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Theory article

Dynamic analysis of SIR epidemic model based on feedback control

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Abstract: In this paper, we study the SIR model and give a strategy to control the epidemic. Based on discrete-time observation, the linear control term is added and the upper bound of time delay is given in order to make the epidemic disappear. Both the deterministic and stochastic cases are considered. The novelty of this paper lies in the fact that it only controls for the infected population. Numerical examples verify the theory results.

Keywords: feedback control; SIR; extinction; stochastic models; noise

1. Introduction

It is well-known that the classical SIR model reads

$$\begin{cases} \frac{dS(t)}{\mathrm{dt}} = \Lambda - \beta S(t)I(t) - \gamma S(t), \\ \frac{dI(t)}{\mathrm{dt}} = \beta S(t)I(t) - \mu I(t) - \gamma I(t), \\ \frac{dR(t)}{\mathrm{dt}} = \mu I(t) - \gamma R(t), \end{cases} \tag{1.1}$$

where $S(t)$, $I(t)$, and $R(t)$ denote the susceptible population, infectious population, and recovered population, respectively. Λ, μ, β, and γ are the entering rate, the rate of recovery, the infection rate, and the death rate, respectively. The SIR model has been studied by many authors. Now we only review some of the many works. [1] gave some difference scheme for the SIR model; [2] obtained the global analysis of a network-based SIR epidemic model with a saturated treatment function; geometric analysis is applied in the SIRS model with secondary infections [3]; [4] studied the nonlocal infectious SIR epidemic models; [5] considered a nonlocal diffusive SIR epidemic model with nonlocal infection; [6] obtained the Turing-Hopf bifurcation and inhomogeneous pattern for a reaction-diffusion SIR epidemic model with chemotaxis and delay; [7] studied a SI epidemic-like propagation model with non-smooth control; [8] considered the spatiotemporal dynamics analysis and optimal control method for an SI reaction-diffusion propagation model. The threshold of SIR model has been studied by many authors. [9] got the threshold of a stochastic SIRS epidemic model with a general incidence; [10] obtained the threshold of a stochastic SIRS epidemic model with a general incidence; [11] studied the fractional SIR model and obtained the threshold. Apart from the above references, the stability analysis of the rumor propagation model has been studied by [12, 13].

For the stochastic SIR model, [14, 15] considered the SIR model driven by Brownian motion and Lévy noise; [16] studied the stochastic SIR model on random bond-diluted lattices; [17] is interested in stochastic multi-group epidemic SVIR models, also see [18]; [19] considered the final and peak epidemic sizes of immuno-epidemiological SIR models. From the viewpoint of the control point, [20] studied the optimal control of the SIR-w infectious disease model; also see [21]. Recently, statistical methods have been used to study the population models: [22] considered the green behavior propagation analysis based on statistical theory and intelligent algorithms in a data-driven environment; [23] studied the West Nile virus spatiotemporal models based on higher-order network topology. In our further work, we will try to study epidemic models using this method.

From a medical perspective, we hope to control the spread of diseases. There are many methods to control the spread of diseases. Among those methods, feedback control is very efficient. Azouani & Titi introduced a simple finite-dimensional feedback control scheme for stabilizing solutions of infinite-dimensional dissipative evolution equations [24]

$$\begin{cases} \frac{\partial u}{\partial t} - \nu\Delta u - \beta u + u^3 = 0, \\ u|_{x=0,L} = 0, \end{cases}$$

where $\alpha, \beta, \nu > 0$. Azouani et al. used a similar idea to consider the data assimilation issue, which is to tackle the challenge of approximating the state of a specific physical system; see [25]. The method is used to study data assimilation for nonlocal diffusion equations and stochastic equations [26, 27]. The idea of time-delay feedback control is to add control terms (usually linear control terms) to the equation, and then use perturbation methods to prove that when the time-delay is very small, the solution of the equation will tend to zero. In this paper, we will use the point of feedback control to consider the SIR model with the aim of making the disease disappear.

It is well known that the basic reproduction number

$$R_0 = \frac{\beta\Lambda}{\gamma(\mu+\gamma)}.$$

is the standard for determining whether a disease is prevalent. More precisely, when $R_0 < 1$, then the disease will disappear; when $R_0 > 1$, then the disease will be popular. Thus, the role of the basic reproduction number is so important. If we want to reduce the value of R_0, we can let β become smaller or let γ and μ be larger. However, considering the significance in reality, it is impossible to let γ and μ be larger. Since the meaning of β is infection rate, we can use isolation, medication, and other methods to reduce the infection rate. Of course, this will also cause certain damage to the economy and scope of activities. What level of control is needed to reach the critical value, which is difficult to achieve in real life. In this paper, ***borrowing the idea of feedback control, we provide a feasible strategy***.

More precisely, let $\delta(t) = [\frac{t}{\tau}]\tau$, τ is the observation time interval, and we consider the following system

$$\begin{cases} \frac{dS(t)}{\mathrm{dt}} = \Lambda - \beta S(t)I(t) - \gamma S(t), \\ \frac{dI(t)}{\mathrm{dt}} = \beta S(t)I(t) - \mu I(t) - \gamma I(t) - kI(\delta(t)), \\ \frac{dR(t)}{\mathrm{dt}} = \mu I(t) - \gamma R(t), \end{cases} \tag{1.2}$$

where $[a]$ is the integer part of a. [28] first introduced the discrete-time stochastic feedback control to stabilize the solution of ordinary differential equations; [29] stabilized the highly nonlinear hybrid systems by feedback control based on discrete-time state observations; [26] considered the fractional reaction-diffusion equations and obtained the continuous data assimilation by using feedback control; moreover, [27] obtained the data assimilation algorithm for evolution equations based on discrete-time observation. The meaning of the control term $-kI(\delta(t))$ is that the infectious population should be isolated. Indeed, the result is certainly right. The reason is as follows. Consider

$$\begin{cases} \frac{dS(t)}{\mathrm{dt}} = \Lambda - \beta S(t)I(t) - \gamma S(t) \\ \frac{dI(t)}{\mathrm{dt}} = \beta S(t)I(t) - \mu I(t) - \gamma I(t) - kI(t), \\ \frac{dR(t)}{\mathrm{dt}} = \mu I(t) - \gamma R(t), \end{cases} \tag{1.3}$$

then we know that the basic reproduction number is changed as

$$\bar{R}_0 = \frac{\beta\Lambda}{\gamma(\mu + \gamma + k)}.$$

Therefore, if $k > 0$ is large enough, we have $\hat{R}_0 < 1$, and so the disease will disappear. By using the continuous method, we have that if the disease in system (1.3) disappears, then for small enough τ, the disease in system (1.2) will also disappear. In this paper, the upper bound of τ will be given. Therefore, ***a strategy for the disease to disappear is given***.

