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https://www.aimspress.com/journal/era

ERA, 33(12): 7902–7917.
DOI: 10.3934/era.2025348
Received: 07 October 2025
Revised: 19 November 2025
Accepted: 22 December 2025
Published: 25 December 2025

Research article

Goldbach-Linnik type problems for mixed powers of primes and powers of 2

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Abstract: Let $p_1, p_2, \ldots, p_7$ be primes. In this paper, we first show that when $k_1 = 23$, every sufficiently large odd integer can be represented as the sum of one prime square, five prime cubes, one prime biquadrate and at most k_1 powers of 2. We further prove that for $k_2 = 45$, every pair of sufficiently large odd integers satisfying certain necessary conditions can be represented as a pair of equations involving one prime square, five prime cubes, one prime biquadrate and at most k_2 powers of 2.

Keywords: circle method; exponential sums; Goldbach-Linnik type problem; powers of 2; primes

1. Introduction

The Goldbach conjecture concerns the representation of every even integer greater than two as the sum of two primes, but a direct proof remains unavailable. Researchers have therefore studied various modified versions of the problem. Among these, Linnik [1, 2] establishes that every sufficiently large even integer n_1 can be expressed as the sum of two primes and at most K_1 powers of 2, namely

$$n_1 = p_1 + p_2 + 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_{K_1}}. \tag{1.1}$$

Many scholars have investigated the value of K_1, and the minimum acceptable value so far is $K_1 = 8$ established by Pintz and Ruzsa [3].

Besides this linear relationship, the representation of integers as sums of mixed prime powers has also attracted the attention of many scholars. For instance, Liu [4], Lü and Cai [5] investigated the equations

$$n_2 = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^3 + p_6^3, \tag{1.2}$$

$$n_3 = p_1^2 + p_2^4 + p_3^4 + \cdots + p_{10}^4, \tag{1.3}$$

respectively. A natural hybrid problem arises by combining the forms of (1.2) and (1.3), leading to the equation

$$n_4 = p_1^2 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^3 + p_7^4. \tag{1.4}$$

However, current techniques remain insufficient to establish the solvability of (1.4) directly. In 2017, motivated by the works of Linnik, Liu [6] proved that every sufficiently large odd integer N_1 can be expressed as the sum of one prime square, five prime cubes, one prime biquadrate and a bounded number of powers of 2, i.e.,

$$N_1 = p_1^2 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^3 + p_7^4 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_{k_1}}, \tag{1.5}$$

where k_1 is an unspecified constant. Further references on this topic include [7–10]. In this work, we provide an explicit bound for k_1 resolving the question of solvability for (1.5), as presented in the following theorem.

Theorem 1.1. *For any integer $k_1 \geq 23$ and every sufficiently large odd integer N_1, Eq* (1.5) *is solvable.*

Kong [11] first studied the Eq (1.1) in a extended way. Specifically, Kong [11] considered the simultaneous representation of every pair of positive even integers B_1, B_2 with $B_2 \gg B_1 > B_2$, in the form

$$\begin{cases} B_1 = p_1 + p_2 + 2^{u_1} + 2^{u_2} + \cdots + 2^{u_{K_2}}, \\ B_2 = p_3 + p_4 + 2^{u_1} + 2^{u_2} + \cdots + 2^{u_{K_2}}, \end{cases} \tag{1.6}$$

where $K_2 \geq 63$ is an integer. Note that in the system of equations, the primes p_1, p_2, p_3, p_4 are not necessarily same, while the powers of 2 share the same tuple $(u_1, u_2, \ldots, u_{K_2})$. Subsequently, Kong and Liu [12] improved this bound to $K_2 \geq 34$.

We follow the idea of Kong [11] and study the simultaneous representation of every pair of sufficiently large odd integers N_2 and N_3 satisfying $N_2 \asymp N_3$, in the form

$$\begin{cases} N_2 = p_1^2 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^3 + p_7^4 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_{k_2}}, \\ N_3 = p_8^2 + p_9^3 + p_{10}^3 + p_{11}^3 + p_{12}^3 + p_{13}^3 + p_{14}^4 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_{k_2}}. \end{cases} \tag{1.7}$$

Our second theorem establishes an explicit bound for k_2 that guarantees the solvability of Eq (1.7).

Theorem 1.2. *For any integer $k_2 \geq 45$ and every pair of sufficiently large odd integers N_2 and N_3 satisfying*

$$N_2 \asymp N_3, \tag{1.8}$$

the system (1.7) *is solvable.*

In this paper, we employ the circle method combined with analytic techniques to establish our results. The proofs of Theorems 1.1 and 1.2 follow similar arguments, thus we focus on outlining the strategy for Theorem 1.1. We begin by decomposing the integral representation of the weighted solution number into the major arcs and minor arcs (see (2.7) below). By carefully computing the singular integral and the singular series (see Lemmas 2.2 and 3.1 below), we establish a sharp lower bound for the major arcs contribution (see Proposition 3.1 below). When handling $\mathrm{R}_2(N_1)$, we utilize integral mean value estimates of exponential sums (see Proposition 3.2 below), which allows us to exploit the value λ appearing in Lemma 2.3. Combining these with more precise upper bounds for $\mathrm{R}_3(N_1)$ (see Proposition 3.3 below), we ultimately establish the conclusion of Theorem 1.1.

Notation. Throughout this paper, we take the following conventions:

· The letters p and k, with or without a subscript, always represent a prime and a positive integer, respectively.

· The function $e(x)$ is defined as $e^{2\pi ix}$.

· The letter ϵ represents an arbitrarily small positive constant, which may have different values in different occurrences.

· $M \asymp N$ means that both $N \ll M$ and $N \gg M$ hold.

· $i = 1, 2, 3$, $r = 1, 2$, $j = 2, 3$.

2. Outline and preliminary lemmas

To apply the circle method, we first introduce some symbols. Define the parameters

$$P_i = N_i^{\frac{3}{20}}(\log N_i)^{-8}, \quad Q_i = N_i^{\frac{17}{20}}, \quad L_r = \log_2 \frac{N_r}{\log N_r},$$

which satisfy the condition

$$Q_i > 2P_i \geq 2.$$

For $1 \leq a_i \leq q_i \leq P_i$ and $(a_i, q_i) = 1$, we define

$$\mathfrak{M}_i(a_i, q_i) = \left\{\alpha_i \in [0,1] : \left|\alpha_i - \frac{a_i}{q_i}\right| \leq \frac{1}{q_i Q_i}\right\}. \tag{2.1}$$

