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Research article

Analytical pricing of maximum and minimum options based on a class of partial differential-integral equations: Applications of Mellin transform

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Abstract: In this work, the analytical pricing of maximum and minimum options under the integrated scenario of interdependent stochastic volatility and jump, and the stochastic interest rate, were investigated by means of the composite Mellin transform approach. The analytic expressions of Mellin transform functions of the price of the maximum put option, minimum call option, and the exchange option were derived by different partial differential-integral equations (PDIEs). Meanwhile, the explicit price of other maximum and minimum options were obtained by means of the payoff decomposition technique and the parity relation of max-min and exchange options. In addition, the convergence of solutions of PDIEs was further demonstrated by transform techniques and decomposition skills. The simulation of the price process of two underlying assets was given to present the effectiveness and uniqueness of the proposed model. Finally, numerical analysis was implemented to examine the accuracy of the PDIE method and the validity of key parameters.

Keywords: Mellin transform; PDIE; stochastic volatility; dependent jump; stochastic interest rate

1. Introduction

With the rapid development of the global financial market, new types of financial derivatives are constantly emerging. As an important and flexible bivariate option, the maximum or minimum options have always attracted the attention of scholars and practitioners. Compared with standard European options, the payoff functions of maximum or minimum options are affected by the two underlying assets that drive different stochastic processes. Therefore, the payoff structure of the option is richer, and it can satisfy the requirement of financial investment and risk hedging for more traders. However, the pricing of the maximum or minimum options is much more complex than the standard European option, especially in the integrated scenarios of stochastic volatility, stochastic interest rate, and interdependent jumps.

Because bivariate options are considered an excellent product for hedging risk in practice, their pricing method plays a crucial role in academic circles. Following the great structure of Black and Scholes (B-S) [1], Stulz [2] first discussed the pricing of options on the minimum or maximum of two risky assets. Johnson [3] further studied the valuation of maximum or minimum options of several assets. Kim [4] studied the pricing problem of exchange options under geometric Brownian motion and provided an exact analytical solution with credit risk. However, it is ideal to assume that the underlying assets are subject to geometric Brownian motion. On the one hand, by embedding the compound Poisson process to the B-S model, Menton [5] pioneered the jump diffusion process to simulate asset prices. It described the effect of rare events on asset prices through a stochastic jump. Wenhan [6] further investigated the pricing of bivariate exchange options under jump-diffusion processes. On the other hand, by replacing the constant volatility with the mean-revert stochastic volatility, Heston [7] established the square-root stochastic volatility model to illustrate the leptokurtic, fat-tail, and smile properties of the volatility rate. Based on Merton and Heston's work, a great deal of literature has shown that the extended and improved form of jump diffusion and stochastic volatility can effectively explain various characteristics of option data in real financial markets [8].

However, it is very difficult to obtain the analytical solution of the improved model through the traditional probabilistic technique [9]. A reliable alternative is the characteristic function approach or Fourier transform. Carr and Madan [10] deduced the characteristic function of the log-price of the underlying asset, and the asymptotic solution of the European option was obtained by the FFT for the first time. Zhang and Wang [11] applied a similar method to derive an approximated pricing expression of European options in cases with stochastic interest rates. Further, under the assumption of the regime-swithing stochastic volatility model, Lin [12] and Xie [13] solved the pricing of forward options and vulnerable options separately by establishing discounted characteristic functions and implementing the algorithm of the FFT. A better choice is to obtain the solution to the partial differential equation (PDIE) or partial integro-differential equation (PDIE) that the option price satisfies. Fortunately, the Mellin transform approach has excellent performance with simplifying PDE and PDIE. Brychkov [14] provided the general introduction on several transformations including the Mellin transform. Frontczak [15] first applied the Mellin transform to the option pricing problem driven by the jump diffusion process. Through single and double Mellin transform approaches, Yoon [16, 17] investigated European options and vulnerable options under the stochastic interest rate and verified the feasibility of this method. Further, Jeon [18] derived the analytic closed-form solution for lookback options, and expanded the application domain of the Mellin transform. Li [19] applied the Mellin transform to the jump-diffusion model, and presented the analytical price of vulnerable options under two jump structures.

Empirical studies have consistently confirmed that the volatility and instantaneous jump of the two interacting underlying assets are interdependent, and the market interest rate has certain random variability. Inspired by the above thinking, we intend to apply the Mellin transform to further explore the analytical pricing problem of maximum and minimum options in complex cases (stochastic volatility, stochastic interest rate, interdependent jump). It is difficult to price directly through expectation computation with partial differential equation theory. An analytical expression of the joint transform function for the price of two underlying assets is derived by utilizing the Mellin transform. Further, the decomposition technique is applied to verify the property of convergence for the pricing formula (inverse Mellin transform). On account of its availability and efficiency, this method draws special attention of many scholars and practitioners.

In this paper, we concentrate on the valuation of maximum and minimum options with stochastic volatility, stochastic interest, and an interdependent jump. Since their design and proposal, maximum and minimum options have been favored by financial practitioners and market investors. On the one hand, they can meet the personalized needs of investors with different risk preferences in the market. For example, high-risk-preference investors can choose maximum call options or minimum put options for investment and trading; low-risk-preference investors can choose minimum call options or maximum put options for investment and trading. On the other hand, for market risk hedgers, extreme value options are powerful and reliable risk hedging tools, and they can also hedge nonlinear risks in the financial market to a certain extent. Therefore, it is of profound significance to study the pricing problem of extreme value options in a relatively complex financial market environment. The remainder of this work is presented in the following order. First, the evolution mechanism of all stochastic factors involved in option pricing is given, including the price process of the underlying asset, interest rate process, and volatility process in Section 2. Second, the single and double Mellin transforms and the option parity relations are used jointly to generate the analytic pricing formula for the above options and the absolute integrability of solutions is demonstrated by a decomposition technique in Section 3. Third, numerical experiments are designed to examine the uniqueness of price model and the stability of the Mellin transform in Section 4. Finally, a summary of this study is given in Section 5.

2. Model formulation

In this section, we use the framework of the interdependent stochastic volatility jump-diffusion model for maximum and minimum options but replace the constant interest rate by the stochastic interest rate, and replace independent with interdependent jumps. Meanwhile, the partial differential equation satisfied by the option price is derived below.

2.1. *A brief description of the price dynamics*

Let $S_{it}(i = 1, 2)$ be the values of the underlying assets at time t. Under the no-arbitrage condition and risk-neutral measure $\mathbb{P}$, the price of assets S_{it} consists of three components: the drift term $r_t dt$ of the risk-free interest rate, the Itô integral term $\sigma_i \sqrt{v_t} dW_{it}$ of stochastic volatility, and the compensatory compound Poisson process term $(e^{J_i} - 1)d(N_{0t} + N_{1t}) - (\lambda_0 + \lambda_i)m_i dt$ of stochastic jumps, satisfying the following stochastic differential equations:

$$
\begin{aligned}
\frac{dS_{1t}}{S_{1t}} &= (r_t - (\lambda_0 + \lambda_1)m_1)dt + \sigma_1 \sqrt{v_t} dW_{1t} + (e^{J_1} - 1)d(N_{0t} + N_{1t}), && (2.1)\\
\frac{dS_{2t}}{S_{2t}} &= (r_t - (\lambda_0 + \lambda_2)m_2)dt + \sigma_2 \sqrt{v_t} dW_{2t} + (e^{J_2} - 1)d(N_{0t} + N_{2t}), && (2.2)\\
dv_t &= k_v(\theta_v - v_t)dt + \sigma_v \sqrt{v_t} dW_{vt}, && (2.3)\\
dr_t &= k_r(\theta_r - r_t)dt + \sigma_r dW_{rt}. && (2.4)
\end{aligned}
$$

where r_t is the riskless expected return rate of the two assets and v_t is the stochastic volatility rate of the two assets. r_t and v_t are the processes driving the mean-reverting rate. Here, we have σ_i (the coefficient of volatility of S_{it}'s return), k_r (the coefficient of mean-reversion speed of the interest process r_t), k_v (the coefficient of the mean-reversion speed of the variance process v_t), θ_r (the coefficient of the mean level of the interest process r_t), θ_v (the coefficient of the mean level of the variance process v_t), and $k_v\theta_v > \sigma_v^2$. The correlation structure of the Brownian motions involved in the above model is given by

$$\begin{aligned} d\langle W_{1t}, W_{2t}\rangle &= \rho_{12}dt, \\ d\langle W_{1t}, W_{vt}\rangle &= \rho_{1v}dt, \\ d\langle W_{2t}, W_{vt}\rangle &= \rho_{2v}dt, \\ d\langle W_{1t}, W_{rt}\rangle &= d\langle W_{2t}, W_{rt}\rangle = d\langle W_{vt}, W_{rt}\rangle = 0, \end{aligned}$$

$\lambda_j(j = 0, 1, 2)$ is the intensity of the Poisson process N_{jt}, $m_i = E[e^{J_i} - 1](i = 1, 2)$ is the average jump amplitude of the underlying asset price process S_{it}. The joint jump vector (J_1, J_2) is independent of all Brownian motions $(W_{1t}, W_{2t}, W_{vt}, W_{rt})$, Poisson processes (N_{0t}, N_{1t}, N_{2t}), and v_t, r_t. To characterize the interdependence of jumps, we assume that the jump vector (J_1, J_2) follows two possible distributions. Situation 1 is that the jump size (J_1, J_2) is subject to joint log-normal distribution $N(\mu_1, \mu_2; \sigma_{J_1}^2, \sigma_{J_1}^2; \rho)$; Situation 2 is that it is subject to joint asymmetric double-exponential distribution [20] with the following marginal density and distribution functions:

$$f_{J_i}(x) = p_i\xi_i e^{\xi_i x}1_{\{x<0\}} + q_i\eta_i e^{-\eta_i x}1_{\{x>0\}}, p_i + q_i = 1, \eta_i > 1, \xi_i > 0.$$

$$F_{J_i}(x) = p_i e^{\xi_i x}1_{\{x<0\}} + (1 - q_i e^{-\eta_i x})1_{\{x>0\}}, p_i + q_i = 1.$$

Since the analytical density and analytical expectation of the double-exponential jump distribution do not exist under general correlation, in the following chapters of this paper, we derived the analytical expectation formulas for the double-exponential jump distribution in three special cases (perfect positive correlation, independence, and perfect negative correlation), providing the upper and lower bounds for pricing the extremum options under the double-exponential distribution. Notice that the interdependence of the double-exponential distributions will be further clarified in the next section of this paper. Meanwhile, in this paper, the two types of jump distribution functions are used separately and not in combination.

2.2. *PDIE of maximum and minimum options*

In this work, the payoff function of maximum or minimum options is given by $h_i(S_{1T}, S_{2T})$. The payoff function of the maximum (call and put) option is given by

$$\begin{aligned} h_1(S_{1T}, S_{2T}) &= (\max\{S_{1T}, S_{2T}\} - K)^+, \\ h_2(S_{1T}, S_{2T}) &= (K - \max\{S_{1T}, S_{2T}\})^+. \end{aligned}$$

Similarly, the payoff function of the minimum (call and put) option is given by

$$\begin{aligned} h_3(S_{1T}, S_{2T}) &= (\min\{S_{1T}, S_{2T}\} - K)^+, \\ h_4(S_{1T}, S_{2T}) &= (K - \min\{S_{1T}, S_{2T}\})^+. \end{aligned}$$

The payoff function of the max-min option is given by

$$h_5(S_{1T}, S_{2T}) = \max\{S_{1T}, S_{2T}\} - \min\{S_{1T}, S_{2T}\}.$$

Without loss of generality, let $P(t, T; r)$ denote the value of a zero coupon bond at time t paying one unit at the maturity T, and then define $P(t, T; r)$ as

$$P(t, T; r) := E^{\mathbb{Q}}[e^{-\int_t^T r_s ds}|r_t = r]. \tag{2.5}$$

Lemma 2.1. *Suppose that the dynamics of process r_t is a mean-reverting process given by Eq (2.4), and then $P(t,T;r)$ is given by*

$$P(t,T;r) = e^{E(\tau)+F(\tau)r}, \tag{2.6}$$

where $\tau = T - t$, $E(\tau) = -\theta_r(F(\tau)+\tau) - \frac{\sigma_r^2}{4k_r}F^2(\tau) + \frac{\sigma_r^2}{2k_r^2}(F(\tau)+\tau)$, $F(\tau) = \frac{1}{k_r}(e^{-k_r\tau}-1)$. Meanwhile, the conditions for the boundedness of the expression $P(t,T,r)$ are $k_r > 0$ and $\theta_r > 0$.

Proof. If the interest process r_t is a mean-reverting process driven by Eq (2.4), then from the Feynman-Kac formula, the zero coupon bond price $P(t,T;r)$ satisfies a partial differential equation given by $(\tau = T - t)$

$$-rP - \frac{\partial P}{\partial \tau} + k_r(\theta_r - r)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma_r^2\frac{\partial^2 P}{\partial r^2} = 0$$

with the boundary condition $P(T,T;r) = 1$.

Assume that $P(t,T;r)$ is of the form $P(t,T;r) = e^{E(\tau)+F(\tau)r}$. Then, we obtain

$$\frac{\partial E}{\partial \tau} = k_r\theta_r F(\tau) + \frac{1}{2}\sigma_r^2 F^2(\tau),$$
$$\frac{\partial F}{\partial \tau} = -k_r F(\tau) - 1$$

with the boundary condition $E(0) = F(0) = 0$. Their solutions are given by the above Lemma 2.1.

From the pricing formula of zero coupon bonds $P(t,T;r)$, the condition for the boundedness of $P(t,T;r)$ can be obtained as the boundedness of $E(\tau)$ and $F(\tau)$. $k_r > 0$ ensures the boundedness of $F(\tau)$, and $\theta_r > 0$ ensures the boundedness of $E(\tau)$. □

According to the principle of risk-neutral pricing, the option price $P(t,s_1,s_2,v,r)$ is defined by

$$P(t,s_1,s_2,v,r) := E[e^{-\int_t^T r_s ds}h(S_{1T},S_{2T})|S_{1t} = s_1, S_{2t} = s_2, v_t = v, r_t = r].$$

Theorem 2.1. *If the prices S_{1t}, S_{2t} of the two assets follow Eqs (2.1)–(2.4), $P(t,s_1,s_2,v,r)$ satisfies a partial differential-integral equation given by $(\tau = T - t)$*

$$\begin{aligned}
&-\frac{\partial P}{\partial \tau} + (r-(\lambda_0+\lambda_1)m_1)s_1\frac{\partial P}{\partial s_1} + (r-(\lambda_0+\lambda_2)m_2)s_2\frac{\partial P}{\partial s_2} + k_v(\theta_v - v)\frac{\partial P}{\partial v}\\
&+k_r(\theta_r - r)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma_1^2 v s_1^2\frac{\partial^2 P}{\partial s_1^2} + \frac{1}{2}\sigma_2^2 v s_2^2\frac{\partial^2 P}{\partial s_2^2} + \frac{1}{2}\sigma_v^2 v\frac{\partial^2 P}{\partial v^2} + \frac{1}{2}\sigma_r^2\frac{\partial^2 P}{\partial r^2}\\
&+\rho_{12}\sigma_1\sigma_2 v s_1 s_2\frac{\partial^2 P}{\partial s_1\partial s_2} + \rho_{1v}\sigma_1\sigma_v v s_1\frac{\partial^2 P}{\partial s_1\partial v} + \rho_{2v}\sigma_2\sigma_v v s_2\frac{\partial^2 P}{\partial s_2\partial v}\\
&+\lambda_1 E[P(t,s_1e^{J_1},s_2,v,r)] + \lambda_2 E[P(t,s_1,s_2e^{J_2},v,r)] + \lambda_0 E[P(t,s_1e^{J_1},s_2e^{J_2},v,r)]\\
&-(r+\lambda_0+\lambda_1+\lambda_2)P = 0
\end{aligned} \tag{2.7}$$

with the initial condition $P(T,s_1,s_2,v,r) = h_i(s_1,s_2)$, $i = 1,2,3,4,5$.

