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Research article

Global solutions for inhomogeneous micropolar equations with density-dependent coefficients and large data

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Abstract: In this paper, we consider the Dirichlet problem of three-dimensional inhomogeneous incompressible micropolar equations with density-dependent viscosity. Under the assumption that the coefficients are power functions of the density, we establish the global existence of strong solutions as long as the initial density is linearly equivalent to a large constant state. There is no restriction on the size of the initial velocity and micro-rotational velocity. As a byproduct, we prove the exponential decay for the solution.

Keywords: inhomogeneous incompressible micropolar equations; density-dependent coefficients; global strong solutions; large initial data; exponential decay

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. In this paper, we are concerned with an initial boundary-value problem of three-dimensional (3D for short) inhomogeneous micropolar fluid equations with density-dependent viscosity in $\Omega \times \mathbb{R}_+$:

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) + \nabla P - 2\mathrm{div}((\mu(\rho) + \xi(\rho))\mathcal{D}(u)) = 2\xi(\rho)\nabla \times w, \\ (\rho w)_t + \mathrm{div}(\rho u \otimes w) + 4\xi(\rho)w - \mathrm{div}\,(2\eta(\rho)\mathcal{D}(w)) - \nabla(\lambda(\rho)\mathrm{div} w) = 2\xi(\rho)\nabla \times u, \\ \mathrm{div} u = 0, \end{cases} \tag{1.1}$$

where $t \geq 0$ and $x = (x_1, x_2, x_3) \in \Omega$ are time and space variables, respectively. ρ, $u = (u^1, u^2, u^3)$, $w = (w^1, w^2, w^3)$, and P represent the density, velocity, micro-rotational velocity, and the pressure of the fluid, respectively.

$$\mathcal{D}(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^{\top}\right), \quad \mathcal{D}(w) = \frac{1}{2}\left(\nabla w + (\nabla w)^{\top}\right), \tag{1.2}$$

is the deformation tensor relying on u and w. $\mu(\rho)$ stands for the viscosity and is a function of the density, satisfying

$$\mu(\rho) = \bar{\mu}\rho^{\alpha},\ \bar{\mu} > 0,\ \alpha \geq 0. \tag{1.3}$$

The coefficients $\xi(\rho)$, $\eta(\rho)$, and $\lambda(\rho)$ are assumed to satisfy

$$\eta(\rho) = \bar{\eta}\rho^{\alpha},\ \lambda(\rho) = \bar{\lambda}\rho^{\alpha},\ \xi(\rho) = \bar{\xi}\rho^{\beta}, \quad \bar{\xi} > 0, \quad \bar{\eta} > 0, \quad 2\bar{\eta} + 3\bar{\lambda} \geq 0,\ \beta \geq 0. \tag{1.4}$$

In this paper, we study the initial boundary value problem to the system (1.1)–(1.4) with the following initial data and Dirichlet boundary condition:

$$\begin{aligned} &(\rho, u, w)(x, 0) = (\rho_0(x), u_0(x), w_0(x)), && \text{in } \Omega, \\ &u(x, t) = 0,\ w(x, t) = 0, && \text{on } \partial\Omega \times [0, T]. \end{aligned} \tag{1.5}$$

The micropolar equations were first proposed by Eringen [1] in 1966, which can describe many phenomena that appear in a large number of complex fluids such as suspensions, animal blood, and liquid crystals. For more background on micropolar fluids, we refer to [2–4] and the references therein. Because of their physical applications and mathematical significance, the well-posedness problem of the micropolar equations has attracted considerable attention in recent years. In 2019, Zhang and Zhu [5] considered the 3D Cauchy problem with constant viscosity, and proved the global well-posedness of strong and classical solutions under the condition that $\bar{\mu}$ is large enough or the initial data is sufficiently small. In 2023, Qian et al. [6] studied the global existence of weak and strong solutions. When initial velocities are sufficiently small in the critical Besov space and the initial density has positive lower and upper bounds, global Fujita-Kato-type solutions are obtained. When the viscosity depends on the density, Qian and Qu [7] studied the 3D Cauchy problem. By the assumption of the smallness of initial velocity in the critical Besov space, the authors obtained the local and global well-posedness of the system. Zhong [8] studied the 3D initial-boundary-value problem, and established the global well-posedness of strong solutions when the initial energy is sufficiently small. Moreover, the velocity and the micro-rotational velocity converge exponentially to zero in H^2 as time goes to infinity. Zhou and Tang [9] proved the global well-posedness of strong solutions to the 3D Cauchy problem, provided that the initial mass is sufficiently small. Moreover, the gradients of velocity and micro-rotational velocity converge exponentially to zero in H^1 as time goes to infinity. For more related results in the two-dimensional case, we refer to [10–14].

If there is no microstructure ($\xi = 0$ and $w = 0$), system (1.1) reduces to inhomogeneous incompressible Navier–Stokes equations. When the viscosity is constant, Choe and Kim [15] proved the well-posedness of local strong solutions to the Cauchy problem or the initial boundary value problem. In 2008, Germain [16] proved weak-strong uniqueness to the Cauchy problem. In 2013, Craig et al. [17] established the global well-posedness of strong solutions for initial data with small $\dot{H}^{\frac{1}{2}}$-norm. In 2016, Chen et al. [18] proved the global well-posedness of solution under the condition that the initial density is bounded and bounded away from zero, and the initial velocity is small enough in H^s for some $s > 1/2$. In 2021, Guo et al. [19] established global strong axisymmetric solutions in the exterior of a cylinder subject to the Dirichlet boundary conditions, where the vacuum is allowed. When the viscosity depends on the density, Huang and Wang [20] studied the 2D initial boundary value problem in bounded domains, and proved that strong solutions exist globally when $\|\nabla\mu(\rho_0)\|_{L^p}$ is sufficiently small, even in the presence of vacuum. In 2015, Liang [21] established the local well-posedness of strong solution for the 2D Cauchy problem, Huang and Wang [22] and Zhang [23] proved that the 3D initial boundary value problem admits unique strong solutions provided that $\|\nabla u_0\|_{L^2}$ is sufficiently small. For more related results in critical Besov spaces, we refer to [24–28] and the references therein.

