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Research article

Perturbed Bott algebraic Schouten solitons on 3D Lorentzian Lie groups

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Abstract: In this paper, we defined and classified the algebraic Schouten solitons that are associated with the perturbed Bott connection on three-dimensional Lorentzian Lie groups possessing three distinct distributions. By transforming equations of perturbed Bott algebraic Schouten solitons into algebraic equations, we found that G_3, G_5, G_6, and G_7 have perturbed Bott algebraic Schouten solitons under the first distribution. G_3, G_6, and G_7 have such solitons under the second distribution. Additionally, G_2, G_3, G_4, G_5, G_6, and G_7 have such solitons under the third distribution.

Keywords: algebraic Schouten solitons; perturbed Bott connection; Lorentzian Lie groups

1. Introduction

Einstein metrics play a crucial role in many areas of mathematical physics and differential geometry. In geometry, Einstein metrics have been investigated within three - dimensional Lorentzian manifolds. The definition of the Ricci soliton, as a natural generalization of Einstein metrics, was put forward by Hamilton in [1]. Furthermore, Hamilton proposed that if a Ricci soliton has only one modulus up to diffeomorphism and a parameter family on the space of Riemannian metrics, then it can serve as a self-similar solution to the Ricci flow. Since then, numerous mathematicians have turned their attention to Ricci solitons, among which the study of Ricci solitons on different manifolds has become one of the interesting topics in geometry and mathematical physics. In [2], Wang studied Einstein manifolds related to semisymmetric nonmetric connections and semisymmetric metric connections, respectively. Naturally, mathematicians began to study Ricci solitons associated with different affine connections.

With the continuous advancement of research, the generalization of Ricci solitons has been widely applied in Lie groups [3–6]. In 2013, Calvino-Louza introduced the concept of the Schouten tensor [7]. Thereafter, mathematicians attempted to follow the definition pattern of Ricci solitons and construct a new type of soliton based on the Schouten tensor, namely Schouten solitons. In 2023, Azami introduced a generalization of generalized Ricci solitons, namely algebraic Schouten solitons [8, 9]. In the same year, Liu defined and classified the algebraic Schouten solitons associated with Levi–Civita connections, canonical connections, and Kobayashi–Nomizu connections on three-dimensional Lorentzian Lie groups that have some product structure [10]. In the same year, Sardar et al. conducted research on almost Schouten solitons and almost gradient Schouten solitons in the spacetime of general relativity [11]. Miao and Yang conducted an investigation into the existence conditions of algebraic Schouten solitons linked to Yano connections, with their research set against the backdrop of three-dimensional Lorentzian Lie groups [12]. Subsequently, in 2025, Jin put forward the definition of algebraic Schouten solitons associated with the Bott connection and carried out a classification of such solitons. This classification work targeted three-dimensional Lorentzian Lie groups equipped with three distinct distributions [13]. Drawing inspiration from the research findings presented in the aforementioned studies, our focus of this paper lies on algebraic Schouten solitons related to the perturbed Bott connection (under the setting of three distributions). In this paper, we define perturbed Bott algebraic Schouten solitons by some algebraic systems. To solve the algebraic systems arising from these definitions, we achieve a complete classification, which indicates that G_3, G_5, G_6, and G_7 have perturbed Bott algebraic Schouten solitons under the first distribution. G_3, G_6, and G_7 have such solitons under the second distribution. Additionally, G_2, G_3, G_4, G_5, G_6, and G_7 have such solitons under the third distribution.

In Section 2, we introduce the fundamental concepts related to three-dimensional Lie groups and perturbed Bott algebraic Schouten solitons. As for Sections 3 to 5, we focus on discussing and presenting algebraic Schouten solitons associated with the perturbed Bott connection on three-dimensional Lorentzian Lie groups, with each section centering on a distinct type of distribution.

2. Preliminaries

The classification of three-dimensional unimodular Lie groups in the context of Lorentzian geometry was completed by Rahmani in [14], and the non-unimodular cases received treatment in [15,16]. Throughout this paper, $\{G_i\}_{i=1}^{7}$ stands for connected and simply connected three-dimensional Lie groups, each endowed with a left-invariant Lorentzian metric g. Their associated Lie algebras are denoted by $\{\mathfrak{g}_i\}_{i=1}^{7}$, and each of these Lie algebras has a pseudo-orthonormal basis $\{\tilde{e_1}, \tilde{e_2}, \tilde{e_3}\}$ (where $\tilde{e_3}$ is timelike [17]). Let ∇^L be the Levi-Civita connection on $\{G_i\}_{i=1,\cdots,7}$, and R^L be the curvature tensor of ∇^L. Then, for any vector fields X, Y, and Z, we have

$$R^L(X, Y)Z = \nabla^L_X \nabla^L_Y Z - \nabla^L_Y \nabla^L_X Z - \nabla^L_{[X,Y]} Z.$$

Then, the Ricci tensor of $(G_i, g)_{i=1,\cdots,7}$ is defined as

$$\rho^L(X, Y) = -g(R^L(X, \tilde{e_1})Y, \tilde{e_1}) - g(R^L(X, \tilde{e_2})Y, \tilde{e_2}) + g(R^L(X, \tilde{e_3})Y, \tilde{e_3}),$$

where $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ is a set of standard pseudo-orthonormal basis, and $\tilde{e_3}$ is timelike. Next, we introduce the perturbed Bott connection $\nabla^{\tilde{B}_1}$ for the first distribution. The Bott connection ∇^{B_1} associated with distribution D_1 is defined as follows:

$$\nabla_X^{B_1} Y = \begin{cases} \pi_{D_1}(\nabla_X^L Y), & X, Y \in \Gamma^\infty(D_1), \\ \pi_{D_1}([X, Y]), & X \in \Gamma^\infty(D_1^\perp),\ Y \in \Gamma^\infty(D_1), \\ \pi_{D_1^\perp}([X, Y]), & X \in \Gamma^\infty(D_1),\ Y \in \Gamma^\infty(D_1^\perp), \\ \pi_{D_1^\perp}(\nabla_X^L Y), & X, Y \in \Gamma^\infty(D_1^\perp), \end{cases} \tag{2.1}$$

where π_{D_1} (resp. $\pi_{D_1^\perp}$) is the projection on D_1 (resp. $D_1^\perp$). Then, we consider the perturbed Bott connection $\nabla^{\tilde{B}_1}$. Let $\tilde{e_3}^*$ be the dual basis of $\tilde{e_3}$, and define on $G_{i=1,...,7}$

$$\nabla_X^{\tilde{B}_1} Y = \nabla_X^{B_1} Y + a_0 \tilde{e_3}^*(X)\tilde{e_3}^*(Y)\tilde{e_3},$$

where a_0 is a non-zero number. Then $\nabla_{\tilde{e_3}}^{\tilde{B}_1} \tilde{e_3} = a_0\tilde{e_3}, \nabla_{\tilde{e_s}}^{\tilde{B}_1} \tilde{e_t} = \nabla_{\tilde{e_s}}^{B_1} \tilde{e_t}$, where $(s, t) \neq (3, 3)$.
We define

$$R^{\tilde{B}}(X, Y)Z = \nabla_X^{\tilde{B}} \nabla_Y^{\tilde{B}} Z - \nabla_Y^{\tilde{B}} \nabla_X^{\tilde{B}} Z - \nabla_{[X,Y]}^{\tilde{B}} Z. \tag{2.2}$$

The Ricci tensor of (G_i, g) associated to the perturbed Bott connection $\nabla^{\tilde{B}}$ is defined by

$$\rho^{\tilde{B}}(X, Y) = -g(R^{\tilde{B}}(X, \tilde{e_1})Y, \tilde{e_1}) - g(R^{\tilde{B}}(X, \tilde{e_2})Y, \tilde{e_2}) + g(R^{\tilde{B}}(X, \tilde{e_3})Y, \tilde{e_3}).$$

Let

$$\tilde{\rho}^{\tilde{B}}(X, Y) = \frac{\rho^{\tilde{B}(X,Y)} + \rho^{\tilde{B}(Y,X)}}{2}.$$

Using the Ricci tensor $\tilde{\rho}^{\tilde{B}}$, the Ricci operator $\mathrm{Ric}^{\tilde{B}}$ is given by:

$$\tilde{\rho}^{\tilde{B}}(X, Y) = g(\mathrm{Ric}^{\tilde{B}}(X), Y). \tag{2.3}$$

Then, we have the definition of the Schouten tensor as follows:

$$S^{\tilde{B}}(\tilde{e_i}, \tilde{e_j}) = \tilde{\rho}^{\tilde{B}}(\tilde{e_i}, \tilde{e_j}) - \frac{s^{\tilde{B}}}{4} g(\tilde{e_i}, \tilde{e_j}),$$

where $s^{\tilde{B}}$ represents the scalar curvature. Moreover, we generalized the Schouten tensor to:

$$S^{\tilde{B}}(\tilde{e_i}, \tilde{e_j}) = \tilde{\rho}^{\tilde{B}}(\tilde{e_i}, \tilde{e_j}) - s^{\tilde{B}} \lambda_0 g(\tilde{e_i}, \tilde{e_j}),$$

where λ_0 is a real number. By [18], we obtain the expression of $s^{\tilde{B}}$ as

$$s^{\tilde{B}} = \tilde{\rho}^{\tilde{B}}(\tilde{e_1}, \tilde{e_1}) + \tilde{\rho}^{\tilde{B}}(\tilde{e_2}, \tilde{e_2}) - \tilde{\rho}^{\tilde{B}}(\tilde{e_3}, \tilde{e_3}).$$

Definition 1. *A manifold (G_i, g) is called an algebraic Schouten soliton associated to the connection $\nabla^{\tilde{B}}$ if it satisfies:*

$$Ric^{\tilde{B}} = (s^{\tilde{B}} \lambda_0 + c)Id + D^{\tilde{B}},$$

where c is a constant, and $D^{\tilde{B}}$ is a derivation of $\mathfrak{g}_i$, i.e.,

$$D^{\tilde{B}}[X_1, X_2] = [D^{\tilde{B}} X_1, X_2] + [X_1, D^{\tilde{B}} X_2], \quad \textit{for}\ \ X_1, X_2 \in \mathfrak{g}_i. \tag{2.4}$$

3. An algebraic Schouten soliton concerning connection $\nabla^{\tilde{B}_1}$

3.1. A perturbed Bott algebraic Schouten soliton of G_1

According to [17], we have the following Lie algebra of G_1 that satisfies

$$[\tilde{e}_1, \tilde{e}_2] = \lambda\tilde{e}_1 - \mu\tilde{e}_3,\ [\tilde{e}_1, \tilde{e}_3] = -\lambda\tilde{e}_1 - \mu\tilde{e}_2,\ [\tilde{e}_2, \tilde{e}_3] = \mu\tilde{e}_1 + \lambda\tilde{e}_2 + \lambda\tilde{e}_3,\ \lambda \neq 0.$$

From this, we derive the following theorem:

Theorem 1. (G_1, g) *is not an algebraic Schouten soliton associated to the perturbed Bott connection* $\nabla^{\tilde{B}_1}$.