Proof. As is well known, the discounted price of financial derivatives is a martingale. According to the Ito formula,

$$\mathrm{d}(\mathrm{e}^{-\int_0^t r_s ds}P(t,s_1,s_2,v,r))$$

$$
\begin{aligned}
&= P(t,s_1,s_2,v,r)\mathrm{d}e^{-\int_0^t r_s ds} + e^{-\int_0^t r_s ds}\mathrm{d}P(t,s_1,s_2,v,r)\\
&= e^{-\int_0^t r_s ds}\Bigg[-rP(t,s_1,s_2,v,r) - \frac{\partial P}{\partial \tau} + (r-(\lambda_0+\lambda_1)m_1)s_1\frac{\partial P}{\partial s_1} + (r-(\lambda_0+\lambda_2)m_2)s_2\frac{\partial P}{\partial s_2}\\
&\quad +k_v(\theta_v - v)\frac{\partial P}{\partial v} + k_r(\theta_r - r)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma_1^2 v s_1^2\frac{\partial^2 P}{\partial s_1^2} + \frac{1}{2}\sigma_2^2 v s_2^2\frac{\partial^2 P}{\partial s_2^2} + \frac{1}{2}\sigma_v^2 v\frac{\partial^2 P}{\partial v^2} + \frac{1}{2}\sigma_r^2\frac{\partial^2 P}{\partial r^2}\\
&\quad +\rho_{12}\sigma_1\sigma_2 v s_1 s_2\frac{\partial^2 P}{\partial s_1\partial s_2} + \rho_{1v}\sigma_1\sigma_v v s_1\frac{\partial^2 P}{\partial s_1\partial v} + \rho_{2v}\sigma_2\sigma_v v s_2\frac{\partial^2 P}{\partial s_2\partial v}\Bigg]\mathrm{d}t\\
&\quad +e^{-\int_0^t r_s ds}(\sigma_1\sqrt{v_t}\frac{\partial P}{\partial s_1}dW_{1t} + \sigma_2\sqrt{v_t}\frac{\partial P}{\partial s_2}dW_{2t} + \sigma_v\sqrt{v_t}\frac{\partial P}{\partial v}dW_{vt} + \sigma_r\sqrt{r_t}\frac{\partial P}{\partial r}dW_{rt})\\
&\quad +e^{-\int_0^t r_s ds}(P(t,s_1e^{J_1},s_2e^{J_2},v,r) - P(t,s_1,s_2,v,r))dN_{0t}\\
&\quad +e^{-\int_0^t r_s ds}(P(t,s_1e^{J_1},s_2,v,r) - P(t,s_1,s_2,v,r))dN_{1t}\\
&\quad +e^{-\int_0^t r_s ds}(P(t,s_1,s_2e^{J_2},v,r) - P(t,s_1,s_2,v,r))dN_{2t}\\
&= e^{-\int_0^t r_s ds}\Bigg[-(r+\lambda_0+\lambda_1+\lambda_2)P - \frac{\partial P}{\partial \tau} + (r-(\lambda_0+\lambda_1)m_1)s_1\frac{\partial P}{\partial s_1} + (r-(\lambda_0+\lambda_2)m_2)s_2\frac{\partial P}{\partial s_2}\\
&\quad +k_v(\theta_v - v)\frac{\partial P}{\partial v} + k_r(\theta_r - r)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma_1^2 v s_1^2\frac{\partial^2 P}{\partial s_1^2} + \frac{1}{2}\sigma_2^2 v s_2^2\frac{\partial^2 P}{\partial s_2^2} + \frac{1}{2}\sigma_v^2 v\frac{\partial^2 P}{\partial v^2} + \frac{1}{2}\sigma_r^2\frac{\partial^2 P}{\partial r^2}\\
&\quad +\rho_{12}\sigma_1\sigma_2 v s_1 s_2\frac{\partial^2 P}{\partial s_1\partial s_2} + \rho_{1v}\sigma_1\sigma_v v s_1\frac{\partial^2 P}{\partial s_1\partial v} + \rho_{2v}\sigma_2\sigma_v v s_2\frac{\partial^2 P}{\partial s_2\partial v}\\
&\quad +\lambda_1 E[P(t,s_1e^{J_1},s_2,v,r)] + \lambda_2 E[P(t,s_1,s_2e^{J_2},v,r)] + \lambda_0 E[P(t,s_1e^{J_1},s_2e^{J_2},v,r)]\Bigg]\mathrm{d}t\\
&\quad +e^{-\int_0^t r_s ds}(\sigma_1\sqrt{v_t}\frac{\partial P}{\partial s_1}dW_{1t} + \sigma_2\sqrt{v_t}\frac{\partial P}{\partial s_2}dW_{2t} + \sigma_v\sqrt{v_t}\frac{\partial P}{\partial v}dW_{vt} + \sigma_r\sqrt{r_t}\frac{\partial P}{\partial r}dW_{rt})\\
&\quad +e^{-\int_0^t r_s ds}(P(t,s_1e^{J_1},s_2e^{J_2},v,r)dN_{0t} - \lambda_0 E[P(t,s_1e^{J_1},s_2e^{J_2},v,r)]dt)\\
&\quad +e^{-\int_0^t r_s ds}(P(t,s_1e^{J_1},s_2,v,r)dN_{1t} - \lambda_1 E[P(t,s_1e^{J_1},s_2,v,r)]dt)\\
&\quad +e^{-\int_0^t r_s ds}(P(t,s_1,s_2e^{J_2},v,r)dN_{2t} - \lambda_2 E[P(t,s_1,s_2e^{J_2},v,r)]dt)\\
&\quad -e^{-\int_0^t r_s ds}P(t,s_1,s_2,v,r))\left[(dN_{0t}-\lambda_0 dt) + (dN_{1t}-\lambda_1 dt) + (dN_{2t}-\lambda_2 dt)\right].
\end{aligned}
$$

Finally, since the Ito integral, the compensating Poisson process, and the compensating compound Poisson process are all martingales, the core content of the theorem can be derived. □

Unfortunately, the direct Mellin transform of $h_1(S_{1T}, S_{2T})$, $h_4(S_{1T}, S_{2T})$, and $h_5(S_{1T}, S_{2T})$ does not exist. In order to price these options, it is required to price the exchange option with the option parity relation. Considering that $h_6(S_{1T}, S_{2T}) = (S_{1T} - S_{2T})^+$ and $h_7(S_{1T}, S_{2T}) = (S_{2T} - S_{1T})^+$ have the linear property, i.e., $h_6(S_{1T}, S_{2T}) = S_{2T}h_6(\frac{S_{1T}}{S_{2T}}, 1)$ and $h_7(S_{1T}, S_{2T}) = S_{2T}h_7(1, \frac{S_{1T}}{S_{2T}})$, let

$$
Z_t := \frac{S_{1t}}{S_{2t}}, z := \frac{s_1}{s_2},
$$

and then, according to pricing unit conversion technology, we define and derive the exchange option price $P(t, s_1, s_2, v, r)$ given by

$$
P(t,s_1,s_2,v,r) \;=\; E[e^{-\int_t^T r_s ds}(S_{1T} - S_{2T})^+ | S_{1t} = s_1, S_{2t} = s_2, v_t = v, r_t = r]
$$

$$
\begin{aligned}
&= s_2 E^{P_2}[(Z_T-1)^+|Z_t=z, v_t=v, r_t=r] \\
&= s_2 p(t,z,v,r),
\end{aligned}
$$

where the measure transformation is

$$
\begin{aligned}
\frac{d\mathbb{P}_2}{d\mathbb{P}}\Big|_{\mathcal{F}_t} &= \frac{S_{2T}/S_{2t}}{e^{\int_0^T r_s ds}/e^{\int_0^t r_s ds}} \\
&= \exp\{\sigma_2\int_t^T\sqrt{v_s}dW_{2s} - \frac{1}{2}\sigma_2^2\int_t^T v_s ds + \int_t^T J_2 d(N_{0s}+N_{2s}) - (\lambda_0+\lambda_2)m_2(T-t)\} \\
&= \exp\{\sigma_2\int_t^T\sqrt{v_s}dW_{2s} - \frac{1}{2}\sigma_2^2\int_t^T v_s ds\}\cdot\exp\{(\lambda_0-\lambda_0(m_2+1))(T-t)\}\prod_{i=N_{0t}+1}^{N_{0T}}\frac{\lambda_0(m_2+1)}{\lambda_0}\frac{\frac{e^{J_2}}{m_2+1}f(J_2)}{f(J_2)} \\
&\quad\cdot\exp\{(\lambda_2-\lambda_2(m_2+1))(T-t)\}\prod_{i=N_{2t}+1}^{N_{2T}}\frac{\lambda_2(m_2+1)}{\lambda_2}\frac{\frac{e^{J_2}}{m_2+1}f(J_2)}{f(J_2)}
\end{aligned}
$$

Under the new measure P_2, $\widetilde{W}_{2t} = W_{2t} - \int_0^t \sigma_2\sqrt{v_s}ds$ is a standard Brownian motion, the intensity of the compound Poisson process N_{0t} is $\lambda_0(m_2+1)$, the intensity of the compound Poisson process N_{2t} is $\lambda_2(m_2+1)$, and the jumping amplitude follows the probability density $\frac{e^x}{m_2+1}f(x)$.

Theorem 2.2. *If the prices S_{1t}, S_{2t} of the two assets follow Eqs (2.1)–(2.4), $p(t,z,v,r)$ satisfies a partial differential-integral equation given by ($\tau = T-t$)*

$$
\begin{aligned}
&-\frac{\partial p}{\partial\tau} + ((\lambda_0+\lambda_2)m_2 - (\lambda_0+\lambda_1)m_1)z\frac{\partial p}{\partial z} + \frac{1}{2}(\sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2)vz^2\frac{\partial^2 p}{\partial z^2} \\
&+(k_v\theta_v - (k_v - \rho_{2v}\sigma_2\sigma_v)v)\frac{\partial p}{\partial v} + \frac{1}{2}\sigma_v^2 v\frac{\partial^2 p}{\partial v^2} + k_r(\theta_r - r)\frac{\partial p}{\partial r} + \frac{1}{2}\sigma_r^2\frac{\partial^2 p}{\partial r^2} \\
&+(\rho_{1v}\sigma_1 - \rho_{2v}\sigma_2)\sigma_v vz\frac{\partial^2 p}{\partial z\partial v} + \lambda_1 E[p(t,ze^{J_1},v,r)] + \lambda_2 E[e^{J_2}p(t,ze^{-J_2},v,r)] \\
&+\lambda_0 E[e^{J_2}p(t,ze^{J_1-J_2},v,r)] - (\lambda_0+\lambda_1+\lambda_2+(\lambda_0+\lambda_2)m_2)p = 0
\end{aligned}
\tag{2.8}
$$

with the initial condition $p(T,z,v,r) = h_6(z,1)$ or $h_7(1,z)$.

Proof. First, under the risk-neutral measure P, by applying Itô's formula to derive the stochastic differential equation that Z_t satisfies, we obtain

$$
\begin{aligned}
dZ_t &= \frac{1}{S_{2t}}dS_{1t} - \frac{S_{1t}}{S_{2t}^2}dS_{2t} + \frac{S_{1t}}{S_{2t}^3}dS_{2t}dS_{2t} - \frac{1}{S_{2t}^2}dS_{2t}dS_{1t} \\
&= Z_t[(r_t - (\lambda_0+\lambda_1)m_1)dt + \sigma_1\sqrt{v_t}dW_{1t} - (r_t - (\lambda_0+\lambda_2)m_2)dt - \sigma_2\sqrt{v_t}dW_{2t} + \sigma_2^2 v_t dt \\
&\quad -\rho_{12}\sigma_1\sigma_2 v_t dt] + Z_t[(e^{J_1-J_2}-1)dN_{0t} + (e^{J_1}-1)dN_{1t} + (e^{-J_2}-1)dN_{2t}] \\
&= Z_t\{[((\lambda_0+\lambda_2)m_2 - (\lambda_0+\lambda_1)m_1)dt + \sigma_2(\sigma_2 - \rho_{12}\sigma_1)v_t dt + \sigma_1\sqrt{v_t}dW_{1t} - \sigma_2\sqrt{v_t}dW_{2t}] \\
&\quad +[(e^{J_1-J_2}-1)dN_{0t} + (e^{J_1}-1)dN_{1t} + (e^{-J_2}-1)dN_{2t}]\}.
\end{aligned}
$$

Based on the relevant equations of Brownian motion, that is, $d\langle W_{1t}, W_{2t}\rangle = \rho_{12}dt$, $d\langle W_{1t}, W_{vt}\rangle = \rho_{1v}dt$, $d\langle W_{2t}, W_{vt}\rangle = \rho_{2v}dt$, we can quickly obtain the following decomposition:

$$
W_{2t}, W_{1t} = \rho_{12}W_{2t} + \sqrt{1-\rho_{12}^2}\widetilde{W}_{1t},\ W_{vt} = \rho_{2v}W_{2t} + \frac{\rho_{1v} - \rho_{2v}\rho_{12}}{\sqrt{1-\rho_{12}^2}}\widetilde{W}_{1t} + \frac{\sqrt{1-\rho_{12}^2-\rho_{1v}^2-\rho_{2v}^2+2\rho_{1v}\rho_{2v}\rho_{12}}}{\sqrt{1-\rho_{12}^2}}\widetilde{W}_{vt}.
$$

