} \langle A^T \nabla\rho + \rho \mathbf{H}_1, \nabla\varphi\rangle dx - \int_{B_r(x_0)} \Big(\langle \mathbf{H}_2, \nabla\rho\rangle + \tilde{h}\rho\Big)\varphi\, dx = 0 \quad \text{for all } \varphi \in H_0^{1,2}(B_r(x_0)). \tag{4.7}$$

Since $\rho(x_1) = 1$, by applying the Harnack inequality ([2, Théorème 8.1]) to (4.7), the assertion follows. □

Remark 4.2. *Whether the conclusion of Theorem 4.1 can be derived under the assumption of* **(T1)** *remains an open question. However, at the very least, our current proof method for Theorem 4.1 is not sufficient to establish the result under assumption* **(T1)**. *The main difficulty arises from the fact that the assumption* $\mathbf{H} \in L^d(B_r(x_0), \mathbb{R}^d)$ *does not allow the solution to be locally bounded. To illustrate this point, consider* $d \geq 3$ *and the function*

$$w(x) := \frac{1}{\ln 2} \ln\left(1 + \frac{1}{\|x\|}\right), \quad x \in B_1 := \{x \in \mathbb{R}^d : \|x\| < 1\}. \tag{4.8}$$

Then, $w(x) > 0$ *for all* $x \in B_1 \setminus \{0\}$, *and* $w \in H^{1,2}(B_1) \cap C(B_1 \setminus \{0\})$. *Moreover, we have*

$$\mathcal{T}(w) = 1 \quad \text{in } L^2(\partial B_1),$$

where $\mathcal{T} : H_0^{1,2}(B_1) \to L^2(\partial B_1)$ *is the trace operator as in [11, Theorem 4.6]. Now define the vector field* $\mathbf{H} : B_1 \to \mathbb{R}^d$ *by*

$$\mathbf{H}(x) := -\nabla \ln w(x), \quad x \in B_1.$$

Then, $\mathbf{H} \in L^d(B_1, \mathbb{R}^d)$, *but* $\mathbf{H} \notin \bigcup_{p \in (d,\infty)} L^p(B_1, \mathbb{R}^d)$. *Direct computation shows that* w *satisfies* (4.4) *with* $B_r(x_0)$ *replaced by* B_1, *and that* w *is in fact the unique function satisfying both* $\mathcal{T}(w) = 1$ *and* (4.4). *However, the function* w *defined in* (4.8) *does not admit a locally bounded version in* B_1. *This demonstrates that the local boundedness of the solution cannot, in general, be deduced under the sole assumption* $\mathbf{H} \in L^d(B_1, \mathbb{R}^d)$. *(If* $\mathbf{H} \in L^p(B_1, \mathbb{R}^d)$ *with* $p \in (d, \infty)$, *then the local boundedness of a solution follows by [4, Theorem 5.1]. Indeed, one can check that* $\operatorname{div} \mathbf{H} \in L^{\frac{d}{2}}(B_1)$, *but* $\operatorname{div} \mathbf{H} \notin \bigcup_{p \in (\frac{d}{2},\infty)} L^p(B_1)$. *Therefore, to obtain the local boundedness of solutions in case of* $\mathbf{H} \in L^d(B_1, \mathbb{R}^d)$, *the condition* **(T)** *regarding* $\mathbf{H}$ *is essential.*

Below, we present the core method of this paper, which transforms a general vector field into a divergence-free vector field. We shall refer to this as the divergence-free transformation. In particular, we present here a simplified form of [1, Theorem 3.2].

Theorem 4.3. **(Divergence-free transformation)** *Assume that* **(T)** *holds. Let* $\rho \in H^{1,2}(U) \cap C(\overline{U})$ *be a strictly positive function on* $\overline{U}$ *constructed as in Theorem 4.1. Define the vector field*

$$\mathbf{B} := \mathbf{H} + \frac{1}{\rho} A^T \nabla\rho \quad \text{in } U. \tag{4.9}$$

Then $\rho\mathbf{B} \in L^2(U, \mathbb{R}^d)$ *and satisfies*

$$\int_U \langle \rho\mathbf{B}, \nabla\varphi \rangle \, dx = 0 \quad \text{for all } \varphi \in C_0^\infty(U). \tag{4.10}$$

Let $f \in L^1(U)$, *and* $u \in H_0^{1,2}(U)$ *with* $cu \in L^1(U)$. *Then the following two statements are equivalent:*

(i) The function u satisfies (2.1).
(ii) The function u satisfies

$$\int_U \langle \rho A \nabla u, \nabla\varphi \rangle + \langle \rho\mathbf{B}, \nabla u \rangle \varphi + \rho c u \varphi \, dx = \int_U \rho f \varphi \, dx \quad \text{for all } \varphi \in C_0^\infty(U).$$

In other words, u is a weak solution to (1.1)*, if and only if u is a weak solution to* (1.2).