Proof. According to [19], the expression for $Ric^{\tilde{B}_1}$ is derived as follows:

$$\mathrm{Ric}^{\tilde{B}_1}\begin{pmatrix}\tilde{e}_1\\ \tilde{e}_2\\ \tilde{e}_3\end{pmatrix} = \begin{pmatrix} -(\lambda^2+\mu^2) & \lambda\mu & \frac{1}{2}\lambda\mu \\ \lambda\mu & -(\lambda^2+\mu^2) & -\frac{1}{2}(\lambda^2+a_0\lambda) \\ -\frac{1}{2}\lambda\mu & \frac{1}{2}(\lambda^2+a_0\lambda) & 0 \end{pmatrix}\begin{pmatrix}\tilde{e}_1\\ \tilde{e}_2\\ \tilde{e}_3\end{pmatrix}.$$

The scalar curvature is $s^{\tilde{B}_1} = -2(\lambda^2 + \mu^2)$. Now, we can express $D^{\tilde{B}_1}$ as follows:

$$\begin{cases} D^{\tilde{B}_1}\tilde{e}_1 = -(\lambda^2+\mu^2+s^B\lambda_0+c)\tilde{e}_1 + \lambda\mu\tilde{e}_2 + \frac{1}{2}\lambda\mu\tilde{e}_3, \\ D^{\tilde{B}_1}\tilde{e}_2 = \lambda\mu\tilde{e}_1 - (\lambda^2+\mu^2+s^B\lambda_0+c)\tilde{e}_2 - \frac{1}{2}(\lambda^2+a_0\lambda)\tilde{e}_3, \\ D^{\tilde{B}_1}\tilde{e}_3 = -\frac{1}{2}\lambda\mu\tilde{e}_1 + \frac{1}{2}(\lambda^2+a_0\lambda)\tilde{e}_2 - (s^B\lambda_0+c)\tilde{e}_3. \end{cases}$$

Consequently, in light of Eq (2.4), an algebraic Schouten soliton corresponding to $\nabla^{\tilde{B}_1}$ exists on the manifold (G_1, g) if and only if the subsequent condition holds:

$$\begin{cases} \frac{1}{2}\lambda^3 + 2\lambda\mu^2 - 2\lambda_0\lambda(\lambda^2+\mu^2) + \lambda c - \frac{1}{2}a_0\lambda^2 = 0, \\ \frac{1}{2}\lambda^2\mu - a_0\lambda = 0, \\ \lambda^2\mu + 2\mu^3 - 2\mu\lambda_0(\lambda^2+\mu^2) + \mu c = 0, \\ \frac{1}{2}\lambda^3 + 2\lambda\mu^2 - 2\lambda\lambda_0(\lambda^2+\mu^2) + \lambda c + \frac{1}{2}a_0\lambda^2 = 0, \\ -\frac{1}{2}\lambda^2\mu + a_0\lambda\mu = 0, \\ \lambda^2\mu - 2\mu\lambda_0(\lambda^2+\mu^2) + \mu c = 0, \\ \lambda^3 + 2\lambda\mu^2 - 4\lambda\lambda_0(\lambda^2+\mu^2) + 2\lambda c = 0. \end{cases}$$

Recall that $\lambda \neq 0$, and we now analyze the system under different assumptions.

Assume first that $\mu = 0$, so we get $a_0\lambda = 0$, which is a contradiction. Next, suppose that $\mu \neq 0$, we get $a_0\lambda = 0$, which is a contradiction.

3.2. A perturbed Bott algebraic Schouten soliton of G_2

According to [17], we have the following Lie algebra of G_2, which satisfies:

$$[\tilde{e}_1, \tilde{e}_2] = \rho\tilde{e}_2 - \mu\tilde{e}_3, [\tilde{e}_1, \tilde{e}_3] = -\mu\tilde{e}_2 - \rho\tilde{e}_3, [\tilde{e}_2, \tilde{e}_3] = \lambda\tilde{e}_1, \rho \neq 0.$$

From this, we derive the following theorem:

Theorem 2. *(G_2, g) is not an algebraic Schouten soliton associated with the perturbed Bott connection $\nabla^{\tilde{B}_1}$.*

Proof. According to [19], the expression for $Ric^{\tilde{B}_1}$ is derived as follows:

$$\mathrm{Ric}^{\tilde{B}_1}\begin{pmatrix}\tilde{e_1}\\ \tilde{e_2}\\ \tilde{e_3}\end{pmatrix}=\begin{pmatrix}-(\mu^2+\rho^2) & 0 & \frac{1}{2}a_0\rho\\ 0 & -(\rho^2+\lambda\mu) & \frac{1}{2}\lambda\rho\\ -\frac{1}{2}a_0\rho & -\frac{1}{2}\lambda\rho & 0\end{pmatrix}\begin{pmatrix}\tilde{e_1}\\ \tilde{e_2}\\ \tilde{e_3}\end{pmatrix}.$$

The scalar curvature is $s^{\tilde{B}_1}=-(\mu^2+2\rho^2+\lambda\mu)$. Now, we can express $D^{\tilde{B}_1}$ as follows:

$$\begin{cases}D^{\tilde{B}}\tilde{e_1}=-(\mu^2+\rho^2+s^{\tilde{B}_1}\lambda_0+c)\tilde{e_1}+\frac{1}{2}a_0\rho\tilde{e_2},\\ D^{\tilde{B}}\tilde{e_2}=-(\rho^2+\lambda\mu+s^{\tilde{B}_1}\lambda_0+c)\tilde{e_2}+\frac{1}{2}\lambda\rho\tilde{e_3},\\ D^{\tilde{B}}\tilde{e_3}=-\frac{1}{2}a_0\rho\tilde{e_1}-\frac{1}{2}\lambda\rho\tilde{e_2}-(s^{\tilde{B}_1}\lambda_0+c)\tilde{e_3}.\end{cases}$$

Consequently, in light of Eq (2.4), an algebraic Schouten soliton corresponding to $\nabla^{\tilde{B}_1}$ exists on the manifold (G_2, g) if and only if the subsequent condition holds:

$$\begin{cases}\lambda\rho^2-\mu^3+\lambda\mu^2+(\mu^2+2\rho^2+\lambda\mu)\mu\lambda_0-c\mu=0,\\ \rho(\mu^2+\rho^2+\lambda\mu-(\mu^2+2\rho^2+\lambda\mu)\lambda_0+c)=0,\\ \lambda\rho^2-\mu^3-2\mu\rho^2-\lambda\mu^2+(\mu^2+2\rho^2+\lambda\mu)\mu\lambda_0-c\mu=0,\\ \lambda(-\mu^2+\lambda\mu-(\mu^2+2\rho^2+\lambda\mu)\lambda_0+c)=0,\\ \frac{1}{2}a_0(\lambda+\mu)\rho=0,\\ \frac{1}{2}a_0\rho^2=0.\end{cases}$$

Since $\rho\neq 0$, we have $a_0=0$, which is a contradiction.

3.3. *A perturbed Bott algebraic Schouten soliton of G_3*

According to [17], we have the following Lie algebra of G_3, which satisfies:

$$[\tilde{e}_1,\tilde{e}_2]=-\rho\tilde{e}_3,\ [\tilde{e}_1,\tilde{e}_3]=-\mu\tilde{e}_2,\ [\tilde{e}_2,\tilde{e}_3]=\lambda\tilde{e}_1.$$

From this, we derive the following theorem:

The existence conditions for Schouten solitons in this part are consistent with those under the Bott connection. For the sake of ensuring the completeness of the paper, the conclusion is provided herein (refer to [13] for the specific proof process).

Theorem 3. *If (G_3, g) is an algebraic Schouten soliton concerning connection $\nabla^{\tilde{B}_1}$, then one of the following cases holds:*

i. $\lambda=\mu=\rho=0$, *for all c.*

ii. $\lambda\neq 0, \mu=\rho=0, c=0.$

iii. $\lambda=\rho=0, \mu\neq 0, c=0.$

iv. $\lambda \neq 0, \mu \neq 0, \rho = 0, c = 0$.

v. $\lambda = \mu = 0, \rho \neq 0, c = 0$.

vi. $\lambda \neq 0, \mu = 0, \rho \neq 0, c = -\lambda\rho + \lambda\rho\lambda_0$.

vii. $\lambda = 0, \mu \neq 0, \rho \neq 0, c = -\mu\rho + \mu\rho\lambda_0$.

3.4. A perturbed Bott algebraic Schouten soliton of G_4

According to [17], we have the following Lie algebra of G_4, which satisfies:

$$[\tilde{e}_1, \tilde{e}_2] = -\tilde{e}_2 + (2\sigma - \mu)\tilde{e}_3, [\tilde{e}_1, \tilde{e}_3] = \tilde{e}_3 - \mu\tilde{e}_2, [\tilde{e}_2, \tilde{e}_3] = \lambda\tilde{e}_1, \sigma \neq 1 or - 1.$$

From this, we derive the following theorem:

Theorem 4. *(G_4, g) is not an algebraic Schouten soliton associated with the perturbed Bott connection $\nabla^{\tilde{B}_1}$.*

Proof. According to [19], the expression for $Ric^{\tilde{B}_1}$ is derived as follows:

$$\mathrm{Ric}^{\tilde{B}_1}\begin{pmatrix}\tilde{e_1}\\ \tilde{e_2}\\ \tilde{e_3}\end{pmatrix} = \begin{pmatrix}-(\mu-\sigma)^2 & 0 & -\frac{1}{2}a_0\\ 0 & 2\lambda\sigma - \lambda\mu - 1 & -\frac{1}{2}\lambda\\ \frac{1}{2}a_0 & \frac{1}{2}\lambda & 0\end{pmatrix}\begin{pmatrix}\tilde{e_1}\\ \tilde{e_2}\\ \tilde{e_3}\end{pmatrix}.$$

The scalar curvature is $s^{\tilde{B}_1} = -(\mu - \sigma)^2 + 2\lambda\sigma - \lambda\mu - 1$. Now, we can express $D^{\tilde{B}_1}$ as follows:

$$\begin{cases} D^{\tilde{B}_1}\tilde{e_1} = (- (\mu-\sigma)^2 - (s^{\tilde{B}_1}\lambda_0 + c))\tilde{e_1} - \frac{1}{2}a_0\tilde{e_3}, \\ D^{\tilde{B}_1}\tilde{e_2} = (2\lambda\sigma - \lambda\mu - 1 - (s^{\tilde{B}_1}\lambda_0 + c))\tilde{e_2} - \frac{1}{2}\lambda\tilde{e_3}, \\ D^{\tilde{B}_1}\tilde{e_3} = \frac{1}{2}a_0\tilde{e_1} + \frac{1}{2}\lambda\tilde{e_2} - (s^{\tilde{B}_1}\lambda_0 + c)\tilde{e_3}. \end{cases}$$

Consequently, in light of Eq (2.4), an algebraic Schouten soliton corresponding to $\nabla^{\tilde{B}_1}$ exists on the manifold (G_4, g) if and only if the subsequent condition holds:

$$\begin{cases} \frac{1}{2}a_0 = 0, \\ a_0(2\sigma - \mu - \lambda) = 0, \\ -(2\sigma-\mu)((\mu-\sigma)^2 - 2\lambda\sigma + \lambda\mu + 1 - ((\mu-\sigma)^2 + \lambda\mu - 2\lambda\sigma + 1)\lambda_0 + c) = \lambda, \\ \mu((\mu-\sigma)^2 + 2\lambda\sigma - \lambda\mu - 1 - ((\mu-\sigma)^2 + \lambda\mu - 2\lambda\sigma + 1)\lambda_0 + c) = \lambda, \\ (\mu-\sigma)^2 - ((\mu-\sigma)^2 + \lambda\mu - 2\lambda\sigma + 1)\lambda_0 + c = \lambda(\sigma - \mu), \\ \lambda((\mu-\sigma)^2 + 2\lambda\sigma - \lambda\mu - 1 + ((\mu-\sigma)^2 + \lambda\mu - 2\lambda\sigma + 1)\lambda_0 - c) = 0. \end{cases} \tag{3.1}$$

By solving (3.1), we have $a_0 = 0$, which is a contradiction.