Immediately afterward, under the new measurement P_2, it can be concluded that

$$\begin{aligned} dW_{2t} =& \ d\widetilde{W}_{2t} + \sigma_2\sqrt{v_t}dt, \\ dW_{1t} =& \ \rho_{12}d\widetilde{W}_{2t} + \rho_{12}\sigma_2\sqrt{v_t}dt + \sqrt{1-\rho_{12}^2}d\widetilde{W}_{1t}, \\ dW_{vt} =& \ \rho_{2v}d\widetilde{W}_{2t} + \rho_{2v}\sigma_2\sqrt{v_t}dt + \tfrac{\rho_{1v}-\rho_{2v}\rho_{12}}{\sqrt{1-\rho_{12}^2}}d\widetilde{W}_{1t} + \tfrac{\sqrt{1-\rho_{12}^2-\rho_{1v}^2-\rho_{2v}^2+2\rho_{1v}\rho_{2v}\rho_{12}}}{\sqrt{1-\rho_{12}^2}}d\widetilde{W}_{vt}. \end{aligned}$$

By substituting it into the stochastic differential equation satisfied by Z_t and v_t, we obtain

$$\begin{aligned} dZ_t =& \ Z_t\{[((\lambda_0+\lambda_2)m_2 - (\lambda_0+\lambda_1)m_1)dt \\ & +\rho_{12}\sigma_1\sqrt{v_t}d\widetilde{W}_{2t} + \sqrt{1-\rho_{12}^2}\sigma_1\sqrt{v_t}d\widetilde{W}_{1t} - \sigma_2\sqrt{v_t}d\widetilde{W}_{2t}] \\ & +[(e^{J_1-J_2}-1)dN_{0t} + (e^{J_1}-1)dN_{1t} + (e^{-J_2}-1)dN_{2t}]\}, \\ dv_t =& \ k_v(\theta_v - v_t)dt + \rho_{2v}\sigma_2\sigma_v v_t dt + \rho_{2v}\sigma_v\sqrt{v_t}d\widetilde{W}_{2t} + \tfrac{\rho_{1v}-\rho_{2v}\rho_{12}}{\sqrt{1-\rho_{12}^2}}\sigma_v\sqrt{v_t}d\widetilde{W}_{1t} \\ & + \tfrac{\sqrt{1-\rho_{12}^2-\rho_{1v}^2-\rho_{2v}^2+2\rho_{1v}\rho_{2v}\rho_{12}}}{\sqrt{1-\rho_{12}^2}}\sigma_v\sqrt{v_t}d\widetilde{W}_{vt}. \end{aligned}$$

Finally, according to Ito's lemma, the differential form $p(t, z, v, r)$ satisfies is

$$\begin{aligned} & \mathrm{d}(p(t,z,v,r)) \\ =& \left[-\frac{\partial p}{\partial \tau} + ((\lambda_0+\lambda_2)m_2 - (\lambda_0+\lambda_1)m_1)z\frac{\partial p}{\partial z} + \frac{1}{2}(\sigma_1^2 - 2\rho_{12}\sigma_1\sigma_1 + \sigma_2^2)z^2\frac{\partial^2 p}{\partial z^2}\right. \\ & +k_v((\theta_v - v) + \rho_{2v}\sigma_2\sigma_v v_t)\frac{\partial p}{\partial v} + \frac{1}{2}\sigma_v^2 v\frac{\partial^2 P}{\partial v^2} + k_r(\theta_r - r)\frac{\partial p}{\partial r} + \frac{1}{2}\sigma_r^2\frac{\partial^2 p}{\partial r^2} \\ & \left.(\rho_{1v}\sigma_1 - \rho_{2v}\sigma_2)\sigma_v vz\frac{\partial^2 p}{\partial z\partial v}\right]\mathrm{d}t \\ & +(\sigma_1\sqrt{v_t}\frac{\partial p}{\partial z}d(\rho_{12}d\widetilde{W}_{2t} + \sqrt{1-\rho_{12}^2}d\widetilde{W}_{1t}) - \sigma_2\sqrt{v_t}\frac{\partial p}{\partial z}d\widetilde{W}_{2t} + \sigma_r\sqrt{r_t}\frac{\partial p}{\partial r}dW_{rt}) \\ & +\sigma_v\sqrt{v_t}\frac{\partial p}{\partial v}d(\rho_{2v}d\widetilde{W}_{2t} + \frac{\rho_{1v}-\rho_{2v}\rho_{12}}{\sqrt{1-\rho_{12}^2}}d\widetilde{W}_{1t} + \frac{\sqrt{1-\rho_{12}^2-\rho_{1v}^2-\rho_{2v}^2+2\rho_{1v}\rho_{2v}\rho_{12}}}{\sqrt{1-\rho_{12}^2}}d\widetilde{W}_{vt}) \\ & +(p(t,ze^{J_1-J_2},v,r) - p(t,z,v,r))dN_{0t} + (p(t,ze^{J_1},v,r) - p(t,z,v,r))dN_{1t} \\ & +(p(t,ze^{-J_2},v,r) - p(t,z,v,r))dN_{2t} \\ =& \left[-\frac{\partial p}{\partial \tau} + ((\lambda_0+\lambda_2)m_2 - (\lambda_0+\lambda_1)m_1)z\frac{\partial p}{\partial z} + \frac{1}{2}(\sigma_1^2 - 2\rho_{12}\sigma_1\sigma_1 + \sigma_2^2)z^2\frac{\partial^2 p}{\partial z^2}\right. \\ & +k_v((\theta_v - v) + \rho_{2v}\sigma_2\sigma_v v_t)\frac{\partial p}{\partial v} + \frac{1}{2}\sigma_v^2 v\frac{\partial^2 P}{\partial v^2} + k_r(\theta_r - r)\frac{\partial p}{\partial r} + \frac{1}{2}\sigma_r^2\frac{\partial^2 p}{\partial r^2} \\ & (\rho_{1v}\sigma_1 - \rho_{2v}\sigma_2)\sigma_v vz\frac{\partial^2 p}{\partial z\partial v} + \lambda_1 E^{P_2}[p(t,ze^{J_1},v,r)] + \lambda_2(m_2+1)E^{P_2}[p(t,ze^{-J_2},v,r)] \\ & \left.+\lambda_0(m_2+1)E^{P_2}[p(t,ze^{J_1-J_2},v,r)] - (\lambda_0+\lambda_1+\lambda_2+(\lambda_0+\lambda_2)m_2)p\right]\mathrm{d}t \\ & +(\sigma_1\sqrt{v_t}\frac{\partial p}{\partial z}d(\rho_{12}d\widetilde{W}_{2t} + \sqrt{1-\rho_{12}^2}d\widetilde{W}_{1t}) - \sigma_2\sqrt{v_t}\frac{\partial p}{\partial z}d\widetilde{W}_{2t} + \sigma_r\sqrt{r_t}\frac{\partial p}{\partial r}dW_{rt}) \end{aligned}$$

$$
\begin{aligned}
&+\sigma_v\sqrt{v_t}\frac{\partial p}{\partial v}d(\rho_{2v}d\widetilde{W}_{2t}+\frac{\rho_{1v}-\rho_{2v}\rho_{12}}{\sqrt{1-\rho_{12}^2}}d\widetilde{W}_{1t}+\frac{\sqrt{1-\rho_{12}^2-\rho_{1v}^2-\rho_{2v}^2+2\rho_{1v}\rho_{2v}\rho_{12}}}{\sqrt{1-\rho_{12}^2}}d\widetilde{W}_{vt})\\
&+(p(t,ze^{J_1-J_2},v,r)dN_{0t}-\lambda_0(m_2+1)E^{P_2}[p(t,ze^{J_1-J_2},v,r)]dt)\\
&+(p(t,ze^{J_1},v,r)dN_{1t}-\lambda_1E^{P_2}[p(t,ze^{J_1},v,r)]dt)\\
&+(p(t,ze^{-J_2},v,r)dN_{2t}-\lambda_2(m_2+1)E^{P_2}[p(t,ze^{-J_2},v,r)]dt)\\
&-p(t,s_1,s_2,v,r))\left[(dN_{0t}-\lambda_0(m_2+1)dt)+(dN_{1t}-\lambda_1dt)+(dN_{2t}-\lambda_2(m_2+1)dt)\right].
\end{aligned}
$$

Based on the relationship between the risk-neutral measure P and the new measure P_2, the following equation can be derived:

$$
\begin{aligned}
\lambda_1E^{P_2}[p(t,ze^{J_1},v,r)] &= \lambda_1E[p(t,ze^{J_1},v,r)],\\
\lambda_2(m_2+1)E^{P_2}[p(t,ze^{-J_2},v,r)] &= \lambda_2E[e^{J_2}p(t,ze^{-J_2},v,r)],\\
\lambda_0(m_2+1)E^{P_2}[p(t,ze^{J_1-J_2},v,r)] &= \lambda_0(m_2+1)E[e^{J_2}p(t,ze^{J_1-J_2},v,r)].
\end{aligned}
$$

Finally, since the Ito integral, the compensating Poisson process, and the compensating compound Poisson process are all martingales, the core content of the theorem can be derived. □

3. Option price formulas

In this section, we solve the above PDIE by using the Mellin transform method. Let $\hat{P}(t,u_1,u_2,v,r)$ and $\hat{p}(t,u,v,r)$ be Mellin transform functions of $P(t,s_1,s_2,v,r)$ and $p(t,z,v,r)$, respectively, so that

$$\hat{P}(t,u_1,u_2,v,r) = \int_0^{+\infty}\int_0^{+\infty}P(t,s_1,s_2,v,r)s_1^{u_1-1}s_2^{u_2-1}ds_1ds_2, \tag{3.1}$$

$$\hat{p}(t,u,v,r) = \int_0^{+\infty}p(t,z,v,r)z^{u-1}dz, \tag{3.2}$$

where u_1, u_2, u are complex variables.

The reason why the Mellin transform is indispensable in the pricing of maximum and minimum options is that the Mellin transform does not exist for the prices of many other exotic options, while the compound Mellin transform of maximum and minimum options maintains continuous existence, which is a concentrated characteristic not possessed by other options. Additionally, compared with traditional characteristic functions or Fourier methods, the operation of the Mellin transformation is easier. This is mainly reflected in the following two aspects. The first aspect, when applied to complex options like maximum and minimum options, is that the characteristic function or Fourier transform methods require the construction of multiple different and complicated new measures, and also need to utilize the transformation relationship between the characteristic function and probability, making the representation of option prices relatively complex. However, the result of the Mellin transform is relatively simple. The second aspect is that the Fourier method generally requires the construction of an appropriate fast Fourier transform algorithm, but due to the fact that the Fourier transform of some functions does not exist and a damping coefficient needs to be added, the algorithm needs to consider more issues. The Mellin transform does not require too many operations. It only needs to consider the specific integral truncation interval. Therefore, from the implementation perspective, the Mellin transform has simplicity and an ease of use.

3.1. Some key theoretical preparation

First, we have the Mellin transforms of expectation terms.
According to the interchange property of integration, we have

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} E[P(t, s_1e^{J_1}, s_2, v, r)]s_1^{u_1-1}s_2^{u_2-1}ds_1ds_2 \\
= \;& E[\int_0^{+\infty}\int_0^{+\infty} P(t, s_1e^{J_1}, s_2, v, r)s_1^{u_1-1}s_2^{u_2-1}ds_1ds_2] \\
= \;& E[e^{-u_1J_1}\int_0^{+\infty}\int_0^{+\infty} P(t, s_1e^{J_1}, s_2, v, r)(s_1e^{J_1})^{u_1-1}s_2^{u_2-1}d(s_1e^{J_1})ds_2] \\
= \;& E[e^{-u_1J_1}]\hat{P}(t, u_1, u_2, v, r).
\end{aligned}$$

Similarly, the other equations are given by

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} E[P(t, s_1, s_2e^{J_2}, v, r)]s_1^{u_1-1}s_2^{u_2-1}ds_1ds_2 = E[e^{-u_2J_2}]\hat{P}(t, u_1, u_2, v, r), \\
&\int_0^{+\infty}\int_0^{+\infty} E[P(t, s_1e^{J_1}, s_2e^{J_2}, v, r)]s_1^{u_1-1}s_2^{u_2-1}ds_1ds_2 = E[e^{-u_1J_1-u_2J_2}]\hat{P}(t, u_1, u_2, v, r), \\
&\int_0^{+\infty} E[p(t, ze^{J_1}, v, r)]z^{u-1}dz = E[e^{-uJ_1}]\hat{p}(t, z, v, r), \\
&\int_0^{+\infty} E[e^{J_2}p(t, ze^{-J_2}, v, r)]z^{u-1}dz = E[e^{(u+1)J_2}]\hat{p}(t, z, v, r), \\
&\int_0^{+\infty} E[e^{J_2}p(t, ze^{J_1-J_2}, v, r)]z^{u-1}dz = E[e^{-uJ_1+(u+1)J_2}]\hat{p}(t, z, v, r).
\end{aligned}$$

Second, we have the Mellin transform of different payoff functions.
When $P(T, s_1, s_2, v, r) = h_2(s_1, s_2)$, then there is

$$\hat{P}(T, u_1, u_2, v, r) = \frac{K^{u_1+u_2+1}}{u_1u_2(u_1+u_2+1)}, Re(u_1) \in (0, +\infty), Re(u_2) \in (0, +\infty);$$

When $P(T, s_1, s_2, v, r) = h_3(s_1, s_2)$, then there is

$$\hat{P}(T, u_1, u_2, v, r) = -\frac{K^{u_1+u_2+1}}{u_1u_2(u_1+u_2+1)}, Re(u_1) \in (-\infty, -1), Re(u_2) \in (-\infty, -1);$$

When $p(T, z, v, r) = h_6(z, 1)$, then there is

$$\hat{p}(T, u, v, r) = \frac{1}{u(u+1)}, Re(u) \in (-\infty, -1);$$

When $p(T, z, v, r) = h_7(1, z)$, then there is

$$\hat{p}(T, u, v, r) = \frac{1}{u(u+1)}, Re(u) \in (0, +\infty),$$

where $Re(\cdot)$ represents the real part of a complex variable.