Recently, Huang et al. [29] studied the Dirichlet problem of 3D inhomogeneous Navier-Stokes equations with density-dependent viscosity, and proved that the system admits a unique global strong solution as long as the initial density is sufficiently large. This is the first result concerning the existence of large strong solutions for the inhomogeneous Navier-Stokes equations in three dimensions. Motivated by [29], we consider the global well-posedness problem of the inhomogeneous incompressible micropolar equations with density-dependent viscosity and large initial data.

Before stating the main results, we explain the notation and conventions used throughout this paper. For a given function $f(x)$ or $f(x,t)$, we denote

$$\int f\mathrm{d}x = \int_\Omega f\mathrm{d}x,$$

For a positive integer k and $p \geq 1$, we denote the standard Lebesgue and Sobolev spaces as follows:

$$\begin{gathered}\|f\|_{L^p} = \|f\|_{L^p(\Omega)},\ \|f\|_{W^{k,p}} = \|f\|_{W^{k,p}(\Omega)},\ \|f\|_{H^k} = \|f\|_{W^{k,2}(\Omega)},\\ C_{0,\sigma}^\infty = \{f \in C_0^\infty(\Omega) : \mathrm{div} f = 0\},\ H_0^1 = \overline{C_0^\infty},\\ H_{0,\sigma}^1 = \overline{C_{0,\sigma}^\infty},\ \text{closure in the norm of } H^1.\end{gathered}$$

Our main result can be stated as follows.

Theorem 1.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^3$. Assume that*

$$\alpha > 1, \quad 0 < \beta \leq \frac{\alpha+1}{2}. \tag{1.6}$$

Given the constants $\bar\rho > 1$ and $C_0 > 1$, suppose that the initial data (ρ_0, u_0, w_0) satisfies

$$\bar\rho \leq \rho_0 \leq C_0\bar\rho, \quad \rho_0 \in W^{1,q},\ 3 < q < 6, \quad u_0 \in H_{0,\sigma}^1 \cap H^2, \quad w_0 \in H_0^1 \cap H^2. \tag{1.7}$$

Then, there exists a positive constant M depending only on $C_0, \bar\mu, \bar\xi, \bar\eta, \bar\lambda, \alpha$, $\|\nabla\rho_0\|_{L^q}, \|u_0\|_{H^2}, \|w_0\|_{H^2}$, and Ω such that if

$$\bar\rho \geq M(\Omega, C_0, \bar\mu, \bar\xi, \bar\eta, \bar\lambda, \alpha, \|\nabla\rho_0\|_{L^q}, \|u_0\|_{H^2}, \|w_0\|_{H^2}), \tag{1.8}$$

then the problem (1.1)–(1.5) *admits a unique global strong solution (ρ, u, w, P) in $\Omega \times (0,\infty)$ satisfying*

$$\begin{cases} \rho \in C([0,\infty); W^{1,q}), \quad \rho_t \in C([0,\infty); L^q), \\ \nabla u, P \in C([0,\infty); H^1) \cap L^2((0,\infty); W^{1,q}), \quad \nabla w \in C([0,\infty); H^1), \\ (u_t, w_t) \in L^\infty(0,\infty; L^2) \cap L^2(0,\infty; H_0^1). \end{cases} \tag{1.9}$$

Moreover, there exist a positive constant $\kappa(\bar\rho)$ depending on $\bar\rho$, C_0, $\bar\mu$, $\bar\xi$, $\bar\eta$, $\bar\lambda$, α, $\|\nabla\rho_0\|_{L^q}$, $\|u_0\|_{H^2}$, $\|w_0\|_{H^2}$, and Ω such that

$$\|u\|_{H^1}^2 + \|w\|_{H^1}^2 \leq Ce^{\kappa(\bar\rho)t}. \tag{1.10}$$

Remark 1.1. *As was pointed out by [29], Theorem 1.1 implies that the flow is globally stable when the initial Reynolds number is sufficiently small.*

Remark 1.2. *Theorem 1.1 can also be applied to the periodic case. Moreover, after slight modifications, we can also treat the case $\alpha > 1$ and $\beta = 0$.*

Let us briefly sketch the proof. To extend the local solution to a global one, we need to establish uniform a priori estimates on solutions in suitable higher-order norms, where some basic ideas are borrowed from [29]. However, due to the strong coupling between the velocity u and the micro-rotational velocity w, the proof of Theorem 1.1 is much more complicated. Compared with the case of Navier-Stokes equations, the main difficulty is caused by the extra terms $\xi(\rho)\nabla \times w$ and $\xi(\rho)\nabla \times u$ on the right-hand side of $(1.1)_2$ and $(1.1)_3$, there are extra linear terms $\bar{\rho}^{\beta-\alpha}\|\nabla u\|_{L^q}$ and $\bar{\rho}^{\beta-\alpha}\|\nabla \times w\|_{L^q}$ in the high-order estimates of u and w (see Lemma 3.3), which makes it impossible for us to follow the idea of [29] to prove the time-weighted estimates (3.18)–(3.19). Inspired by [8], we find that in the L^2-estimates of u and w, all the terms on the right-hand side can be absorbed into the left-hand side with the help of Poincaré inequality. This leads to the exponential decay of $\|(u, w)\|_{L^2}$ and the integrability of $e^{\kappa t}(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2)$ on $[0, T]$ for some $\kappa(\bar{\rho}) > 0$, with an order of $\bar{\rho}^{1-\alpha}$ (see (3.9)). With (3.9) at hand, we are able to derive a $e^{\kappa t}$-weighted estimate of ∇u and ∇w, which leads to the H^1-exponential decay of u and w. The time-weighted estimates are essential in establishing the uniform bound of $\|\nabla u\|_{L_t^1 L_x^\infty}$, while the control of $\|\nabla u\|_{L_t^1 L_x^\infty}$ leads to uniform estimates for other higher-order norms. As a result, we are able to extend the local strong solutions by a standard bootstrap argument.

The rest of the paper is organized as follows. In Section 2, we collect some elementary facts and inequalities which will be needed in later analysis. Section 3 is devoted to the proof of Theorem 1.1.

2. Preliminaries

First, the following local existence theory, where the initial density is strictly away from vacuum, can be shown by similar arguments as in Cho and Kim [30].