The major arcs $\mathfrak{M}_i$ and the minor arcs $\mathfrak{m}_i$ are defined as

$$\mathfrak{M}_i = \bigcup_{1\leq q_i\leq P_i} \bigcup_{\substack{1\leq a_i\leq q_i \\ (a_i,q_i)=1}} \mathfrak{M}_i(a_i, q_i), \quad \mathfrak{m}_i = [0,1]\backslash\mathfrak{M}_i, \tag{2.2}$$

respectively. Observe that for distinct rationals $\frac{a_i}{q_i} \neq \frac{a_i'}{q_i'}$, we have

$$\left|\frac{a_i}{q_i} - \frac{a_i'}{q_i'}\right| \geq \frac{1}{q_i q_i'} \geq \frac{1}{2}\left(\frac{1}{q_i P_i} + \frac{1}{q_i' P_i}\right) > \frac{1}{q_i Q_i} + \frac{1}{q_i' Q_i}.$$

By combining this inequality with (2.1), it follows that $\mathfrak{M}_i(a_i, q_i)$ are mutually disjoint. Then we further define

$$\mathfrak{M}' = \mathfrak{M}_2 \times \mathfrak{M}_3 = \left\{(\alpha_2, \alpha_3) \in [0,1]^2 : \alpha_2 \in \mathfrak{M}_2, \alpha_3 \in \mathfrak{M}_3\right\}, \quad \mathfrak{m}' = [0,1]^2\backslash\mathfrak{M}'. \tag{2.3}$$

Let

$$\begin{aligned} &A_i = \frac{1}{2}\left((1-\eta)N_i\right)^{\frac{1}{2}}, \ B_i = \frac{1}{2}\left(\frac{\eta N_i}{2}\right)^{\frac{1}{3}}, \ C_i = \frac{1}{2}\left(\frac{\eta N_i}{2}\right)^{\frac{1}{4}}, \\ &U_i = \left(\frac{N_i}{16(1+\eta)}\right)^{\frac{1}{3}}, \ V_i = U_i^{\frac{5}{6}}, \end{aligned} \tag{2.4}$$

where η is a sufficiently small positive constant. We shall consider the sums

$$\mathrm{R}(N_1) = \sum_{N_1=p_1^2+p_2^3+p_3^3+p_4^3+p_5^3+p_6^3+p_7^4+2^{\nu_1}+2^{\nu_2}+\cdots+2^{\nu_{k_1}}} (\log p_1)(\log p_2)\cdots(\log p_7), \tag{2.5}$$

which is the weighted number of solutions of (1.5) in $(p_1, p_2, \ldots, p_7, v_1, v_2, \ldots, v_{k_1})$ such that

$$A_1 < p_1 \le 2A_1,\ B_1 < p_2 \le 2B_1,\ U_1 < p_3, p_4 \le 2U_1,\ V_1 < p_5, p_6 \le 2V_1,$$
$$C_1 < p_7 \le 2C_1,\ 4 \le v_1, v_2, \ldots, v_{k_1} \le L_1;$$

$$\mathrm{R}(N_2, N_3) = \sum_{\substack{N_2 = p_1^2+p_2^3+p_3^3+p_4^3+p_5^3+p_6^3+p_7^4+2^{v_1}+2^{v_2}+\cdots+2^{v_{k_2}} \\ N_3 = p_8^2+p_9^3+p_{10}^3+p_{11}^3+p_{12}^3+p_{13}^3+p_{14}^4+2^{v_1}+2^{v_2}+\cdots+2^{v_{k_2}}}} (\log p_1)(\log p_2)\cdots(\log p_{14}), \tag{2.6}$$

which is the weighted number of solutions of (1.7) in $(p_1, p_2, \ldots, p_{14}, v_1, v_2, \ldots, v_{k_2})$ such that

$$A_2 < p_1 \le 2A_2,\ B_2 < p_2 \le 2B_2, U_2 < p_3, p_4 \le 2U_2,\ V_2 < p_5, p_6 \le 2V_2,\ C_2 < p_7 \le 2C_2,$$
$$A_3 < p_8 \le 2A_3,\ B_3 < p_9 \le 2B_3, U_3 < p_{10}, p_{11} \le 2U_3,\ V_3 < p_{12}, p_{13} \le 2V_3,\ C_3 < p_{14} \le 2C_3,$$
$$4 \le v_1, v_2, \ldots, v_{k_2} \le L_2.$$

Let λ be a constant to be determined later. We define the following exponential sums

$$f_2(\alpha_i) = \sum_{A_i < p \le 2A_i} (\log p) e(p^2\alpha_i), \quad f_3(\alpha_i) = \sum_{B_i < p \le 2B_i} (\log p) e(p^3\alpha_i),$$

$$S(\alpha_i) = \sum_{U_i < p \le 2U_i} (\log p) e\left(p^3\alpha_i\right), \quad T(\alpha_i) = \sum_{V_i < p \le 2V_i} (\log p) e\left(p^3\alpha_i\right),$$

$$f_4(\alpha_i) = \sum_{C_i < p \le 2C_i} (\log p) e(p^4\alpha_i), \quad G_r(\alpha_i) = \sum_{4 \le v \le L_r} e\left(2^v\alpha_i\right),$$

and the exceptional sets

$$\mathscr{E}_{1\lambda} = \{\alpha_1 \in [0,1] : |G_1(\alpha_1)| \ge \lambda L_1\}, \quad \mathscr{E}_{2\lambda} = \left\{(\alpha_2, \alpha_3) \in [0,1]^2 : |G_2(\alpha_2 + \alpha_3)| \ge \lambda L_2\right\}.$$

By (2.2), (2.3), (2.5), (2.6) and orthogonality, we have

$$\begin{aligned} \mathrm{R}(N_1) = & \left(\int_{\mathfrak{M}_1} + \int_{\mathfrak{m}_1 \cap \mathscr{E}_{1\lambda}} + \int_{\mathfrak{m}_1 \setminus \mathscr{E}_{1\lambda}}\right) f_2(\alpha_1) f_3(\alpha_1) f_4(\alpha_1) S^2(\alpha_1) T^2(\alpha_1) G_1^{k_1}(\alpha_1) e(-\alpha_1 N_1) \mathrm{d}\alpha_1 \\ := & \mathrm{R}_1(N_1) + \mathrm{R}_2(N_1) + \mathrm{R}_3(N_1) \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \mathrm{R}(N_2, N_3) = & \left(\iint_{\mathfrak{M}'} + \iint_{\mathfrak{m}' \cap \mathscr{E}_{2\lambda}} + \iint_{\mathfrak{m}' \setminus \mathscr{E}_{2\lambda}}\right) f_2(\alpha_2) f_3(\alpha_2) f_4(\alpha_2) S^2(\alpha_2) T^2(\alpha_2) f_2(\alpha_3) f_3(\alpha_3) \\ & \times f_4(\alpha_3) S^2(\alpha_3) T^2(\alpha_3) G_2^{k_2}(\alpha_2 + \alpha_3) e(-\alpha_2 N_2 - \alpha_3 N_3) \mathrm{d}\alpha_2 \mathrm{d}\alpha_3 \\ := & \mathrm{R}_1(N_2, N_3) + \mathrm{R}_2(N_2, N_3) + \mathrm{R}_3(N_2, N_3). \end{aligned} \tag{2.8}$$