Notice that the above four payoff functions can be directly solved by single and double Mellin transforms. However, when $P(T, s_1, s_2, v, r) = h_1(s_1, s_2)$, $h_4(s_1, s_2)$, or $h_5(s_1, s_2)$, then the Mellin transform functions of $P(T, s_1, s_2, v, r)$ do not exist. It is required to combine the Mellin transform with the price of exchange option. The parity relation of the max-min and exchange options is given by

$$h_1(s_1, s_2) = h_2(s_1, s_2) + h_6(s_1, s_2) + s_2 - K; \quad (3.3)$$
$$h_4(s_1, s_2) = h_3(s_1, s_2) + h_6(s_1, s_2) + K - s_1; \quad (3.4)$$
$$h_5(s_1, s_2) = h_6(s_1, s_2) + h_7(s_1, s_2). \quad (3.5)$$

Third, we have the expectation Expectation computation under two different jump dependence distributions.
If (J_1, J_2) is subject to joint log-normal distribution, we have

$$\begin{aligned} E[e^{-u_1 J_1}] &= e^{-\mu_1 u_1 + \frac{1}{2}\sigma_{J_1}^2 u_1^2}, \\ E[e^{-u_2 J_2}] &= e^{-\mu_2 u_2 + \frac{1}{2}\sigma_{J_2}^2 u_2^2}, \\ E[e^{-u_1 J_1 - u_2 J_2}] &= e^{-(\mu_1 u_1 + \mu_2 u_2) + \frac{1}{2}(\sigma_{J_1}^2 u_1^2 + 2\rho_J \sigma_{J_1}\sigma_{J_2} u_1 u_2 + \sigma_{J_2}^2 u_2^2)}. \end{aligned} \quad (3.6)$$

If (J_1, J_2) is subject to asymmetric double-exponential distribution, we have

$$\begin{aligned} E[e^{-u_1 J_1}] &= \frac{p_1\xi_1}{\xi_1 - u_1} + \frac{q_1\eta_1}{\eta_1 - u_1}, -\eta_1 < Re(u_1) < \xi_1, \\ E[e^{-u_2 J_2}] &= \frac{p_2\xi_2}{\xi_2 - u_2} + \frac{q_2\eta_2}{\eta_2 - u_2}, -\eta_2 < Re(u_2) < \xi_2. \end{aligned} \quad (3.7)$$

Case 1. (J_1, J_2) is an independent structure, and then

$$E[e^{-u_1 J_1 - u_2 J_2}] = (\frac{p_1\xi_1}{\xi_1 - u_1} + \frac{q_1\eta_1}{\eta_1 - u_1})(\frac{p_2\xi_2}{\xi_2 - u_2} + \frac{q_2\eta_2}{\eta_2 - u_2}). \quad (3.8)$$

Case 2. (J_1, J_2) is a completely positive dependent structure, and then
when $p_1 > p_2$,

$$\begin{aligned} E[e^{-u_1 J_1 - u_2 J_2}] &= (\frac{p_1}{p_2})^{\frac{u_1}{\xi_1}} \frac{p_2\xi_1\xi_2}{\xi_1\xi_2 - u_1\xi_2 - u_2\xi_1} + p_1^{\frac{u_1}{\xi_1}} q_2^{-\frac{u_2}{\eta_2}} \int_{q_1}^{q_2} (1-x)^{-\frac{u_1}{\xi_1}} x^{\frac{u_2}{\eta_2}} dx \\ &+ (\frac{q_2}{q_1})^{-\frac{u_2}{\eta_2}} \frac{q_1\eta_1\eta_2}{\eta_1\eta_2 + u_1\eta_2 + u_2\eta_1}, \frac{Re(u_1)}{\xi_1} + \frac{Re(u_2)}{\xi_2} < 1, \frac{Re(u_1)}{\eta_1} + \frac{Re(u_2)}{\eta_2} > -1; \end{aligned} \quad (3.9)$$

when $p_1 < p_2$,

$$\begin{aligned} E[e^{-u_1 J_1 - u_2 J_2}] &= (\frac{p_2}{p_1})^{\frac{u_2}{\xi_2}} \frac{p_1\xi_1\xi_2}{\xi_1\xi_2 - u_1\xi_2 - u_2\xi_1} + p_2^{\frac{u_2}{\xi_2}} q_1^{-\frac{u_1}{\eta_1}} \int_{q_2}^{q_1} (1-x)^{-\frac{u_2}{\xi_2}} x^{\frac{u_1}{\eta_1}} dx \\ &+ (\frac{q_1}{q_2})^{-\frac{u_1}{\eta_1}} \frac{q_2\eta_1\eta_2}{\eta_1\eta_2 + u_1\eta_2 + u_2\eta_1}, \frac{Re(u_1)}{\xi_1} + \frac{Re(u_2)}{\xi_2} < 1, \frac{Re(u_1)}{\eta_1} + \frac{Re(u_2)}{\eta_2} > -1; \end{aligned} \quad (3.10)$$

when $p_1 = p_2$,

$$\begin{aligned} E[e^{-u_1 J_1 - u_2 J_2}] &= \frac{p_1\xi_1\xi_2}{\xi_1\xi_2 - u_1\xi_2 - u_2\xi_1} + \frac{q_2\eta_1\eta_2}{\eta_1\eta_2 + u_1\eta_2 + u_2\eta_1}, \\ \frac{Re(u_1)}{\xi_1} + \frac{Re(u_2)}{\xi_2} &< 1, \frac{Re(u_1)}{\eta_1} + \frac{Re(u_2)}{\eta_2} > -1. \end{aligned} \quad (3.11)$$

Case 3. (J_1, J_2) is a completely negative dependent structure, and then
when $p_1 > q_2$,

$$\begin{aligned} E[e^{-u_1 J_1 - u_2 J_2}] &= (\frac{q_1}{p_2})^{-\frac{u_2}{\xi_2}} \frac{q_1\eta_1\xi_2}{\eta_1\xi_2 + u_1\xi_2 - u_2\eta_1} + p_1^{\frac{u_1}{\xi_1}} p_2^{\frac{u_2}{\eta_2}} \int_{q_1}^{p_2} (1-x)^{-\frac{u_1}{\xi_1}} x^{-\frac{u_2}{\xi_2}} dx \\ &+ (\frac{q_2}{p_1})^{-\frac{u_1}{\xi_1}} \frac{q_2\xi_1\eta_2}{\xi_1\eta_2 - u_1\eta_2 + u_2\xi_1}, \frac{Re(u_1)}{\xi_1} - \frac{Re(u_2)}{\eta_2} < 1, \frac{Re(u_1)}{\eta_1} - \frac{Re(u_2)}{\xi_2} > -1; \end{aligned} \quad (3.12)$$

when $p_1 < q_2$,

$$E[e^{-u_1J_1-u_2J_2}] = (\frac{q_1}{p_2})^{-\frac{u_1}{\eta_1}} \frac{p_2\eta_1\xi_2}{\eta_1\xi_2+u_1\xi_2-u_2\eta_1} + q_1^{-\frac{u_1}{\eta_1}} q_2^{-\frac{u_2}{\eta_2}} \int_{p_1}^{q_2} (1-x)^{\frac{u_1}{\eta_1}} x^{\frac{u_2}{\eta_2}} dx$$
$$+(\frac{q_2}{p_1})^{-\frac{u_2}{\eta_2}} \frac{p_1\xi_1\eta_2}{\xi_1\eta_2-u_1\eta_2+u_2\xi_1}, \frac{Re(u_1)}{\xi_1} - \frac{Re(u_2)}{\eta_2} < 1, \frac{Re(u_1)}{\eta_1} - \frac{Re(u_2)}{\xi_2} > -1; \tag{3.13}$$

when $p_1 = q_2$,

$$E[e^{-u_1J_1-u_2J_2}] = \frac{p_2\eta_1\xi_2}{\eta_1\xi_2+u_1\xi_2-u_2\eta_1} + \frac{p_1\xi_1\eta_2}{\xi_1\eta_2-u_1\eta_2+u_2\xi_1},$$
$$\frac{Re(u_1)}{\xi_1} - \frac{Re(u_2)}{\eta_2} < 1, \frac{Re(u_1)}{\eta_1} - \frac{Re(u_2)}{\xi_2} > -1. \tag{3.14}$$

Notice that the above Eqs (3.9)–(3.14) are proved in the Appendix. Due to the complexity of the double-exponential distribution, it is difficult to give specific analytical expressions for other types of dependence structures.

3.2. The solution of Mellin transform functions

Lemma 3.1. *Let A, B, C, E be constants, and $A \neq 0, B^2 - 4AC \neq 0$. Then the following ordinary differential equations*

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}x} = Ay^2 + By + C, \\ y|_{x=0} = E \end{cases} \tag{3.15}$$

have the solution that

$$y = \frac{y_1 - \frac{E-y_1}{E-y_2} y_2 \mathrm{e}^{dx}}{1 - \frac{E-y_1}{E-y_2}\mathrm{e}^{dx}} = \frac{(E-y_2)y_1 - (E-y_1)y_2\mathrm{e}^{dx}}{E - y_2 - (E-y_1)\mathrm{e}^{dx}},$$

where

$$d = \sqrt{B^2 - 4AC}, \quad y_1 = \frac{-B+d}{2A}, \quad y_2 = \frac{-B-d}{2A}.$$

Proof. Because of the condition $B^2 - 4AC \neq 0$, obviously, the inequality $y_1 \neq y_2$ holds. Furthermore, at least one of y_1 and y_2 is not equal to E. Let us assume that $y_2 \neq E$. Since

$$Ay^2 + By + C = A(y-y_1)(y-y_2),$$

then, when $y \neq y_1, y_2$,

$$\frac{1}{(y-y_1)(y-y_2)}\mathrm{d}y = A\mathrm{d}x,$$

and when $y \neq 0$, integrating both sides of the equation gives

$$\frac{1}{y_1 - y_2} \ln \frac{y-y_1}{y-y_2} = Ax + D.$$

Here, D is a constant. From the above equation, we can obtain

$$\frac{y-y_1}{y-y_2}=\mathrm{e}^{D(y_1-y_2)}\mathrm{e}^{Ax(y_1-y_2)}.$$

Let $c=\mathrm{e}^{D(y_1-y_2)}$. Considering that $A(y_1-y_2)=d$, then

$$\frac{y-y_1}{y-y_2}=c\mathrm{e}^{dx}.$$

Therefore, we can obtain

$$y=\frac{y_1-y_2c\mathrm{e}^{dx}}{1-c\mathrm{e}^{dx}}. \tag{3.16}$$

Since $y_2\neq E$, it follows that $y=y_2$ is not a solution to Eq (3.16). If $c=0$ is allowed, then when $c=0$, $y=y_1$, this equation is the general solution of differential equations. Using the initial condition $y|_{x=0}=E$, we can solve for the constant $c=\frac{E-y_1}{E-y_2}$. Substituting this into the equation gives the solution to differential equations as

$$y=\frac{y_1-\frac{E-y_1}{E-y_2}y_2\mathrm{e}^{dx}}{1-\frac{E-y_1}{E-y_2}\mathrm{e}^{dx}}=\frac{(E-y_2)y_1-(E-y_1)y_2\mathrm{e}^{dx}}{E-y_2-(E-y_1)\mathrm{e}^{dx}}.$$

The lemma has been proved. □

According to the above three steps, full preparation has been made for the subsequent work. Next, in the following Theorems 3.1 and 3.2, we will obtain the solution $\hat{P}(t,u_1,u_2,v,r)$, $\hat{p}(t,u,v,r)$ by variable substitution and differential equation techniques, respectively.

Theorem 3.1. *A solution of $\hat{P}(t,u_1,u_2,v,r)$ is given by*

$$\hat{P}(t,u_1,u_2,v,r)=\hat{P}(T,u_1,u_2,v,r)e^{F(u_1,u_2)\tau+A(\tau,u_1,u_2)+B(\tau,u_1,u_2)r+C(\tau,u_1,u_2)v}, \tag{3.17}$$

where $\tau=T-t$, $F(u_1,u_2), A(\tau,u_1,u_2), B(\tau,u_1,u_2)$, and $C(\tau,u_1,u_2)$ are given by
$F(u_1,u_2)=-(1-m_1u_1-m_2u_2-E[e^{-u_1J_1-u_2J_2}])\lambda_0-(1-m_1u_1-E[e^{-u_1J_1}])\lambda_1$
$-(1-m_2u_2-E[e^{-u_2J_2}])\lambda_2$,
$A(\tau,u_1,u_2)=A_1(\tau,u_1,u_2)+A_2(\tau,u_1,u_2)+A_3(\tau,u_1,u_2)$,
$A_1(\tau,u_1,u_2)=\frac{\theta_r}{k_r}(1-e^{-k_r\tau}-k_r\tau)(1+u_1+u_2)$,
$A_2(\tau,u_1,u_2)=\frac{\sigma_r^2}{4k_r^3}(-3-e^{-k_r\tau}+4e^{-k_r\tau}+2k_r\tau)(1+u_1+u_2)^2$,
$A_3(\tau,u_1,u_2)=\frac{k_v\theta_v}{\sigma_v^2}[(b(u_1,u_2)+\delta(u_1,u_2))\tau-2\ln(\frac{1-g(u_1,u_2)e^{\delta(u_1,u_2)\tau}}{1-g(u_1,u_2)})]$,
$B(\tau,u_1,u_2)=\frac{e^{-k_r\tau}-1}{k_r}(1+u_1+u_2)$,
$C(\tau,u_1,u_2)=\frac{b(u_1,u_2)+\delta(u_1,u_2)}{\sigma_v^2}(\frac{1-e^{\delta(u_1,u_2)\tau}}{1-g(u_1,u_2)e^{\delta(u_1,u_2)\tau}})$,
$b(u_1,u_2)=k_v+\rho_{1v}\sigma_1\sigma_vu_1+\rho_{2v}\sigma_2\sigma_vu_2$,
$c(u_1,u_2)=\sigma_1^2u_1(u_1+1)+2\rho_{12}\sigma_1\sigma_2u_1u_2+\sigma_2^2u_2(u_2+1)$,
$\delta(u_1,u_2)=\sqrt{b^2(u_1,u_2)-\sigma_v^2c(u_1,u_2)}$, $g(u_1,u_2)=\frac{b(u_1,u_2)+\delta(u_1,u_2)}{b(u_1,u_2)-\delta(u_1,u_2)}$.

Proof. From the property of the double Mellin transform, we can obtain

$$-\frac{\partial\hat{P}}{\partial\tau}+k_r(\theta_r-r)\frac{\partial\hat{P}}{\partial r}+\frac{1}{2}\sigma_r^2\frac{\partial^2\hat{P}}{\partial r^2}+(k_v\theta_v-(k_v+\rho_{1v}\sigma_1\sigma_v u_1+\rho_{2v}\sigma_2\sigma_v u_2)v)\frac{\partial\hat{P}}{\partial v}+\frac{1}{2}\sigma_v^2 v\frac{\partial^2\hat{P}}{\partial v^2}$$
$$+\{-(1-m_1u_1-m_2u_2-E[e^{-u_1J_1-u_2J_2}])\lambda_0-(1-m_1u_1-E[e^{-u_1J_1}])\lambda_1-(1-m_2u_2-E[e^{-u_2J_2}])\lambda_2$$
$$-(1+u_1+u_2)r+\frac{1}{2}(\sigma_1^2u_1(u_1+1)+2\rho_{12}\sigma_1\sigma_2u_1u_2+\sigma_2^2u_2(u_2+1))v\}\hat{P}=0,$$

with the boundary condition $\hat{P}(T,u_1,u_2,v,r)=\hat{h}(u_1,u_2)$.