Proof. The proof is identical to that of [1, Theorem 3.2] in the case where $\mathbf{F} = 0$. □

The following two lemmas, which play a supporting role in the proof of the main result, are adapted from [1] and [18], respectively.

Lemma 4.4. *Assume* $d \geq 3$. *Let* $\hat{\lambda} > 0$ *be a constant, and let* $\hat{A} = (\hat{a}_{ij})_{1 \leq i,j \leq d}$ *be a matrix of bounded and measurable functions on* $\mathbb{R}^d$ *such that*

$$\langle \hat{A}(x)\xi, \xi \rangle \geq \hat{\lambda} \|\xi\|^2 \quad \text{for a.e. } x \in \mathbb{R}^d \text{ and all } \xi \in \mathbb{R}^d. \tag{4.11}$$

Let $\hat{\mathbf{B}} \in L^2(U, \mathbb{R}^d)$ *be a vector field satisfying*

$$\int_U \langle \hat{\mathbf{B}}, \nabla\varphi \rangle \, dx = 0 \quad \text{for all } \varphi \in C_0^\infty(U). \tag{4.12}$$

Let $\hat{c} \in L^1(U)$ *with* $\hat{c} \geq 0$, *and let* $\hat{f} \in L^q(U)$ *for some* $q \in \left(\frac{d}{2}, \infty\right)$. *Then, the following statements hold:*

(i) There exists a weak solution $\hat{u} \in H_0^{1,2}(U) \cap L^\infty(U)$ *to*

$$\begin{cases} -\operatorname{div}(\hat{A}\nabla\hat{u}) + \langle \hat{\mathbf{B}}, \nabla\hat{u} \rangle + \hat{c}\hat{u} = \hat{f} & \text{in } U, \\ \hat{u} = 0 & \text{on } \partial U, \end{cases} \tag{4.13}$$

i.e., $\hat{u} \in H_0^{1,2}(U)$ *with* $\hat{c}\hat{u} \in L^1(U)$ *satisfies*

$$\int_U \langle \hat{A}\nabla\hat{u}, \nabla\varphi \rangle + \langle \hat{\mathbf{B}}, \nabla\hat{u} \rangle \varphi + \hat{c}\hat{u}\varphi \, dx = \int_U \hat{f}\varphi \, dx \quad \text{for all } \varphi \in C_0^\infty(U).$$

Moreover, the following estimates hold:

$$\|\hat{u}\|_{H_0^{1,2}(U)} \leq \hat{K}_3 \|\hat{f}\|_{L^{\frac{2d}{d+2}}(U)}, \tag{4.14}$$

$$\|\hat{u}\|_{L^\infty(U)} \leq \hat{K}_4 \|\hat{f}\|_{L^q(U)}, \tag{4.15}$$

where $\hat{K}_3 > 0$ *depends only on d,* $\hat{\lambda}$*, and* $|U|$*, and* $\hat{K}_4 > 0$ *depends only on d,* $\hat{\lambda}$*, q, and* $|U|$.

(ii) *Let* $\hat{v} \in H_0^{1,2}(U)$ *with* $\hat{c}\hat{v} \in L^1(U)$ *be such that*

$$\int_U \langle \hat{A}\nabla\hat{v}, \nabla\varphi\rangle + \langle \hat{\mathbf{B}}, \nabla\hat{v}\rangle\varphi + \hat{c}\hat{v}\varphi\, dx = 0 \quad \text{for all } \varphi \in C_0^\infty(U).$$

Then $\hat{v} = 0$ *in* U. *In particular, the solution* $\hat{u}$ *in (i) is unique.*

(iii) *Let* $\alpha > 0$ *and* $\theta \in [1, \infty]$, *and assume that* $\hat{c} \geq \alpha$ *and* $\hat{f} \in L^\theta(U) \cap L^q(U)$. *Then* $\hat{u}$ *in (i) satisfies*

$$\|\hat{u}\|_{L^\theta(U)} \leq \frac{1}{\alpha}\|\hat{f}\|_{L^\theta(U)}. \tag{4.16}$$

Proof. The assertion follows from [1, Theorem 3.3] in the case where $\mathbf{F} = 0$. □