3.5. A perturbed Bott algebraic Schouten soliton of G_5

According to [17], we have the following Lie algebra of G_5, which satisfies:

$$[\tilde{e}_1, \tilde{e}_2] = 0,\ [\tilde{e}_1, \tilde{e}_3] = \lambda\tilde{e}_1 + \mu\tilde{e}_2,\ [\tilde{e}_2, \tilde{e}_3] = \rho\tilde{e}_1 + \theta\tilde{e}_2, \lambda + \theta \neq 0, \lambda\rho - \mu\theta = 0.$$

From this, we derive the following theorem:

The existence conditions for Schouten solitons in this part are consistent with those under the Bott connection. For the sake of ensuring the completeness of the paper, the conclusion is provided herein (refer to [13] for the specific proof process).

Theorem 5. *If (G_5, g) constitutes an algebraic Schouten soliton concerning connection $\nabla^{\tilde{B}_1}$, then $c = 0$.*

3.6. A perturbed Bott algebraic Schouten soliton of G_6

According to [17], we have the following Lie algebra of G_6, which satisfies:

$$[\tilde{e}_1, \tilde{e}_2] = \lambda\tilde{e}_2 + \mu\tilde{e}_3, [\tilde{e}_1, \tilde{e}_3] = \rho\tilde{e}_2 + \theta\tilde{e}_3, [\tilde{e}_2, \tilde{e}_3] = 0, \lambda + \theta \neq 0, \lambda\rho - \mu\theta = 0.$$

From this, we derive the following theorem:

Theorem 6. *If* (G_6, g) *constitutes an algebraic Schouten soliton associated with the perturbed Bott connection* $\nabla^{\tilde{B}_1}$*, then it fulfills the conditions:* $\lambda \neq 0, \mu = \rho = \theta = 0$*, and* $c = -\lambda^2 + 2\lambda^2\lambda_0$.

Proof. According to [19], the expression for $Ric^{\tilde{B}_1}$ is derived as follows:

$$\mathrm{Ric}^{\tilde{B}_1}\begin{pmatrix}\tilde{e}_1\\ \tilde{e}_2\\ \tilde{e}_3\end{pmatrix} = \begin{pmatrix}-(\lambda^2+\mu\rho) & 0 & -\frac{1}{2}a_0\theta\\ 0 & -\lambda^2 & 0\\ \frac{1}{2}a_0\theta & 0 & 0\end{pmatrix}\begin{pmatrix}\tilde{e}_1\\ \tilde{e}_2\\ \tilde{e}_3\end{pmatrix}.$$

The scalar curvature is $s^{\tilde{B}_1} = -(2\lambda^2 + \mu\rho)$. Now, we can express $D^{\tilde{B}}$ as follows:

$$\begin{cases} D^{\tilde{B}_1}\tilde{e}_1 = -(\lambda^2 + \mu\rho + (s^{\tilde{B}_1}\lambda_0 + c))\tilde{e}_1 - \frac{1}{2}a_0\theta\tilde{e}_3,\\ D^{\tilde{B}_1}\tilde{e}_2 = -(\alpha^2 + s^{\tilde{B}_1}\lambda_0 + c)\tilde{e}_2,\\ D^{\tilde{B}_1}\tilde{e}_3 = \frac{1}{2}a_0\theta\tilde{e}_1 - (s^{\tilde{B}_1}\lambda_0 + c)\tilde{e}_3.\end{cases}$$

Consequently, in light of Eq (2.4), an algebraic Schouten soliton corresponding to $\nabla^{\tilde{B}_1}$ exists on the manifold (G_6, g) if and only if the subsequent condition holds:

$$\begin{cases} \frac{1}{2}a_0\mu\theta = 0,\\ \frac{1}{2}a_0\lambda\theta = 0,\\ \frac{1}{2}a_0\theta^2 = 0,\\ \mu\left(2\lambda^2 + \mu\rho - (2\lambda^2 + \mu\rho)\lambda_0 + c\right) = 0,\\ \lambda\left(\lambda^2 + \mu\rho - (2\lambda^2 + \mu\rho)\lambda_0 + c\right) = 0,\\ \rho\left(\mu\rho - (2\lambda^2 + \mu\rho)\lambda_0 + c\right) = 0,\\ \theta\left(\lambda^2 + \mu\rho - (2\lambda^2 + \mu\rho)\lambda_0 + c\right) = 0.\end{cases} \tag{3.2}$$

From the first equation above, we have either $\mu = 0$ or $\mu \neq 0$. We now analyze the system under different assumptions.

Assuming that $\mu = 0$, we have:

$$\begin{cases} \frac{1}{2}a_0\lambda\theta = 0,\\ \frac{1}{2}a_0\theta^2 = 0,\\ \lambda(\lambda^2 - 2\lambda^2\lambda_0 + c) = 0,\\ \rho\left(-2\lambda^2\lambda_0 + c\right) = 0,\\ \theta\left(\lambda^2 - 2\lambda^2\lambda_0 + c\right) = 0.\end{cases} \tag{3.3}$$

Given $\lambda\rho - \mu\theta = 0$ and $\lambda + \theta = 0$, we assume first that $\lambda = 0$, so we get $a_0 = 0$, which is a contradiction. Assuming $\lambda \neq 0$, in this case, system (3.3) can be simplified to: $\lambda^2 - 2\lambda^2 + c = 0$. Then, we have $c = -\lambda^2 + 2\lambda^2\lambda_0$.

If $\mu \neq 0$, system (3.2) becomes:

$$\begin{cases} \mu\left(2\lambda^2 + \mu\rho - (2\lambda^2 + \mu\rho)\lambda_0 + c\right) = 0, \\ \lambda\left(\lambda^2 + \mu\rho - (2\lambda^2 + \mu\rho)\lambda_0 + c\right) = 0. \end{cases}$$

This is a contradiction.

3.7. A perturbed Bott algebraic Schouten soliton of G_7

According to [17], we have the following Lie algebra of G_7, which satisfies:

$$[\tilde{e}_1, \tilde{e}_2] = -\lambda\tilde{e}_1 - \mu\tilde{e}_2 - \mu\tilde{e}_3, [\tilde{e}_1, \tilde{e}_3] = \lambda\tilde{e}_1 + \mu\tilde{e}_2 + \mu\tilde{e}_3, [\tilde{e}_2, \tilde{e}_3] = \rho\tilde{e}_1 + \theta\tilde{e}_2 + \theta\tilde{e}_3, \lambda + \theta \neq 0, \lambda\rho = 0.$$

From this, we derive the following theorem:

Theorem 7. *If (G_7, g) is an algebraic Schouten soliton concerning connection $\nabla^{\tilde{B}_1}$, then one of the following cases holds:*

i. $\lambda \neq 0, \mu = \rho = 0, c = -\frac{1}{2}\lambda^2 - s^B\lambda_0.$

ii. $\lambda = 0, \mu \neq 0, a_0 = -3\theta, c = \frac{1}{2}\theta^2 - s^B\lambda_0.$

iii. $\lambda = \mu = 0, a_0 = -2\theta, c = -s^B\lambda_0.$

Proof. According to [19], the expression for $Ric^{\tilde{B}}$ is derived as follows:

$$\mathrm{Ric}^{\tilde{B}_1}\begin{pmatrix}\tilde{e_1}\\ \tilde{e_2}\\ \tilde{e_3}\end{pmatrix} = \begin{pmatrix} -\lambda^2 & \frac{1}{2}\mu(\theta-\lambda) & -\mu(\lambda+\theta+a_0) \\ \frac{1}{2}\mu(\theta-\lambda) & -(\lambda^2+\mu^2+\mu\rho) & -\theta^2 - \frac{1}{2}(\mu\rho+\lambda\theta+a_0\theta) \\ \mu(\lambda+\theta+a_0) & \theta^2+\frac{1}{2}(\mu\rho+\lambda\theta+a_0\theta) & 0 \end{pmatrix}\begin{pmatrix}\tilde{e_1}\\ \tilde{e_2}\\ \tilde{e_3}\end{pmatrix}.$$

The scalar curvature is $s^{\tilde{B}_1} = -(2\lambda^2 + \mu^2 + \mu\rho)$. Now, we can express $D^{\tilde{B}_1}$ as follows:

$$\begin{cases} D^{\tilde{B}_1}\tilde{e_1} = -(\lambda^2 + (s^{\tilde{B}_1}\lambda_0 + c))\tilde{e_1} + \frac{1}{2}\mu(\theta-\lambda)\tilde{e_2} - \mu(\lambda+\theta+\frac{1}{2}a_0)\tilde{e_3}, \\ D^{\tilde{B}_1}\tilde{e_2} = \frac{1}{2}\mu(\theta-\lambda)\tilde{e_1} - (\lambda^2+\mu^2+\mu\rho+(s^{\tilde{B}_1}\lambda_0+c))\tilde{e_2} - (\delta^2+\frac{1}{2}(\mu\rho+\lambda\theta+a_0\theta))\tilde{e_3}, \\ D^{\tilde{B}_1}\tilde{e_3} = \mu(\lambda+\theta+\frac{1}{2}a_0)\tilde{e_1} + (\theta^2+\frac{1}{2}(\mu\rho+\lambda\theta+a_0\theta))\tilde{e_2} - (s^{\tilde{B}_1}\lambda + c)\tilde{e_3}. \end{cases}$$

Consequently, in light of Eq (2.4), an algebraic Schouten soliton corresponding to $\nabla^{\tilde{B}_1}$ exists on the manifold (G_7, g) if and only if the subsequent condition holds:

$$\begin{cases} \lambda(\lambda^2+\mu^2+\mu\rho+s^B\lambda_0+c) + (\rho+\mu)(\lambda\mu+\mu\theta+\frac{1}{2}a_0\mu) + \frac{1}{2}\mu^2(\theta-\lambda) = \lambda(\theta^2+\frac{1}{2}(\mu\rho+\lambda\theta+a_0\theta)), \\ \mu(\lambda^2+s^B\lambda_0+c) + \theta(\lambda\mu+\mu\theta+\frac{1}{2}a_0\mu) + \frac{1}{2}\lambda\mu(\theta-\lambda) = 0, \\ \mu(2\lambda^2+\mu^2+\mu\rho+s^B\lambda_0+c) + \mu(\theta-\lambda)(\lambda+\theta+\frac{1}{2}a_0\mu) = 2\mu(\theta^2+\frac{1}{2}(\mu\rho+\lambda\theta+a_0\theta)), \\ \lambda(s^B\lambda_0+c) + \lambda(\theta^2+\frac{1}{2}(\mu\rho+\lambda\theta+a_0\theta)) + \frac{1}{2}\mu(\mu-\rho)(\theta-\lambda) = 0, \\ -\mu(\mu^2+\mu\rho+s^B\lambda_0+c) - \frac{1}{2}\mu(\theta-\lambda)^2 + 2\mu(\theta^2+\frac{1}{2}(\mu\rho+\lambda\theta+a_0\theta)) = 0, \\ \mu(\lambda^2+s^B\lambda_0+c) = \lambda\mu(\lambda+\theta) + \frac{1}{2}\mu\theta(\theta-\lambda), \\ \rho(\mu^2+\mu\lambda+s^B\lambda_0+c) = \mu(\lambda-\theta)(\lambda+\theta) - \frac{1}{2}\mu(\theta-\lambda)^2, \\ \theta(s^B\lambda_0+c) + \theta(\theta^2+\frac{1}{2}(\mu\rho+\lambda\theta+a_0\theta)) = \frac{1}{2}\mu(\mu-\rho)(\theta-\lambda), \\ \theta(\lambda^2+\mu^2+\mu\rho+s^B\lambda_0+c) - \theta(\theta^2+\frac{1}{2}(\mu\rho+\lambda\theta+a_0\theta)) = (\mu+\rho)(\lambda\mu+\mu\theta+\frac{1}{2}a_0\mu) + \frac{1}{2}\mu^2(\theta-\lambda). \end{cases} \tag{3.4}$$