Lemma 2.1. *Assume that the initial data (ρ_0, u_0, w_0) satisfies the regularity condition (1.7). Then, there exists a small time T_0 and a unique strong solution (ρ, u, P) to the initial boundary value problem (1.1)–(1.5) such that*

$$\begin{cases} \rho \in C([0, T_0]; W^{1,q}), \quad \rho_t \in C([0, T_0]; L^q), \\ \nabla u, P \in C([0, T_0]; H^1) \cap L^2(0, T_0; W^{1,q}), \quad \nabla w \in C([0, T_0]; H^1), \\ (\sqrt{\rho} u_t, \sqrt{\rho} w_t) \in L^\infty(0, T_0; L^2), \quad (u_t, w_t) \in L^2(0, T_0; H_0^1). \end{cases} \tag{2.1}$$

Furthermore, if T^* is the maximal existence time of the local strong solution (ρ, u), then either $T^* = \infty$ or

$$\sup_{0 \le t \le T^*} (\|\nabla \rho\|_{L^q} + \|\nabla u\|_{L^2} + \|\nabla w\|_{L^2}) = \infty. \tag{2.2}$$

In this paper, we will employ Bovosgii's theory which can be found in [31].

Lemma 2.2. *Let Ω be a bounded domain with Lipschitz boundary, $1 < p < \infty$, and $b(x) \in L^p(\Omega)$ with $\int_\Omega b(x)\mathrm{d}x = 0$. Then, there exists $v(x) \in W_0^{1,p}(\Omega)$ satisfying:*

$$\mathrm{div} v = b, \qquad \text{in } \Omega,$$

and

$$\|\nabla v\|_{L^p(\Omega)} \le C(p)\|b\|_{L^p(\Omega)}. \tag{2.3}$$

Also, the well-known Gagliardo-Nirenberg inequality [32] will be frequently used in this paper.

Lemma 2.3. *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^3$. Let $1 \le q \le \infty$ be a positive extended real quantity. Let j and m be non-negative integers such that $j < m$. Furthermore, let $1 \le r \le \infty$ be a positive extended real quantity, $p \ge 1$ be real and $\theta \in [0, 1]$ such that the relations*

$$\frac{1}{p} = \frac{j}{n} + \theta\left(\frac{1}{r} - \frac{m}{n}\right) + \frac{1-\theta}{q}, \qquad \frac{j}{m} \le \theta \le 1 \tag{2.4}$$

hold. Then,

$$\|\nabla^j u\|_{L^p(\Omega)} \le C\|\nabla^m u\|^{\theta}_{L^r(\Omega)}\|u\|^{1-\theta}_{L^q(\Omega)} + C_1\|u\|_{L^q(\Omega)} \tag{2.5}$$

where $u \in L^q(\Omega)$ such that $\nabla^m u \in L^r(\Omega)$. Moreover, if $q > 1$ and $r > 3$,

$$\|u\|_{C(\bar{\Omega})} \le C\|u\|^{q(r-3)/(3r+q(r-3))}_{L^q}\|\nabla u\|^{3r/(3r+q(r-3))}_{L^r} + C_2\|u\|_{L^q}. \tag{2.6}$$

where $u \in L^q(\Omega)$ such that $\nabla u \in L^r(\Omega)$. In any case, the constant $C > 0$ depends on the parameters j, m, n, q, r, and θ on the domain Ω, but not on u. In addition, if $u \cdot n|_{\partial\Omega} = 0$ or $\int_\Omega u\mathrm{d}x = 0$, we can choose $C_1 = C_2 = 0$.

High-order a priori estimates rely on the following regularity results for density-dependent Stokes equations. The proof can be found in [29].

Lemma 2.4. *Assume that $\rho \in W^{1,q}$, $\alpha > 1$, $3 < q < 6$, and $\bar{\rho} \le \rho \le C_0\bar{\rho}$. Let $(u, P) \in H^1_{0,\sigma} \times L^2$ be the unique weak solution to the boundary value problem*

$$\begin{cases} -\mathrm{div}(2\bar{\mu}\rho^{\alpha}\mathcal{D}(u)) + \nabla P = F, \\ \mathrm{div} u = 0, \\ \int \frac{P}{\rho^{\alpha}}\mathrm{d}x = 0. \end{cases} \tag{2.7}$$

Then, we have the following regularity results:

(1) If $F \in L^2$, then $(u, P) \in H^2 \times H^1$ and

$$\|u\|_{H^2} + \left\|\frac{P}{\rho^{\alpha}}\right\|_{H^1} \le C(\bar{\rho}^{-\alpha} + \bar{\rho}^{-\alpha-\frac{q}{q-3}}\|\nabla\rho\|^{\frac{q}{q-3}}_{L^q})\|F\|_{L^2}; \tag{2.8}$$

(2) If $F \in L^q$ for some $q \in (3, 6)$, then $(u, P) \in W^{2,q} \times W^{1,q}$ and

$$\|u\|_{W^{2,q}} + \left\|\frac{P}{\rho^{\alpha}}\right\|_{W^{1,q}} \le C(\bar{\rho}^{-\alpha} + \bar{\rho}^{-\alpha-\frac{5q-6}{2(q-3)}}\|\nabla\rho\|^{\frac{5q-6}{2(q-3)}}_{L^q})\|F\|_{L^q}, \tag{2.9}$$

where the constant C in (2.8) and (2.9) depends on Ω, q.

Proof. The proof of (2.8) can be found in [29]. Below, we prove (2.9). Similarly, rewrite $(2.7)_1$ as

$$-\mu\Delta u + \nabla\left(\frac{P}{\rho^{\alpha}}\right) = \frac{F}{\rho^{\alpha}} + 2\mu\alpha\rho^{-1}\nabla\rho \cdot d + \alpha\rho^{-1}\nabla\rho\left(\frac{P}{\rho^{\alpha}}\right).$$

The L^p-estimate yields that

$$
\begin{aligned}
&\|\nabla^2 u\|_{L^q} + \Big\|\nabla\Big(\frac{P}{\rho^\alpha}\Big)\Big\|_{L^q}\\
\leq\ & C\bar{\rho}^{-\alpha}\|F\|_{L^q} + C\bar{\rho}^{-1}\|\nabla\rho\|_{L^q}\left(\|\nabla u\|_{L^q}^{\frac{2q-6}{5q-6}}\|\nabla^2 u\|_{L^q}^{\frac{3q}{5q-6}} + \|\frac{P}{\rho^\alpha}\|_{L^q}^{\frac{2q-6}{5q-6}}\Big\|\nabla\Big(\frac{P}{\rho^\alpha}\Big)\Big\|_{L^q}^{\frac{3q}{5q-6}}\right)\\
\leq\ & \frac{1}{4}\left(\|\nabla^2 u\|_{L^q} + \Big\|\nabla\Big(\frac{P}{\rho^\alpha}\Big)\Big\|_{L^q}\right) + C\bar{\rho}^{-\alpha}\|F\|_{L^q} + C\bar{\rho}^{-\frac{5q-6}{2q-6}}\|\nabla\rho\|_{L^q}^{\frac{5q-6}{2q-6}}\left(\|\nabla u\|_{L^2} + \Big\|\frac{P}{\rho^\alpha}\Big\|_{L^2}\right).
\end{aligned}
$$