Notice that in the real world, there are many factors that have an effect on disease. It is well-known that Brownian motion is a continuous process, which can be used to describe many kinds of perturbation. In the epidemic model, the Brownian motion $B(t)$ represents a certain degree of uncertainty. As a result, the parameter β should be random:

$$\beta dt \to \beta dt + \sigma dB(t).$$

Then system (1.1) becomes

$$\begin{cases} dS(t) = [\Lambda - \beta S(t)I(t) - \gamma S(t)]\mathrm{dt} - \sigma \mathrm{S(t)I(t)dB(t)}, \\ dI(t) = [\beta S(t)I(t) - \mu I(t) - \gamma I(t) - kI(\delta(t))]\mathrm{dt} + \sigma \mathrm{S(t)I(t)dB(t)}, \\ \frac{dR(t)}{\mathrm{dt}} = \mu I(t) - \gamma R(t). \end{cases} \tag{1.4}$$

We will use a similar idea to deal with the stochastic case (1.4). But there is a significant difference from the deterministic case; see Lemma 2.2.

Contributions:

1) The sufficient condition making disease extinction is given for both deterministic case and stochastic case;

2) A new strategy that the disease disappears is given;

3) The difference between deterministic case and stochastic case is given.

In the next section, we consider the deterministic case. In Section 3, the stochastic case is studied. In the last section, some numerical examples are given.

2. Deterministic SIR model

In this section, we consider the deterministic SIR model

$$\begin{cases} \frac{dS(t)}{\mathrm{dt}} = \Lambda - \beta S(t)I(t) - \gamma S(t), \\ \frac{dI(t)}{\mathrm{dt}} = \beta S(t)I(t) - \mu I(t) - \gamma I(t) - kI(\delta(t)), \\ \frac{dR(t)}{\mathrm{dt}} = \mu I(t) - \gamma R(t), \end{cases} \tag{2.1}$$

where k is a positive constant. In order to make the disease extinct, we firstly study the auxiliary system

$$\begin{cases} \frac{d\tilde{S}(t)}{\mathrm{dt}} = \Lambda - \beta \tilde{S}(t)\tilde{I}(t) - \gamma \tilde{S}(t), \\ \frac{d\tilde{I}(t)}{\mathrm{dt}} = \beta \tilde{S}(t)\tilde{I}(t) - \mu \tilde{I}(t) - \gamma \tilde{I}(t) - k\tilde{I}(t), \\ \frac{d\tilde{R}(t)}{\mathrm{dt}} = \mu \tilde{I}(t) - \gamma \tilde{R}(t), \end{cases} \tag{2.2}$$

It follows from the result of [1] that Eq (2.2) admits a unique positive solution. The spread of infectious diseases involves a large number of short-term random factors, such as the onset time of individual cases and occasional clustering in local areas. Point-by-point existence (such as the number of cases at a certain moment) will be masked by these noises, making it difficult to see the true trend. Time averaging can filter short-term fluctuations and highlight the core characteristics of disease transmission, such as epidemic peaks and spread speeds. In order to do that, denote

$$\langle x(t) \rangle = \frac{1}{t} \int_0^t x(s)\mathrm{ds}.$$

Assume the initial data of (2.1) is $(S(0), I(0), R(0))$; we have the first result.

Lemma 2.1. *Let $k > 0$ satisfy $\beta\Lambda \leq \gamma(\mu + \gamma + k)$. Then the solution of (2.2) satisfies*

$$\lim_{t\to 0} \frac{\tilde{S}(t)}{t} = \lim_{t\to 0} \frac{\tilde{I}(t)}{t} = \lim_{t\to 0} \frac{\tilde{R}(t)}{t} = 0, \ \lim_{t\to\infty} \langle \tilde{S}(t) \rangle = \frac{\Lambda}{\gamma}, \ \lim_{t\to\infty} \tilde{I}(t) = \lim_{t\to\infty} \tilde{R}(t) = 0.$$

More precisely, for large enough t, $\tilde{I}(t) \leq e^{(\mu+\eta+k)(\hat{R}_0 - 1)t}$.

Proof. Let $\tilde{Z}(t) = \tilde{S}(t) + \tilde{I}(t) + \tilde{R}(t)$; we have

$$\begin{aligned} \frac{d\tilde{Z}(t)}{\mathrm{dt}} &= \Lambda - \gamma \tilde{Z}(t) - k\tilde{I}(t) \\ &\leq \Lambda - \gamma \tilde{Z}(t), \end{aligned}$$

which yields that

$$\tilde{Z}(t) \leq \tilde{Z}(0) + \frac{\Lambda}{\gamma}(1 - e^{-\gamma t}) < \infty, \ \ \forall t \geq 0.$$

Thus

$$\frac{\tilde{S}(t)}{t} \vee \frac{\tilde{I}(t)}{t} \vee \frac{\tilde{R}(t)}{t} \leq \frac{\tilde{S}(t) + \tilde{I}(t)}{t} \leq \frac{S(0) + I(0) + R(0)}{t} + \frac{\Lambda}{\gamma t}(1 - e^{-\gamma t}) \to 0. \tag{2.3}$$

Integrating Eq (2.2), we have

$$\begin{aligned}
& \frac{\tilde{S}(t) + \tilde{I}(t) + \tilde{R}(t)}{t} - \frac{S(0) + I(0) + R(0)}{t} \\
= & \frac{\int_0^t (\Lambda - \gamma \tilde{S}(r) - (\gamma + k)\tilde{I}(r) - \gamma \tilde{R}(r))\mathrm{dr}}{t} \\
= & \Lambda - \frac{\gamma}{t}\int_0^t \tilde{S}(r)\mathrm{dr} - \frac{\gamma + \mathrm{k}}{\mathrm{t}}\int_0^{\mathrm{t}} \tilde{\mathrm{I}}(\mathrm{r})\mathrm{dr} - \frac{\gamma}{\mathrm{t}}\int_0^{\mathrm{t}} \tilde{\mathrm{R}}(\mathrm{r})\mathrm{dr} \\
= & \Lambda - \gamma\langle \tilde{S}(t)\rangle - (\gamma + k)\langle \tilde{I}(t)\rangle - \gamma\langle \tilde{R}(t)\rangle.
\end{aligned}$$