Recall that $\lambda+\theta \neq 0$ and $\lambda\rho = 0$, and we now analyze the system under different assumptions, assuming first that $\lambda \neq 0, \rho = 0$. Then, the above system (3.4) becomes:

$$\begin{cases} \lambda(\lambda^2 + \mu^2 + s^B\lambda_0 + c) + \mu^2(\lambda + \theta + \frac{1}{2}a_0) + \frac{1}{2}\mu^2(\theta - \lambda) = \lambda(\theta^2 + \frac{1}{2}(\lambda\theta + a_0\theta)), \\ \mu(\lambda^2 + s^B\lambda_0 + c) + \theta(\lambda\mu + \mu\theta + \frac{1}{2}a_0\mu) + \frac{1}{2}\lambda\mu(\theta - \lambda) = 0, \\ \mu(2\lambda^2 + \mu^2 + s^B\lambda_0 + c) + \mu(\theta - \lambda)(\lambda + \theta + \frac{1}{2}a_0\mu) = 2\mu(\theta^2 + \frac{1}{2}(\lambda\theta + a_0\theta)), \\ \lambda(s^B\lambda_0 + c) + \lambda(\theta^2 + \frac{1}{2}(\lambda\theta + a_0\theta)) + \frac{1}{2}\mu^2(\theta - \lambda) = 0, \\ -\mu(\mu^2 + s^B\lambda_0 + c) - \frac{1}{2}\mu(\theta - \lambda)^2 + 2\mu(\theta^2 + \frac{1}{2}(\lambda\theta + a_0\theta)) = 0, \\ \mu(\lambda^2 + s^B\lambda_0 + c) = \lambda\mu(\lambda + \theta) + \frac{1}{2}\mu\theta(\theta - \lambda), \\ \mu(\lambda - \theta)(\lambda + \theta) - \frac{1}{2}\mu(\theta - \lambda)^2 = 0, \\ \theta(s^B\lambda_0 + c) + \theta(\theta^2 + \frac{1}{2}(\lambda\theta + a_0\theta))= \frac{1}{2}\mu^2(\theta - \lambda), \\ \theta(\lambda^2 + \mu^2 + s^B\lambda_0 + c) - \theta(\theta^2 + \frac{1}{2}(\lambda\theta + a_0\theta)) = \mu^2(\lambda + \theta + \frac{1}{2}a_0) + \frac{1}{2}\mu^2(\theta - \lambda). \end{cases} \tag{3.5}$$

Next, suppose that $\mu = 0$, and the first, second, seventh, and eighth expressions in Eq (3.5) can be transformed into:

$$\begin{cases} \lambda(\lambda^2 + s^B\lambda_0 + c) - \lambda(\theta^2 + \frac{1}{2}(\lambda\theta + a_0\theta)) = 0, \\ \lambda(s^B\lambda_0 + c) + \lambda(\theta^2 + \frac{1}{2}(\lambda\theta + a_0\theta)) = 0, \\ \theta(\lambda^2 + s^B\lambda_0 + c) - \theta(\theta^2 + \frac{1}{2}(\lambda\theta + a_0\theta)) = 0, \\ \theta(s^B\lambda_0 + c) + \theta(\theta^2 + \frac{1}{2}(\lambda\theta + a_0\theta)) = 0. \end{cases}$$

Since $\lambda \neq 0$, we have $\lambda^2 + 2s^B\lambda_0 + 2c = 0$, and we then get $c = -\frac{1}{2}\lambda^2 - s^B\lambda_0$.

If $\mu \neq 0$, we further assume that $\theta = 0$. Under this assumption, the last equation in (3.5) yields $\frac{1}{2}a_0\mu^3 = 0$, which is a contradiction. Additionally, if we assume that $\theta \neq 0$ and $\theta \neq -\lambda$, then we get $-\frac{1}{2}\mu(\theta - \lambda)^2 = 0$, which is a contradiction. If we presume $\lambda = \theta$, then we get $2\lambda^2\mu = 0$, which is a contradiction.

Second, let $\lambda = 0, \theta \neq 0$. Then, the above system (3.4) becomes:

$$\begin{cases} (\rho + \mu)(\mu\theta + \frac{1}{2}a_0\mu) + \frac{1}{2}\mu^2\theta = 0, \\ \mu(s^B\lambda_0 + c) + \theta(\mu\theta + \frac{1}{2}a_0\mu) = 0, \\ \mu(\mu^2 + \mu\rho + s^B\lambda_0 + c) + \mu\theta(\theta + \frac{1}{2}a_0\mu) = 2\mu(\theta^2 + \frac{1}{2}(\mu\rho + a_0\theta)), \\ \frac{1}{2}\mu(\mu - \rho)\theta = 0, \\ -\mu(\mu^2 + \mu\rho + s^B\lambda_0 + c) - \frac{1}{2}\mu\theta^2 + 2\mu(\theta^2 + \frac{1}{2}(\mu\rho + a_0\theta)) = 0, \\ \mu(s^B\lambda_0 + c) = \frac{1}{2}\mu\theta^2, \\ \rho(\mu^2 + s^B\lambda_0 + c) = -\frac{3}{2}\mu\theta^2, \\ \theta(s^B\lambda_0 + c) + \theta(\theta^2 + \frac{1}{2}(\mu\rho + a_0\theta))= \frac{1}{2}\mu(\mu - \rho)\theta. \\ \theta(\mu^2 + \mu\rho + s^B\lambda_0 + c) - \theta(\theta^2 + \frac{1}{2}(\mu\rho + a_0\theta)) = (\mu + \rho)\mu(\theta + \frac{1}{2}a_0\mu) + \frac{1}{2}\mu^2\theta. \end{cases} \tag{3.6}$$

Then, if $\mu \neq 0$, the second and sixth equation in (3.6) exists as $c = \frac{1}{2}\theta^2 - s^B\lambda_0$ if and only if $a_0 = -3\theta$. If $\mu = 0$, we have :

$$\begin{cases} \rho(s^B\lambda_0 + c) = 0, \\ \theta(s^B\lambda_0 + c) + \theta(\theta^2 + \frac{1}{2}a_0\theta)= 0, \\ \theta(s^B\lambda_0 + c) - \theta(\theta^2 + \frac{1}{2}a_0\theta) = 0. \end{cases}$$

Then, $c = -s^B\lambda_0$ exists if and only if $a_0 = -2\theta$.

4. An algebraic Schouten soliton concerning connection $\nabla^{\tilde{B}_2}$

In this section, we present the algebraic criterion that is necessary for a three-dimensional Lorentzian Lie group to possess an algebraic Schouten soliton associated with the perturbed Bott connection $\nabla^{\tilde{B}_2}$. Next, we introduce the Perturbed Bott connection $\nabla^{\tilde{B}_2}$ for the second distribution. The Bott connection $\nabla^{\tilde{B}_2}$ associated with distribution D_2 is defined as follows:

$$\nabla^{B_2}_X Y = \begin{cases} \pi_{D_2}(\nabla^L_X Y), & X, Y \in \Gamma^\infty(D_2), \\ \pi_{D_2}([X, Y]), & X \in \Gamma^\infty({D_2}^\perp),\ Y \in \Gamma^\infty(D_2), \\ \pi_{{D_2}^\perp}([X, Y]), & X \in \Gamma^\infty(D_2),\ Y \in \Gamma^\infty({D_2}^\perp), \\ \pi_{{D_2}^\perp}(\nabla^L_X Y), & X, Y \in \Gamma^\infty(D_2^\perp), \end{cases}$$

where π_{D_2} (resp. $\pi_{D_2^\perp}$) the projection on D_2 (resp. $D_2^\perp$). Then, we consider the perturbed Bott connection $\nabla^{\tilde{B}_2}$. Let $\tilde{e_2}^*$ be the dual basis of e_2, and define on $G_{i=1,...,7}$

$$\nabla^{\tilde{B}_2}_X Y = \nabla^{B_2}_X Y + a_0 \tilde{e_2}^*(X)\tilde{e_2}^*(Y)\tilde{e_2},$$

where a_0 is a non-zero number. Then $\nabla^{\tilde{B}_2}_{\tilde{e_2}}\tilde{e_2} = a_0\tilde{e_2}, \nabla^{\tilde{B}_2}_{\tilde{e_s}}\tilde{e_t} = \nabla^{B_2}_{\tilde{e_s}}\tilde{e_t}$, where $(s, t) \neq (2, 2)$.

4.1. A perturbed Bott algebraic Schouten soliton of G_1

Lemma 1. *The Ricci tensor $\rho^{\tilde{B}_2}$ related to the connection $\nabla^{\tilde{B}_2}$ of (G_1, g) is presented as follows:*

$$\rho^{\tilde{B}_2}(\tilde{e}_i, \tilde{e}_j) = \begin{pmatrix} \lambda^2 - \mu^2 & \frac{1}{2}\lambda\mu & -\lambda\mu \\ \frac{1}{2}\lambda\mu & 0 & \frac{1}{2}\lambda^2 - a_0\lambda \\ -\lambda\mu & \frac{1}{2}\lambda^2 - a_0\lambda & \mu^2 - \lambda^2 \end{pmatrix}.$$

From this lemma, the subsequent theorem is deduced.

Theorem 8. (G_1, g) *is not an algebraic Schouten soliton associated with the perturbed Bott connection* $\nabla^{\tilde{B}_2}$.

Proof. According to [19], the expression for $Ric^{\tilde{B}_2}$ is derived as follows:

$$\mathrm{Ric}^{\tilde{B}_2}\begin{pmatrix} \tilde{e_1} \\ \tilde{e_2} \\ \tilde{e_3} \end{pmatrix} = \begin{pmatrix} \lambda^2 - \mu^2 & \frac{1}{2}\lambda\mu & \lambda\mu \\ \frac{1}{2}\lambda\mu & 0 & a_0\lambda - \frac{1}{2}\lambda^2 \\ -\lambda\mu & \frac{1}{2}\lambda^2 - a_0\lambda & \lambda^2 - \mu^2 \end{pmatrix}\begin{pmatrix} \tilde{e_1} \\ \tilde{e_2} \\ \tilde{e_3} \end{pmatrix}.$$

The scalar curvature is $s^{\tilde{B}_2} = 2(\lambda^2 - \mu^2)$. Now, we can express $D^{\tilde{B}_2}$ as follows:

$$\begin{cases} D^{\tilde{B}_2}\tilde{e_1} = (\lambda^2 - \mu^2 - (s^{\tilde{B}_2}\lambda_0 + c))\tilde{e_1} + \frac{1}{2}\lambda\mu\tilde{e_2} + \lambda\mu\tilde{e_3}, \\ D^{\tilde{B}_2}\tilde{e_2} = \frac{1}{2}\lambda\mu\tilde{e_1} - (s^{\tilde{B}_2}\lambda_0 + c)\tilde{e_2} + \left(a_0\lambda - \frac{1}{2}\lambda^2\right)\tilde{e_3}, \\ D^{\tilde{B}_2}\tilde{e_3} = -\lambda\mu\tilde{e_1} + \left(\frac{1}{2}\lambda^2 - a_0\lambda\right)\tilde{e_2} + \left(\lambda^2 - \mu^2 - (s^{\tilde{B}_2}\lambda_0 + c)\right)\tilde{e_3}. \end{cases}$$

Consequently, in light of Eq (2.4), an algebraic Schouten soliton corresponding to $\nabla^{\tilde{B}_2}$ exists on the manifold (G_1, g) if and only if the subsequent condition holds:

$$\begin{cases} \lambda\left(2(\lambda^2-\mu^2)\lambda_0+c\right)+2\lambda\mu^2+\frac{1}{2}\lambda^3-a_0\lambda^2=0, \\ \lambda^2\mu=0, \\ \mu\left(2(\lambda^2-\mu^2)\lambda_0+c\right)-2\lambda^2\mu=0, \\ \lambda\left(\lambda^2-\mu^2-2(\lambda^2-\mu^2)\lambda_0-c\right)-\lambda\mu^2+\frac{1}{2}\lambda^3-a_0\lambda^2=0, \\ \mu\left(\lambda^2-2\mu^2-2(\lambda^2-\mu^2)\lambda_0-c\right)=0, \\ 2a_0\lambda\mu-\frac{1}{2}\lambda\mu^2=0, \\ \mu\left(2(\lambda^2-\mu^2)\lambda_0+c\right)-\lambda^2\mu=0, \\ \lambda\left(2(\lambda^2-\mu^2)\lambda_0+c\right)-\frac{1}{2}\lambda^3+2\lambda\mu^2-a_0\lambda^2=0, \\ \lambda\left(2(\lambda^2-\mu^2)\lambda_0+c\right)-\frac{1}{2}\lambda^3+2\lambda\mu^2+a_0\lambda^2=0. \end{cases} \tag{4.1}$$

Since $\lambda \neq 0$, then $\mu = 0$, the eighth and ninth equation in (4.1) yields $a_0\lambda^2 = 0$, which is a contradiction.