Let l be a positive integer throughout this section. Let also

$$C_l(q, a) = \sum_{\substack{m=1 \\ (m,q)=1}}^{q} e\left(\frac{am^l}{q}\right),$$

$$A(n,q)=\frac{1}{\varphi^7(q)}\sum_{\substack{a=1\\(a,q)=1}}^{q}C_2(q,a)C_3^5(q,a)C_4(q,a)e\left(-\frac{an}{q}\right),\tag{2.9}$$

$$\mathfrak{S}(n)=\sum_{q=1}^{\infty}A(n,q).\tag{2.10}$$

Lemma 2.1. *Let $\mathfrak{M}_i$ be defined as in* (2.2)*. Then for* $2\le n_i\le N_i$*, we have*

$$\int_{\mathfrak{M}_1}f_2(\alpha_1)f_3(\alpha_1)f_4(\alpha_1)S^2(\alpha_1)T^2(\alpha_1)e(-\alpha_1N_1)\mathrm{d}\alpha_1=\frac{1}{1944}\mathfrak{S}(n_1)\mathfrak{J}(n_1)+O\left(N_1^{\frac{47}{36}}L_1^{-1}\right),$$

$$\int_{\mathfrak{M}_j}f_2(\alpha_j)f_3(\alpha_j)f_4(\alpha_j)S^2(\alpha_j)T^2(\alpha_j)e(-\alpha_jN_j)\mathrm{d}\alpha_j=\frac{1}{1944}\mathfrak{S}(n_j)\mathfrak{J}(n_j)+O\left(N_j^{\frac{47}{36}}L_2^{-1}\right).$$

Here $\mathfrak{S}(n_i)$ *is defined as* (2.10) *and satisfies* $\mathfrak{S}(n_i)\gg 1$ *for* $n_i\equiv 1\pmod 2$*,* $\mathfrak{J}(n_i)$ *is defined as*

$$\mathfrak{J}(n_i)=\sum_{(m_1,m_2,\dots,m_7)\in\mathfrak{D}}m_1^{-\frac12}(m_2m_3m_4m_5m_6)^{-\frac23}m_7^{-\frac34}\tag{2.11}$$

with

$$\mathfrak{D}=\left\{(m_1,m_2,\dots,m_7):\begin{array}{l}m_1+m_2+\cdots+m_7=n_i,A_i^2<m_1\le(2A_i)^2\\B_i^3<m_2\le(2B_i)^3,U_i^3<m_3,m_4\le(2U_i)^3\\V_i^3<m_5,m_6\le(2V_i)^3,C_i^4<m_7\le(2C_i)^4\end{array}\right\},$$

and satisfies $\mathfrak{J}(n_i)\asymp N_i^{\frac{47}{36}}$*.*

Proof. The proof of this lemma is a standard application of handling enlarged major arcs in the circle method (see [13, 14], etc.). Thus we omit its proof here. □

Lemma 2.2. *For* $(1-\eta)N_i\le n_i\le N_i$*, we have*

$$\mathfrak{J}(n_i)\ge 489.952(1+\eta)^{\frac12}N_i^{-\frac12}B_iC_iU_i^2V_i^2.$$

Proof. Let

$$\mathfrak{D}^*=\left\{(m_1,m_2,\dots,m_7):\begin{array}{l}A_i^2<m_1\le(2A_i)^2,B_i^3<m_2\le(2B_i)^3,U_i^3<m_3,m_4\le 5U_i^3\\V_i^3<m_5,m_6\le(2V_i)^3,C_i^4<m_7\le(2C_i)^4\end{array}\right\}.$$

For $(m_1,m_2,\dots,m_7)\in\mathfrak{D}^*$, we deduce from $(1-\eta)N_i\le n_i\le N_i$ that

$$A_i^2<m_1=n_i-m_2-m_3-m_4-m_5-m_6-m_7\le(2A_i)^2.$$

Thus $\mathfrak{D}^*$ is a subset of $\mathfrak{D}$. Then by (2.11), we have

$$\begin{aligned}\mathfrak{J}(n_i)\ge&(1+\eta)^{\frac12}N_i^{-\frac12}\sum_{B_i^3<m_2\le(2B_i)^3}m_2^{-\frac23}\sum_{U_i^3<m_3\le 5U_i^3}m_3^{-\frac23}\sum_{U_i^3<m_4\le 5U_i^3}m_4^{-\frac23}\\&\times\sum_{V_i^3<m_5\le(2V_i)^3}m_5^{-\frac23}\sum_{V_i^3<m_6\le(2V_i)^3}m_6^{-\frac23}\sum_{C_i^4<m_7\le(2C_i)^4}m_7^{-\frac34}\\\ge&3\times 3(\sqrt[3]{5}-1)\times 3(\sqrt[3]{5}-1)\times 3\times 3\times 4(1+\eta)^{\frac12}N_i^{-\frac12}B_iC_iU_i^2V_i^2\\\ge&489.952(1+\eta)^{\frac12}N_i^{-\frac12}B_iC_iU_i^2V_i^2,\end{aligned}$$

where $\sum_{a<m\leq b} m^{-c}$ is well approximated by the corresponding integral $\int_a^b x^{-c}\mathrm{d}x$. Thus we complete the proof of the lemma. □

Lemma 2.3. *Let* meas$(\mathscr{E}_{r\lambda})$ *denote the Lebesgue measure of* $\mathscr{E}_{r\lambda}$. *Then for* $\lambda = 0.83372$*, we have*

$$\text{meas}\,(\mathscr{E}_{r\lambda}) \ll N_r^{-\frac{2}{3}-10^{-20}}.$$

Proof. This lemma follows from Lemma 5 and (3.10) in [15]. □

Remark. The above lemma was established through an optimized implementation of the Pintz and Ruzsa algorithm [16] by Languasco and Zaccagnini [15], which is sharper than the one of Heath-Brown and Puchta [17]. The refined algorithm achieves improvements to existing results, particularly enhancing the findings in Huang's [18] and Lü's [19] works.

The complete algorithmic implementation is accessible in the code ocean capsule [20] of Professor Alessandro Languasco's homepage.