Combining (2.3), we get

$$\langle \tilde{S}(t)\rangle = \frac{\Lambda}{\gamma} - \frac{\gamma + k}{\gamma}\langle \tilde{I}(t)\rangle - \langle \tilde{R}(t)\rangle - \frac{1}{\gamma}\phi(t), \tag{2.4}$$

where $\phi(t) = \frac{\tilde{S}(t)+\tilde{I}(t)+\tilde{R}(t)}{t} - \frac{S(0)+I(0)+R(0)}{t}$. On the other hand, it holds that

$$(\ln \tilde{I}(t))' = \beta\tilde{S}(t) - (\mu + \gamma + k). \tag{2.5}$$

Integrating (2.5) and using (2.4), we have

$$\begin{aligned}
\frac{\ln \tilde{I}(t)}{t} &= \beta\langle \tilde{S}(t)\rangle - (\mu + \eta + k) + \frac{\ln I_0}{t} \\
&= \frac{\beta\Lambda}{\gamma} - \frac{\gamma + k}{\gamma}\langle \tilde{I}(t)\rangle - \langle \tilde{R}(t)\rangle - (\mu + \gamma + k) + \frac{\ln I(0)}{t} - \frac{1}{\mu}\phi(t) \\
&\leq (\mu + \gamma + k)(\hat{R}_0 - 1) + \frac{\ln I(0)}{t} - \frac{1}{\mu}\phi(t).
\end{aligned}$$

Notice that $\lim_{t\to\infty}\left(\frac{\ln I(0)}{t} - \frac{1}{\mu}\phi(t)\right) = 0$; we immediately get that

$$\limsup_{t\to\infty} \frac{\ln \tilde{I}(t)}{t} \leq (\mu + \gamma + k)(\hat{R}_0 - 1) < 0,$$

which yields that

$$\tilde{I}(t) \leq e^{(\mu+\gamma+k)(\hat{R}_0-1)t}, \quad for\ t \gg 1 \Rightarrow \lim_{t\to\infty} \tilde{I}(t) = 0 \Rightarrow \lim_{t\to\infty}\langle \tilde{I}(t)\rangle = 0. \tag{2.6}$$

Submitting (2.6) into (2.4), we obtain $\lim_{t\to\infty}\langle \tilde{S}(t)\rangle = \frac{\Lambda}{\gamma}$. Note that

$$\tilde{R}(t) = \int_0^t e^{-\gamma(t-s)} I(s)\mathrm{ds} + \mathrm{e}^{-\gamma \mathrm{t}}\mathrm{I}(0),$$

we have $\lim_{t\to\infty}\tilde{R}(t) = 0$. The proof is complete. □

Next, we consider Eq (2.1). If $0 \le S(0) < \frac{A}{\mu}$, then we have

$$S'(t) \le \Lambda - \gamma S(t) \Rightarrow S(t) \le S(0)e^{-\gamma t} + \frac{\Lambda}{\gamma}(1 - e^{-\gamma t}) \le \frac{\Lambda}{\gamma}.$$

Then we have the following relationship between the solutions of (2.1) and (2.2).

Lemma 2.2. *Assume that* $0 \le S(0) + I(0) + R(0) \le \frac{\Lambda}{\gamma}$. *Then*

$$|I(t) - \tilde{I}(t)| \le \frac{k\vartheta_1}{\vartheta_2} I(0)\tau \left[e^{\frac{\beta\Lambda}{\gamma}t} - 1\right],$$

where

$$\vartheta_1 = \frac{\beta\Lambda}{\mu} + \mu + \gamma + k, \quad \vartheta_2 = \mu + \gamma + \frac{2\beta\Lambda}{\gamma}.$$

Proof. It follows from (2.1) that

$$\begin{aligned} [S(t) + I(t) + R(t)]' &= \Lambda - \gamma[S(t) + I(t) + R(t)] - kI(\delta(t)) \\ &\le \Lambda - \gamma[S(t) + I(t) + R(t)] \end{aligned}$$

which yields that

$$S(t) + I(t) + R(t) \le (S(0) + I(0) + R(0))e^{-\gamma t} + \frac{\Lambda}{\gamma}(1 - e^{-\gamma t}) \le \frac{\Lambda}{\gamma},$$

provided that $0 \le S(0) + I(0) + R(0) \le \frac{\Lambda}{\gamma}$. Note that $S(t), I(t), R(t) \ge 0$; we have $S(t), I(t), R(t) \le \frac{A}{\mu}$. Using the second equation of (2.1), we get

$$\begin{aligned} I(t) &= I(0)e^{-(\mu+\gamma)t} + \beta\int_0^t e^{-(\mu+\gamma)(t-\tau)}S(r)I(r)\mathrm{d}r - \mathrm{k}\int_0^{\mathrm{t}} \mathrm{e}^{-(\mu+\gamma)(\mathrm{t-r})}\mathrm{I}(\delta(\mathrm{r}))\mathrm{dr} \\ &\le I(0)e^{-(\mu+\gamma)t} + \frac{\beta\Lambda}{\gamma}\int_0^t e^{-(\mu+\gamma)(t-r)}I(r)\mathrm{d}r - \mathrm{k}\int_0^{\mathrm{t}} \mathrm{e}^{-(\mu+\gamma)(\mathrm{t-r})}\mathrm{I}(\delta(\mathrm{r}))\mathrm{dr}, \end{aligned}$$

which implies that

$$\begin{aligned} \max_{0\le r\le t} I(r) &\le I(0)e^{-(\mu+\gamma)t} + \frac{\beta\Lambda}{\gamma}\int_0^t e^{-(\mu+\gamma)(t-\iota)} \max_{0\le r\le \iota} I(r)\mathrm{d}\iota \\ &\le I(0) + \frac{\beta\Lambda}{\gamma}\int_0^t \max_{0\le r\le \iota} I(r)\mathrm{d}\iota. \end{aligned}$$