Moreover, from (2.8), we have

$$
\|u\|_{W^{2,q}} + \Big\|\frac{P}{\rho^\alpha}\Big\|_{W^{1,q}} \leq C\Big(\bar{\rho}^{-\alpha} + \bar{\rho}^{-\alpha-\frac{5q-6}{2(q-3)}}\|\nabla\rho\|_{L^q}^{\frac{5q-6}{2(q-3)}}\Big)\|F\|_{L^q}.
$$

This completes the proof. □

3. A priori estimates

For any fixed time $T > 0$, (ρ, u, w, P) is the unique local strong solution to (1.1)–(1.5) on $\Omega\times(0, T]$ with initial data (ρ_0, u_0, w_0) satisfying (1.7), which is guaranteed by Lemma 2.1. Define

$$
\mathcal{E}_\rho(T) := \sup_{t\in[0,T]}\|\nabla\rho\|_{L^q}, \tag{3.1}
$$

$$
\mathcal{E}_w(T) := \bar{\rho}^\alpha \sup_{t\in[0,T]}\|\nabla w\|_{L^2}^2 + \int_0^T \|\sqrt{\rho}w_t\|_{L^2}^2 \mathrm{d}t, \tag{3.2}
$$

$$
\mathcal{E}_u(T) := \bar{\rho}^\alpha \sup_{t\in[0,T]}\|\nabla u\|_{L^2}^2 + \int_0^T \|\sqrt{\rho}u_t\|_{L^2}^2 \mathrm{d}t. \tag{3.3}
$$

We have the following key proposition.

Proposition 3.1. *Let $\alpha > 1$ and $0 < \beta \leq \frac{\alpha+1}{2}$. Under the conditions of Theorem 1.1, there exists positive constants K and M depending on $C_0, \bar{\mu}, \bar{\xi}, \bar{\eta}, \bar{\lambda}, \alpha, \beta$, $\|\nabla\rho_0\|_{L^q}, \|u_0\|_{H^2}, \|w_0\|_{H^2}$, and Ω such that if (ρ, u, w, P) is a smooth solution to the problem (1.1)–(1.5) on $\Omega\times(0, T]$ satisfying*

$$
\mathcal{E}_\rho(T) \leq 3\mathcal{E}_\rho(0), \quad \mathcal{E}_u(T) + \mathcal{E}_w(T) \leq 3K\bar{\rho}^\alpha, \tag{3.4}
$$

then the following estimates hold:

$$
\mathcal{E}_\rho(T) \leq 2\mathcal{E}_\rho(0), \quad \mathcal{E}_u(T) + \mathcal{E}_w(T) \leq 2K\bar{\rho}^\alpha, \tag{3.5}
$$

provided

$$
\bar{\rho} \geq M(\Omega, C_0, \bar{\mu}, \bar{\xi}, \bar{\eta}, \bar{\lambda}, \alpha, \beta, \|\nabla\rho_0\|_{L^q}, \|u_0\|_{H^2}, \|w_0\|_{H^2}). \tag{3.6}
$$

First, since the density satisfies the transport equation $(1.1)_1$, and using $(1.1)_4$, one has the following lemma.

Lemma 3.1. *Under the conditions of Proposition 3.1, it holds that*

$$
\bar{\rho} \leq \rho \leq C_0\bar{\rho}, \quad (x, t) \in \Omega\times[0, T]. \tag{3.7}
$$

Next, we establish the basic energy inequality of (1.1).

Lemma 3.2. *Under the conditions of Proposition 3.1, there exists a positive constant M_1 such that*

$$\sup_{0\le t\le T}\bar\rho\left(\|u\|_{L^2}^2+\|w\|_{L^2}^2\right)+\bar\rho^\alpha\int_0^T\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right)\mathrm{d}t\le C\bar\rho, \tag{3.8}$$

and

$$\sup_{0\le t\le T}\bar\rho e^{\kappa(\bar\rho)t}\left(\|u\|_{L^2}^2+\|w\|_{L^2}^2\right)+\bar\rho^\alpha\int_0^T e^{\kappa(\bar\rho)t}\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right)\mathrm{d}t\le C\bar\rho, \tag{3.9}$$

provided $\bar\rho\ge M_1(\Omega,C_0,\bar\mu,\bar\xi,\bar\eta,\bar\lambda,\alpha,\beta,\|\nabla\rho_0\|_{L^q},\|u_0\|_{H^2},\|w_0\|_{H^2})$, where $\kappa(\bar\rho)=\bar\kappa\bar\rho^{\alpha-1}$ for some constant $\bar\kappa$ independent of $\bar\rho$.

Proof. Making the L^2-inner product of $(1.1)_2$–$(1.1)_3$ by $(u,w)^\top$, we obtain after integration by parts that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\sqrt{\rho}u\|_{L^2}^2+\|\sqrt{\rho}w\|_{L^2}^2\right)+\int(\bar\mu\rho^\alpha+\bar\xi\rho^\beta)|\mathcal{D}(u)|^2\mathrm{d}x\\
&\quad+2\bar\eta\int\rho^\alpha|\mathcal{D}(w)|^2\mathrm{d}x+4\bar\xi\int\rho^\beta|w|^2\mathrm{d}x+\bar\lambda\int\rho^\alpha(\mathrm{div}w)^2\mathrm{d}x\\
=\;&2\bar\xi\int\left(\rho^\beta w\cdot(\nabla\times u)+w\cdot\nabla\times(\rho^\beta u)\right)\mathrm{d}x\\
\le\;&C\bar\rho^\beta\left(\|u\|_{H^1}\|w\|_{H^1}+\|u\|_{L^{\frac{2q}{q-1}}}\|w\|_{L^{\frac{2q}{q-1}}}\|\nabla\rho\|_{L^q}\right)\\
\le\;&C\bar\rho^\beta\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right),
\end{aligned}\tag{3.10}$$

where we have used (3.4), Poincaré's inequality, and the fact $\frac{2q}{q-1}\in(2,3)$.