Lemma 4.5. *Assume* $d \geq 3$. *Let* $\hat{\lambda} > 0$ *be a constant, and let* $\hat{A} = (\hat{a}_{ij})_{1\leq i,j\leq d}$ *be a matrix of bounded and measurable functions on* $\mathbb{R}^d$ *satisfying* (4.11). *Let* $\hat{\mathbf{B}} \in L^2(U, \mathbb{R}^d)$ *be a vector field satisfying* (4.12). *Let* $\hat{c} \in L^{\frac{2d}{d+2}}(U)$ *with* $\hat{c} \geq 0$, *and let* $\hat{f} \in L^{\frac{2d}{d+2}}(U)$. *Then the following statements hold:*

(i) *There exists a unique solution* $\hat{u} \in H_0^{1,2}(U)$ *to* (4.13), *and* $\hat{u}$ *satisfies the estimate* (4.14).

(ii) *Let* $\alpha > 0$ *and* $\theta \in [1, \infty]$, *and assume that* $\hat{c} \geq \alpha$ *and* $\hat{f} \in L^\theta(U) \cap L^{\frac{2d}{d+2}}(U)$. *Then the solution* $\hat{u}$ *in (i) satisfies* (4.16).

Proof. (i) The existence and uniqueness of the solution $\hat{u}$ to (4.13), as well as the estimate (4.14), follow from [18, Theorem 1.1(i)].
(ii) The assertion follows from [18, Theorem 1.1(ii)]. □

Now, we present the proof of the main result stated in the Introduction.

Proof of Theorem 1.1

(i) Let $\rho \in H^{1,2}(U) \cap C(\overline{U})$ be a strictly positive function on $\overline{U}$ constructed as in Theorem 4.1, and define the vector field $\mathbf{B}$ as in (4.9). Then (4.10) is satisfied. Let $v \in H_0^{1,2}(U)$ with $cv \in L^1(U)$ be such that (1.4) holds. By Theorem 4.3, we obtain

$$\int_U \langle \rho A\nabla v, \nabla\varphi\rangle + \langle \rho\mathbf{B}, \nabla v\rangle\varphi + \rho c v\varphi\, dx = 0 \quad \text{for all } \varphi \in C_0^\infty(U).$$

Then, by Lemma 4.4(ii), it follows that $v = 0$ in U.
(ii) Let $f \in L^q(U)$ for some $q \in (\frac{d}{2}, \infty)$. By Lemma 4.4(i), there exists a unique function $u \in H_0^{1,2}(U) \cap L^\infty(U)$ satisfying

$$\int_U \langle \rho A\nabla u, \nabla\varphi\rangle + \langle \rho\mathbf{B}, \nabla u\rangle\varphi + \rho c u\varphi\, dx = \int_U \rho f\varphi\, dx \quad \text{for all } \varphi \in C_0^\infty(U), \tag{4.17}$$

and (4.14), (4.15) and Theorem 4.1(ii) imply

$$\begin{aligned} \|u\|_{H_0^{1,2}(U)} &\leq K_3\|f\rho\|_{L^{\frac{2d}{d+2}}(U)} \leq K_3 \max_{\overline{U}} \rho \cdot \|f\|_{L^{\frac{2d}{d+2}}(U)} \leq K_3\tilde{K}_1\|f\|_{L^{\frac{2d}{d+2}}(U)}, \\ \|u\|_{L^\infty(U)} &\leq K_4\|f\rho\|_{L^q(U)} \leq K_4 \max_{\overline{U}} \rho \cdot \|f\|_{L^q(U)} \leq K_4\tilde{K}_1\|f\|_{L^q(U)}. \end{aligned} \tag{4.18}$$

By Theorem 4.3, u is a weak solution to (1.1), so that (1.5) and (1.6) follow. The uniqueness follows from (i). Note that since $\rho c \geq \alpha \min_{\overline{U}} \rho > 0$, we may apply Lemma 4.4(iii) to (4.17) to obtain

$$\|u\|_{L^{\theta}(U)} \leq \frac{1}{\alpha \min_{\overline{U}} \rho} \|\rho f\|_{L^{\theta}(U)} \leq \frac{\max_{\overline{U}} \rho}{\alpha \min_{\overline{U}} \rho} \|f\|_{L^{\theta}(U)} \leq \frac{\tilde{K}_1}{\alpha} \|f\|_{L^{\theta}(U)}.$$

(iii) By Lemma 4.5(i), there exists $u \in H^{1,2}_0(U)$ satisfying both (4.17) and the estimate (4.18). Again, by Theorem 4.3, u is a weak solution to (1.1), and the uniqueness follows from part (i). Since $\rho c \geq \alpha \min_{\overline{U}} \rho > 0$, the contraction estimate follows from Lemma 4.5(ii).