The Gronwall's inequality yields

$$\max_{0\le r\le t} I(r) \le I(0)e^{\frac{\beta\Lambda}{\gamma}t}. \tag{2.7}$$

Consequently, we have

$$|I(t) - I(\delta(t))| = |\int_{\delta(t)}^t \mathrm{dI(r)}|$$

$$
\begin{aligned}
&= |\int_{\delta(t)}^{t}[\beta S(r)I(r)-(\mu+\gamma)I(r)-kI(\delta(r))]\,\mathrm{dr} \\
&\leq I(0)\tau\vartheta_1 e^{\frac{\beta\Lambda}{\gamma}t},
\end{aligned}
\tag{2.8}
$$

where $\vartheta_1 = \frac{\beta\Lambda}{\mu} + \mu + \gamma + k$. It follows from (2.1) and (2.2) that

$$
[S(t)-\tilde{S}(t)]' = -\beta(S(t)I(t)-\tilde{S}(t)\tilde{I}(t)) - \gamma(S(t)-\tilde{S}(t))
$$

and

$$
\begin{aligned}
[I(t)-\tilde{I}(t)]' &= \beta(S(t)I(t)-\tilde{S}(t)\tilde{I}(t)) - (\mu+\gamma)(I(t)-\tilde{I}(t)) \\
&\quad -k(I(\delta(t))-\tilde{I}(t)),
\end{aligned}
$$

which imply

$$
\begin{aligned}
&|S(t)-\tilde{S}(t)| + |I(t)-\tilde{I}(t)| \\
\leq\ & \left(\mu+\gamma+\frac{2\beta\Lambda}{\gamma}\right)\int_0^t |S(r)-\tilde{S}(r)| + |I(r)-\tilde{I}(r)|\mathrm{dr} \\
&+k\int_0^t |I(r)-I(\delta(r))|\mathrm{dr}.
\end{aligned}
$$

The Gronwall's inequality yields that

$$
|S(t)-\tilde{S}(t)| + |I(t)-\tilde{I}(t)| \leq \frac{k\vartheta_1}{\mu+\gamma+\frac{2\beta\Lambda}{\gamma}} I(0)\tau\left[e^{\frac{\beta\Lambda}{\gamma}t}-1\right].
$$

The proof is complete. □

In the proof of Lemma 2.2, the information of $R(t)$ is not used, which is similar to the rumor model [30]. Next, we will obtain the main result of the deterministic case.

Theorem 2.1. *Let the assumptions of Lemma 2.2 hold. Then there exists a $\tau^* > 0$ such that for any $\tau \in (0, \tau^*)$, the solution of (2.1) will decay exponentially, where τ^* is the unique root of*

$$
\frac{k\vartheta_1}{\vartheta_2}\tau\left(e^{\frac{\beta\Lambda}{\gamma}\left(\tau+\log(\frac{1}{\varepsilon})/\nu\right)}-1\right) = 1-\varepsilon,
\tag{2.9}
$$

and $\varepsilon \in (0, 1)$, $\nu = (\mu + \eta + k)(1 - \hat{R}_0)$.

Proof. Notice that the left-hand side of (2.9) is a continuously increasing function of $\tau \geq 0$ and equals zero when $\tau = 0$. Thus (2.9) admits a unique root τ^*. Set $\tau \in (0, \tau^*)$. Take a positive integer m such that

$$
\frac{\log(\frac{1}{\varepsilon})}{\nu\tau} \leq m < 1 + \frac{\log(\frac{1}{\varepsilon})}{\nu\tau},
$$

where ν is defined as in Theorem 2.1. Thus $e^{-\nu m\tau} \leq \varepsilon$. Lemma 2.1 yields that

$$
|\tilde{I}(m\tau)| \leq I(0)e^{-\nu m\tau}, \quad t \geq 0.
$$

Notice that

$$\begin{aligned}|I(m\tau)| &\leq |\tilde{I}(m\tau)| + |I(m\tau) - \tilde{I}(m\tau)| \\ &\leq I(0)\left[\varepsilon + \frac{k\vartheta_1}{\vartheta_2}\tau\left(e^{\frac{\beta\Lambda}{\gamma}m\tau} - 1\right)\right].\end{aligned}$$

It follows from the definition of m that

$$\varepsilon + \frac{k\vartheta_1}{\vartheta_2}\tau\left(e^{\frac{\beta\Lambda}{\gamma}m\tau} - 1\right) \leq \varepsilon + \frac{k\vartheta_1}{\vartheta_2}\tau\left(e^{\frac{\beta\Lambda}{\gamma}\left(\tau+\log(\frac{1}{\varepsilon})/\nu\right)} - 1\right) < 1.$$

Thus, we can rewrite

$$\varepsilon + \frac{k\vartheta_1}{\vartheta_2}\tau\left(e^{\frac{\beta\Lambda}{\gamma}\left(\tau+\log(\frac{1}{\varepsilon})/\nu\right)} - 1\right) = e^{-\lambda\bar{k}\tau}.$$

Consequently, we obtain

$$|I(m\tau)| \leq e^{-\lambda m\tau}.$$

Due to the time-homogeneous property of system (2.1), we get

$$|I(im\tau)| \leq |I((i-1)m\tau)||e^{-\lambda m\tau} \leq e^{-\lambda im\tau}, \quad \forall i = 1, 2, \ldots.$$

(2.7) yields that

$$\max_{0\leq r\leq m\tau} |I(s)| \leq NI(0), \quad t \geq 0.$$

where $N = e^{\frac{\beta\Lambda}{\gamma}m\tau}$. Similarly, we have

$$\max_{im\tau\leq r\leq(i+1)m\tau} |I(r)| \leq N|I(im\tau)|, \quad t \geq 0.$$

Hence, for any $t > 0$, there exists i such that $im\tau \leq t \leq (i+1)m\tau$ and

$$|I(t)| \leq Ne^{-\lambda t}, \quad t \geq 0.$$

The proof is complete. □

3. Stochastic case

In this section, we will consider the stochastic rumor model

$$\begin{cases} dS(t) = [\Lambda - \beta S(t)I(t) - \gamma S(t)]\mathrm{dt} - \sigma \mathrm{S(t)I(t)dB(t)}, \\ dI(t) = [\beta S(t)I(t) - \mu I(t) - \gamma I(t) - kI(\delta(t))]\mathrm{dt} + \sigma \mathrm{S(t)I(t)dB(t)}, \\ \frac{dR(t)}{\mathrm{dt}} = \mu I(t) - \gamma R(t), \end{cases} \tag{3.1}$$

where $\delta(t)$ is defined as in (1.2), $B(t)$ is a standard Brownian motion, and $k > 0$. We assume that when $k = 0$, then the basic reproduction number $R_0 > 1$. Let $k > 0$ such that

$$\hat{R}_0 = \frac{\beta\Lambda}{\gamma(\mu + \gamma + k)} < 1.$$

The classical results show that when $k = 0$, the disease will be popular, and when $k > 0$ and $\delta(t) = t$, the disease will disappear. Similar to the deterministic case, we first study the system

$$\begin{cases} d\tilde{S}(t) = [\Lambda - \beta\tilde{S}(t)\tilde{I}(t) - \gamma\tilde{S}(t)]\mathrm{dt} - \sigma\tilde{\mathrm{S}}(\mathrm{t})\tilde{\mathrm{I}}(\mathrm{t})\mathrm{dB}(\mathrm{t}), \\ d\tilde{I}(t) = [\beta\tilde{S}(t)\tilde{I}(t) - (\mu + \gamma)\tilde{I}(t) - k\tilde{I}(t)]\mathrm{dt} + \sigma\tilde{\mathrm{S}}(\mathrm{t})\tilde{\mathrm{I}}(\mathrm{t})\mathrm{dB}(\mathrm{t}), \\ \frac{dR(t)}{\mathrm{dt}} = \mu I(t) - \gamma R(t). \end{cases} \tag{3.2}$$

It follows from the classical results [1] that system (3.2) has the following property.