Integrating the above inequality over $(0,T]$ and taking M_1 large enough, such that

$$4C\bar\rho^\beta\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right)\le\bar\mu\bar\rho^\alpha\int|\mathcal{D}(u)|^2\mathrm{d}x+2\bar\eta\bar\rho^\alpha\int|\mathcal{D}(w)|^2\mathrm{d}x,$$

we obtain

$$\sup_{0\le t\le T}\bar\rho\left(\|u\|_{L^2}^2+\|w\|_{L^2}^2\right)+\bar\rho^\alpha\int_0^T\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right)\mathrm{d}t\le C\bar\rho\left(\|u_0\|_{L^2}^2+\|w_0\|_{L^2}^2\right)\le C\bar\rho. \tag{3.11}$$

Multiplying (3.10) by $e^{\kappa t}$ and using Poincaré's inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\kappa t}\left(\|\sqrt{\rho}u\|_{L^2}^2+\|\sqrt{\rho}w\|_{L^2}^2\right)\right)+e^{\kappa t}\int(\bar\mu\rho^\alpha+\bar\xi\rho^\beta)|\mathcal{D}(u)|^2\mathrm{d}x\\
&\quad+2\bar\eta e^{\kappa t}\int\rho^\alpha|\mathcal{D}(w)|^2\mathrm{d}x+4\bar\xi e^{\kappa t}\int\rho^\beta|w|^2\mathrm{d}x+\bar\lambda e^{\kappa t}\int\rho^\alpha(\mathrm{div}w)^2\mathrm{d}x\\
\le\;&C\bar\rho^\beta e^{\kappa t}\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right)+\frac{\kappa}{2}e^{\kappa t}\left(\|\sqrt{\rho}u\|_{L^2}^2+\|\sqrt{\rho}w\|_{L^2}^2\right)\\
\le\;&C\bar\rho^\beta e^{\kappa t}\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right)+\frac{C_0\bar\kappa}{2}\bar\rho^\alpha e^{\kappa t}\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right).
\end{aligned}\tag{3.12}$$

Integrating the above inequality over $(0,T]$ and taking $\bar\kappa$ sufficiently small, such that

$$4C_0\bar\kappa\bar\rho^\alpha\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right)\le\bar\mu\bar\rho^\alpha\int|\mathcal{D}(u)|^2\mathrm{d}x+2\bar\eta\bar\rho^\alpha\int|\mathcal{D}(w)|^2\mathrm{d}x,$$

we complete the proof of Lemma 3.2. □

Next, we have the following high-order estimate of the velocities, which will be used frequently.

Lemma 3.3. *Under the conditions of Proposition 3.1, it holds that*

$$\|u\|_{H^2} \le C\left(\bar\rho^{\frac12-\alpha}\|\sqrt{\rho}u_t\|_{L^2} + \bar\rho^{2-2\alpha}\|\nabla u\|_{L^2}^3 + \bar\rho^{\beta-\alpha}\|\nabla\times w\|_{L^2}\right), \tag{3.13}$$

$$\|w\|_{H^2} \le C\left(\bar\rho^{\frac12-\alpha}\|\sqrt{\rho}w_t\|_{L^2} + \bar\rho^{2-2\alpha}\|\nabla u\|_{L^2}^2\|\nabla w\|_{L^2} + \bar\rho^{\beta-\alpha}\|\nabla u\|_{L^2}\right), \tag{3.14}$$

$$\|u\|_{W^{2,q}} \le C\Big(\bar\rho^{-\alpha}\|\rho u_t\|_{L^q} + \bar\rho^{(1-\alpha)\frac{5q-6}{q}}\|\nabla u\|_{L^2}^{\frac{6(q-1)}{q}} + \bar\rho^{\beta-\alpha}\|\nabla\times w\|_{L^q}\Big), \tag{3.15}$$

and

$$\|w\|_{W^{2,q}} \le C\Big(\bar\rho^{-\alpha}\|\rho w_t\|_{L^q} + \bar\rho^{(1-\alpha)\frac{5q-6}{q}}\|\nabla u\|_{L^2}^{\frac{5q-6}{q}}\|\nabla w\|_{L^2} + \bar\rho^{\beta-\alpha}\|\nabla u\|_{L^q}\Big). \tag{3.16}$$

Proof. Let $F = -\rho u_t - \rho(u\cdot\nabla)u + 2\xi(\rho)\nabla\times w$ in Lemma 2.4. Then, the proof of (3.15) is similar to that of (3.29) in [29], and we sketch it here for completeness. From (2.9), (3.4), and the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}\|u\|_{W^{2,q}} \le\ & C(\bar\rho^{-\alpha} + \bar\rho^{-\alpha-\frac{5q-6}{2(q-3)}}\mathcal{E}_\rho(0)^{\frac{5q-6}{2(q-3)}})(\|\rho u_t\|_{L^q} + \|\rho(u\cdot\nabla)u\|_{L^q} + \|2\xi(\rho)\nabla\times w\|_{L^q})\\ \le\ & C(\bar\rho^{-\alpha} + \bar\rho^{-\alpha-\frac{5q-6}{2(q-3)}})(\|\rho u_t\|_{L^q} + \bar\rho\|\nabla u\|_{L^2}^{\frac{6(q-1)}{5q-6}}\|\nabla u\|_{W^{1,q}}^{\frac{4q-6}{5q-6}} + \bar\rho^{\beta}\|\nabla\times w\|_{L^q})\\ \le\ & \frac12\|\nabla u\|_{W^{1,q}} + C\bar\rho^{-\alpha}\|\rho u_t\|_{L^q} + C\bar\rho^{(1-\alpha)\frac{5q-6}{q}}\|\nabla u\|_{L^2}^{\frac{6(q-1)}{q}} + C\bar\rho^{\beta-\alpha}\|\nabla\times w\|_{L^q},\end{aligned}$$

provided $\bar\rho \ge 1$. Next, we rewrite $(1.1)_3$ as

$$\bar\mu\Delta w + (\bar\mu+\bar\lambda)\nabla\mathrm{div}w = -2\bar\mu\rho^{-1}\nabla\rho\cdot\mathcal{D}(w) - \bar\lambda\rho^{-1}\nabla\rho(\mathrm{div}w) + \rho w_t + \rho(u\cdot\nabla)w + 4\xi(\rho)w - 2\xi(\rho)\nabla\times u.$$