4.2. *A perturbed Bott algebraic Schouten soliton of G_2*

Lemma 2. *The Ricci tensor $\rho^{\tilde{B}_2}$ related to the connection $\nabla^{\tilde{B}_2}$ of (G_2, g) is presented as follows:*

$$\rho^{\tilde{B}_2}(\tilde{e}_i,\tilde{e}_j)=\begin{pmatrix} -(\mu^2+\rho^2) & \frac{1}{2}a_0\gamma & 0 \\ \frac{1}{2}a_0\gamma & 0 & -\frac{1}{2}\lambda\rho \\ 0 & -\frac{1}{2}\lambda\rho & -\lambda\mu-\rho^2 \end{pmatrix}.$$

From this lemma, the subsequent theorem is deduced.

Theorem 9. (G_2, g) *is not an algebraic Schouten soliton associated with the perturbed Bott connection* $\nabla^{\tilde{B}_2}$.

Proof. According to [19], the expression for $Ric^{\tilde{B}_2}$ is derived as follows:

$$\mathrm{Ric}^{\tilde{B}_2}\begin{pmatrix}\tilde{e}_1\\ \tilde{e}_2\\ \tilde{e}_3\end{pmatrix}=\begin{pmatrix} -(\mu^2+\rho^2) & \frac{1}{2}a_0\gamma & 0 \\ \frac{1}{2}a_0\gamma & 0 & \frac{1}{2}\lambda\rho \\ 0 & -\frac{1}{2}\lambda\rho & \lambda\mu+\rho^2 \end{pmatrix}\begin{pmatrix}\tilde{e}_1\\ \tilde{e}_2\\ \tilde{e}_3\end{pmatrix}.$$

The scalar curvature is $s^{\tilde{B}_2} = -(\mu^2 + 2\rho^2 + \lambda\mu)$. Now, we can express $D^{\tilde{B}_2}$ as follows:

$$\begin{cases} D^{\tilde{B}_2}\tilde{e}_1=-(\mu^2+\rho^2+s^{\tilde{B}_2}\lambda_0+c)\tilde{e}_1+\frac{1}{2}a_0\rho\tilde{e}_2, \\ D^{\tilde{B}_2}\tilde{e}_2=\frac{1}{2}a_0\rho\tilde{e}_1-(s^{\tilde{B}_2}\lambda_0+c)\tilde{e}_2+\frac{1}{2}\lambda\rho\tilde{e}_3, \\ D^{\tilde{B}_2}\tilde{e}_3=-\frac{1}{2}\lambda\rho\tilde{e}_2-(\lambda\mu+\rho^2+s^{\tilde{B}_2}\lambda_0+c)\tilde{e}_3. \end{cases}$$

Consequently, in light of Eq (2.4), an algebraic Schouten soliton corresponding to $\nabla^{\tilde{B}_2}$ exists on the manifold (G_2, g) if and only if the subsequent condition holds:

$$\begin{cases}\lambda\left(\mu^2+\rho^2-(\mu^2+\rho^2+\lambda\mu)\lambda_0+c\right)+\frac{1}{2}\lambda\mu\rho=0,\\ \mu\left((\mu^2+\rho^2+\lambda\mu)\lambda_0-c\right)+\lambda\rho^2=0,\\ \mu\left(\mu^2+2\rho^2+\lambda\mu-(\mu^2+\rho^2+\lambda\mu)\lambda_0+c\right)-\lambda\rho^2=0,\\ \rho\left(\mu^2+\rho^2-(\mu^2+\rho^2+\lambda\mu)\lambda_0+c\right)+\frac{1}{2}\lambda\mu\rho=0,\\ \lambda\left(\lambda\mu-\mu^2-(\mu^2+\rho^2+\lambda\mu)\lambda_0+c\right)=0,\\ \frac{1}{2}(a_0\rho^2-\lambda^2\rho)=0,\\ -\frac{1}{2}(a_0\mu\rho+a_0\lambda\rho)=0,\\ -\frac{1}{2}a_0\rho^2=0.\end{cases}$$

Since $\rho \neq 0$, we have $a_0 = 0$, which is a contradiction.

4.3. A perturbed Bott algebraic Schouten soliton of G_3

Lemma 3. *The Ricci tensor $\rho^{\tilde{B}_2}$ related to the connection $\nabla^{\tilde{B}_2}$ of (G_3, g) is presented as follows:*

$$\rho^{\tilde{B}_2}(\tilde{e}_i, \tilde{e}_j) = \begin{pmatrix} -\mu\rho & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda\mu \end{pmatrix}.$$

The existence conditions for Schouten solitons in this part are consistent with those under the Bott connection. For the sake of ensuring the completeness of the paper, the conclusion is provided herein (refer to [13] for the specific proof process).

Theorem 10. *If (G_3, g) constitutes an algebraic Schouten soliton concerning connection $\nabla^{\tilde{B}_2}$, then one of the following cases holds:*

i. $\lambda = \mu = \rho = 0$, *for all c.*

ii. $\lambda \neq 0, \mu = \rho = 0, c = 0.$

iii. $\lambda = 0, \mu \neq 0, \rho = 0, c = 0.$

iv. $\lambda \neq 0, \mu \neq 0, \rho = 0, c = \lambda\mu\lambda_0.$

v. $\lambda = \mu = 0, \rho \neq 0, c = 0.$

vi. $\lambda \neq 0, \mu = 0, \rho \neq 0, c = 0.$

vii. $\lambda = 0, \mu \neq 0, \rho \neq 0, c = -\mu\rho + \mu\rho\lambda_0.$

4.4. A perturbed Bott algebraic Schouten soliton of G_4

Lemma 4. *The Ricci tensor $\rho^{\tilde{B}_2}$ related to the connection $\nabla^{\tilde{B}_2}$ of (G_4, g) is presented as follows:*

$$\rho^{\tilde{B}_2}(\tilde{e}_i, \tilde{e}_j) = \begin{pmatrix} -(\mu-\sigma)^2 & -\frac{1}{2}a_0 & 0 \\ -\frac{1}{2}a_0 & 0 & \frac{1}{2}\lambda \\ 0 & \frac{1}{2}\lambda & \lambda\mu+1 \end{pmatrix}.$$

From this lemma, the subsequent theorem is deduced.

Theorem 11. *(G_4, g) is not an algebraic Schouten soliton associated with the perturbed Bott connection $\nabla^{\tilde{B}_2}$.*

Proof. According to [19], the expression for $Ric^{\tilde{B}_2}$ is derived as follows:

$$Ric^{\tilde{B}_2}\begin{pmatrix}\tilde{e_1}\\ \tilde{e_2}\\ \tilde{e_3}\end{pmatrix}=\begin{pmatrix}-(\mu-\sigma)^2 & -\frac{1}{2}a_0 & 0\\ -\frac{1}{2}a_0 & 0 & -\frac{1}{2}\lambda\\ 0 & \frac{1}{2}\lambda & -\lambda\mu-1\end{pmatrix}\begin{pmatrix}\tilde{e_1}\\ \tilde{e_2}\\ \tilde{e_3}\end{pmatrix}.$$

The scalar curvature is $s^{\tilde{B}_2}=-((\mu-\eta)^2+\lambda\mu+1)$. Now, we can express $D^{\tilde{B}_2}$ as follows:

$$\begin{cases}D^{\tilde{B}_2}\tilde{e_1}=-\left((\mu-\sigma)^2+s^{\tilde{B}_2}\lambda_0+c\right)\tilde{e_1}-\frac{1}{2}a_0\tilde{e_2},\\ D^{\tilde{B}_2}e_2=-\frac{1}{2}a_0\tilde{e_1}-\left(s^{\tilde{B}_2}\lambda_0+c\right)\tilde{e_2}-\frac{1}{2}\lambda\tilde{e_3},\\ D^{\tilde{B}_2}\tilde{e_3}=\frac{1}{2}\lambda\tilde{e_2}-\left(\lambda\mu+1+\left(s^{\tilde{B}_2}\lambda_0+c\right)\right)e_3.\end{cases}$$

Consequently, in light of Eq (2.4), an algebraic Schouten soliton corresponding to $\nabla^{\tilde{B}_2}$ exists on the manifold (G_4, g) if and only if the subsequent condition holds:

$$\begin{cases}\frac{1}{2}a_0=0,\\ \frac{1}{2}a_0(\lambda-\mu)=0,\\ (\mu-\sigma)^2-((\mu-\sigma)^2+\lambda\mu+1)\lambda_0+c-\lambda(\sigma-\mu)=0,\\ (2\sigma-\mu)((\mu-\sigma)^2-\lambda\mu-1-((\mu-\sigma)^2+\lambda\mu+1)\lambda_0+c)+\lambda=0,\\ \mu((\mu-\sigma)^2+\lambda\mu+1-((\mu-\sigma)^2+\lambda\mu+1)\lambda_0+c)-\lambda=0,\\ \lambda(\lambda\mu+1-(\mu-\sigma)^2-((\mu-\sigma)^2+\lambda\mu+1)\lambda_0+c)=0.\end{cases}\tag{4.2}$$

By solving (4.2), we have $a_0=0$, which is a contradiction.

4.5. A perturbed Bott algebraic Schouten soliton of G_5

Lemma 5. *The Ricci tensor $\rho^{\tilde{B}_2}$ related to the connection $\nabla^{\tilde{B}_2}$ of (G_5, g) is presented as follows:*

$$\rho^{\tilde{B}_2}(\tilde{e}_i,\tilde{e}_j)=\begin{pmatrix}\lambda^2 & 0 & 0\\ 0 & 0 & -\frac{1}{2}a_0\delta\\ 0 & -\frac{1}{2}a_0\delta & -(\mu\rho+\lambda^2)\end{pmatrix}.$$

From this lemma, the subsequent theorem is deduced.