Lemma 2.4. *We have*

$$\int_0^1 |f_2^2(\alpha_i)f_3^2(\alpha_i)f_4^2(\alpha_i)|\mathrm{d}\alpha_i \leq 6.52263(1+12\eta)^{\frac{1}{2}}B_i^2C_i^2.$$

Proof. Following the argument in Section 4 of [9], we have

$$\begin{aligned}\int_0^1 |f_2^2(\alpha_i)f_3^2(\alpha_i)f_4^2(\alpha_i)|\mathrm{d}\alpha_i &\leq 2\frac{\log 2A_i}{\log D_1D_2}(1+o(1))\mathfrak{S}(4)\mathfrak{J}(4)\\ &\leq 8\log 2(1+o(1))\mathfrak{S}(4)\mathfrak{J}(4),\end{aligned} \tag{2.12}$$

where $D_1 = A_i^{\frac{1}{6}-\delta}$, $D_2 = \sqrt{D_1}$ with a positive constant $\delta < 10^{-100}$, $\mathfrak{S}(4)$ is defined by (3.3) in [9], and

$$\mathfrak{J}(4) = \frac{1}{2^2\times 3^2\times 4^2}\sum_{\substack{m_1+m_2+m_3=n_1+n_2+n_3\\ A_i^2<m_1,n_1\leq(2A_i)^2\\ B_i^3<m_2,n_2\leq(2B_i)^3\\ C_i^4<m_3,n_3\leq(2C_i)^4}}(m_1n_1)^{\frac{1}{2}-1}(m_2n_2)^{\frac{1}{3}-1}(m_3n_3)^{\frac{1}{4}-1}.$$

Noting that

$$\begin{aligned}m_1 &= n_1+n_2+n_3-m_2-m_3\\ &\geq n_1+B_i^3+C_i^4-(2B_i)^3-(2C_i)^4\\ &\geq (1-12\eta)n_1,\end{aligned}$$

and we have

$$\begin{aligned}\mathfrak{J}(4) &\leq \frac{1}{2^2\times 3^2\times 4^2}\sum_{\substack{m_1+m_2+m_3=n_1+n_2+n_3\\ A_i^2<m_1,n_1\leq(2A_i)^2\\ B_i^3<m_2,n_2\leq(2B_i)^3\\ C_i^4<m_3,n_3\leq(2C_i)^4}}(1-12\eta)^{-\frac{1}{2}}n_1^{-1}(m_2n_2)^{\frac{1}{3}-1}(m_3n_3)^{\frac{1}{4}-1}\\ &\leq \frac{2\log 2}{2^2\times 3^2\times 4^2}(3B_i)^2(4C_i)^2(1+12\eta)^{\frac{1}{2}}(1+o(1))\\ &\leq \left(\frac{\log 2}{2}+o(1)\right)(1+12\eta)^{\frac{1}{2}}B_i^2C_i^2.\end{aligned}$$

This in combination with (2.12) and Lemma 3.1 in [9] leads to

$$\int_0^1 |f_2^2(\alpha_i) f_3^2(\alpha_i) f_4^2(\alpha_i)| \mathrm{d}\alpha_i \le 6.52263(1+12\eta)^{\frac{1}{2}} B_i^2 C_i^2,$$

which completes the proof of the lemma. □

Lemma 2.5. *We have*

$$\int_0^1 |S^4(\alpha_i) T^4(\alpha_i)| \mathrm{d}\alpha_i \le 7.39088 U_i V_i^4, \tag{2.13}$$

$$\int_0^1 |f_2(\alpha_i) f_3(\alpha_i) f_4(\alpha_i) S^2(\alpha_i) T^2(\alpha_i)| \mathrm{d}\alpha_i \le 27.77279(1+\eta)^{\frac{1}{2}}(1+12\eta)^{\frac{1}{4}} N_i^{-\frac{1}{2}} B_i C_i U_i^2 V_i^2. \tag{2.14}$$

Proof. The estimate (2.13) is Lemma 3.6 in [21]. Next, we give the proof of the estimate (2.14). By Cauchy's inequality, (2.4), (2.13) and Lemma 2.4 we have

$$\begin{aligned}\int_0^1 |f_2(\alpha_i) f_3(\alpha_i) f_4(\alpha_i) S^2(\alpha_i) T^2(\alpha_i)| \mathrm{d}\alpha_i &\le \left(\int_0^1 |f_2^2(\alpha_i) f_3^2(\alpha_i) f_4^2(\alpha_i)| \mathrm{d}\alpha_i\right)^{\frac{1}{2}} \left(\int_0^1 |S^4(\alpha_i) T^4(\alpha_i)| \mathrm{d}\alpha_i\right)^{\frac{1}{2}} \\ &\le 27.77279(1+\eta)^{\frac{1}{2}}(1+12\eta)^{\frac{1}{4}} N_i^{-\frac{1}{2}} B_i C_i U_i^2 V_i^2.\end{aligned}$$

Thus the lemma follows. □

Lemma 2.6. *We have*

$$\int_{\mathfrak{m}_i} |f_2^{\frac{9}{4}}(\alpha_i) f_3^{\frac{9}{2}}(\alpha_i)| \mathrm{d}\alpha_i \ll N_i^{\frac{19}{12}+\epsilon}, \tag{2.15}$$

$$\int_0^1 |f_2^2(\alpha_i) f_4^4(\alpha_i)| \mathrm{d}\alpha_i \ll N_i^{1+\epsilon}. \tag{2.16}$$

Proof. The estimate (2.15) is Lemma 3.1 in [22]. By Hua's well-known theorem on mean value estimates for exponential sums [23], we can get the estimate (2.16). □

3. The proof of Theorem 1.1

Lemma 3.1. *Let* $\Xi(N_1, k_1) = \{n_1 \ge 2 : n_1 = N_1 - 2^{\nu_1} - 2^{\nu_2} - \cdots - 2^{\nu_{k_1}}, 4 \le \nu_1, \nu_2, \ldots, \nu_{k_1} \le L_1\}$ *with* $k_1 \ge 20$. *Then for* $N_1 \equiv 1 \pmod 2$, *we have*

$$\sum_{\substack{n_1 \in \Xi(N_1, k_1) \\ n_1 \equiv 1 \pmod 2}} \mathfrak{S}(n_1) \ge 1.97616 L_1^{k_1}.$$

Proof. For $k \ge 2$, by (2.9) we have

$$A(n_i, p^k) = 0$$

and $A(n_i, p)$ is multiplicative. Then

$$\mathfrak{S}(n_i) = \prod_{p \ge 2} (1 + A(n_i, p)). \tag{3.1}$$

For $p = 2$, we have

$$1 + A(n_i, 2) = \begin{cases} 0, & n_i \equiv 0 \pmod 2, \\ 2, & n_i \equiv 1 \pmod 2. \end{cases} \tag{3.2}$$

For $3 \le p \le 200$, with help of a mathematical software, we can get

$$1 + A(n_i, 3) \ge 0.9375, \quad 1 + A(n_i, 7) \ge 0.704861, \quad 1 + A(n_i, 13) \ge 0.926757$$

and

$$(1 + A(n_i, 5))(1 + A(n_i, 11)) \prod_{17 \le p \le 200} (1 + A(n_i, p)) \ge 0.99045. \tag{3.3}$$