The standard L^p-estimate and (3.4) yield that

$$\begin{aligned}\|w\|_{W^{2,q}} \le\ & C\bar\rho^{-\alpha}(\|\rho w_t\|_{L^q} + \|\rho(u\cdot\nabla)w\|_{L^q} + \|2\xi(\rho)\nabla\times w\|_{L^q})\\ \le\ & C\bar\rho^{-\alpha}(\|\rho w_t\|_{L^q} + \bar\rho\|\nabla u\|_{L^2}\|\nabla w\|_{L^2}^{\frac{q}{5q-6}}\|\nabla w\|_{W^{1,q}}^{\frac{4q-6}{5q-6}} + \bar\rho^{\beta}\|\nabla\times w\|_{L^q})\\ \le\ & \frac12\|\nabla w\|_{W^{1,q}} + C\bar\rho^{-\alpha}\|\rho w_t\|_{L^q} + C\bar\rho^{(1-\alpha)\frac{5q-6}{q}}\|\nabla u\|_{L^2}^{\frac{5q-6}{q}}\|\nabla w\|_{L^2} + C\bar\rho^{\beta-\alpha}\|\nabla u\|_{L^q}.\end{aligned}$$

The proof of (3.13) and (3.14) are similar to those of (3.15) and (3.16), respectively, and thus omitted. Thus, the proof is complete. □

We are now ready to derive the estimates of $\mathcal{E}_u(T)$ and $\mathcal{E}_w(T)$.

Lemma 3.4. *Under the conditions of Proposition 3.1, there exists a positive constant M_2 such that*

$$\mathcal{E}_u(T) + \mathcal{E}_w(T) \le 2K\bar\rho^{\alpha}, \tag{3.17}$$

$$\sup_{t\in[0,T]}\bar\rho^{\alpha}t\left(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right) + \int_0^T t\left(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}w_t\|_{L^2}^2\right)\mathrm{d}t \le C\bar\rho, \tag{3.18}$$

$$\sup_{t\in[0,T]}\bar\rho^{\alpha}t^2\left(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right) + \int_0^T t^2\left(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}w_t\|_{L^2}^2\right)\mathrm{d}t \le C\bar\rho, \tag{3.19}$$

and

$$\sup_{t\in[0,T]}\bar\rho^{\alpha}e^{\kappa(\bar\rho)t}\left(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right) + \int_0^T e^{\kappa(\bar\rho)t}\left(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}w_t\|_{L^2}^2\right)\mathrm{d}t \le C\bar\rho^{\alpha}, \tag{3.20}$$

provided $\bar\rho \ge M_2(\Omega, C_0, \bar\mu, \bar\xi, \bar\eta, \bar\lambda, \alpha, \beta, \|\nabla\rho_0\|_{L^q}, \|u_0\|_{H^2}, \|w_0\|_{H^2})$.

Proof. Making the L^2-inner product of $(1.1)_2$–$(1.1)_3$ with $(u_t, w_t)^\top$ and integrating by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int\left(\bar{\mu}\rho^\alpha+\bar{\xi}\rho^\beta\right)|\mathcal{D}(u)|^2\mathrm{d}x+2\bar{\eta}\int\rho^\alpha|\mathcal{D}(w)|^2\mathrm{d}x+\bar{\lambda}\int\rho^\alpha(\mathrm{div}w)^2\mathrm{d}x\right)\\
&+2\bar{\xi}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho^\beta|w|^2\mathrm{d}x+\int\rho(|u_t|^2+|w_t|^2)\mathrm{d}x\\
=\;&-\int\rho u\cdot\nabla u\cdot u_t\mathrm{d}x-\int\rho u\cdot\nabla w\cdot w_t\mathrm{d}x+2\bar{\xi}\int\rho^\beta\left(u_t\cdot\nabla\times w+w_t\cdot\nabla\times u\right)\mathrm{d}x\\
&+\frac{1}{2}\int(\alpha\bar{\mu}\rho^{\alpha-1}+\beta\bar{\xi}\rho^{\beta-1})\rho_t|\mathcal{D}(u)|^2\mathrm{d}x+\frac{\bar{\eta}\alpha}{2}\int\rho^{\alpha-1}\rho_t|\mathcal{D}(w)|^2\mathrm{d}x\\
&+\frac{\bar{\lambda}\alpha}{2}\int\rho^{\alpha-1}\rho_t(\mathrm{div}w)^2\mathrm{d}x+2\bar{\xi}\beta\int\rho^{\beta-1}\rho_t|w|^2\mathrm{d}x\\
:=\;&\sum_{j=1}^{7}I_j.
\end{aligned}\tag{3.21}$$

It follows from the Hölder and Sobolev inequalities that

$$\begin{aligned}
I_1+I_2=\;&-\int\rho u\cdot\nabla u\cdot u_t\mathrm{d}x-\int\rho u\cdot\nabla w\cdot w_t\mathrm{d}x\\
\leq\;&C\bar{\rho}^{\frac{1}{2}}\|u\|_{L^6}\left(\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u\|_{L^3}+\|\sqrt{\rho}w_t\|_{L^2}\|\nabla w\|_{L^3}\right)\\
\leq\;&C\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{H^1}^{\frac{1}{2}}+C\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}w_t\|_{L^2}\|\nabla u\|_{L^2}\|\nabla w\|_{L^2}^{\frac{1}{2}}\|\nabla w\|_{H^1}^{\frac{1}{2}}\\
\leq\;&C\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u\|_{L^2}^{\frac{3}{2}}\left(\bar{\rho}^{\frac{1}{2}-\alpha}\|\sqrt{\rho}u_t\|_{L^2}+\bar{\rho}^{2-2\alpha}\|\nabla u\|_{L^2}^3+\bar{\rho}^{\beta-\alpha}\|\nabla w\|_{L^2}\right)^{\frac{1}{2}}\\
&+C\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}w_t\|_{L^2}\|\nabla u\|_{L^2}\|\nabla w\|_{L^2}^{\frac{1}{2}}\left(\bar{\rho}^{\frac{1}{2}-\alpha}\|\sqrt{\rho}w_t\|_{L^2}+\bar{\rho}^{2-2\alpha}\|\nabla u\|_{L^2}^2\|\nabla w\|_{L^2}+\bar{\rho}^{\beta-\alpha}\|\nabla u\|_{L^2}\right)^{\frac{1}{2}}\\
\leq\;&\frac{1}{6}\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2\right)+C\bar{\rho}^{3-2\alpha}\left(\|\nabla u\|_{L^2}^6+\|\nabla u\|_{L^2}^4\|\nabla w\|_{L^2}^2\right)+C\bar{\rho}^{1+\beta-\alpha}\|\nabla u\|_{L^2}^3\|\nabla w\|_{L^2}\\
\leq\;&\frac{1}{6}\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2\right)+C\left(\bar{\rho}^{3-2\alpha}+\bar{\rho}^{1+\beta-\alpha}\right)\|\nabla u\|_{L^2}^2,
\end{aligned}\tag{3.22}$$