Proposition 3.1. *Let $(\tilde{S}(t), \tilde{I}(t))$ be the solution of system (3.2) with any initial value $S(0) > 0$ and $I(0) > 0$. If $\hat{R}_0 < 1$, then*

$$\lim_{t\to\infty} \frac{\ln \tilde{I}(t)}{t} \leq (\mu + \eta)(\hat{R}_0 - 1) < 0 \;\; a.s.$$

In addition,

$$\lim_{t\to\infty} \langle \tilde{S}(t) \rangle = \frac{\Lambda}{\gamma}, \;\; a.s.$$

That is to say, $I(t)$ tends to zero exponentially a.s.

We will use the method of Section 2 to prove the disease will disappear almost surely for system (3.1). Let

$$\varpi(t) = t - k\tau, \quad \text{for } t \in [k\tau, (k+1)\tau), \;\; k = 0, 1, \ldots,$$

then we have $\delta(t) = t - \varpi(t)$. Thus, system (3.1) can be rewritten as

$$\begin{cases} dS(t) = [\Lambda - \beta S(t)I(t) - \gamma S(t)]\mathrm{dt} - \sigma\mathrm{S}(\mathrm{t})\mathrm{I}(\mathrm{t})\mathrm{dB}(\mathrm{t}), \\ dI(t) = [\beta S(t)I(t) - \mu I(t) - \gamma I(t) - kI(t - \varpi(t))]\mathrm{dt} + \sigma\mathrm{S}(\mathrm{t})\mathrm{I}(\mathrm{t})\mathrm{dB}(\mathrm{t}), \\ \frac{dR(t)}{\mathrm{dt}} = \mu I(t) - \gamma R(t), \end{cases} \tag{3.3}$$

where $\varpi(t) \in [0, \tau]$. Obviously, systems (3.1) and (3.3) are equivalent. In fact, it is easy to prove the existence of a solution to system (3.3), and we only give the outline of proof.

Proposition 3.2. *Let $S(0) > 0$ and $I(\varsigma) \geq 0$ for all $\varsigma \in [-\tau, 0)$ with $I(0) > 0$. Then system (3.3) admits a unique positive solution $(S(t), I(t), R(t))$ on $t > 0$ and the solution will remain in $\mathbb{R}^2_+$ with probability one.*

Proof. We only give the outline of the proof; see [31, Theorem 3.1] for more details. Note that if system (3.3) satisfies the local Lipschitz condition, then a unique local positive solution $(S(t), I(t))$ exists on $t \in [-\tau, \tau_e)$, where τ_e represents the explosion time ([32]). To prove the solution is global, we only need to show $\tau_e = \infty$ a.s. Note that the first two equations are independent of the third equation. So we only consider the first two equations. Let's define $V(S, I) = \ln(SI)$. By using Itô's formula the Lyapunov method, one can complete the proof. □

Now, similar to Lemma 2.2, we calculate the difference between $I(t)$ and $\tilde{I}(t)$.

Lemma 3.1. *Let the assumptions of Proposition 3.2 hold and assume that* $0 \le S(0)+I(0)+R(0) \le \frac{\Lambda}{\gamma}$. *For* $0 < p < 2$*, we have*

$$\mathbb{E}\left[|I(t)-\tilde{I}(t)|^p\right] \le K_2^p e^{(K_1+1)tp}\tau^p, \tag{3.4}$$

where

$$\begin{aligned} K_1 &= \frac{2\Lambda}{\gamma}\left(2\beta + 2\sigma^2\frac{\Lambda}{\gamma} + k - \gamma\right), \\ K_2 &= \frac{k\Lambda}{\gamma}\left(\frac{\sigma^2\Lambda^3}{\gamma^3} + \frac{\beta\Lambda}{\gamma} + \mu + \gamma + k\right). \end{aligned}$$

Proof. Similar to Lemma 2.2, from (3.1), we have

$$\begin{aligned} d[S(t)+I(t)] &= [\Lambda - \gamma[S(t)+I(t)] - \mu I(t) - kI(\delta(t))]\mathrm{dt} \\ &\le \Lambda\mathrm{dt} - \gamma[\mathrm{S(t)}+\mathrm{I(t)}]\mathrm{dt}, \end{aligned}$$

almost surely, which implies that

$$S(t)+I(t) \le (S(0)+I(0))e^{-\gamma t} + \frac{\Lambda}{\gamma}(1-e^{-\gamma t}) \le \frac{\Lambda}{\gamma},\ a.s.$$

It follows from $S(t), I(t) \ge 0$ that $S(t), I(t) \le \frac{\Lambda}{\gamma}$. We can also obtain $\tilde{S}(t), \tilde{I}(t) \le \frac{\Lambda}{\gamma}$. The Itô formula implies that