Theorem 12. *(G_5, g) is not an algebraic Schouten soliton associated with the perturbed Bott connection $\nabla^{\tilde{B}_2}$.*

Proof. According to [19], the expression for $Ric^{\tilde{B}_2}$ is derived as follows:

$$Ric^{\tilde{B}_2}\begin{pmatrix}\tilde{e_1}\\ \tilde{e_2}\\ \tilde{e_3}\end{pmatrix}=\begin{pmatrix}\lambda^2 & 0 & 0\\ 0 & 0 & \frac{1}{2}a_0\delta\\ 0 & -\frac{1}{2}a_0\delta & (\mu\rho+\lambda^2)\end{pmatrix}\begin{pmatrix}\tilde{e_1}\\ \tilde{e_2}\\ \tilde{e_3}\end{pmatrix}.$$

The scalar curvature is $s^{\tilde{B_2}} = (2\lambda^2 + \mu\rho)$. Now, we can express $D^{\tilde{B_2}}$ as follows:

$$\begin{cases} D^{\tilde{B_2}}\tilde{e_1} = \left(\lambda^2 - (s^{\tilde{B_2}}\lambda_0 + c)\right)\tilde{e_1}, \\ D^{\tilde{B_2}}\tilde{e_2} = -\left(s^{\tilde{B_2}}\lambda_0 + c\right)\tilde{e_2} + \frac{1}{2}a_0\delta\tilde{e_3}, \\ D^{\tilde{B_2}}\tilde{e_3} = -\frac{1}{2}a_0\delta\tilde{e_2} + \left(\mu\rho + \lambda^2 - (s^{\tilde{B_2}}\lambda_0 + c)\right)\tilde{e_3}. \end{cases}$$

Consequently, in light of Eq (2.4), an algebraic Schouten soliton corresponding to $\nabla^{\tilde{B_2}}$ exists on the manifold (G_5, g) if and only if the subsequent condition holds:

$$\begin{cases} \lambda(\mu\rho + \lambda^2 - (\mu\rho + 2\lambda^2)\lambda_0 - c) = 0, \\ \mu(\mu\rho + 2\lambda^2 - (\mu\rho + 2\lambda^2)\lambda_0 - c) = 0, \\ \rho(\mu\rho - (\mu\rho + 2\lambda^2)\lambda_0 - c) = 0, \\ \sigma(\mu\rho + \lambda^2 - (\mu\rho + 2\lambda^2)\lambda_0 - c) = 0, \\ \lambda\left(\mu\rho + \lambda^2 - (\mu\rho + 2\lambda^2)\lambda_0 - c\right) - \frac{1}{2}a_0 r\theta = 0, \\ \mu\left(\mu\rho + \lambda^2 + (\mu\rho + 2\lambda^2)\lambda_0 - c\right) - \frac{1}{2}a_0\theta^2 = 0, \\ \mu\left(\mu\rho + 2\lambda^2 - 2(\mu\rho + 2\lambda^2)\lambda_0 - c\right) - \frac{1}{2}a_0\mu\theta = 0. \end{cases} \tag{4.3}$$

Recall that $\lambda + \theta \neq 0$ and $\lambda\rho - \mu\theta = 0$. Thus, we now analyze the system under different assumptions: Assume first that $\lambda = 0$. Then, the above system becomes:

$$\begin{cases} \mu(\mu\rho - \mu\rho\lambda_0 - c) - \frac{1}{2}a_0\theta = 0, \\ \mu(\mu\rho - 2\mu\rho\lambda_0 - c) - \frac{1}{2}a_0\mu\theta = 0, \\ \rho(\mu\rho - \mu\rho\lambda_0 - c) = 0, \\ \theta(\mu\rho - \mu\rho\lambda_0 - c) = 0. \end{cases}$$

We first analyze the third equation $\rho(\mu\rho - \mu\rho\lambda_0 - c) = 0$, which gives two cases: $\rho = 0$ or $\mu\rho(1-\lambda_0) - c = 0$, and then the fourth equation $\theta(\mu\rho - \mu\rho\lambda_0 - c) = 0$ also has two cases: $\theta = 0$ or $\mu\rho - \mu\rho\lambda_0 - c = 0$ (matching the second case of the third equation). Next, we discuss the value of μ: When $\mu = 0$, substituting into the first equation leads to $-\frac{1}{2}a_0\sigma = 0$, which is a contradiction. When $\mu \neq 0$, we get $\theta = 0$, which is a contradiction.

Assume that $\lambda \neq 0$ and $\theta = 0$, from the third equations in (4.3), we get $-\frac{1}{2}a_0\rho\theta = 0$ and $-\frac{1}{2}a_0\theta = 0$, which is a contradiction. When $\theta \neq 0$, from the first and third equations, we get $\lambda^3 = 0$, which is a contradiction.

4.6. A perturbed Bott algebraic Schouten soliton of G_6

Lemma 6. *The Ricci tensor $\rho^{\tilde{B_2}}$ related to the connection $\nabla^{\tilde{B_2}}$ of (G_6, g) is presented as follows:*

$$\rho^{\tilde{B_2}}(\tilde{e_i}, \tilde{e_j}) = \begin{pmatrix} -(\theta^2 + \mu\rho) & \frac{1}{2}a_0\lambda & 0 \\ \frac{1}{2}a_0\lambda & 0 & 0 \\ 0 & 0 & \theta^2 \end{pmatrix}.$$

From this lemma, the subsequent theorem is deduced.

Theorem 13. *If (G_6, g) constitutes an algebraic Schouten soliton associated with the perturbed Bott connection $\nabla^{\tilde{B}_2}$, then one of the following cases holds:*

i. $\lambda \neq 0, \mu = \theta = \rho = 0, c = 0.$

ii. $\lambda \neq 0, \mu \neq 0, \theta = \rho = 0, c = 0.$

Proof. According to [19], the expression for $Ric^{\tilde{B}_2}$ is derived as follows:

$$Ric^{\tilde{B}_2}\begin{pmatrix}\tilde{e_1}\\ \tilde{e_2}\\ \tilde{e_3}\end{pmatrix} = \begin{pmatrix}-(\theta^2+\mu\rho) & \frac{1}{2}a_0\lambda & 0\\ \frac{1}{2}a_0\lambda & 0 & 0\\ 0 & 0 & -\theta^2\end{pmatrix}\begin{pmatrix}\tilde{e_1}\\ \tilde{e_2}\\ \tilde{e_3}\end{pmatrix}.$$

The scalar curvature is $s^{\tilde{B}_2} = -(2\theta^2 + \mu\rho)$. Now, we can express $D^{\tilde{B}_2}$ as follows:

$$\begin{cases} D^{\tilde{B}_2}\tilde{e_1} = -(\theta^2 + \mu\rho + s^{\tilde{B}_2}\lambda_0 + c)\tilde{e_1} + \frac{1}{2}a_0\lambda\tilde{e_2},\\ D^{\tilde{B}_2}\tilde{e_2} = \frac{1}{2}a_0\lambda\tilde{e_1} - (s^{\tilde{B}_2}\lambda_0 + c)\tilde{e_2},\\ D^{\tilde{B}_2}\tilde{e_3} = -(\theta^2 + s^{\tilde{B}_2}\lambda_0 + c)\tilde{e_3}. \end{cases}$$

Consequently, in light of Eq (2.4), an algebraic Schouten soliton corresponding to $\nabla^{\tilde{B}_2}$ exists on the manifold (G_6, g) if and only if the subsequent condition holds:

$$\begin{cases} \lambda(\theta^2 + \mu\rho - (2\theta^2 + \mu\rho)\lambda_0 + c) = 0,\\ \mu(\mu\rho - (2\theta^2 + \mu\rho)\lambda_0 + c) = 0,\\ \rho(2\theta^2 + \mu\rho - (2\theta^2 + \mu\rho)\lambda_0 + c) = 0,\\ \theta(\theta^2 + \mu\rho - (2\theta^2 + \mu\rho)\lambda_0 + c) = 0,\\ \frac{1}{2}a_0\lambda^2 = 0,\\ \frac{1}{2}a_0\lambda\rho = 0,\\ \frac{1}{2}a_0\lambda\theta = 0. \end{cases} \tag{4.4}$$

From (4.4), we get $\lambda \neq 0$. Suppose $\theta = 0$, and from the equations in (4.4), we can derive that $\rho = 0$. Then, we have $c = 0$. Consequently, we have Case *i* and *ii*. If $\theta \neq 0$, which is a contradiction.

4.7. A perturbed Bott algebraic Schouten soliton of G_7

Lemma 7. *The Ricci tensor $\rho^{\tilde{B}_2}$ related to the connection $\nabla^{\tilde{B}_2}$ of (G_7, g) is presented as follows:*

$$\rho^{\tilde{B}_2}(\tilde{e}_i, \tilde{e}_j) = \begin{pmatrix}\lambda^2 & \frac{1}{2}(2\mu\lambda\theta - a_0\mu) & -\frac{1}{2}\mu(\theta-\lambda)\\ \frac{1}{2}(2\mu\lambda\theta - a_0\mu) & 0 & -\frac{1}{2}(2\theta^2 + \mu\rho + \lambda\theta - a_0\theta)\\ \frac{1}{2}\mu(\theta-\lambda) & \frac{1}{2}(2\theta^2 + \mu\rho + \lambda\theta - a_0\theta) & -\mu^2 + \lambda^2 + \mu\rho\end{pmatrix}\begin{pmatrix}\tilde{e_1}\\ \tilde{e_2}\\ \tilde{e_3}\end{pmatrix}.$$

From this lemma, the subsequent theorem is deduced.

Theorem 14. *If (G_7, g) constitutes an algebraic Schouten soliton associated with the perturbed Bott connection $\nabla^{\tilde{B}_2}$, then one of the following cases holds:*

i. $\lambda = \mu = \rho = 0, \theta \neq 0,$ *for all* c.

ii. $\lambda = \mu = 0, \rho \neq 0, \theta \neq 0, c = -s^B\lambda_0$.

iii. $\lambda = 0, \mu \neq 0, \theta \neq 0, c = \frac{1}{2}\theta^2 - s^B\lambda_0$.

iv. $\lambda \neq 0, \rho = \mu = 0, c = -s^B\lambda_0 + \frac{1}{2}\lambda^2$.

Proof. According to [19], the expression for $Ric^{\tilde{B_2}}$ is derived as follows:

$$Ric^{\tilde{B_2}}\begin{pmatrix}\tilde{e_1}\\ \tilde{e_2}\\ \tilde{e_3}\end{pmatrix} = \begin{pmatrix}\lambda^2 & \frac{1}{2}(2\mu\lambda\theta - a_0\mu) & \frac{1}{2}\mu(\theta-\lambda)\\ \frac{1}{2}(2\mu\lambda\theta - a_0\mu) & 0 & \frac{1}{2}(2\theta^2+\mu\rho+\lambda\theta-a_0\theta)\\ \frac{1}{2}\mu(\theta-\lambda) & \frac{1}{2}(2\theta^2+\mu\rho+\lambda\theta-a_0\theta) & \mu^2-\lambda^2-\mu\rho\end{pmatrix}\begin{pmatrix}\tilde{e_1}\\ \tilde{e_2}\\ \tilde{e_3}\end{pmatrix}.$$

The scalar curvature is $s^{\tilde{B_2}} = 2\lambda^2 - \mu^2 + \mu\rho$. Now, we can express $D^{\tilde{B_2}}$ as follows:

$$\begin{cases} D^{\tilde{B_2}}\tilde{e_1} = (\lambda^2 - (s^{\tilde{B_2}}\lambda_0 + c))\tilde{e_1} + \frac{1}{2}(2\lambda\mu\theta - a_0\mu)\tilde{e_2} - \frac{1}{2}\mu(\theta-\lambda)\tilde{e_3},\\ D^{\tilde{B_2}}\tilde{e_2} = \frac{1}{2}(2\lambda\mu\theta - a_0\mu)\tilde{e_1} - (s^{\tilde{B_2}}\lambda_0 + c)\tilde{e_2} - \frac{1}{2}(2\theta^2+\lambda\theta+\mu\rho-a_0\theta)\tilde{e_3},\\ D^{\tilde{B_2}}\tilde{e_3} = \frac{1}{2}\mu(\theta-\lambda)\tilde{e_1} + \frac{1}{2}(2\theta^2+\lambda\theta+\mu\rho-a_0\theta)\tilde{e_2} + (\lambda^2-\mu^2+\mu\rho-(s^{\tilde{B_2}}\lambda_0+c))\tilde{e_3}. \end{cases}$$