We use the following change of variable,

$$\hat{P}(t,u_1,u_2,v,r)=\hat{P}(T,u_1,u_2,v,r)\exp\{F(u_1,u_2)\}Q(t,u_1,u_2,v,r),$$

and substitute it into the above PDIE. Then, the PDIE problem becomes a problem for $Q(t,u_1,u_2,v,r)$ as follows:

$$-\frac{\partial\hat{Q}}{\partial\tau}+k_r(\theta_r-r)\frac{\partial\hat{Q}}{\partial r}+\frac{1}{2}\sigma_r^2\frac{\partial^2\hat{Q}}{\partial r^2}+(k_v\theta_v-(k_v+\rho_{1v}\sigma_1\sigma_v u_1+\rho_{2v}\sigma_2\sigma_v u_2)v)\frac{\partial\hat{Q}}{\partial v}+\frac{1}{2}\sigma_v^2 v\frac{\partial^2\hat{Q}}{\partial v^2}$$
$$+\{-(1+u_1+u_2)r+\frac{1}{2}(\sigma_1^2u_1(u_1+1)+2\rho_{12}\sigma_1\sigma_2u_1u_2+\sigma_2^2u_2(u_2+1))v\}\hat{Q}=0,$$

with the boundary condition $\hat{Q}(T,u_1,u_2,v,r)=1$.

Next, we assume that $\hat{Q}$ is of the form

$$\hat{Q}(t,u_1,u_2,v,r)=\exp\{A(\tau,u_1,u_2)+B(\tau,u_1,u_2)r+C(\tau,u_1,u_2)v\}.$$

Then, $A(\tau,u_1,u_2)$, $B(\tau,u_1,u_2)$, and $C(\tau,u_1,u_2)$ become solutions of Riccati differential equations

$$\frac{\partial A}{\partial\tau}=k_r\theta_rB(\tau,u_1,u_2)+\frac{1}{2}\sigma_r^2B^2(\tau,u_1,u_2)+k_v\theta_vC(\tau,u_1,u_2),A(\tau,u_1,u_2)=0;$$
$$\frac{\partial B}{\partial\tau}=-k_rB(\tau,iu_1,u_2)-(1+u_1+u_2),B(\tau,u_1,u_2)=0;$$
$$\frac{\partial C}{\partial\tau}=\frac{1}{2}\sigma_v^2C^2(\tau,u_1,u_2)-b(u_1,u_2)C(\tau,u_1,u_2)+\frac{1}{2}c(u_1,u_2),C(\tau,u_1,u_2)=0,$$

respectively, whose solutions are given by the above. □

Theorem 3.2. *A solution of $\hat{p}(t,u,v,r)$ is given by*

$$\hat{p}(t,u,v,r)=\hat{p}(T,u,v,r)e^{\acute{F}(u)\tau+\acute{A}(\tau,u)+\acute{B}(\tau,u)r+\acute{C}(\tau,u)v},\tag{3.18}$$

where $\tau=T-t$, $\acute{F}(u),\acute{A}(\tau,u),\acute{B}(\tau,u)$, and $\acute{C}(\tau,u)$ are given by
$\acute{F}(u)=-(1+m_2+(m_2-m_1)u-E[e^{-uJ_1+(u+1)J_2}])\lambda_0-(1-m_1u-E[e^{-uJ_1}])\lambda_1$
$-(1+m_2+m_2u-E[e^{(u+1)J_2}])\lambda_2,$
$\acute{A}(\tau,u)=\acute{A}_1(\tau,u)+\acute{A}_2(\tau,u)+\acute{A}_3(\tau,u),$
$\acute{A}_1(\tau,u)=\acute{A}_2(\tau,u)=\acute{B}(\tau,u)=0,$
$\acute{A}_3(\tau,u)=\frac{k_v\theta_v}{\sigma_v^2}[(\acute{b}(u)+\acute{\delta}(u))\tau-2\ln(\frac{1-\acute{g}(u)e^{\acute{\delta}(u)\tau}}{1-\acute{g}(u)})],$
$\acute{C}(\tau,u)=\frac{\acute{b}(u)+\acute{\delta}(u)}{\sigma_v^2}(\frac{1-e^{\acute{\delta}(u)\tau}}{1-\acute{g}(u)e^{\acute{\delta}(u)\tau}}),$
$\acute{b}(u)=k_v-(\rho_{2v}\sigma_2-(\rho_{1v}\sigma_1-\rho_{2v}\sigma_2)u)\sigma_v,$
$\acute{c}(u)=(\sigma_1^2-2\rho_{12}\sigma_1\sigma_2+\sigma_2^2)u(u+1),$
$\acute{\delta}(u)=\sqrt{\acute{b}^2(u)-\sigma_v^2\acute{c}(u)},\ \acute{g}(u)=\frac{\acute{b}(u)+\acute{\delta}(u)}{\acute{b}(u)-\acute{\delta}(u)}.$

Proof. The proof of this theorem is similar to Theorem 3.1. □

From inverse transform of the Mellin transform, we obtain

$$P(t, s_1, s_2, v, r) = \frac{1}{(2\pi i)^2} \int_{c_2-i\infty}^{c_2+i\infty} \int_{c_1-i\infty}^{c_1+i\infty} \hat{P}(t, u_1, u_2, v, r) s_1^{-u_1} s_2^{-u_2} du_1 du_2, \quad (3.19)$$

$$p(t, z, v, r) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \hat{p}(t, u, v, r) z^{-u} du. \quad (3.20)$$

3.3. The convergence of the inverse Mellin transform

In the following Theorems 3.3 and 3.4, we will check that the right-hand sides in the above Eqs (3.17) and (3.18) have the rigorous solutions of the proposed PDIES (Eqs (2.5) and (2.6)) and prove their convergence to the left-hand side by term-by-term decomposition and an equivalent infinite-small technique.

Theorem 3.3. *Let c_1 and c_2 be real numbers such that*

$$E[e^{-c_1 J_1}] < +\infty, E[e^{-c_2 J_2}] < +\infty, E[e^{-c_1 J_1 - c_2 J_2}] < +\infty,$$

and then the rigorous solution to the above PDIE for maximum or minimum options at any time $t \leq T$ is given by

$$P(t, s_1, s_2, v, r) = \frac{1}{(2\pi i)^2} \int_{c_2-i\infty}^{c_2+i\infty} \int_{c_1-i\infty}^{c_1+i\infty} \hat{P}(t, u_1, u_2, v, r) s_1^{-u_1} s_2^{-u_2} du_1 du_2. \quad (3.21)$$

Proof. According to the inverse of the Mellin transform, it is easy to see that the boundary condition $P(T, s_1, s_2, v, r) = h_2(s_1, s_2)$ or $h_3(s_1, s_2)$ holds. Next, we will finish the proof in two sections.

On the one hand, we prove the absolute integrability of $P(t, u_1, u_2, v, r)$.

First, the following change of variables is used to $P(t, u_1, u_2, v, r)$, that is, $u_1 = c_1 + id_1, u_2 = c_2 + id_2$. Then, we obtain

$$\frac{1}{(2\pi i)^2} \int_{c_2-i\infty}^{c_2+i\infty} \int_{c_1-i\infty}^{c_1+i\infty} \hat{P}(t, u_1, u_2, v, r) s_1^{-u_1} s_2^{-u_2} du_1 du_2$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{P}(T, c_1 + id_1, c_2 + id_2, v, r) \exp\{F(c_1 + id_1, c_2 + id_2)\tau + A(\tau, c_1 + id_1, c_2 + id_2)$$

$$+ B(\tau, c_1 + id_1, c_2 + id_2)r + C(\tau, c_1 + id_1, c_2 + id_2)v\} s_1^{-c_1 - id_1} s_2^{-c_1 - id_1} dd_1 dd_2.$$

Second, we prove the boundedness of terms $\hat{P}(T, u_1, u_2, v, r), e^{F(u_1, u_2)\tau}, e^{A_1(\tau, u_1, u_2)}, e^{A_2(\tau, u_1, u_2)}$, and $e^{B(\tau, u_1, u_2)r}$.

(1) The boundedness of $\hat{P}(T, u_1, u_2, v, r)$.

According to the absolute value inequality for the integral, the following inequality holds:

$$\begin{aligned}
& |\hat{P}(T, c_1 + id_1, c_2 + id_2, v, r)| \\
= & \left| \int_0^{+\infty} \int_0^{+\infty} h(s_1, s_2) s_1^{c_1 + id_1 - 1} s_2^{c_1 + id_1 - 1} ds_1 ds_2 \right| \\
\leq & \int_0^{+\infty} \int_0^{+\infty} |h(s_1, s_2) s_1^{c_1 + id_1 - 1} s_2^{c_1 + id_1 - 1}| ds_1 ds_2 \\
= & \hat{P}(T, c_1, c_2, v, r).
\end{aligned}$$

So $\hat{P}(T, c_1, c_2, v, r)$ is the boundary of $\hat{P}(T, u_1, u_2, v, r)$.

(2) The boundedness of $e^{F(u_1,u_2)\tau}$, $e^{A_1(\tau,u_1,u_2)}$, and $e^{B(\tau,u_1,u_2)r}$.

It is easy to see

$$\begin{aligned} Re(B(\tau, c_1 + id_1, c_2 + id_2)) &= B(\tau, c_1, c_2), \\ Re(A_1(\tau, c_1 + id_1, c_2 + id_2)) &= A_1(\tau, c_1, c_2), \\ Re(F(c_1 + id_1, c_2 + id_2)) &\leq F(c_1, c_2). \end{aligned}$$

So it is clear that $e^{F(u_1,u_2)\tau}$, $e^{A_1(\tau,u_1,u_2)}$, and $e^{B(\tau,u_1,u_2)r}$ have a limit boundary.

(3) The boundedness of $e^{A_2(\tau,u_1,u_2)}$.

Notice that

$$\begin{aligned} & Re(A_2(\tau, c_1 + id_1, c_2 + id_2)) \\ = \; & \tfrac{\sigma_r^2}{2k_r^2} \int_0^\tau (1 - e^{-k_r\omega})^2 d\omega \; \cdot ((1 + c_1 + c_2)^2 - (d_1 + d_2)^2) \\ \leq \; & A_2(\tau, c_1, c_2). \end{aligned}$$

So $e^{A_2(\tau,c_1,c_2)}$ is the boundary of $e^{A_2(\tau,u_1,u_2)}$.

Third, we prove the boundedness and absolute integrability of $e^{C(\tau,u_1,u_2)v}$ and $e^{A_3(\tau,u_1,u_2)}$.

1) The absolute integrability of $e^{C(\tau,u_1,u_2)v}$.

According to the above equation and the conjugate property of complex numbers, the following equation is given by

$$\frac{\partial \bar{C}}{\partial \tau} = \frac{1}{2}\sigma_v^2 \bar{C}^2 - b(\bar{u}_1, \bar{u}_2)\bar{C} + \frac{1}{2}c(\bar{u}_1, \bar{u}_2), \bar{C}(0) = 0.$$

Denote $Re(C) = \frac{c+\bar{c}}{2}$, the imaginary part $\varepsilon = Im(C) = \frac{c-\bar{c}}{2i}$, and arrange the above equation to yield

$$\frac{\partial Re(C)}{\partial \tau} = \frac{1}{2}\sigma_v^2 Re^2(C) - b(c_1, c_2)Re(C) + \frac{1}{2}c(c_1, c_2) - \frac{1}{2}l(\varepsilon), Re(C)(0) = 0,$$

with

$$l(\varepsilon) = \sigma_v^2 \varepsilon^2 - 2(\rho_{1v}\sigma_1\sigma_v d_1 + \rho_{2v}\sigma_2\sigma_v d_2)\varepsilon + \sigma_1^2 d_1^2 + 2\rho_{12}\sigma_1\sigma_2 d_1 d_2 + \sigma_1^2 d_2^2.$$

i) Analyze the change law of function $l(\varepsilon)$.

By calculating

$$\begin{aligned} \Delta(d_1, d_2) &= 4(\rho_{1v}\sigma_1\sigma_v d_1 + \rho_{2v}\sigma_2\sigma_v d_2)^2 - 4\sigma_v^2(\sigma_1^2 d_1^2 + 2\rho_{12}\sigma_1\sigma_2 d_1 d_2 + \sigma_1^2 d_2^2) \\ &= 4\sigma_v^2((\rho_{1v}^2 - 1)\sigma_1^2 d_1^2 + 2(\rho_{1v}\rho_{2v} - \rho_{12})\sigma_1\sigma_2 d_1 d_2 + (\rho_{2v}^2 - 1)\sigma_2^2 d_2^2), \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} \frac{\partial \Delta}{\partial d_1} \\ \frac{\partial \Delta}{\partial d_2} \end{bmatrix} &= 8\sigma_v^2 \begin{bmatrix} (\rho_{1v}^2 - 1)\sigma_1^2 & (\rho_{1v}\rho_{2v} - \rho_{12})\sigma_1\sigma_2 \\ (\rho_{1v}\rho_{2v} - \rho_{12})\sigma_1\sigma_2 & (\rho_{2v}^2 - 1)\sigma_2^2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \end{aligned}$$

$$\Lambda = \begin{bmatrix} \frac{\partial^2 \Delta}{\partial d_1^2} & \frac{\partial^2 \Delta}{\partial d_1 \partial d_2} \\ \frac{\partial^2 \Delta}{\partial d_1 \partial d_2} & \frac{\partial^2 \Delta}{\partial d_2^2} \end{bmatrix}$$

$$= \begin{bmatrix} (\rho_{1v}^2 - 1)\sigma_1^2 & (\rho_{1v}\rho_{2v} - \rho_{12})\sigma_1\sigma_2 \\ (\rho_{1v}\rho_{2v} - \rho_{12})\sigma_1\sigma_2 & (\rho_{2v}^2 - 1)\sigma_2^2 \end{bmatrix}.$$

Then we have $\det(\Lambda) = 8\sigma_v^4 \det(\rho) > 0$, $\Delta(d_1, d_2) < 0, \forall d_1, \forall d_1$, and the following conclusion

$$\nabla(d_1, d_2) = \min l(\varepsilon) = (1 - \rho_{1v}^2)\sigma_1^2 d_1^2 - 2(\rho_{1v}\rho_{2v} - \rho_{12})\sigma_1\sigma_2 d_1 d_2 + (1 - \rho_{2v}^2)\sigma_2^2 d_2^2 \geq 0.$$

ii) Analyze the structure of function $Re(C(\tau, u_1, u_2))$.

It is clear that $Re(C(\tau, u_1, u_2))$ also is solution of Riccati differential equations. So we have the following result:

$$Re(C(\tau, c_1 + id_1, c_2 + id_2)) = \frac{b(c_1, c_2) + \delta_R}{\sigma_v^2} \frac{1 - e^{\delta_R \tau}}{1 - g_R e^{\delta_R \tau}},$$

where

$$\delta_R = \sqrt{b^2(c_1, c_2) - \sigma_v^2(c(c_1, c_2) - l(\varepsilon))} = \sqrt{b^2(c_1, c_2) - \sigma_v^2 c(c_1, c_2) + \sigma_v^2 l(\varepsilon)}, g_R = \frac{b + \delta_R}{b - \delta_R}.$$

Obviously, $\delta_R \geq \sqrt{\delta(c_1, c_2)^2 + \sigma_v^2 \nabla(d_1, d_2)}$. So when $d_1 \to \pm\infty$ or $d_2 \to \pm\infty$, then $\nabla(d_1, d_2) \to +\infty, \delta_R \to +\infty$. Notice that $Re(C)$ is a continuous function with respect to δ_R , and $\delta_R \to +\infty, Re(C) \to -\infty$. So it is clear that $e^{Re(C(\tau,u_1,u_2))v}$ is a bounded continuous function with respect to d_1, d_2, and when $d_1 \to \pm\infty$ or $d_2 \to \pm\infty$, then $e^{Re(C(\tau,u_1,u_2))v} \to 0$.

iii) Prove the absolute integrability of $e^{C(\tau,u_1,u_2)v}$.