For $p \ge 200$ and $p \equiv 1 \pmod 3$, we have

$$1 + A(n_i, p) \ge 1 - \frac{\sum_{a=1}^{p-1} |C_2(p, a) C_3^5(p, a) C_4(p, a)|}{(p-1)^7} \ge 1 - \frac{(\sqrt{p} + 1)(2\sqrt{p} + 1)^5 (3\sqrt{p} + 1)}{(p-1)^6}, \tag{3.4}$$

where the elementary estimate $|C_l(p, a)| \le (l-1)\sqrt{p} + 1$ are used. For $p \ge 200$, if $p \equiv 2 \pmod 3$ and $(a, p) = 1$, we can deduce that $C_3(p, a) = -1$. Therefore, we have

$$1 + A(n_i, p) \ge 1 - \frac{\sum_{a=1}^{p-1} |C_2(p, a) C_4(p, a)|}{(p-1)^7} \ge 1 - \frac{(\sqrt{p} + 1)(3\sqrt{p} + 1)}{(p-1)^6}. \tag{3.5}$$

By (3.4) and (3.5), we again apply a mathematical software and get

$$\begin{aligned} \prod_{200 \le p < 10^6} (1 + A(n_i, p)) \ge & \prod_{\substack{200 \le p < 10^6 \\ p \equiv 1 \pmod 3}} \left(1 - \frac{(\sqrt{p} + 1)(2\sqrt{p} + 1)^5 (3\sqrt{p} + 1)}{(p-1)^6}\right) \\ & \times \prod_{\substack{200 \le p < 10^6 \\ p \equiv 2 \pmod 3}} \left(1 - \frac{(\sqrt{p} + 1)(3\sqrt{p} + 1)}{(p-1)^6}\right) \\ \ge & 0.99789, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \prod_{p \ge 10^6} (1 + A(n_i, p)) \ge & \prod_{p \ge 10^6} \left(1 - \frac{(\sqrt{p} + 1)(2\sqrt{p} + 1)^5 (3\sqrt{p} + 1)}{(p-1)^6}\right) \\ \ge & \prod_{p \ge 10^6} \left(1 - \frac{1}{(p-1)^2}\right)^{17} \\ \ge & \ 0.99998. \end{aligned} \tag{3.7}$$

From (3.3), (3.6) and (3.7), we have

$$(1 + A(n_i, 5))(1 + A(n_i, 11)) \prod_{p \ge 17} (1 + A(n_i, p)) \ge 0.98834 := C. \tag{3.8}$$

Let $q = 3 \times 7 \times 13 = 273$, by (3.1), (3.2) and (3.8), we get

$$\begin{aligned}\sum_{\substack{n\in\Xi(N_1,k_1)\\ n_1\equiv 1 \pmod 2}} \mathfrak{S}(n_1) &\geq 2C \sum_{\substack{n_1\in\Xi(N_1,k_1)\\ n_1\equiv 1 \pmod 2}} \prod_{p=3,7,13} (1 + A(n_1, p))\\ &\geq 2C \sum_{1\leq h\leq q} \prod_{p=3,7,13} (1 + A(h, p)) \sum_{\substack{n_1\in\Xi(N_1,k_1)\\ n_1\equiv 1 \pmod 2\\ n_1\equiv h \pmod q}} 1. \end{aligned} \tag{3.9}$$

To estimate the innermost sum at the right side of (3.9), we take a similar argument to Lemma 4.4 in [24]. We can deduce that

$$S := \sum_{\substack{n_1\in\Xi(N_1,k_1)\\ n_1\equiv 1 \pmod 2\\ n_1\equiv j \pmod q}} 1 = \left(\frac{L_1}{\delta(q)} + O(1)\right)^{k_1} \sum_{\substack{1\leq \nu_1,\nu_2,\ldots,\nu_{k_1}\leq\delta(q)\\ 2^{\nu_1}+2^{\nu_2}+\cdots+2^{\nu_{k_1}}\equiv N-h \pmod q}} 1,$$

where $\delta(q)$ denotes the smallest positive integer δ such that $2^\delta \equiv 1 \pmod q$. Noting that

$$S = \frac{1}{q}\left(\frac{L_1}{\delta(q)} + O(1)\right)^{k_1} \sum_{t=0}^{q-1} e\left(\frac{t(N-h)}{q}\right)\theta^{k_1}(t),$$

we get

$$\begin{aligned} S &\geq \frac{1}{q}\left(\frac{L_1}{\delta(q)} + O(1)\right)^{k_1}\left(\delta(q)^{k_1} - (q-1)\left(\max_{0<t\leq q-1}|\theta(t)|\right)^{k_1}\right)\\ &\geq \frac{L_1^{k_1}}{q}\left(1 - (q-1)\left(\frac{\max_{0<t\leq q-1}|\theta(t)|}{\delta(q)}\right)^{k_1}\right) + O\left(L_1^{k_1-1}\right), \end{aligned}$$

where $\theta(t) = \sum_{1\leq s\leq\delta(q)} e\left(\frac{t2^s}{q}\right)$. Recalling the definition of $\delta(q)$, we have

$$\delta(q) = 12 \quad \text{and} \quad \max_{0<t\leq q-1}|\theta(t)| \approx 6.$$

Therefore, we can get

$$S \geq 3.66205 \times 10^{-3} L_1^{k_1}. \tag{3.10}$$

From (3.9) and

$$\sum_{1\leq h\leq p}(1 + A(h,p)) = p + \sum_{1\leq h\leq p} A(h,p) = p,$$

we have

$$\sum_{\substack{n_1\in\Xi(N_1,k_1)\\ n_1\equiv 1 \pmod 2}} \mathfrak{S}(n_1) \geq 2\times 3.66205\times 10^{-3} CqL_1^{k_1} \geq 1.97616 L_1^{k_1},$$

which completes the proof of this lemma. □

Proposition 3.1. *Let* $\mathrm{R}_1(N_1)$ *be defined as in* (2.7). *We have*

$$\mathrm{R}_1(N_1) \geq 0.49805(1+\eta)^{\frac{1}{2}} N_1^{-\frac{1}{2}} B_1 C_1 U_1^2 V_1^2 L_1^{k_1} + O\left(N_1^{\frac{47}{36}} L_1^{k_1-1}\right).$$

Proof. Let $\Xi(N_1, k_1)$ be defined as in Lemma 3.1. We have

$$\begin{aligned}G_1^{k_1}(\alpha_1)e(-\alpha_1 N_1)=&\sum_{4\le v_1,v_2,\ldots,v_{k_1}\le L_1} e((2^{v_1}+2^{v_2}+\cdots+2^{v_{k_1}})\alpha_1)e(-\alpha_1 N_1)\\ =&\sum_{4\le v_1,v_2,\ldots,v_{k_1}\le L_1} e((2^{v_1}+2^{v_2}+\cdots+2^{v_{k_1}}-N_1)\alpha_1)\\ =&\sum_{\substack{n_1\in\Xi(N_1,k_1)\\ n_1\equiv 1 \pmod 2}} e(-n_1\alpha_1).\end{aligned}$$