Consequently, in light of Eq (2.4), an algebraic Schouten soliton corresponding to $\nabla^{\tilde{B_2}}$ exists on the manifold (G_7, g) if and only if the subsequent condition holds:

$$\begin{cases} \lambda(s^B\lambda_0+c) + \frac{1}{2}\mu(\mu+\rho)(\theta-\lambda) + \frac{1}{2}\mu^2(2\lambda\theta-a_0) - \frac{1}{2}\lambda(2\theta^2+\lambda\theta^2+\mu\rho-a_0\theta) = 0,\\ \mu(-\lambda^2-(s^B\lambda_0+c)) + \frac{1}{2}\mu\theta(\theta-\lambda) + \frac{1}{2}\lambda\mu(2\rho\theta-a_0) = 0,\\ \mu(-\mu^2+\mu\rho+s^B\lambda_0+c) + \frac{1}{2}\mu(\theta-\lambda)^2 - \mu(\theta^2+\lambda\theta+\mu\rho-a_0\theta) = 0,\\ \lambda(\lambda^2+\mu\rho-\mu^2-s^B\lambda_0-c) + \frac{1}{2}\mu(\rho-\mu)(2\lambda\theta-a_0) - \frac{1}{2}\mu^2(\theta-\lambda) - \frac{1}{2}\lambda(2\theta^2+\lambda\theta+\mu\rho-a_0\theta) = 0,\\ \mu(2\lambda^2+\mu\rho-\mu^2-s^B\lambda_0-c) + \frac{1}{2}\mu(\theta-\lambda)(2\lambda\theta-a_0) - \mu(2\theta^2+\lambda\theta+\mu\rho-a_0\theta) = 0,\\ \mu(\lambda^2-s^B\lambda_0-c) + \frac{1}{2}\mu\theta(2\lambda\theta-a_0) + \frac{1}{2}\lambda\mu(\theta-\lambda) = 0,\\ \rho(-\mu+\mu\rho-s^B\lambda_0-c) + \frac{1}{2}\mu(\lambda-\theta)(2\lambda\theta-a_0) - \frac{1}{2}\mu(\lambda-\theta)^2 = 0,\\ \theta(\lambda^2-\mu^2+\mu\rho-s^B\lambda_0-c) + \frac{1}{2}\mu(\mu-\rho)(2\lambda\theta-a_0) + \frac{1}{2}\mu^2(\theta-\lambda) - \frac{1}{2}\rho(2\theta^2+\lambda\theta+\mu\rho-a_0\theta) = 0,\\ -\theta(s^B\lambda_0+c) + \frac{1}{2}\mu^2(2\lambda\theta-a_0) + \frac{1}{2}\mu(\mu+\rho)(\theta-\lambda) + \frac{1}{2}\rho(2\theta^2+\lambda\theta+\mu\rho-a_0\theta) = 0. \end{cases} \tag{4.5}$$

Since $\lambda + \rho = 0$ and $\lambda + \theta \neq 0$, we analyze the system under different assumptions.

First, if $\lambda = 0, \mu = 0, \theta \neq 0$, the last three expressions in Eq (4.5) can be transformed into:

$$\begin{cases} \rho(-s^B\lambda_0 - c) = 0,\\ -\theta(s^B\lambda_0+c) - \frac{1}{2}\theta(2\theta^2-a_0\theta) = 0,\\ -\theta(s^B\lambda_0+c) + \frac{1}{2}\theta(2\theta^2-a_0\theta) = 0. \end{cases}$$

From the equations above, we have $\rho \neq 0, c = -s^B\lambda_0$. If $\mu \neq 0$, the first equations provide $\mu(\frac{1}{2}\theta^2 - s^B\lambda_0 - c) = 0$, we get $c = \frac{1}{2}\theta^2 - s^B\lambda_0$.

Second, if $\lambda \neq 0$ and $\rho = 0$. The seventh equation gives rise to two possible subcases: $\mu = 0, \lambda = \theta$. Assume $\mu = 0$, we have:

$$\begin{cases} \lambda(s^B\lambda_0+c) - \frac{1}{2}\lambda(2\theta^2+\lambda\theta-a_0\theta) = 0,\\ \lambda(\lambda^2-s^B\lambda_0-c) - \frac{1}{2}\lambda(2\theta^2+\lambda\theta-a_0\theta) = 0,\\ \theta(-s^B\lambda_0-c) + \frac{1}{2}\lambda(2\theta^2+\lambda\theta-a_0\theta) = 0,\\ \theta(\lambda^2-s^B\lambda_0-c) - \frac{1}{2}\lambda(2\theta^2+\lambda\theta-a_0\theta) = 0. \end{cases}$$

From the equations above, we have $c = -s^B\lambda_0 + \frac{1}{2}\lambda^2$. Next, we consider the subcase where $\lambda = \theta$, after simple calculation, which leads to a contradiction.

5. An algebraic Schouten soliton concerning connection $\nabla^{\tilde{B}_3}$

In this section, we present the algebraic criterion that is necessary for a three-dimensional Lorentzian Lie group to possess an algebraic Schouten soliton associated with the perturbed Bott connection $\nabla^{\tilde{B}_3}$. Next, we introduce the perturbed Bott connection $\nabla^{\tilde{B}_3}$ for the second distribution. The Bott connection $\nabla^{\tilde{B}_3}$ associated with distribution D_3 is defined as follows:

$$\nabla^{B_3}_X Y = \begin{cases} \pi_{D_3}(\nabla^L_X Y), & X, Y \in \Gamma^\infty(D_3), \\ \pi_{D_3}([X,Y]), & X \in \Gamma^\infty(D_3{}^\perp),\ Y \in \Gamma^\infty(D_3), \\ \pi_{D_3{}^\perp}([X,Y]), & X \in \Gamma^\infty(D_3),\ Y \in \Gamma^\infty(D_3{}^\perp), \\ \pi_{D_3{}^\perp}(\nabla^L_X Y), & X, Y \in \Gamma^\infty(D_3^\perp), \end{cases}$$

where π_{D_3} (resp. $\pi_{D_3^\perp}$) is the projection on D_3 (resp. $D_3^\perp$). Then, we consider the perturbed Bott connection $\nabla^{\tilde{B}_3}$. Let $\tilde{e_3}^*$ be the dual basis of e_3, and define on $G_{i=1,...,7}$

$$\nabla^{\tilde{B}_3}_X Y = \nabla^{B_3}_X Y + a_0 \tilde{e}_1^*(X)\tilde{e_1}^*(Y)\tilde{e}_1,$$

where a_0 is a non-zero number. Then, $\nabla^{\tilde{B}_3}_{\tilde{e}_1}\tilde{e}_1 = a_0\tilde{e}_1$ and $\nabla^{\tilde{B}_3}_{\tilde{e}_s}\tilde{e}_t = \nabla^{B_3}_{\tilde{e}_s}\tilde{e}_t$, where $(s,t) \neq (1,1)$.

5.1. A perturbed Bott algebraic Schouten soliton of G_1

Lemma 8. *The Ricci tensor $\rho^{\tilde{B}_3}$ related to the connection $\nabla^{\tilde{B}_3}$ of (G_1, g) is presented as follows:*

$$\rho^{\tilde{B}_3}(\tilde{e}_i, \tilde{e}_j) = \begin{pmatrix} 0 & \frac{1}{2}(\lambda\mu - a_0\lambda) & -\frac{1}{2}(\lambda\mu - a_0\lambda) \\ \frac{1}{2}(\lambda\mu - a_0\lambda) & -\mu^2 & 0 \\ -\frac{1}{2}(\lambda\mu - a_0\lambda) & 0 & \mu^2 \end{pmatrix}.$$

From this lemma, the subsequent theorem is deduced.

Theorem 15. (G_1, g) *is not an algebraic Schouten soliton associated with the perturbed Bott connection $\nabla^{\tilde{B}_3}$.*

Proof. According to [19], the expression for $Ric^{\tilde{B}_3}$ is derived as follows:

$$Ric^{\tilde{B}_3}\begin{pmatrix} \tilde{e_1} \\ \tilde{e_2} \\ \tilde{e_3} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}(\lambda\mu - a_0\lambda) & \frac{1}{2}(\lambda\mu - a_0\lambda) \\ \frac{1}{2}(\lambda\mu - a_0\lambda) & -\mu^2 & 0 \\ -\frac{1}{2}(\lambda\mu - a_0\lambda) & 0 & -\mu^2 \end{pmatrix}\begin{pmatrix} \tilde{e_1} \\ \tilde{e_2} \\ \tilde{e_3} \end{pmatrix}.$$

The scalar curvature is $s^{\tilde{B}_3} = -2\mu^2$. Now, we can express $D^{\tilde{B}_3}$ as follows:

$$\begin{cases} D^{\tilde{B}_3}\tilde{e_1} = -(s^{\tilde{B}_3}\lambda_0 + c)\tilde{e_1} + \frac{1}{2}(\lambda\mu - a_0\lambda)\tilde{e_2} + \frac{1}{2}(\lambda\mu - a_0\lambda\tilde{e_3}, \\ D^{\tilde{B}_3}\tilde{e_2} = \frac{1}{2}(\lambda\mu - a_0\lambda)\tilde{e_1} - (\mu^2 + s^{\tilde{B}_3}\lambda_0 + c)\tilde{e_2}, \\ D^{\tilde{B}_3}\tilde{e_3} = \frac{1}{2}(a_0\lambda - \lambda\mu)\tilde{e_1} - (\mu^2 + s^{\tilde{B}_3}\lambda_0 + c)\tilde{e_3}. \end{cases}$$

Consequently, in light of Eq (2.4), an algebraic Schouten soliton corresponding to $\nabla^{\tilde{B}_3}$ exists on the manifold (G_1, g) if and only if the subsequent condition holds:

$$\begin{cases} \lambda(\mu^2 - 2\mu^2\lambda_0 + c) + \lambda\mu^2 - a_0\sigma\mu = 0, \\ \lambda^2\mu - a_0\lambda^2 = 0, \\ \theta(-2\mu^2\lambda_0 + c) - \lambda\mu^2 + a_0\lambda^2 = 0, \\ \theta(-2\mu^2\lambda_0 + c) + \lambda\mu^2 - a_0\lambda^2 = 0, \\ \theta(2\mu^2 - 2\mu^2\lambda_0 + c) = 0, \\ \lambda(\mu^2 - 2\mu^2\lambda_0 + c) + \lambda\mu^2 - a_0\lambda^2 = 0. \end{cases} \tag{5.1}$$

Since $\lambda \neq 0$, we assume that $\mu = 0$, and from the second equations in (5.1), we get $a_0 = 0$, which is a contradiction. If $\mu \neq 0$, after simple calculation, we have $2\mu(-2\mu^2\lambda_0 + c) = 0$, which is a contradiction.

5.2. A perturbed Bott algebraic Schouten soliton of G_2

Lemma 9. *The Ricci tensor $\rho^{\tilde{B}_3}$ concerning connection $\nabla^{\tilde{B}_3}$ of (G_2, g) is given by:*

$$\rho^{\tilde{B}_3}(\tilde{e}_i, \tilde{e}_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\lambda\mu & -\lambda\rho \\ 0 & -\lambda\rho & \lambda\mu \end{pmatrix}.$$

The existence conditions for Schouten solitons in this part are consistent with those under the Bott connection. For the sake of ensuring the completeness of the paper, the conclusion is provided herein (refer to [13] for the specific proof process).