where we have also used (3.13), (3.14), and $\bar{\rho}>1$. Next, we have

$$\begin{aligned}
I_3=\;&2\bar{\xi}\int\rho^\beta\left(u_t\cdot\nabla\times w+w_t\cdot\nabla\times u\right)\mathrm{d}x\\
\leq\;&C\bar{\rho}^{\beta-\frac{1}{2}}\left(\|\sqrt{\rho}u_t\|_{L^2}\|\nabla w\|_{L^2}+\|\sqrt{\rho}w_t\|_{L^2}\|\nabla u\|_{L^2}\right)\\
\leq\;&\frac{1}{6}\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2\right)+C\bar{\rho}^{2\beta-1}\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right).
\end{aligned}\tag{3.23}$$

Using $(1.1)_1$, (3.13), and the Hölder and Sobolev inequalities, we get

$$\begin{aligned}
I_4=\;&-\frac{1}{2}\int(\alpha\bar{\mu}\rho^{\alpha-1}+\beta\bar{\xi}\rho^{\beta-1})u\cdot\nabla\rho|\mathcal{D}(u)|^2\mathrm{d}x\\
\leq\;&C\bar{\rho}^{\alpha-1}\|u\|_{L^{\frac{2q}{q-2}}}\|\nabla\rho\|_{L^q}\|\nabla u\|_{L^4}^2\\
\leq\;&C\bar{\rho}^{\alpha-1}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2u\|_{L^2}^{\frac{3}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\le C\bar{\rho}^{\alpha-1}\|\nabla u\|_{L^2}^{\frac{1}{2}}\left(\bar{\rho}^{\frac{1}{2}-\alpha}\|\sqrt{\rho}u_t\|_{L^2}+\bar{\rho}^{2-2\alpha}\|\nabla u\|_{L^2}^{3}+\bar{\rho}^{\beta-\alpha}\|\nabla w\|_{L^2}\right)^{\frac{3}{2}}\\
&\le \frac{1}{6}\|\sqrt{\rho}u_t\|_{L^2}^2+C\bar{\rho}^{-1-2\alpha}\|\nabla u\|_{L^2}^2+\bar{\rho}^{2-2\alpha}\|\nabla u\|_{L^2}^5+C\bar{\rho}^{\frac{3}{2}\beta-\frac{\alpha}{2}-1}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla w\|_{L^2}^{\frac{3}{2}}\\
&\le \frac{1}{6}\|\sqrt{\rho}u_t\|_{L^2}^2+C\bar{\rho}^{2-2\alpha}\|\nabla u\|_{L^2}^2+C\bar{\rho}^{\frac{3}{2}\beta-\frac{\alpha}{2}-1}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla w\|_{L^2}^{\frac{3}{2}}.
\end{aligned}
\tag{3.24}
$$

Similarly, it follows from $(1.1)_1$ and (3.14) that

$$
\begin{aligned}
I_5+I_6=&\ \frac{\bar{\eta}\alpha}{2}\int\rho^{\alpha-1}\rho_t|\mathcal{D}(w)|^2\mathrm{d}x+\frac{\bar{\lambda}\alpha}{2}\int\rho^{\alpha-1}\rho_t(\mathrm{div}w)^2\mathrm{d}x\\
\le&\ C\bar{\rho}^{\alpha-1}\|u\|_{L^{\frac{2q}{q-2}}}\|\nabla\rho\|_{L^q}\|\nabla w\|_{L^4}^2\\
\le&\ C\bar{\rho}^{\alpha-1}\|\nabla w\|_{L^2}^{\frac{1}{2}}\|\nabla^2 w\|_{L^2}^{\frac{3}{2}}\\
\le&\ C\bar{\rho}^{\alpha-1}\|\nabla w\|_{L^2}^{\frac{1}{2}}\left(\bar{\rho}^{\frac{1}{2}-\alpha}\|\sqrt{\rho}w_t\|_{L^2}+\bar{\rho}^{2-2\alpha}\|\nabla u\|_{L^2}^2\|\nabla w\|_{L^2}+\bar{\rho}^{\beta-\alpha}\|\nabla u\|_{L^2}\right)^{\frac{3}{2}}\\
\le&\ \frac{1}{6}\|\sqrt{\rho}w_t\|_{L^2}^2+C\bar{\rho}^{-1-2\alpha}\|\nabla w\|_{L^2}^2+\bar{\rho}^{2-2\alpha}\|\nabla u\|_{L^2}^3\|\nabla w\|_{L^2}^2+C\bar{\rho}^{\frac{3}{2}\beta-\frac{\alpha}{2}-1}\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla w\|_{L^2}^{\frac{1}{2}}\\
\le&\ \frac{1}{6}\|\sqrt{\rho}w_t\|_{L^2}^2+C\bar{\rho}^{2-2\alpha}\|\nabla w\|_{L^2}^2+C\bar{\rho}^{\frac{3}{2}\beta-\frac{\alpha}{2}-1}\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla w\|_{L^2}^{\frac{1}{2}}.
\end{aligned}
\tag{3.25}
$$

Next, it follows from $(1.1)_1$ and Poincaré's inequality that

$$
I_7\le C\bar{\rho}^{\alpha-1}\|u\|_{L^{\frac{2q}{q-2}}}\|\nabla\rho\|_{L^q}\|w\|_{L^4}^2\le C\bar{\rho}^{\alpha-1}\|w\|_{L^2}^{\frac{1}{2}}\|\nabla w\|_{L^2}^{\frac{3}{2}}\le C\bar{\rho}^{\alpha-1}\|\nabla w\|_{L^2}^2.
\tag{3.26}
$$