We deduce from Lemmas 2.1, 2.2 and 3.1 that

$$\begin{aligned}\mathrm{R}_1(N_1)\ge&\frac{1}{1944}\sum_{\substack{n_1\in\Xi(N_1,k_1)\\ n_1\equiv 1 \pmod 2}}\left(\mathfrak{S}(n_1)\mathfrak{J}(n_1)+O\left(N_1^{\frac{47}{36}}L_1^{-1}\right)\right)\\ \ge&0.49805(1+\eta)^{\frac12}N_1^{-\frac12}B_1C_1U_1^2V_1^2L_1^{k_1}+O\left(N_1^{\frac{47}{36}}L_1^{k_1-1}\right).\end{aligned}$$

Thus this proposition follows. □

Proposition 3.2. *Let* $\mathrm{R}_2(N_1)$ *be defined as in* (2.7). *We have*

$$\mathrm{R}_2(N_1)\ll N_1^{\frac{47}{36}}L_1^{k_1-1}.$$

Proof. By the trivial bound $G_1(\alpha_1)\ll L_1$, Hölder's inequality, (2.4), (2.13), (2.15), (2.16) and Lemma 2.3, we have

$$\begin{aligned}\mathrm{R}_2(N_1)\ll& L_1^{k_1}\left(\int_{\mathfrak{m}_1}|f_2^{\frac94}(\alpha_1)f_3^{\frac92}(\alpha_1)|\mathrm{d}\alpha_1\right)^{\frac29}\left(\int_0^1|f_2^2(\alpha_1)f_4^4(\alpha_1)|\mathrm{d}\alpha_1\right)^{\frac14}\\ &\times\left(\int_0^1|S^4(\alpha_1)T^4(\alpha_1)|\mathrm{d}\alpha_1\right)^{\frac12}\left(\int_{\mathscr{E}_{1\lambda}}1\mathrm{d}\alpha_1\right)^{\frac{1}{36}}\\ \ll& N_1^{\frac{47}{36}+\frac{1}{54}+\epsilon}(\mathrm{meas}(\mathscr{E}_{1\lambda}))^{\frac{1}{36}}L_1^{k_1}\\ \ll& N_1^{\frac{47}{36}}L_1^{k_1-1}.\end{aligned}$$

Thus this proposition follows. □

Proposition 3.3. *Let* $\mathrm{R}_3(N_1)$ *be defined as in* (2.7). *We have*

$$|\mathrm{R}_3(N_1)|\le 27.77279(1+\eta)^{\frac12}(1+12\eta)^{\frac14}\lambda^{k_1}N_1^{-\frac12}B_1C_1U_1^2V_1^2L_1^{k_1}$$

with $\lambda = 0.83372$.

Proof. By (2.14) and the definition of $\mathscr{E}_{1\lambda}$, we get

$$\begin{aligned}|\mathrm{R}_3(N_1)|\le&(\lambda L_1)^{k_1}\int_0^1|f_2(\alpha_1)f_3(\alpha_1)f_4(\alpha_1)S^2(\alpha_1)T^2(\alpha_1)|\mathrm{d}\alpha_1\\ \le&27.77279(1+\eta)^{\frac12}(1+12\eta)^{\frac14}\lambda^{k_1}N_1^{-\frac12}B_1C_1U_1^2V_1^2L_1^{k_1}.\end{aligned}$$

Thus this proposition follows. □

By (2.7), Propositions 3.1–3.3, we have

$$\begin{aligned}\mathrm{R}(N_1) &\geq \mathrm{R}_1(N_1) - |\mathrm{R}_3(N_1)| + O\left(N_1^{\frac{47}{36}} L_1^{k_1-1}\right)\\ &> \left(0.49805(1+\eta)^{\frac{1}{2}} - 27.77279(1+\eta)^{\frac{1}{2}}(1+12\eta)^{\frac{1}{4}}\lambda^{k_1}\right) N_1^{-\frac{1}{2}} B_1 C_1 U_1^2 V_1^2 L_1^{k_1}.\end{aligned}$$

Recall that $\lambda = 0.83372$ in Lemma 2.3, then $\mathrm{R}(N_1) > 0$ with $k_1 \geq 23$, which proves Theorem 1.1.

4. The proof of Theorem 1.2

Lemma 4.1. *Let* $\Xi(N_j, k_2) = \{n_j \geq 2 : n_j = N_j - 2^{v_1} - 2^{v_2} - \cdots - 2^{v_{k_2}}, 4 \leq v_1, v_2, \ldots, v_{k_2} \leq L_2\}$ *with* $k_2 \geq 20$. *Then for* $N_2 \equiv N_3 \equiv 1 \pmod 2$, *we have*

$$\sum_{\substack{n_2 \in \Xi(N_2,k_2)\\ n_3 \in \Xi(N_3,k_2)\\ n_2 \equiv n_3 \equiv 1 \pmod 2}} \mathfrak{S}(n_2)\,\mathfrak{S}(n_3) \geq 3.90624 L_2^{k_2}.$$

Proof. Let $q = 273$, by (3.1), (3.2) and (3.8), we get

$$\begin{aligned}&\sum_{\substack{n_2 \in \Xi(N_2,k_2)\\ n_3 \in \Xi(N_3,k_2)\\ n_2 \equiv n_3 \equiv 1 \pmod 2}} \mathfrak{S}(n_2)\,\mathfrak{S}(n_3)\\ &\geq (2C)^2 \sum_{\substack{n_2 \in \Xi(N_2,k_2)\\ n_3 \in \Xi(N_3,k_2)\\ n_2 \equiv n_3 \equiv 1 \pmod 2}} \prod_{2 \leq j \leq 3} \prod_{p_j = 3,7,13} \left(1 + A\left(n_j, p_j\right)\right)\\ &\geq (2C)^2 \sum_{1 \leq h \leq q} \sum_{\substack{n_2 \in \Xi(N_2,k_2)\\ n_3 \in \Xi(N_3,k_2)\\ n_2 \equiv n_3 \equiv 1 \pmod 2\\ n_2 \equiv n_3 \equiv h \pmod q}} \prod_{2 \leq j \leq 3} \prod_{p_j = 3,7,13} \left(1 + A\left(h, p_j\right)\right)\\ &\geq (2C)^2 \sum_{1 \leq h \leq q} \prod_{p = 3,7,13} (1 + A(h,p))^2 \sum_{\substack{n_2 \in \Xi(N_2,k_2)\\ n_2 \equiv 1 \pmod 2\\ n_2 \equiv h \pmod q}} 1.\end{aligned} \tag{4.1}$$