By applying the equivalent infinitesimal theory in calculus, the following equations are given by

$$\begin{aligned} Re(C) \sim \tfrac{-\delta_R}{\sigma_v^2}, \quad e^{Re(C)v} \sim e^{\frac{2b(c_1,c_2)-\delta_R}{\sigma_v^2}v}, \\ d_1 \to \pm\infty, \quad d_2 \to \pm\infty. \end{aligned}$$

So the absolute integrability of $e^{Re(C)v}$ is the same as $e^{\frac{2b(c_1,c_2)-\delta_R}{\sigma_v^2}v}$.

Further, in light of the three following equations:

$$e^{\frac{2b(c_1,c_2)-\delta_R}{\sigma_v^2}v} \leq e^{\frac{2b(c_1,c_2)-\sqrt{\sigma_v^2\nabla(d_1,d_2)}}{\sigma_v^2}v},$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{\frac{2b(c_1,c_2)-\sigma_v\sqrt{\nabla(d_1,d_2)}}{\sigma_v^2}v} dd_1 dd_2 < +\infty,$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d_1^n d_2^m e^{\frac{2b(c_1,c_2)-\sigma_v\sqrt{\nabla(d_1,d_2)}}{\sigma_v^2}v} dd_1 dd_2 < +\infty,$$

then the absolute integrability of $e^{C(\tau,u_1,u_2)v}$ is verified.

2) The boundedness of $e^{A_3(\tau,u_1,u_2)}$.

In consideration of the continuity and convergence of $e^{Re(A_3(\tau,u_1,u_2))}$, and

$$Re(A_3(\tau, u_1, u_2)) = k_v\theta_v \int_0^\tau Re(C(\omega, u_1, u_2)))d\omega,$$

it is clear that $e^{Re(A_3(\tau,u_1,u_2))}$ has a limit boundary.

To sum up the above results, we obtain

$$
\begin{aligned}
& |\tfrac{1}{(2\pi i)^2}\int_{c_2-i\infty}^{c_2+i\infty}\int_{c_1-i\infty}^{c_1+i\infty}\hat{P}(t,u_1,u_2,v,r)s_1^{-u_1}s_2^{-u_2}du_1du_2| \\
\leq\ & \tfrac{1}{(2\pi)^2}\hat{P}(T,c_1,c_2,v,r)s_1^{-c_1}s_2^{-c_2}\exp\{F(c_1,c_2)\tau + A_1(\tau,c_1,c_2) + A_2(\tau,c_1,c_2) + B(\tau,c_1,c_2)r\} \\
& \cdot\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp\{Re(A_3(\tau,c_1+id_1,c_2+id_2)) + Re(C(\tau,c_1+id_1,c_2+id_2))v\}dd_1dd_2 \\
<\ & +\infty.
\end{aligned}
$$

On the other hand, we prove that $P(t,u_1,u_2,v,r)$ satisfies the PDIE.

In the light of the differentiation theorem of a parametric integral and substituting $P(t,u_1,u_2,v,r)$ into the PDIE, the equation can be verified and written:

$$
\begin{aligned}
& \tfrac{1}{(2\pi i)^2}\int_{c_2-i\infty}^{c_2+i\infty}\int_{c_1-i\infty}^{c_1+i\infty}\hat{P}(t,u_1,u_2,v,r)s_1^{-u_1}s_2^{-u_2}[-(F(u_1,u_2)+\tfrac{\partial A}{\partial\tau}+\tfrac{\partial B}{\partial\tau}r+\tfrac{\partial C}{\partial\tau}v)-(r-(\lambda_0+\lambda_1)m_1)u_1 \\
& +\tfrac{1}{2}\sigma_1^2vu_1(u_1+1)-(r-(\lambda_0+\lambda_2)m_2)u_2+\tfrac{1}{2}\sigma_2^2vu_2(u_2+1)+k_v(\theta_v-v)C+\tfrac{1}{2}\sigma_v^2vC^2+k_r(\theta_r-r)B+\tfrac{1}{2}\sigma_r^2B^2 \\
& +\rho_{12}\sigma_1\sigma_2vu_1u_2-\rho_{1v}\sigma_1\sigma_vvu_1C-\rho_{2v}\sigma_2\sigma_vvu_2C+\lambda_1E[e^{-u_1J_1}]+\lambda_2E[e^{-u_2J_2}]+\lambda_0E[e^{-u_1J_1-u_2J_2}] \\
& -(r+\lambda_0+\lambda_1+\lambda_2)]du_1du_2 = 0,
\end{aligned}
$$

by Eq (2.5).

The proof is finished. □

Theorem 3.4. *Let c be a real number such that*

$$
E[e^{-cJ_1}] < +\infty, E[e^{(c+1)J_2}] < +\infty, E[e^{-cJ_1+(c+1)J_2}] < +\infty,
$$

and then the rigorous solution to the above PDIE for some binary options at any time $t \leq T$ is given by

$$
p(t,z,v,r) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\hat{p}(t,z,v,r)z^{-u}du. \tag{3.22}
$$

Proof. The proof of this theorem is similar to Theorem 3.3. □

4. Numerical analysis

In the following section, numerical experiments are applied to examine the performance of the Mellin transform method. The values of the remaining parameters for numerical examples are presented in Table 1. Without loss of generality, the following numerical experiments are implemented in MATLAB R2022a. In addition, it should be pointed out that the values of key parameters in this paper come from Ruijter and Oosterlee [21].

First, to depict the price mechanism of two underlying assets under the proposed model, we utilize the following scheme to discretize stochastic differential equations:

$$
S_{1t_i} = S_{1t_{i-1}} + S_{1t_{i-1}}[(r_{t_{i-1}}-(\lambda_0+\lambda_1)m_1)\triangle t+\sigma_1\sqrt{v_{t_{i-1}}}\sqrt{\triangle t}\cdot\epsilon_1+(e^{J_1}-1)(N_{0\triangle t}+N_{1\triangle t})],
$$

$$
S_{2t_i} = S_{2t_{i-1}} + S_{2t_{i-1}}[(r_{t_{i-1}}-(\lambda_0+\lambda_2)m_2)\triangle t+\sigma_2\sqrt{v_{t_{i-1}}}\sqrt{\triangle t}\cdot(\rho_{12}\epsilon_1+\sqrt{1-\rho_{12}^2}\epsilon_2)
$$

$+ (e^{J_2} - 1)(N_{0\triangle t} + N_{2\triangle t})],$

$$v_{t_i} = v_{t_{i-1}} + k_v(\theta_v - v_{t_{i-1}})\triangle t + \sigma_v \sqrt{v_{t_{i-1}}}\triangle t(\rho_{1v}\epsilon_1 + \frac{\rho_{2v}-\rho_{1v}\rho_{12}}{\sqrt{1-\rho_{12}^2}}\epsilon_2 + \frac{\sqrt{1-\rho_{12}^2-\rho_{1v}^2-\rho_{2v}^2+2\rho_{1v}\rho_{2v}\rho_{12}}}{\sqrt{1-\rho_{12}^2}}\epsilon_3),$$

$$r_{t_i} = k_r(\theta_r - r_{t_{i-1}})\triangle t + \sigma_r \sqrt{\triangle t} \cdot \epsilon_4,$$

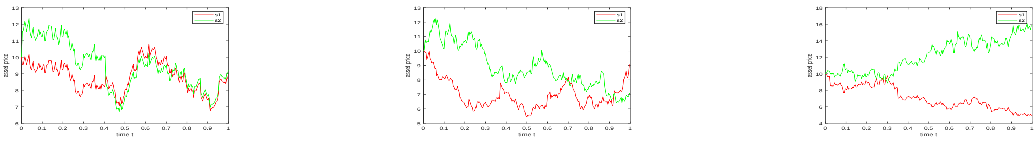
where $\epsilon_i \sim N(0,1), i = 1,2,3,4; N_{j\triangle t} \sim P(\lambda_j), j = 0,1,2; \triangle t = \frac{\tau}{250}$. If $N_{0\triangle t} > 0$, then generate the jump vector (J_1, J_2) randomly using pseudo-random numbers. If $N_{1\triangle t} > 0$, then we generate the jump variable J_1 randomly using pseudo-random numbers. If $N_{2\triangle t} > 0$, then we generate the jump variable J_2. It should be noted that in the entire generation process of the compound Poisson process, specific jump times do not directly appear; instead, attention is focused on the possible number of jumps and the scale of the jumps.

Table 1. Parameter values for the numerical experiments.

Parameter	r	v	s_1	s_2	σ_1	σ_2	k_v	θ_v	σ_v
Value	0.05	0.2	10	10	0.1	0.15	0.3	0.2	0.01
Parameter	k_r	θ_r	σ_r	K	τ	λ_0	λ_1	λ_2	p_1
value	1.2	0.3	0.01	10	1	2	3	3	0.6
Parameter	ξ_1	η_1	p_2	ξ_2	η_2	μ_1	μ_2	$\sigma_{J_1}^2$	$\sigma_{J_2}^2$
Value	12	4	0.64	18	21	0	0	0.1	0.1

To verify the reliability of the Mellin transform, we introduced the Monte Carlo simulation pricing algorithm. Through experiments, it was found that in cases with fewer paths, the convergence of the algorithm was poor. To ensure the accuracy of the Monte Carlo reference algorithm, in the following comparative experiments, we adopted a higher path count $N = 2000$. From the perspective of the initial price, as the price increases, the leverage of the corresponding option also increases, the stability of the price decreases, and the requirements of the algorithm also increase. From the efficiency perspective, the Monte Carlo algorithm often spends more computing time under high-precision requirements, while the Mellin transformation method compared next gives the analytical expression of the option, so the cost of computing time is very small, which is also the main advantage of this paper for conducting corresponding theoretical arguments. It should be noted that this paper uses the command integral2 in MATLAB to implement the inversion of the Mellin transform formulas (3.19) and (3.20).

Before conducting the numerical analysis for pricing the extremum options using the Mellin transform, the last point that needs to be further explained is the selection of c_1 and c_2. Since not all types of extremum options have corresponding Mellin transformations, the selection ranges of c_1 and c_2 vary for different option types. However, they all need to satisfy the conditions of boundary integrability and the existence of jump expectations. Otherwise, it will result in an unfeasible Mellin transformation. For more details, please refer to the preparatory work described in the beginning of Chapter 3 of this article. Additionally, it is best to choose c_1 and c_2 away from the boundaries of the existing range, that is, away from the singular points of the integral.



(a) $\rho_{12} = 1$ (b) $\rho_{12} = 0$ (c) $\rho_{12} = -1$

Figure 1. The sample path of underlying assets s_{1t} and s_{2t} under different correlations.

Figure 1 illustrates the sample path of the underlying assets s_{1t} and s_{2t} under $\rho_{12} = -1$, $\rho_{12} = 0$, and $\rho_{12} = 1$. On the one hand, from the analysis of a single path, the large volatility of each path comes from the stochastic volatility and Poisson jump. Due to the rapid reverting of the Heston model, the large volatility is gradually transformed into small volatility in each path. On the other hand, from the analysis of two paths, it is clear that s_{1t} and s_{2t} show high positive (negative) correlation under $\rho_{12} = 1$ ($\rho_{12} = -1$), but not completely because of the log-normal jump.

Further, we construct the maximum likelihood estimator and solve for the estimators of the relevant parameters. The likelihood function of the parameter group is as follows:

$$l(k_v, k_r; \theta_v, \theta_r; \sigma_v, \sigma_r) = \prod_i \frac{1}{\sqrt{2\pi}} e^{-\frac{\zeta_{1i}^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{\zeta_{2i}^2}{2}},$$

where $\zeta_{1i} = \frac{v_{t_i} - v_{t_{i-1}} - k_v(\theta_v - v_{t_{i-1}})\Delta t}{\sigma_v \sqrt{v_{t_{i-1}}} \sqrt{\Delta t}}$, $\zeta_{2i} = \frac{r_{t_i} - r_{t_{i-1}} - k_r(\theta_r - r_{t_{i-1}})\Delta t}{\sigma_r \sqrt{\Delta t}}$. Then, the log-likelihood function of the parameter group is calculated as well:

$$Lnl(k_v, k_r; \theta_v, \theta_r; \sigma_v, \sigma_r) = -n \ln(2\pi) - \frac{1}{2} \sum_i \zeta_{1i}^2 - \frac{1}{2} \sum_i \zeta_{2i}^2,$$

and the corresponding stability point equation is as follows:

$$\frac{\partial Lnl}{\partial k_v} = \frac{\sqrt{\Delta t}}{\sigma_v} \sum_i \zeta_{1i} \frac{(\theta_v - v_{t_{i-1}})}{\sqrt{v_{t_{i-1}}}} = 0, \frac{\partial Lnl}{\partial \theta_v} = \frac{k_v \sqrt{\Delta t}}{\sigma_v} \sum_i \frac{\zeta_{1i}}{\sqrt{v_{1t_{i-1}}}} = 0, \frac{\partial Lnl}{\partial \sigma_v} = \frac{1}{\sigma_v} \sum_i \zeta_{1i}^2 = 0;$$

$$\frac{\partial Lnl}{\partial k_r} = \frac{\sqrt{\Delta t}}{\sigma_r} \sum_i \zeta_{2i}(\theta_r - r_{t_{i-1}}) = 0, \frac{\partial Lnl}{\partial \theta_r} = \frac{k_r \sqrt{\Delta t}}{\sigma_r} \sum_i \zeta_{2i} = 0, \frac{\partial Lnl}{\partial \sigma_r} = \frac{1}{\sigma_r} \sum_i \zeta_{2i}^2 = 0.$$

Finally, by narrowing down the time interval for discretization, more optimal parameter estimators can be obtained. At the same time, other parameters of the two-asset prices are set, and using nonlinear regression techniques, specific parameter estimates of the asset common volatility can be obtained. It should be noted that other parameters in the two asset price model, such as stochastic scale (σ_1, σ_2), jump intensity $(\lambda_0, \lambda_1, \lambda_2)$, and jump vector (J_1, J_2), can all obtain consistent parameter estimators through the maximum likelihood estimation method, and this will not be elaborated further here.

Next, we consider that in the financial market, traders and practitioners often use the implied volatility [22, 23] to assess the value of options. Therefore, under the method of the Mellin transformation provided in this paper, a calculation method for implied volatility v is established, which offers a practical tool for traders who rely on the volatility surface for hedging.