Theorem 16. *If (G_2, g) constitutes an algebraic Schouten soliton concerning connection $\nabla^{\tilde{B}_3}$, then one of the following cases holds:*

i. $\lambda = 0, \mu = 0, c = 0.$

ii. $\lambda = 0, \mu \neq 0, c = 0.$

5.3. A perturbed Bott algebraic Schouten soliton of G_3

Lemma 10. *The Ricci tensor $\rho^{\tilde{B}_3}$ concerning connection $\nabla^{\tilde{B}_3}$ of (G_3, g) is given by:*

$$\rho^{B_3}(e_i, e_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda\mu \end{pmatrix}.$$

The existence conditions for Schouten solitons in this part are consistent with those under the Bott connection. For the sake of ensuring the completeness of the paper, the conclusion is provided herein (refer to [13] for the specific proof process).

Theorem 17. *If (G_3, g) constitutes an algebraic Schouten soliton concerning connection $\nabla^{\tilde{B}_3}$, then one of the following cases holds:*

i. $\lambda = \mu = \rho = 0$, *for all c.*

ii. $\lambda = \rho = 0, \mu \neq 0, c = 0.$

iii. $\lambda \neq 0, \mu = \rho = 0, c = 0.$

iv. $\lambda \neq 0, \mu \neq 0, \rho = 0, c = -\lambda\mu + \lambda\mu\lambda_0.$

v. $\lambda = \mu = 0, \rho \neq 0, c = 0.$

5.4. A perturbed Bott algebraic Schouten soliton of G_4

Lemma 11. *The Ricci tensor $\rho^{\tilde{B}_3}$ concerning connection $\nabla^{\tilde{B}_3}$ of (G_4, g) is given by:*

$$\rho^{\tilde{B}_3}(\tilde{e}_i, \tilde{e}_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda(2\sigma - \mu) & \lambda \\ 0 & \lambda & \lambda\mu \end{pmatrix}.$$

The existence conditions for Schouten solitons in this part are consistent with those under the Bott connection. For the sake of ensuring the completeness of the paper, the conclusion is provided herein (refer to [13] for the specific proof process).

Theorem 18. *If (G_4, g) constitutes an algebraic Schouten soliton concerning connection $\nabla^{\tilde{B}_3}$, then one of the following cases holds:*

i. $\lambda = 0, c = 0.$

ii. $\lambda \neq 0, \mu = \sigma, c = 0.$

5.5. A perturbed Bott algebraic Schouten soliton of G_5

Lemma 12. *The Ricci tensor $\rho^{\tilde{B}_3}$ related to the connection $\nabla^{\tilde{B}_3}$ of (G_5, g) is presented as follows:*

$$\rho^{\tilde{B}_3}(\tilde{e}_i, \tilde{e}_j) = \begin{pmatrix} 0 & -\frac{1}{2}a_0\lambda & 0 \\ -\frac{1}{2}a_0\lambda & \theta^2 & 0 \\ 0 & 0 & -(\mu\rho + \theta^2) \end{pmatrix}.$$

From this lemma, the subsequent theorem is deduced.

Theorem 19. *If (G_5, g) constitutes an algebraic Schouten soliton concerning connection $\nabla^{\tilde{B}_3}$, then one of the following cases holds:*

i. $\lambda = \mu = \rho = 0, \theta \neq 0, c = \theta^2 - 2\theta^2\lambda_0.$

ii. $\lambda \neq 0, \mu = \theta = \rho = 0, c = 0.$

iii. $\lambda \neq 0, \mu \neq 0, \theta = \rho = 0, c = 0.$

iv. $\lambda \neq 0, \mu = \rho = 0, \theta \neq 0, c = \theta^2 - 2\theta^2\lambda_0.$

Proof. According to [19], the expression for $Ric^{\tilde{B}_3}$ is derived as follows:

$$Ric^{\tilde{B}_3}\begin{pmatrix}\tilde{e_1}\\ \tilde{e_2}\\ \tilde{e_3}\end{pmatrix}=\begin{pmatrix}0 & -\frac{1}{2}a_0\lambda & 0\\ -\frac{1}{2}a_0\lambda & \theta^2 & 0\\ 0 & 0 & \mu\rho+\theta^2\end{pmatrix}\begin{pmatrix}\tilde{e_1}\\ \tilde{e_2}\\ \tilde{e_3}\end{pmatrix}.$$

The scalar curvature is $s^{\tilde{B}_3}=\mu\rho+2\theta^2$. Now, we can express $D^{\tilde{B}_3}$ as follows:

$$\begin{cases}D^{\tilde{B}_3}\tilde{e_1}=-(s^{\tilde{B}_3}\lambda_0+c)\tilde{e_1}-\frac{1}{2}a_0\lambda\tilde{e_2},\\ D^{\tilde{B}_3}\tilde{e_2}=-\frac{1}{2}a_0\lambda\tilde{e_1}-(\theta^2+s^{\tilde{B}_3}\lambda_0+c)\tilde{e_2},\\ D^{\tilde{B}_3}\tilde{e_3}=(\mu\rho+\theta^2-s^{\tilde{B}_3}\lambda_0-c)\tilde{e_3}.\end{cases}$$

Consequently, in light of Eq (2.4), an algebraic Schouten soliton corresponding to $\nabla^{\tilde{B}_3}$ exists on the manifold (G_5,g) if and only if the subsequent condition holds:

$$\begin{cases}\lambda(\mu\rho+\theta^2-(\mu\rho+2\theta^2)\lambda_0-c)+\frac{1}{2}a_0\lambda(\mu-\rho)=0,\\ \mu(\mu\rho-(\mu\rho+2\theta^2)\lambda_0-c)+\frac{1}{2}a_0\lambda(\lambda-\theta)=0,\\ \rho(\mu\rho+2\theta^2-(\mu\rho+2\theta^2)\lambda_0-c)+\frac{1}{2}a_0\lambda(\theta-\lambda)=0,\\ \theta(\mu\rho+\theta^2-(\mu\rho+2\theta^2)\lambda_0-c)+\frac{1}{2}a_0\lambda(\rho-\mu)=0.\end{cases}$$

Assume first that $\lambda=0$, and then we have $\mu=0$ and $\theta\neq 0$. In this case, we get:

$$\begin{cases}\rho(2\theta^2-2\theta^2\lambda_0-c)=0,\\ \theta(\theta^2-2\theta^2-c)=0.\end{cases}$$

Therefore, we conclude that $\rho=0$.

Second, we assume that $\lambda\neq 0$. If $\theta=\rho=0$, then $c=0$. If $\rho=0$ and $\theta\neq 0$, then $\mu=0$.

5.6. A perturbed Bott algebraic Schouten soliton of G_6

Lemma 13. *The Ricci tensor $\rho^{\tilde{B}_3}$ concerning connection $\nabla^{\tilde{B}_3}$ of (G_6,g) is given by:*

$$\rho^{\tilde{B}_3}(\tilde{e}_i,\tilde{e}_j)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}.$$

The existence conditions for Schouten solitons in this part are consistent with those under the Bott connection. For the sake of ensuring the completeness of the paper, the conclusion is provided herein (refer to [13] for the specific proof process).

Theorem 20. *If (G_6,g) constitutes an algebraic Schouten soliton concerning connection $\nabla^{\tilde{B}_3}$, then we have $c=0$.*

5.7. A perturbed Bott algebraic Schouten soliton of G_7

Lemma 14. *The Ricci tensor $\rho^{\tilde{B}_3}$ related to the connection $\nabla^{\tilde{B}_3}$ of (G_7, g) is presented as follows:*

$$\rho^{\tilde{B}_3}(\tilde{e}_i, \tilde{e}_j) = \begin{pmatrix} 0 & \frac{1}{2}a_0\lambda & 0 \\ \frac{1}{2}a_0\lambda & -\mu\rho & -\frac{1}{2}a_0\lambda \\ 0 & -\frac{1}{2}a_0\lambda & -\mu\rho \end{pmatrix}.$$

Theorem 21. *If (G_7, g) constitutes an algebraic Schouten soliton concerning connection $\nabla^{\tilde{B}_3}$, then one of the following cases holds:*

i. $\lambda = \mu = 0, \rho \neq 0, c = 0.$

ii. $\lambda \neq 0, \rho = 0, c = 0.$

iii. $\lambda = \rho = 0, c = 0.$

Proof. According to [19], the expression for $Ric^{\tilde{B}_3}$ is derived as follows:

$$Ric^{\tilde{B}_3}\begin{pmatrix} \tilde{e}_1 \\ \tilde{e}_2 \\ \tilde{e}_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}a_0\lambda & 0 \\ \frac{1}{2}a_0\lambda & -\mu\rho & \frac{1}{2}a_0\lambda \\ 0 & -\frac{1}{2}a_0\lambda & \mu\rho \end{pmatrix}\begin{pmatrix} \tilde{e}_1 \\ \tilde{e}_2 \\ \tilde{e}_3 \end{pmatrix}.$$

The scalar curvature is $s^{\tilde{B}_3} = 0$. Now, we can express $D^{\tilde{B}_3}$ as follows:

$$\begin{cases} D^{\tilde{B}_3}\tilde{e}_1 = -(s^{\tilde{B}_3}\lambda_0 + c)\tilde{e}_1 + \frac{1}{2}a_0\tilde{e}_2, \\ D^{\tilde{B}_3}\tilde{e}_2 = \frac{1}{2}a_0\lambda\tilde{e}_1 - (\mu\rho + s^{\tilde{B}_3}\lambda_0 + c)\tilde{e}_2 + \frac{1}{2}a_0\lambda\tilde{e}_3, \\ D^{\tilde{B}_3}\tilde{e}_3 = -\frac{1}{2}a_0\lambda\tilde{e}_2 + (\mu\rho - s^{\tilde{B}_3}\lambda_0 - c)\tilde{e}_3. \end{cases}$$

Consequently, in light of Eq (2.4), an algebraic Schouten soliton corresponding to $\nabla^{\tilde{B}_3}$ exists on the manifold (G_7, g) if and only if the subsequent condition holds:

$$\begin{cases} \lambda(\mu\rho + c) + \frac{1}{2}a_0\lambda^2 = 0, \\ \mu c + \frac{1}{2}a_0\lambda(\mu + \lambda) = 0, \\ \mu(2\mu\rho + c) + a_0\lambda\mu = 0, \\ \lambda(\mu r - c) + \frac{1}{2}a_0\lambda\rho = 0, \\ \mu c - 2\mu^2 r + \frac{1}{2}a_0\lambda(\lambda - \theta) = 0, \\ \mu c + \frac{1}{2}a_0\lambda(\mu - \theta) = 0, \\ \rho c = 0, \\ \theta(-\mu\rho + c) + \frac{1}{2}a_0\lambda\rho = 0, \\ \theta(\mu r + c) + \frac{1}{2}a_0\lambda\theta = 0. \end{cases}$$

Given that $\lambda\rho = 0$ and $\lambda + \theta \neq 0$, we proceed to analyze the system under distinct suppositions.

First, when $\lambda = 0$ and $\rho \neq 0$, under this premise, which leads to $\mu = c = 0$. Second, if $\lambda \neq 0$ and $\rho = 0$, then we get $c = 0$. Last, in the case where $\lambda = \rho = 0$, it follows that $\theta \neq 0$, and then we get $c = 0$.

6. Conclusions

We establish algebraic criteria for three-dimensional Lorentzian Lie groups to qualify as algebraic Schouten solitons associated with the perturbed Bott connection, taking three distributions into account. The key finding shows that G_3, G_5, G_6, and G_7 have perturbed Bott algebraic Schouten soliton under the first distribution. G_3, G_6, and G_7 have such solitons under the second distribution. Additionally, G_2, G_3, G_4, G_5, G_6, and G_7 have such solitons under the third distribution. In the coming time, we will investigate algebraic Schouten solitons in higher dimensions, similar to what is done in [20, 21].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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