□

The following provides an explicit example of a vector field $\mathbf{H} \in L^d(B_r(x_0), \mathbb{R}^d)$ that satisfies condition **(T)** but does not belong to $\bigcup_{p \in (d,\infty)} L^p(B_r(x_0), \mathbb{R}^d)$.

Example 4.6. *Let* $B_1 := \{x \in \mathbb{R}^d : \|x\| < 1\}$, *and define* $\Phi : B_1 \to \mathbb{R}$ *by*

$$\Phi(x) := \ln \ln \left(1 + \frac{1}{\|x\|}\right), \quad x \in B_1.$$

Then $\nabla\Phi \in L^d(B_1, \mathbb{R}^d)$, *but* $\nabla\Phi \notin \bigcup_{p \in (d,\infty)} L^p(B_1, \mathbb{R}^d)$. *By symmetry, for each* $i \in \{1, \ldots, d\}$,

$$\partial_i \Phi \in L^d(B_1), \quad \text{but} \quad \partial_i \Phi \notin \bigcup_{p \in (d,\infty)} L^p(B_1).$$

Let $\mathbf{H}_1 \in L^p(B_1, \mathbb{R}^d)$ *be an arbitrary vector field, and define* $\mathbf{H}_2 : B_1 \to \mathbb{R}^d$ *by*

$$\mathbf{H}_2 := (\partial_d \Phi, 0, \ldots, -\partial_1 \Phi) \quad \text{on } B_1.$$

Then $\mathbf{H}_2 \in L^d(B_1, \mathbb{R}^d)$, *but* $\mathbf{H}_2 \notin \bigcup_{p \in (d,\infty)} L^p(B_1, \mathbb{R}^d)$. *In particular, for all* $\varphi \in C_0^\infty(B_1)$,

$$\int_{B_1} \langle \mathbf{H}_2, \nabla\varphi \rangle \, dx = \int_{B_1} \partial_d \Phi \, \partial_1 \varphi - \partial_1 \Phi \, \partial_d \varphi \, dx = \int_{B_1} \Phi(-\partial_d \partial_1 \varphi + \partial_1 \partial_d \varphi) \, dx = 0,$$

and hence $\operatorname{div} \mathbf{H}_2 = 0 \in L^{\tilde{q}}(B_1)$ *for any* $\tilde{q} \in (\frac{d}{2}, \infty)$. *Thus, the vector field* $\mathbf{H} := \mathbf{H}_1 + \mathbf{H}_2 \in L^d(B_1, \mathbb{R}^d)$ *satisfies condition* **(T)** *but does not belong to* $\bigcup_{p \in (d,\infty)} L^p(B_1, \mathbb{R}^d)$.

5. Conclusions and discussion

This paper establishes the existence and uniqueness of weak solutions to homogeneous boundary value problems for linear elliptic equations with drift coefficients $\mathbf{H} \in L^d(U, \mathbb{R}^d)$, under the assumption that $\mathbf{H}$ satisfies a suitable divergence-type condition. The argument fundamentally relies on the elliptic regularity results (Hölder regularity and the Harnack inequality) of G. Stampacchia [2], which remain applicable even in the critical case. A key analytical observation is that both the Harnack inequality and Hölder continuity hold despite the limited regularity of the drift term.

In contrast to the framework developed in [1], the present work does not provide quantitative control over the constants appearing in the a priori estimates. For instance, the constant $\tilde{K}_1 \geq 1$ in Theorem 4.1 depends on the drift $\mathbf{H}$ itself rather than its norm in a specific function space. It remains unclear whether such constants remain stable under mollification or other approximation procedures for **H**, and further investigation is needed to address this issue.

Another natural question is whether the results extend beyond the critical case $\mathbf{H} \in L^d(U, \mathbb{R}^d)$ to the subcritical setting $\mathbf{H} \in L^2(U, \mathbb{R}^d)$. Although some special cases have been studied, such as divergence-free drifts [19] and drifts with nonnegative divergence [18], the general case with drifts in L^2 or L^d remains open. Addressing this problem would likely require a more delicate analysis.

Finally, the methods developed in this paper are not confined to the context of linear divergence-form equations. They may also be applicable to regularity theory for double-divergence form equations and to the study of invariant measures for stochastic analysis, as in [20].

Use of AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares there is no conflict of interest.

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