$$\begin{aligned} d|S(t)-\tilde{S}(t)|^2 &= 2(S(t)-\tilde{S}(t))[-\beta(S(t)I(t)-\tilde{S}(t)\tilde{I}(t)) - \gamma(S(t)-\tilde{S}(t))]\mathrm{dt} \\ &\quad +\sigma^2(S(t)I(t)-\tilde{S}(t)\tilde{I}(t))^2\mathrm{dt} - 2\sigma(\mathrm{S(t)}-\tilde{\mathrm{S}}(\mathrm{t}))(\mathrm{S(t)I(t)}-\tilde{\mathrm{S}}(\mathrm{t})\tilde{\mathrm{I}}(\mathrm{t}))\mathrm{dB(t)} \\ &\le \frac{\Lambda}{\gamma}\left(3\beta + 2\sigma^2\frac{\Lambda}{\gamma} - 2\gamma\right)(S(t)-\tilde{S}(t))^2\mathrm{dt} + \frac{\Lambda}{\gamma}\left(\beta + 2\sigma^2\frac{\Lambda}{\gamma}\right)(\mathrm{I(t)}-\tilde{\mathrm{I}}(\mathrm{t}))^2\mathrm{dt} \\ &\quad -2\sigma(S(t)-\tilde{S}(t))(S(t)I(t)-\tilde{S}(t)\tilde{I}(t))\mathrm{dB(t)}, \\ d|I(t)-\tilde{I}(t)|^2 &= -2\beta(I(t)-\tilde{I}(t))(S(t)I(t)-\tilde{S}(t)\tilde{I}(t))\mathrm{dt} - 2(\mu+\gamma)(\mathrm{I(t)}-\tilde{\mathrm{I}}(\mathrm{t}))^2\mathrm{dt} \\ &\quad -2k(I(t)-\tilde{I}(t))(I(\delta(t))-\tilde{I}(t)) + \sigma^2(S(t)I(t)-\tilde{S}(t)\tilde{I}(t))^2\mathrm{dt} \\ &\quad +2\sigma(S(t)I(t)-\tilde{S}(t)\tilde{I}(t))\mathrm{dB(t)} \\ &\le \frac{\Lambda}{\gamma}\left(\beta + 2\sigma^2\frac{\Lambda}{\gamma}\right)(S(t)-\tilde{S}(t))^2\mathrm{dt} + \mathrm{k}(\mathrm{I(t)}-\mathrm{I}(\delta(\mathrm{t})))^2\mathrm{dt} \\ &\quad +\frac{\Lambda}{\gamma}\left(3\beta + 2\sigma^2\frac{\Lambda}{\gamma} + 2k - 2\mu - 2\gamma\right)(I(t)-\tilde{I}(t))^2\mathrm{dt} \\ &\quad +2\sigma(S(t)-\tilde{S}(t))(S(t)I(t)-\tilde{S}(t)\tilde{I}(t))\mathrm{dB(t)}, \end{aligned}$$

where the fact that $0 \le S(t), I(t), \tilde{S}(t), \tilde{I}(t) \le \frac{\Lambda}{\gamma}$ is used. Combining the above two inequalities and taking expectation, we have

$$\mathbb{E}\left[|S(t)-\tilde{S}(t)|^2 + |I(t)-\tilde{I}(t)|^2\right] \le K_1\int_0^t \mathbb{E}\left[|S(r)-\tilde{S}(r)|^2 + |I(r)-\tilde{I}(r)|^2\right]\mathrm{dr}$$

$$+k\int_0^t \mathbb{E}\left[|I(r)-I(\delta(r))|^2\right]\mathrm{dr},$$

where

$$K_1=\frac{2\Lambda}{\gamma}\left(2\beta+2\sigma^2\frac{\Lambda}{\gamma}+k-\gamma\right).$$

By using Gronwall's inequality, we get

$$\mathbb{E}\left[|S(t)-\tilde{S}(t)|^2+|I(t)-\tilde{I}(t)|^2\right]\le ke^{K_1t}\int_0^t \mathbb{E}\left[|I(r)-I(\delta(r))|^2\right]\mathrm{dr}. \tag{3.5}$$

The Hölder inequality and Itô isometry imply that

$$\begin{aligned}
&\mathbb{E}\left[|I(t)-I(\delta(t))|^2\right]\\
=\;&\mathbb{E}\Big|\int_{\delta(t)}^t[\beta S(r)I(r)-(\mu+\gamma)I(r)-kI(\delta(r))]\mathrm{dr}+\sigma \mathrm{S(r)I(r)dB(r)}\Big|^2\\
=\;&\mathbb{E}\Big|\int_{\delta(t)}^t[\beta S(r)I(r)-(\mu+\gamma)I(r)-kI(\delta(r))]\mathrm{dr}\Big|^2\\
&+\mathbb{E}\Big|\sigma\int_{\delta(t)}^t S(r)I(r)\mathrm{dB(r)}\Big|^2\\
\le\;&\frac{\Lambda}{\gamma}\left(\frac{\sigma^2\Lambda^3}{\gamma^3}+\frac{\beta\Lambda}{\gamma}+\mu+\gamma+k\right)\tau.
\end{aligned}$$

Submitting the above inequality into (3.5), we get

$$\mathbb{E}\left[|I(t)-\tilde{I}(t)|^2\right]\le K_2e^{(K_1+1)t}\tau,$$

where

$$K_2=\frac{k\Lambda}{\gamma}\left(\frac{\sigma^2\Lambda^3}{\gamma^3}+\frac{\beta\Lambda}{\gamma}+\mu+\gamma+k\right).$$

Then the Hölder inequality yields (3.4). The proof is complete. □

Next, we state the key lemma.

Lemma 3.2. *Let assumptions of Proposition 3.2 hold and $\tau^*>0$ be the unique root of the following equation:*

$$2^pK_2^pe^{(K_1+1)\left(p\tau+\ln\frac{2^p}{\varepsilon}/\gamma\right)}\tau^p=1-\varepsilon, \tag{3.6}$$

where K_i ($i=1,2$) are as defined in Lemma 3.1. For any $\tau\in(0,\tau^*)$, there exists a pair of positive constants $\bar{m}$ and λ such that for any initial value $0\le S(0)+I(0)\le\frac{\Lambda}{\gamma}$ and $\varepsilon\in(0,1)$, the solution of system (3.1) satisfies

$$\mathbb{E}\,|I(j\bar{m}\tau;I(0))|^p\le|S(0)+I(0)|^p(1+i\lambda\bar{m}\tau)^{-2},\quad\forall\ j=1,2,\cdots.$$

Proof. When p is fixed, the left side of Eq (3.6) is a continuous increasing function with respect to $\tau \geq 0$, and when $\tau = 0$, it is equal to 0. So Eq (3.6) has a unique positive root, which can be denoted as $\tau^* > 0$. For any fixed $\tau \in (0, \tau^*)$ and initial value $0 \leq S(0) + I(0) + R(0) \leq \frac{\Lambda}{\gamma}$. It follows from Proposition 3.1 that there exists a positive number m_* such that $m \geq m_*$, it holds that

$$\left|\tilde{I}(m\tau; I(0))\right| \leq |I(0) + S(0) + R(0)|e^{-\nu m\tau}, \tag{3.7}$$

where $\nu = \frac{(\mu+\gamma)(1-\hat{R}_0)}{2}$. Let $\varepsilon \in (0, 1)$, we take a large positive integer $\bar{m}$ satisfying (3.7) and

$$\frac{\ln\frac{2^p}{\varepsilon}}{\nu p\tau} \leq \bar{m} \leq \frac{\ln\frac{2^p}{\varepsilon}}{\nu p\tau} + 1, \tag{3.8}$$

which yields that

$$2^p e^{-\nu p\bar{m}\tau} \leq \varepsilon. \tag{3.9}$$

By using the Cauchy-Schwarz inequality, we get

$$\mathbb{E}|I(\bar{m}\tau; I(0))|^p \leq 2^p\mathbb{E}|\tilde{I}(\bar{m}\tau; I(0))|^p + 2^p\mathbb{E}|I(\bar{m}\tau; I(0)) - \tilde{I}(\bar{m}\tau; I(0))|^p.$$