For the innermost sum at the right side of (4.1), by (3.10), we can get

$$\sum_{\substack{n_2 \in \Xi(N_2,k_2)\\ n_2 \equiv 1 \pmod 2\\ n_2 \equiv h \pmod q}} 1 \geq 3.66205 \times 10^{-3} L_2^{k_2}.$$

From (4.1) and

$$\begin{aligned}\sum_{1 \leq h \leq p} (1 + A(h,p))^2 &= p + 2 \sum_{1 \leq h \leq p} A(h,p) + \sum_{1 \leq h \leq p} A^2(h,p)\\ &= p + \sum_{1 \leq h \leq p} A^2(h,p)\\ &\geq p,\end{aligned}$$

we have

$$\sum_{\substack{n_2\in\Xi(N_2,k_2)\\ n_3\in\Xi(N_3,k_2)\\ n_2\equiv n_3\equiv 1 \pmod 2}} \mathfrak{S}(n_2)\,\mathfrak{S}(n_3) \geq 4\times 3.66205\times 10^{-3}C^2qL_2^{k_2} \geq 3.90624L_2^{k_2},$$

which completes the proof of this lemma. □

Proposition 4.1. *Let* $\mathrm{R}_1(N_2, N_3)$ *be defined as in* (2.8). *We have*

$$\mathrm{R}_1(N_2,N_3) \geq 0.24812(1+\eta)N_2^{-\frac{1}{2}}N_3^{-\frac{1}{2}}B_2B_3C_2C_3U_2^2U_3^2V_2^2V_3^2L_2^{k_2} + O\left(N_2^{\frac{47}{36}}N_3^{\frac{47}{36}}L_2^{k_2-1}\right).$$

Proof. Let $\Xi(N_j, k_2)$ be defined as in Lemma 4.1. We have

$$\begin{aligned}
&G_2^{k_2}(\alpha_2+\alpha_3)e(-\alpha_2N_2-\alpha_3N_3)\\
&=\sum_{4\leq v_1,v_2,\ldots,v_{k_2}\leq L_2} e((2^{v_1}+2^{v_2}+\cdots+2^{v_{k_2}})(\alpha_2+\alpha_3))e(-\alpha_2N_2-\alpha_3N_3)\\
&=\sum_{4\leq v_1,v_2,\ldots,v_{k_2}\leq L_2} e((2^{v_1}+2^{v_2}+\cdots+2^{v_{k_2}}-N_2)\alpha_2)e((2^{v_1}+2^{v_2}+\cdots+2^{v_{k_2}}-N_3)\alpha_3)\\
&=\sum_{\substack{n_2\in\Xi(N_2,k_2)\\ n_3\in\Xi(N_3,k_2)\\ n_2\equiv n_3\equiv 1 \pmod 2}} e(-n_2\alpha_2)e(-n_3\alpha_3).
\end{aligned}$$

This, in combination with Lemmas 2.1, 2.2 and 4.1, yields

$$\begin{aligned}
\mathrm{R}_1(N_2,N_3) \geq& \frac{1}{1944^2}\sum_{\substack{n_2\in\Xi(N_2,k_2)\\ n_3\in\Xi(N_3,k_2)\\ n_2\equiv n_3\equiv 1 \pmod 2}} \left(\mathfrak{S}(n_2)\mathfrak{J}(n_2)+O\left(N_2^{\frac{47}{36}}L_2^{-1}\right)\right)\left(\mathfrak{S}(n_3)\mathfrak{J}(n_3)+O\left(N_3^{\frac{47}{36}}L_2^{-1}\right)\right)\\
\geq& \frac{3.90624\times(489.952)^2}{1944^2}(1+\eta)N_2^{-\frac{1}{2}}N_3^{-\frac{1}{2}}B_2B_3C_2C_3U_2^2U_3^2V_2^2V_3^2L_2^{k_2}+O\left(N_2^{\frac{47}{36}}N_3^{\frac{47}{36}}L_2^{k_2-1}\right)\\
\geq& 0.24812(1+\eta)N_2^{-\frac{1}{2}}N_3^{-\frac{1}{2}}B_2B_3C_2C_3U_2^2U_3^2V_2^2V_3^2L_2^{k_2}+O\left(N_2^{\frac{47}{36}}N_3^{\frac{47}{36}}L_2^{k_2-1}\right).
\end{aligned}$$

Thus this proposition follows. □

Proposition 4.2. *Let* $\mathrm{R}_2(N_2, N_3)$ *be defined as in* (2.8). *We have*

$$\mathrm{R}_2(N_2,N_3) \ll N_2^{\frac{47}{36}}N_3^{\frac{47}{36}}L_2^{k_2-1}.$$

Proof. By (2.2) and (2.3), we obtain

$$\mathfrak{m}' \subset \{(\alpha_2,\alpha_3):\alpha_2\in\mathfrak{m}_2,\alpha_3\in[0,1]\}\cup\{(\alpha_2,\alpha_3):\alpha_2\in[0,1],\alpha_3\in\mathfrak{m}_3\}.$$

From the trivial bound $G_2(\alpha_1+\alpha_2) \ll L_2$, we have

$$\begin{aligned} \mathrm{R}_2(N_2,N_3) \ll & L_2^{k_2}\left(\iint\limits_{\substack{(\alpha_2,\alpha_3)\in\mathfrak{m}_2\times[0,1]\\ |G_2(\alpha_2+\alpha_3)|\geq\lambda L_2}} + \iint\limits_{\substack{(\alpha_2,\alpha_3)\in[0,1]\times\mathfrak{m}_3\\ |G_2(\alpha_2+\alpha_3)|\geq\lambda L_2}} \right) \Big| f_2(\alpha_2)f_3(\alpha_2)f_4(\alpha_2)S^2(\alpha_2)T^2(\alpha_2) \\ & \times f_2(\alpha_3)f_3(\alpha_3)f_4(\alpha_3)S^2(\alpha_3)T^2(\alpha_3)\Big| \mathrm{d}\alpha_2\mathrm{d}\alpha_3 \\ = & L_2^{k_2}\Big(\int_0^1 \Big|f_2(\alpha_2)f_3(\alpha_2)f_4(\alpha_2)S^2(\alpha_2)T^2(\alpha_2)J_1(\alpha_2)\Big|\,\mathrm{d}\alpha_2 \\ & + \int_0^1 \Big|f_2(\alpha_3)f_3(\alpha_3)f_4(\alpha_3)S^2(\alpha_3)T^2(\alpha_3)J_2(\alpha_3)\Big|\,\mathrm{d}\alpha_3\Big), \end{aligned} \tag{4.2}$$

where

$$J_1(\alpha_2) = \int\limits_{\substack{\alpha_3\in\mathfrak{m}_3\\ |G_2(\alpha_2+\alpha_3)|\geq\lambda L_2}} \Big|f_2(\alpha_3)f_3(\alpha_3)f_4(\alpha_3)S^2(\alpha_3)T^2(\alpha_3)\Big|\,\mathrm{d}\alpha_3,$$

$$J_2(\alpha_3) = \int\limits_{\substack{\alpha_2\in\mathfrak{m}_2\\ |G_2(\alpha_2+\alpha_3)|\geq\lambda L_2}} \Big|f_2(\alpha_2)f_3(\alpha_2)f_4(\alpha_2)S^2(\alpha_2)T^2(\alpha_2)\Big|\,\mathrm{d}\alpha_2.$$