Based on the Mellin transform inversion formula for maximum and minimum option prices and the exchange option price, the volatility is expanded using a Taylor series. The specific formula is as follows:

$$
\begin{aligned}
P(t, s_1, s_2, v, r) &= \frac{1}{(2\pi i)^2} \int_{c_2 - i\infty}^{c_2 + i\infty} \int_{c_1 - i\infty}^{c_1 + i\infty} \hat{P}(t, u_1, u_2, v, r) s_1^{-u_1} s_2^{-u_2} du_1 du_2 \\
&= \frac{1}{(2\pi i)^2} \int_{c_2 - i\infty}^{c_2 + i\infty} \int_{c_1 - i\infty}^{c_1 + i\infty} P(T, s_1, s_2, v, r) e^{F(u_1,u_2)\tau + A(\tau,u_1,u_2) + B(\tau,u_1,u_2) r + C(\tau,u_1,u_2) v} s_1^{-u_1} s_2^{-u_2} du_1 du_2 \\
&= \sum_{n=0}^{+\infty} \frac{v^n}{(2\pi i)^2} \int_{c_2 - i\infty}^{c_2 + i\infty} \int_{c_1 - i\infty}^{c_1 + i\infty} e^{F(u_1,u_2)\tau + A(\tau,u_1,u_2) + B(\tau,u_1,u_2) r} C^n(\tau, u_1, u_2) s_1^{-u_1} s_2^{-u_2} du_1 du_2 \\
&= \sum_{n=0}^{+\infty} P_n(t, s_1, s_2, r) v^n, \\
p(t, z, v, r) &= \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \hat{p}(t, u, v, r) z^{-u} du \\
&= \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \hat{p}(T, u, v, r) e^{\acute{F}(u)\tau + \acute{A}(\tau,u) + \acute{B}(\tau,u) r + \acute{C}(\tau,u) v} z^{-u} du \\
&= \sum_{n=0}^{+\infty} \frac{v^n}{2\pi i} \int_{c - i\infty}^{c + i\infty} \hat{p}(T, u, v, r) e^{\acute{F}(u)\tau + \acute{A}(\tau,u) + \acute{B}(\tau,u) r} \acute{C}^n(\tau, u) z^{-u} du \\
&= \sum_{n=0}^{+\infty} p_n(t, z, r) v^n,
\end{aligned}
$$

where $P_0(t, s_1, s_2, r)$ and $p_0(t, z, r)$ are option prices under constant volatility, and $n!P_n(t, s_1, s_2, r)$ and $n!p_n(t, z, r)$ are the nth derivative $\frac{\partial P}{\partial v}$, $\frac{\partial p}{\partial v}$ of the option, respectively.

Depending on the precision requirements for different estimates of implied volatility, the following specific estimation methods can be employed.

First-order approximate estimator:

$$
\begin{aligned}
v &\approx \frac{P(t,s_1,s_2,v,r) - P_0(t,s_1,s_2,r)}{P_1(t,s_1,s_2,r)}, \\
v &\approx \frac{p(t,z,v,r) - p_1(t,z,r)}{p_1(t,z,r)};
\end{aligned}
$$

Second-order approximate estimator:

$$
\begin{aligned}
v &\approx \frac{-P_1(t,s_1,s_2,r) + \sqrt{P_1^2(t,s_1,s_2,r) - 4P_2(t,s_1,s_2,r)(P_0(t,s_1,s_2,r) - P(t,s_1,s_2,v,r))}}{2P_2(t,s_1,s_2,r)}, \\
v &\approx \frac{-p_1(t,z,r) + \sqrt{p_1^2(t,z,r) - 4p_2(t,z,r)(p_0(t,s_1,s_2,r) - p(t,z,v,r))}}{2p_2(t,z,r)};
\end{aligned}
$$

n-order approximate estimator:

$$
\begin{aligned}
v &\approx \arg\max\{|P(t, s_1, s_2, v, r) - \sum_{i=0}^{+\infty} P_i(t, s_1, s_2, r) v^i|^2\}, \\
v &\approx \arg\max\{|p(t, z, v, r) - \sum_{i=0}^{+\infty} p_i(t, z, r) v^i|^2\};
\end{aligned}
$$

where the optimal problem here can be calculated using the multi-part method that extends the binary method.

Table 2. Option values under the Heston model with different volatility v using MC and Mellin (ML).

$P(v)$	Exchange	Max-Call	Max-Put	Min-Call	Min-Put	Max-Min
P (0.05) (ML)	0.9467	0.9816	0.0349	0.5535	1.5002	1.8934
P (0.05) (MC)	0.9489	0.9795	0.0350	0.5557	1.4883	1.8784
APE (%)	0.23	0.22	0.33	0.41	0.79	0.80
P (0.1) (ML)	1.0463	1.0848	0.0386	0.6117	1.6579	2.0925
P (0.1) (MC)	1.0417	1.0871	0.0386	0.6138	1.6516	2.0781
APE (%)	0.44	0.21	0.02	0.35	0.38	0.69
P (0.15) (ML)	1.1563	1.1989	0.0426	0.6760	1.8323	2.3126
P (0.15) (MC)	1.1549	1.1959	0.0426	0.6771	1.8391	2.3026
APE (%)	0.12	0.25	0.00	0.16	0.37	0.43
P (0.2) (ML)	1.2779	1.3250	0.0471	0.7471	2.0250	2.5558
P (0.2) (MC)	1.2751	1.3191	0.0470	0.7454	2.0321	2.5659
APE (%)	0.22	0.45	0.23	0.23	0.35	0.39
P (0.25) (ML)	1.4123	1.4644	0.0521	0.8257	2.2380	2.8246
P (0.25) (ML)	1.4180	1.4619	0.0519	0.8274	2.2309	2.8301
APE (%)	0.40	0.17	0.23	0.21	0.31	0.19
P (0.3) (ML)	1.5608	1.6184	0.0575	0.9125	2.4733	3.1217
P (0.3) (MC)	1.5587	1.6226	0.0577	0.9112	2.4700	3.1254
APE (%)	0.14	0.26	0.34	0.14	0.13	0.12
P (0.35) (ML)	1.7250	1.7886	0.0636	1.0085	2.7335	3.4500
P (0.35) (MC)	1.7293	1.7835	0.0633	1.0059	2.7352	3.4389
APE (%)	0.25	0.28	0.45	0.26	0.06	0.32
P (0.4) (ML)	1.9064	1.9767	0.0703	1.1145	3.0209	3.8128
P (0.4) (MC)	1.9093	1.9712	0.0705	1.1127	3.0237	3.8114
APE (%)	0.15	0.28	0.39	0.16	0.09	0.04

Table 2 presents the results of the compound Mellin transform pricing and Monte Carlo simulation pricing for six options under the Heston model with different volatility levels. Overall, it can be observed that as the initial volatility changes, the prices of all options show an upward trend, and the difference between the two pricing methods is small, which to some extent verifies the effectiveness of the composite Mellin transform method proposed in this paper. From an individual perspective, the influence of the initial volatility on the pricing behavior of options varies. Specifically, the price of the maximum option increases at a faster rate as the volatility increases. The swap option and the minimum option have the same trend of change, that is, as the volatility increases, the price of the option steadily increases. The minimum option has a characteristic phenomenon; when the volatility increases to an extreme level, it shows an S-shaped form of change. These facts indicate that volatility plays an important role in the pricing of extreme value options and is a highly sensitive parameter. Finally, although the Mellin transform does not exist for some options and their prices are obtained indirectly through the parity relationship with the swap option, from the comparison of Table 2 with the Monte Carlo simulation algorithm, it can be seen that this does not significantly affect the stability of the Mellin transform pricing, which further verifies the reliability and stability of the composite Mellin transform.

Table 3. Option values under the Heston model with different s_1 and s_2 using MC and Mellin (ML).

$P(s_1, s_2)$	Exchange	Max-Call	Max-Put	Min-Call	Min-Put	Max-Min
P (6,6) (ML)	0.2636	0.2608	0.0981	0.1506	4.1568	0.5218
P (6,6) (MC)	0.2659	0.2600	0.0964	0.1495	4.1282	0.5184
APE (%)	0.90	0.31	0.82	0.76	0.69	0.66
P (7,7) (ML)	0.5107	0.5196	0.0904	0.3014	3.5821	1.0071
P (7,7) (MC)	0.5115	0.5130	0.0890	0.3041	3.4965	1.0012
APE (%)	0.17	0.29	0.62	0.88	2.45	0.59
P (8,8) (ML)	0.7977	0.8218	0.0799	0.4667	3.0911	1.5779
P (8,8) (MC)	0.8060	0.8178	0.0780	0.4639	3.0896	1.5579
APE (%)	0.03	0.50	0.32	0.59	0.70	0.29
P (9,9) (ML)	1.0458	1.1011	0.0682	0.5701	2.8745	2.0799
P (9,9) (MC)	1.0413	1.0835	0.0681	0.5782	2.8352	2.0910
APE (%)	0.45	0.63	0.08	0.40	0.39	0.53
P (10,10) (ML)	1.2836	1.3051	0.0581	0.7007	2.4579	2.5071
P (10,10) (MC)	1.2992	1.2959	0.0572	0.7125	2.4453	2.4984
APE (%)	0.77	0.27	0.55	0.66	0.76	0.55
P (11,11) (ML)	1.4975	1.6525	0.0470	0.9018	1.9876	3.2055
P (11,11) (MC)	1.5040	1.6440	0.0461	0.8939	1.9848	3.2580
APE (%)	0.39	0.75	0.65	0.03	0.14	0.61
P (12,12) (ML)	1.8148	1.7679	0.0396	1.0549	1.6069	3.5961
P (12,12) (MC)	1.8117	1.7676	0.0393	1.0522	1.5977	3.6032
APE (%)	0.38	0.11	0.73	0.19	0.34	0.30
P (13,13) (ML)	1.9635	2.1456	0.0294	1.1708	1.2840	3.9935
P (13,13) (MC)	1.9830	2.1397	0.0292	1.1735	1.2708	3.9947
APE (%)	0.98	0.28	0.85	0.08	1.84	0.03
P (14,14) (ML)	2.2955	2.3323	0.0198	1.3459	0.8282	4.6390
P (14,14) (MC)	2.3009	2.3454	0.0197	1.3271	0.8188	4.6560
APE (%)	0.39	-0.41	0.41	0.42	0.40	0.82
P (15,15) (ML)	2.5563	2.7037	0.0100	1.5379	0.4214	4.9955
P (15,15) (MC)	2.5170	2.6967	0.0098	1.5676	0.4167	5.0074
APE (%)	0.41	0.39	0.60	0.90	0.12	0.24

Table 3 presents the comparison results of compound Mellin transform pricing and Monte Carlo simulation under the Heston model, with the price of the underlying asset as the variable. In terms of the pricing error, the difference between the two pricing methods is relatively small, indicating the stability of the compound Mellin transformation pricing method. Considering the computing time of the Monte Carlo simulation, it can be seen that, under the premise of a small pricing distance, the Monte Carlo simulation requires a much greater multiple of computing time consumption. The reason is that the compound Mellin transformation pricing obtains the analytical formula price of the option, while each path simulation in the Monte Carlo method requires process iteration (without counting the countless generation of normal random numbers). In other words, to achieve the same convergence, the Monte Carlo simulation must pay more simulation times and longer computing time. From the content of Table 3, except for the maximum put option and the minimum call option, the prices of other options increase with the increase of the asset price, and overall present different forms of return structures, which is also the intrinsic reason why the extreme value options have become active financial derivatives.

Second, we investigate the influence of some core parameters on the prices of maximum or minimum options, such as ρ_{12}, k_v, θ_v, k_r, θ_r. In the next numerical experiments, the values of maximum or minimum options will be changed to test the effect of one or two parameters taking different values. In the meantime, the values of other parameters are constant in Table 1.

Table 4. Option values under different ρ_{12}.

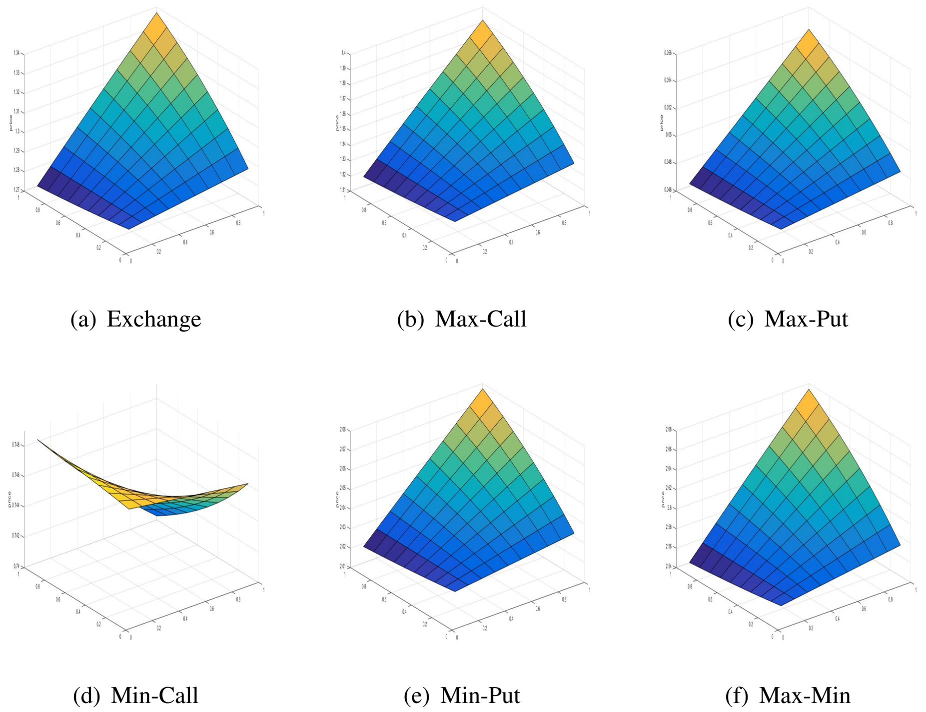
ρ_{12}	Exchange	Max-Call	Max-Put	Min-Call	Min-Put	Max-Min
-1	1.2968	1.3387	0.0419	0.7149	2.0117	2.5936
-0.8	1.2930	1.3359	0.0429	0.7213	2.0143	2.5861
-0.6	1.2893	1.3332	0.0439	0.7277	2.0169	2.5785
-0.4	1.2855	1.3304	0.0450	0.7341	2.0196	2.5710
-0.2	1.2817	1.3277	0.0460	0.7406	2.0223	2.5634
0	1.2779	1.3250	0.0471	0.7471	2.0250	2.5558
0.2	1.2741	1.3223	0.0482	0.7537	2.0277	2.5481
0.4	1.2702	1.3195	0.0493	0.7603	2.0305	2.5405
0.6	1.2664	1.3168	0.0504	0.7669	2.0333	2.5328
0.8	1.2625	1.3141	0.0516	0.7736	2.0362	2.5250
1	1.2586	1.3114	0.0527	0.7804	2.0390	2.5173

Table 4 shows the variation of maximum or minimum option prices with respect to ρ_{12}. Notice that the valuation of the exchange option, maximum-put option, and minimum-call option are calculated by the the Mellin transformation, and the valuation of the maximum-call option, minimum-put option, and maximum-minimum option are calculated by Equation . The data in Table 1 show that the price of the Maximum-Put and Minimum-Call options change in the same direction, while the other options change in the opposite direction as the correlation coefficient changes. In addition, the maximum option price has the lowest order of magnitude, because the driving mechanism of the two underlying asset prices is asymmetric, and the probability of upward breakthrough is much greater than the probability of downward breakthrough.