Combining (3.7)–(3.9) and using lemma 3.1, we have

$$\begin{aligned}\mathbb{E}|I(\bar{m}\tau; I(0))|^p &\leq 2^p|I(0) + S(0)|^p e^{-\nu p\bar{m}\tau} + 2^p|I(0) + S(0)|^p k^p e^{(K+1)\bar{m}\tau p}\tau^p \\ &\leq |I(0) + S(0)|^p \left(\varepsilon + 2^p K_2^p e^{(K_1+1)\bar{m}\tau p}\tau^p\right).\end{aligned} \tag{3.10}$$

(3.8) implies that

$$e^{(K_1+1)\bar{m}\tau p} \leq e^{(K_1+1)\left(p\tau + \ln\frac{2^p}{\varepsilon}/\nu\right)}.$$

Consequently, we get

$$\varepsilon + 2^p K_2^p e^{(K_1+1)\bar{m}\tau p}\tau^p \leq \varepsilon + 2^p K_2^p e^{(K_1+1)\left(p\tau + \ln\frac{2^p}{\varepsilon}/\nu\right)}\tau^p < 1.$$

Thus, there exists $\lambda > 0$, such that

$$\varepsilon + 2^p K_2^p e^{(K_1+1)\left(p\tau + \ln\frac{2^p}{\varepsilon}/\nu\right)}\tau^p = e^{-\lambda\bar{m}\tau}.$$

Submitting the above inequality into (3.10), we get

$$\mathbb{E}|I(\bar{m}\tau; I(0))|^p \leq |I(0) + S(0)|^p e^{-\lambda\bar{m}\tau}.$$

Now, we investigate the solution $I(t)$ of system (3.1) on $t \geq \bar{m}\tau$, which can be seen as the solution of system (3.1) starting from time $t = \bar{m}\tau$ and state $I_{\bar{m}\tau}$. Therefore, from the time homogeneity of system (3.1), we obtain

$$\mathbb{E}\left(|I(2\bar{m}\tau; I(0))|^p|\mathcal{F}_{\bar{m}\tau}\right) \leq |I(\bar{m}\tau; I(0))|^p e^{-\lambda\bar{m}\tau}.$$

Therefore, we have

$$\mathbb{E}|I(2\bar{m}\tau; I(0))|^p \leq |S(0) + I(0)|^p e^{-2\lambda\bar{m}\tau}.$$

By repeating the above process, we can deduce

$$\begin{aligned}\mathbb{E}\,|I(j\bar{m}\tau; I(0))|^p &\leq \mathbb{E}\,|I\,((j-1)\bar{m}\tau; I(0))|^p\, e^{-\lambda\bar{m}\tau}\\ &\leq |S(0) + I(0)|^p e^{-\lambda j\bar{m}\tau}, \quad \forall\ j = 1, 2, \cdots .\end{aligned}$$

The lemma is proofed. □

Now, we get the following main result.

Theorem 3.1. *Let assumptions of Proposition 3.2 hold. There exists a positive constant* τ^**, such that for any initial value* $0 \leq S(0) + I(0) + R(0) \leq \frac{\Lambda}{\gamma}$*, if* $\tau \in (0, \tau^*)$*, the solution of (3.1) satisfies,*

$$\limsup_{t\to\infty} \frac{\log |I(t)|^p}{t} < 0 \quad \text{a.s.} \tag{3.11}$$

Proof. For any fixed $\tau \in (0, \tau^*)$ and initial value $0 \leq S(0) + I(0) + R(0) \leq \frac{\Lambda}{\gamma}$, we denote $I(t; I(0)) = I(t)$. Let $\bar{m}$ be defined as in Lemma 3.2. For $t \in [0, \bar{m}\tau]$, from (3.1) we have

$$I(t) = I(0) + \int_0^t [\beta S(r)I(r) - (\mu + \gamma)I(r)]\, \mathrm{dr} - \mathrm{k}\int_0^{\mathrm{t}} \mathrm{I}(\delta(\mathrm{r}))\mathrm{dr} + \sigma \int_0^{\mathrm{t}} \mathrm{S(r)I(r)dB(r)}.$$

By using Hölder inequality, Doob's inequality and Grönwall's inequality and combining the fact that $0 < S(t) + I(t) + R(t) \leq \frac{A}{\mu}$, we have

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\leq r\leq t} |I(r)|^2\right)\\ \leq\ & 3|I(0)|^2 + 3\sigma^2\mathbb{E}\left(\sup_{0\leq u\leq t}\left|\int_0^u S(r)I(r)\mathrm{dB(r)}\right|^2\right)\\ &+3\mathbb{E}\left(\sup_{0\leq u\leq t}\left|\int_0^u [\beta S(r)I(r) - (\mu+\eta)I(r)]\,\mathrm{dr}\right|^2\right)\\ \leq\ & 3|I(0)|^2 + \left[2\left(\frac{\beta\Lambda}{\gamma} + \mu + \gamma\right)^2 + \frac{12\sigma^2\Lambda^2}{\gamma^2}\right]\int_0^t \mathbb{E}\left(\sup_{0\leq u\leq r} |I(r)|^2\right)\mathrm{dr}.\end{aligned}$$

The Gronwall's inequality yields that

$$\mathbb{E}\left(\sup_{0\leq t\leq\bar{m}\tau} |I(t)|^2\right) \leq 3|I(0)|^2 e^{K_3\bar{m}\tau},$$

where

$$K_3 = 2\left(\frac{\beta\Lambda}{\gamma} + \mu + \gamma\right)^2 + \frac{12\sigma^2\Lambda^2}{\gamma^2}.$$

The Hölder inequality yields that

$$\mathbb{E}\left(\sup_{0\le t\le \bar{m}\tau}|I(t)|^p\right)\le\le 3^{\frac{p}{2}}|I(0)|^p e^{K_3 p\bar{m}\tau/2}\equiv K_4|I(0)|^p. \tag{3.12}$$