We apply Hölder's inequality, (2.13), (2.15), (2.16) and the periodicity of $G_2(\alpha)$ and get

$$\begin{aligned} J_1(\alpha_2) \ll & \left(\int_{\mathfrak{m}_2} |f_2^{\frac{9}{4}}(\alpha_3)f_3^{\frac{9}{2}}(\alpha_3)|\mathrm{d}\alpha_3\right)^{\frac{2}{9}} \left(\int_0^1 |f_2^2(\alpha_3)f_4^4(\alpha_3)|\mathrm{d}\alpha_3\right)^{\frac{1}{4}} \\ & \times \left(\int_0^1 |S^4(\alpha_3)T^4(\alpha_3)|\mathrm{d}\alpha_3\right)^{\frac{1}{2}} \left(\int\limits_{\substack{\alpha_3\in\mathfrak{m}_3\\ |G_2(\alpha_2+\alpha_3)|\geq\lambda L_2}} 1\mathrm{d}\alpha_3 \right)^{\frac{1}{36}} \\ \ll & N_3^{\frac{47}{36}+\frac{1}{54}+\epsilon} \left(\int\limits_{\substack{\omega\in[\alpha_3,1+\alpha_3]\\ |G_2(\omega)|\geq\lambda L_2}} 1\,\mathrm{d}\omega \right)^{\frac{1}{36}}, \end{aligned} \tag{4.3}$$

where $\omega = \alpha_2+\alpha_3$. This combining with (1.8), (2.4), (2.14) and Lemma 2.3 gives

$$\begin{aligned} & \int_0^1 \Big|f_2(\alpha_2)f_3(\alpha_2)f_4(\alpha_2)S^2(\alpha_2)T^2(\alpha_2)J_1(\alpha_2)\Big|\,\mathrm{d}\alpha_2 \\ \ll & N_3^{\frac{47}{36}+\frac{1}{54}+\epsilon} (\mathrm{meas}\,(\mathscr{E}_{2\lambda}))^{\frac{1}{36}} \int_0^1 \Big|f_2(\alpha_2)f_3(\alpha_2)f_4(\alpha_2)S^2(\alpha_2)T^2(\alpha_2)\Big|\,\mathrm{d}\alpha_2 \\ \ll & N_2^{\frac{47}{36}-10^{-22}} N_3^{\frac{47}{36}+\epsilon}. \end{aligned} \tag{4.4}$$

Arguing similarly we can also get

$$\int_0^1 \Big|f_2(\alpha_3)f_3(\alpha_3)f_4(\alpha_3)S^2(\alpha_3)T^2(\alpha_3)J_2(\alpha_3)\Big|\,\mathrm{d}\alpha_3 \ll N_2^{\frac{47}{36}+\epsilon} N_3^{\frac{47}{36}-10^{-22}}. \tag{4.5}$$

Inserting (4.4) and (4.5) into (4.2), we have

$$\begin{aligned}\mathrm{R}_2(N_2,N_3) &\ll L_2^{k_2}\left(N_2^{\frac{47}{36}-10^{-22}}N_3^{\frac{47}{36}+\epsilon}+N_2^{\frac{47}{36}+\epsilon}N_3^{\frac{47}{36}-10^{-22}}\right)\\ &\ll N_2^{\frac{47}{36}}N_3^{\frac{47}{36}}L_2^{k_2-1}.\end{aligned} \tag{4.6}$$

Thus this proposition follows. □

Proposition 4.3. *Let* $\mathrm{R}_3(N_2,N_3)$ *be defined as in* (2.8)*. We have*

$$|\mathrm{R}_3(N_2,N_3)| \le 771.32787(1+\eta)(1+12\eta)^{\frac{1}{2}}\lambda^{k_2}N_2^{-\frac{1}{2}}N_3^{-\frac{1}{2}}B_2B_3C_2C_3U_2^2U_3^2V_2^2V_3^2L_2^{k_2}$$

with $\lambda = 0.83372$.

Proof. By (2.14) and the definition of $\mathscr{E}_{2\lambda}$, we get

$$\begin{aligned}|\mathrm{R}_3\,(N_2,N_3)\,| \le&(\lambda L_2)^{k_2}\int_0^1\left|f_2(\alpha_2)f_3(\alpha_2)f_4(\alpha_2)S^2(\alpha_2)T^2(\alpha_2)\right|\mathrm{d}\alpha_2\\ &\times\int_0^1\left|f_2(\alpha_3)f_3(\alpha_3)f_4(\alpha_3)S^2(\alpha_3)T^2(\alpha_3)\right|\mathrm{d}\alpha_3\\ &\le 771.32787(1+\eta)(1+12\eta)^{\frac{1}{2}}\lambda^{k_2}N_2^{-\frac{1}{2}}N_3^{-\frac{1}{2}}B_2B_3C_2C_3U_2^2U_3^2V_2^2V_3^2L_2^{k_2}.\end{aligned}$$

Thus this proposition follows. □

By (2.8), Propositions 4.1–4.3, we have

$$\begin{aligned}\mathrm{R}(N_2,N_3) &\ge \mathrm{R}_1(N_2,N_3)-|\mathrm{R}_3(N_2,N_3)|+O\left(N_2^{\frac{47}{36}}N_3^{\frac{47}{36}}L_2^{k_2-1}\right)\\ &> \left(0.24812(1+\eta)-771.32787(1+\eta)(1+12\eta)^{\frac{1}{2}}\lambda^{k_2}\right)N_2^{-\frac{1}{2}}N_3^{-\frac{1}{2}}B_2B_3C_2C_3U_2^2U_3^2V_2^2V_3^2L_2^{k_2}.\end{aligned}$$

Recall that $\lambda = 0.83372$ in Lemma 2.3, then $\mathrm{R}\,(N_2,N_3) > 0$ with $k_2 \ge 45$, which proves Theorem 1.2.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

We are grateful to the reviewers for their thorough reviews and constructive comments. Their suggestions regarding the proof structure and methodological improvements have both deepened our understanding of the problem and significantly improved the manuscript's presentation.

Conflict of interest

The authors declare there is no conflicts of interest.

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