Table 5 and Figure 2 illustrate the comprehensive influence of k_v and θ_v with respect to option prices. As is present in Table 3, the price of every maximum or minimum option is increasing with respect to the parameter of long-term volatility θ_v, whereas according to Figure 2, only the price of the minimum-call option is decreasing with respect to the parameter of reverting speed k_v. Notice that when $\theta_v = v = 0.2$, the price of every option is constant with respect to different k_v. The reason is that initial volatility has entered the equilibrium of the Heston volatility, so k_v does not affect the price of the option. This is the peculiar phenomenon of the Heston model.

Table 5. Option values under different k_v, θ_v.

k_v	θ_v	Exchange	Max-Call	Max-Put	Min-Call	Min-Put	Max-Min
0.3	0.2	1.2779	1.3250	0.0471	0.7471	2.0250	2.5558
0.3	0.4	1.2835	1.3313	0.0478	0.7464	2.0299	2.5670
0.3	0.6	1.2891	1.3375	0.0484	0.7457	2.0348	2.5781
0.3	0.8	1.2946	1.3437	0.0491	0.7450	2.0396	2.5892
0.6	0.2	1.2779	1.3250	0.0471	0.7471	2.0250	2.5558
0.6	0.4	1.2881	1.3364	0.0483	0.7458	2.0339	2.5762
0.6	0.6	1.2982	1.3477	0.0495	0.7446	2.0428	2.5964
0.6	0.8	1.3082	1.3589	0.0508	0.7434	2.0516	2.6164
0.9	0.2	1.2779	1.3250	0.0471	0.7471	2.0250	2.5558
0.9	0.4	1.2919	1.3406	0.0488	0.7453	2.0372	2.5837
0.9	0.6	1.3057	1.3561	0.0505	0.7437	2.0494	2.6113
0.9	0.8	1.3193	1.3714	0.0521	0.7421	2.0614	2.6386



(a) Exchange (b) Max-Call (c) Max-Put

(d) Min-Call (e) Min-Put (f) Max-Min

Figure 2. Option value surfaces under different k_v, θ_v (x–k_v, y–θ_v, z–P).

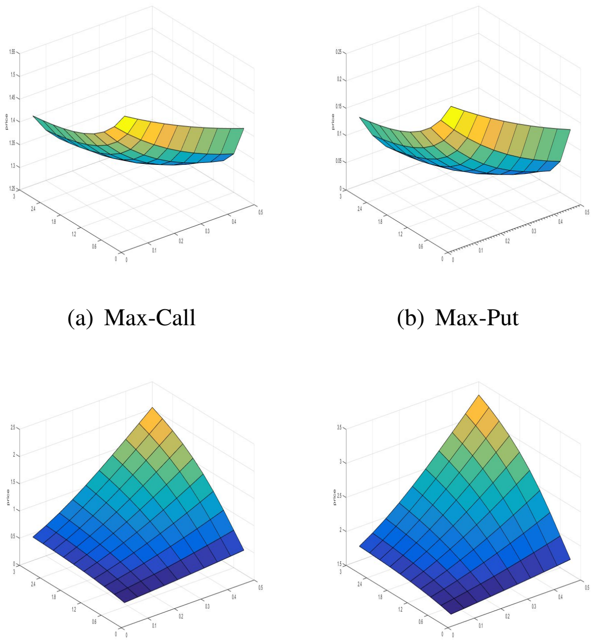
Table 6 and Figure 3 depict the price of maximum or minimum options under different k_r and θ_r. Since the exchange option and maximum-minmum option can be converted through the unit of account, the prices of both are not affected by the interest rate r_t. As is shown in Table 4, the prices of every minimum-call or minimum-put option are increasing with respect to the parameter of long-term interest rate θ_r, whereas according to Figure 3, the prices of maximum-call and maximum-put options are decreasing; with respect to the parameter of reverting speed k_r, the prices of options have a similar conclusion. Moreover, when $\theta_r = r = 0.05$, it does not have the same phenomenon as the above Heston model. The reason is that the mechanism by which stochastic interest rates affect option prices is not the same or even more complex than that of stochastic volatility. This is a point worthy of further research and attention in this paper. The invariance of swap option prices with respect to short-term interest rate parameters is an interesting phenomenon in the pricing of financial derivatives. From a formal perspective, this invariance stems from the linearity of the swap option payoff function. Similar linear option types also exhibit this phenomenon, such as floating Asian options, etc. From an economic perspective, the cause of this invariance is the use of an internal valuation unit conversion technique, which ensures that the price of the option is not directly influenced by short-term interest rates. From a probabilistic perspective, this invariance arises from the strong Markov property of the underlying asset price, meaning that the future price changes of the underlying asset are only related to the current price.

Table 6. Option values under different k_r, θ_r.

k_r	θ_r	Max-Call	Max-Put	Min-Call	Min-Put
0.6	0.05	1.4890	0.2111	0.2963	1.5742
0.6	0.2	1.4175	0.1396	0.4033	1.6812
0.6	0.35	1.3677	0.0898	0.5335	1.8114
0.6	0.5	1.3342	0.0563	0.6860	1.9639
1.2	0.05	1.4458	0.1679	0.3534	1.6313
1.2	0.2	1.3582	0.0803	0.5688	1.8467
1.2	0.35	1.3136	0.0357	0.8456	2.1235
1.2	0.5	1.2927	0.0148	1.1716	2.4495
1.8	0.05	1.4233	0.1454	0.3918	1.6697
1.8	0.2	1.3324	0.0545	0.6965	1.9744
1.8	0.35	1.2961	0.0182	1.0935	2.3714
1.8	0.5	1.2834	0.0055	1.5502	2.8281
2.4	0.05	1.4106	0.1327	0.4172	1.6951
2.4	0.2	1.3194	0.0415	0.7909	2.0688
2.4	0.35	1.2891	0.0112	1.2781	2.5560
2.4	0.5	1.2805	0.0026	1.8256	3.1035

Third, we will research the influence of dependent jumps on the prices of maximum or minimum options. In the next numerical experiments, the values of maximum or minimum options will be changed to test the effect of a log-normal jump in Table 5 and asymmetric double-exponential distribution in Table 6.

Table 7 illustrates the prices of maximum or minimum options with a log-normal jump under different ρ_J. It is worth noting that the correlation of Poisson jumps has important implications for option pricing. To be more specific, the prices of maximum-put or minimum-call options increase with respect to different ρ_J, whereas the prices of other options decrease. It has the same phenomenon as the above ρ_{12} because normal distribution is symmetric.



(a) Max-Call (b) Max-Put

(c) Min-Call (d) Min-Put

Figure 3. Option value surfaces under different k_r, θ_r (x–k_r, y–θ_r, z–P).

Table 7. Option values under different ρ_J.

ρ_J	Exchange	Max-Call	Max-Put	Min-Call	Min-Put	Max-Min
-1	1.4853	1.5033	0.0179	0.5689	2.0542	2.9707
-0.8	1.4469	1.4692	0.0224	0.6029	2.0497	2.8937
-0.6	1.4070	1.4346	0.0276	0.6375	2.0446	2.8141
-0.4	1.3657	1.3991	0.0334	0.6730	2.0387	2.7315
-0.2	1.3228	1.3627	0.0399	0.7094	2.0322	2.6455
0	1.2779	1.3250	0.0471	0.7471	2.0250	2.5558
0.2	1.2308	1.2858	0.0550	0.7863	2.0171	2.4616
0.4	1.1811	1.2447	0.0636	0.8274	2.0085	2.3623
0.6	1.1283	1.2014	0.0730	0.8708	1.9991	2.2567
0.8	1.0716	1.1551	0.0834	0.9171	1.9887	2.1432
1	1.0095	1.1047	0.0951	0.9675	1.9770	2.0191

Table 8. Option values under different ρ_J.

ρ_J	Exchange	Max-Call	Max-Put	Min-Call	Min-Put	Max-Min
-1	1.7217	1.7489	0.0272	0.4442	2.1659	4.6924
0	2.5665	2.6559	0.0894	0.8179	3.3844	5.4602
1	2.1743	2.2913	0.1170	0.9210	3.0953	4.9884

Table 8 depicts the variation of maximum or minimum option prices with an asymmetric double-exponential jump under different ρ_J. Considering the complexity and diversity of the correlation structure of the double-exponential normal distribution, three case of complete negative dependence, mutual independence, and complete positive dependence are derived in this paper. As is shown in Table 6, the prices of minimum-call or minimum-put options are increasing with respect to ρ_J, whereas the prices of other options achieve maximum values under the case of mutual independence. It indicates that the performance of double-exponential jumps is more sensitive and less predictable than log-normal jumps. The correlation of Poisson jumps has important implications for option pricing. To be more specific, the prices of maximum-put or minimum-call options increase with respect to different ρ_J, whereas the prices of other options decrease. It has the same phenomenon as the above ρ_{12} because normal distribution is symmetric. Without doubt, the main source of all this is the asymmetric nature of the double-exponential distribution. In Table 6, the different behaviors under the asymmetric double-exponential jump are related to the asymmetry of the double-exponential distribution. What it reflects is the complex changes in investors' psychology in the financial market. When the price of the underlying asset rises, investors generally show an optimistic attitude; when the price of the underlying asset falls, investors generally show a pessimistic attitude. However, the downward panic effect is more prominent and obvious. That is, the degree of panic that investors feel about the price decline is far greater than the degree of excitement they feel about the price increase. Therefore, the double-exponential reflects this asymmetric effect of the financial market. However, since the joint normal distribution is linear, it does not have the construction ability to describe nonlinear risks, especially asymmetric risks.

5. Conclusions

This paper solves the pricing problem of a class of complex options, namely maximum and minimum options, in complex scenarios based on the composite Mellin transform method, and provides a method reference for the pricing of other complex financial derivatives. In the future, under the condition that the Mellin transform does not exist, various decomposition techniques can be utilized to convert complex payoff functions into cases where the Mellin transform is applicable, to complete the pricing of complex financial derivatives, and further expand the pricing space and pricing capacity of the Mellin transform. Under the combined assumptions of stochastic volatility, the stochastic interest rate, and interdependent jumps, analytical pricing formulas for maximum and minimum options and exchange options are derived. The single and double Mellin transforms are jointly utilized to simplify the option price equation, and the affine structures with variable substitution are applied to deduce the solution of the joint Mellin transform function. An analytical formula of maximum or minimum option prices is obtained by the inverse Mellin transform method and a decomposition technique. Simulation and numerical examples indicate that the price dynamics and pricing method constructed in this paper are unique and effective.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

No potential conflict of interest was reported by the authors.

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Appendix: The proofs of Eqs (3.9)–(3.14)

Given the similarity of the proofs for Eqs (3.9)–(3.14), here we just prove Eq (3.9).

According to J_1 having complete positive correlation with J_2, then there must be a strictly monotonically increasing function g that satisfies $P\{J_1 = g(J_1)\} = 1$. At the same time, we obtain

$$f_{J_1}(g(y))g'(y) = f_{J_2}(y),$$

$$F_{J_1}(x) = P\{J_1 \le x\} = P\{g(J_2) \le x\} = P\{J_2 \le g^{-1}(x)\} = F_{J_2}(g^{-1}(x)),$$

$$F_{J_2}(y) = P\{J_2 \le y\} = P\{g(J_2) \le g(y)\} = P\{J_1 \le g(y)\} = F_{J_1}(g(y)).$$

When $y \in (-\infty, 0]$, due to $F_{J_2}(0) = F_{J_1}(g(0)) = P_2 < p_1$, then $g(0) = \frac{1}{\xi_1} \ln \frac{p_2}{p_1}$. Therefore, we obtain the ordinary differential equation

$$p_1 \xi_1 e^{\xi_1 g(y)} g'(y) = p_2 \xi_2 e^{\xi_2 y},$$

with the boundary condition $g(0) = \frac{1}{\xi_1}\ln\frac{p_2}{p_1}$. The solution of the above equation is

$$g(y) = \frac{1}{\xi_1}\ln\frac{p_2}{p_1} + \frac{\xi_2}{\xi_1}y.$$

When $y \in [0, g^{-1}(0)]$, due to $F_{J_1}(0) = F_{J_2}(g^{-1}(0)) = p_1 > p_2$, then $g^{-1}(0) = \frac{1}{\eta_2}\ln\frac{q_2}{q_1} > 0$. Therefore, we obtain the ordinary differential equation

$$p_1\xi_1 e^{\xi_1 g(y)} g'(y) = q_2\eta_2 e^{-\eta_2 y},$$

with the boundary conditions $g^{-1}(0) = \frac{1}{\eta_2}\ln\frac{q_2}{q_1}$, $g(0) = \frac{1}{\xi_1}\ln\frac{p_2}{p_1}$. The solution of the above equation is

$$g(y) = \frac{1}{\xi_1}\ln\frac{1 - q_2 e^{-\eta_2 y}}{p_1}.$$

When $y \in [g^{-1}(0), +\infty)$, due to $F_{J_1}(0) = F_{J_2}(g^{-1}(0)) = p_1 > p_2$, then $g^{-1}(0) = \frac{1}{\eta_2}\ln\frac{q_2}{q_1} > 0$. Therefore, we obtain the ordinary differential equation

$$q_1\eta_1 e^{-\eta_1 g(y)} g'(y) = q_2\eta_2 e^{-\eta_2 y},$$

with the boundary condition $g^{-1}(0) = \frac{1}{\eta_2}\ln\frac{q_2}{q_1}$. The solution of the above equation is

$$g(y) = \frac{1}{\eta_1}\ln\frac{q_1}{q_2} + \frac{\eta_2}{\eta_1}y.$$

Based on the above results, it can be obtained that

$$g(y) = \begin{cases} \frac{1}{\xi_1}\ln\frac{p_2}{p_1} + \frac{\xi_2}{\xi_1}y & \text{if } y < 0, \\ \frac{1}{\xi_1}\ln\frac{1-q_2e^{-\eta_2 y}}{p_1} & \text{if } \frac{1}{\xi_1}\ln\frac{p_2}{p_1} \le y \le \frac{1}{\eta_2}\ln\frac{q_2}{q_1}, \\ \frac{1}{\eta_1}\ln\frac{q_1}{q_2} + \frac{\eta_2}{\eta_1}y & \text{if } y > \frac{1}{\eta_2}\ln\frac{q_2}{q_1}. \end{cases}$$

Finally, the technique of piecewise integration is applied to the following equation:

$$E[e^{-u_1 J_1 - u_2 J_2}] = E[e^{-u_1 g(J_2) - u_2 J_2}],$$

and it is easy to obtain Eq (3.9).