Now, we consider the solution $I(t)$ of (3.1) on the interval $t\in[j\bar{m}\tau,(j+1)\bar{m}\tau](j=1,2,\cdots)$, which can be regarded as the solution $(I(t); j\bar{k}\tau)$ of (3.1) starting from time $t=j\bar{m}\tau$ and state $I(j\bar{k}\tau)$. It follows from the time homogeneity of (3.1) and (3.12) that

$$\mathbb{E}\left(\sup_{j\bar{m}\tau\le t\le (j+1)\bar{m}\tau}|I(t)|^p|\mathcal{F}_{j\bar{m}\tau}\right)\le K_4\mathbb{E}|I(j\bar{m}\tau;I(0))|^p.$$

By combining Lemma 3.2, for $j\ge 1$, we get

$$\mathbb{E}\left(\sup_{j\bar{m}\tau\le t\le (j+1)\bar{m}\tau}|I(t)|^p\right)\le K_4\mathbb{E}|I(j\bar{m}\tau;I(0))|^p\le K_4|s_0+I(0)|^p. \tag{3.13}$$

Using Chebyshev inequality, for any $j\ge 1$, $\gamma_1>0$, we have

$$\mathbb{P}\left(\sup_{j\bar{m}\tau\le t\le (j+1)\bar{m}\tau}|I(t)|^p\ge e^{-0.5\lambda j\bar{m}\tau}\right)\le K_4|I(0)|^p e^{-0.5\lambda j\bar{m}\tau}.$$

According to Borel-Cantelli lemma, for almost all $\omega\in\Omega$, there exists an integer $j_0=j_0(\omega)$, $\forall j\ge j_0(\omega)$ such that

$$\sup_{j\bar{m}\tau\le t\le (j+1)\bar{m}\tau}|I(t)|^p< e^{-0.5\lambda j\bar{m}\tau}$$

holds. Let $t\to\infty$, we have

$$\limsup_{t\to\infty}\frac{\log|I(t,\omega)|^p}{t}\le-\frac{\lambda}{2p}<0.$$

The proof is complete. □

4. Examples

Let $\Lambda=1,\beta=0.3,\gamma=0.2,\mu=0.1$. Set $S(0)=1.2$, $I(0)=0.6$ and $R(0)=0.2$. Then we have

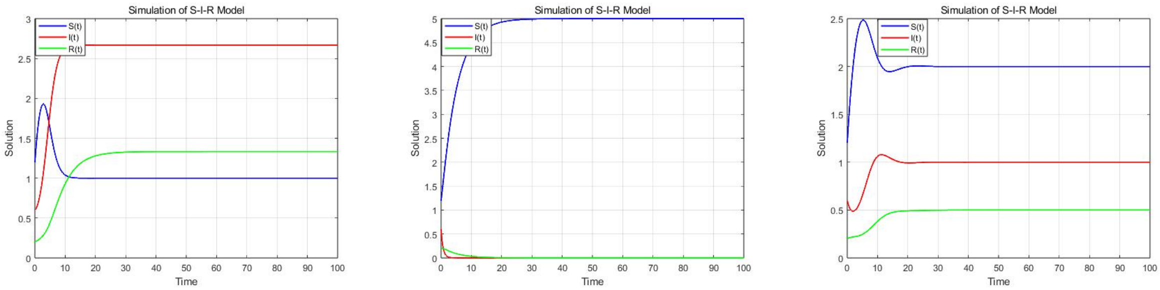
$$R_0=\frac{\beta\Lambda}{\gamma(\mu+\gamma)}=5>1.$$

It follows from the classical results of [1] that the disease will be popular; see Figure 1(a). It follows from the different k that

$$\hat{R}_0=\frac{\beta\Lambda}{\gamma(\mu+\gamma+k)}=\begin{cases}\frac{5}{2}>1, & \text{if } k=0.3,\\ \frac{90}{119}<1, & \text{if } k=\frac{5}{3}.\end{cases}$$

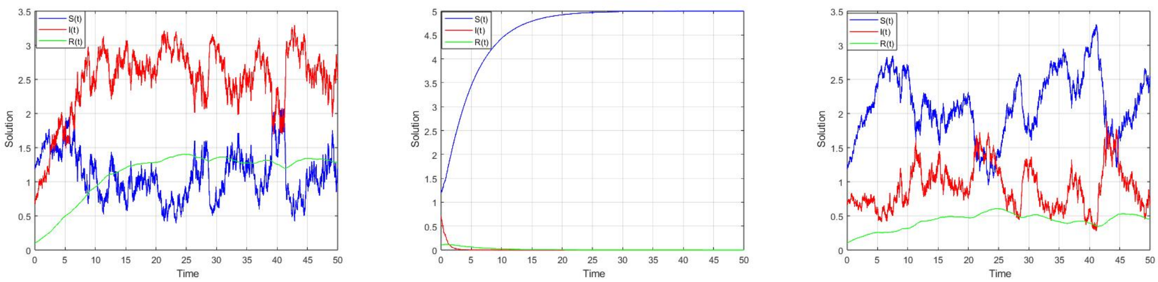
Set $\varepsilon=0.8$, then it is not difficult to get $\tau^*>0.024$. Let $\tau=0.024$ and $k=\frac{5}{3}$, then it is easy to check the assumptions of Theorem 2.1 hold. It follows from Theorem 2.1 that the rumor will disappear exponentially; see Figure 1(b). However, if $k=0.3$, the rumor will be popular; see Figure 1(c).

Set $S(0)=1.2$, $I(0)=0.6$, and $R(0)=0.2$. For the stochastic case, let $\sigma=0.15$ and keep the same parameters as in the deterministic case; if $k=0$, then the solution of (3.1) will be like Figure 3(a). Similar to the deterministic case, let $k=\frac{5}{3}$ and $k=0.3$; then the solution of (3.1) will be like Figure 2(b),(c), which verifies Theorem 3.1.



(a) (b) (c)

Figure 1. (a), (b), (c) denote the solutions of system (2.1) with $k = 0$, $k = \frac{5}{3}$ and $\tau = 0.024$, and $k = 0.3$ and $\tau = 0.024$, respectively.



(a) (b) (c)

Figure 2. (a), (b), (c) denote the solutions of system (3.1) with $k = 0$, $k = \frac{5}{3}$ and $\tau = 0.024$, and $k = 0.3$ and $\tau = 0.024$, respectively.

5. Conclusions

In this paper, we consider the classical SIR model and give a strategy to control the epidemic based on discrete time observation. More precisely, the linear control term is added and by using the perturbation method, we obtain the epidemic will disappear for deterministic and stochastic SIR models. The novelty of this paper lies in the fact that it only controls for the infected population.

The feedback control is often used to stabilize the system industrially. We borrow the idea to consider the epidemic model. In the real world, this strategy is easy to implement. In order to verify our results, numerical simulations are given.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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