Substituting (3.22)–(3.25) into (3.21), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int\left(\bar{\mu}\rho^{\alpha}+\bar{\xi}\rho^{\beta}\right)|\mathcal{D}(u)|^2\mathrm{d}x+2\bar{\eta}\int\rho^{\alpha}|\mathcal{D}(w)|^2\mathrm{d}x+\bar{\lambda}\int\rho^{\alpha}(\mathrm{div}w)^2\mathrm{d}x\right)\\
&\quad+2\bar{\xi}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho^{\beta}|w|^2\mathrm{d}x+\frac{1}{2}\int\rho(|u_t|^2+|w_t|^2)\mathrm{d}x\\
&\le C\left(\bar{\rho}^{3-2\alpha}+\bar{\rho}^{1+\beta-\alpha}\right)\|\nabla u\|_{L^2}^2+C\bar{\rho}^{2\beta-1}\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right)+C\bar{\rho}^{2-2\alpha}\|\nabla u\|_{L^2}^2\\
&\quad+C\bar{\rho}^{\frac{3}{2}\beta-\frac{\alpha}{2}-1}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla w\|_{L^2}^{\frac{3}{2}}+C\bar{\rho}^{2-2\alpha}\|\nabla w\|_{L^2}^2+C\bar{\rho}^{\frac{3}{2}\beta-\frac{\alpha}{2}-1}\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla w\|_{L^2}^{\frac{1}{2}}+C\bar{\rho}^{\alpha-1}\|\nabla w\|_{L^2}^2\\
&\le C\left(\bar{\rho}^{3-2\alpha}+\bar{\rho}^{1+\beta-\alpha}+\bar{\rho}^{2\beta-1}+\bar{\rho}^{\alpha-1}\right)\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right).
\end{aligned}
\tag{3.27}
$$

Integrating (3.27) with respect to t over $(0,T]$, we obtain from (3.7) and (3.8) that

$$
\begin{aligned}
&\sup_{t\in[0,T]}\bar{\rho}^{\alpha}\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right)+\int_0^T\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2\right)\mathrm{d}t\\
&\le C_1\bar{\rho}^{\alpha}+C\left(\bar{\rho}^{3-2\alpha}+\bar{\rho}^{1+\beta-\alpha}+\bar{\rho}^{2\beta-1}+\bar{\rho}^{\alpha-1}\right)\int_0^T\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right)\mathrm{d}t\\
&\le C_1\bar{\rho}^{\alpha}+C_2\left(\bar{\rho}^{4-3\alpha}+\bar{\rho}^{2+\beta-2\alpha}+\bar{\rho}^{2\beta-\alpha}+1\right).
\end{aligned}
$$

Taking $K\ge C_1$ and M_2 sufficiently large, such that

$$
M_2\ge M_1,\quad \text{and}\quad C_2\left(\bar{\rho}^{4-3\alpha}+\bar{\rho}^{2+\beta-2\alpha}+\bar{\rho}^{2\beta-\alpha}+1\right)\le C_1\bar{\rho}^{\alpha},
\tag{3.28}
$$

we complete the proof of (3.17).

For simplicity, we denote

$$\mathcal{E}(t) := \int \left(\bar{\mu}\rho^{\alpha} + \bar{\xi}\rho^{\beta}\right)|\mathcal{D}(u)|^2 \mathrm{d}x + 2\bar{\eta}\int \rho^{\alpha}|\mathcal{D}(w)|^2 \mathrm{d}x + \bar{\lambda}\int \rho^{\alpha}(\mathrm{div} w)^2 \mathrm{d}x + 4\bar{\xi}\int \rho^{\beta}|w|^2 \mathrm{d}x. \quad (3.29)$$

It is easy to see that there exists a positive constant C independent of $\bar{\rho}$, such that

$$C^{-1}\bar{\rho}^{\alpha}\left(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right) \le \mathcal{E}(t) \le C\bar{\rho}^{\alpha}\left(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right), \quad (3.30)$$

and (3.27) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + \left(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}w_t\|_{L^2}^2\right) \le C\left(\bar{\rho}^{3-2\alpha} + \bar{\rho}^{1+\beta-\alpha} + \bar{\rho}^{2\beta-1} + \bar{\rho}^{\alpha-1}\right)\left(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right). \quad (3.31)$$

Multiplying (3.31) by t^k for $k = 1, 2$, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(t^k\mathcal{E}(t)) + t^k\left(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}w_t\|_{L^2}^2\right)\\
\le\ & C(t^k + t^{k-1})\left(\bar{\rho}^{3-2\alpha} + \bar{\rho}^{1+\beta-\alpha} + \bar{\rho}^{2\beta-1} + \bar{\rho}^{\alpha-1}\right)\left(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right)\\
\le\ & C(t^k + t^{k-1})\bar{\rho}^{\alpha}\left(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right). \quad (3.32)
\end{aligned}$$

Integrating (3.32) with respect to t over $(0, T]$ and using (3.9), we obtain (3.18) and (3.19).

On the other hand, multiplying (3.31) by $e^{\kappa(\bar{\rho})t}$, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\kappa(\bar{\rho})t}\mathcal{E}(t)\right) + e^{\kappa(\bar{\rho})t}\left(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}w_t\|_{L^2}^2\right)\\
\le\ & Ce^{\kappa(\bar{\rho})t}(1 + \bar{\kappa}\bar{\rho}^{\alpha-1})\left(\bar{\rho}^{3-2\alpha} + \bar{\rho}^{1+\beta-\alpha} + \bar{\rho}^{2\beta-1} + \bar{\rho}^{\alpha-1}\right)\\
\le\ & Ce^{\kappa(\bar{\rho})t}\left(\bar{\rho}^{2-\alpha} + \bar{\rho}^{\beta} + \bar{\rho}^{\alpha+2\beta-2} + \bar{\rho}^{2\alpha-2}\right)\left(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right). \quad (3.33)
\end{aligned}$$

Integrating (3.33) with respect to t over $(0, T]$ and using (3.9), we obtain (3.20). This completes the proof of Lemma 3.4. □

Remark 3.1. *When we complete the proof of Proposition 3.1, all the a priori estimates, including* (3.9) *and* (3.20)*, hold when $t \in \mathbb{R}_+$. This leads to the exponentially stability* (1.10).

In order to close the estimate of $\mathcal{E}_\rho$, we need the following time-weight estimates.

Lemma 3.5. *Under the conditions of Proposition 3.1, there exists a positive constant M_3 such that*

$$\sup_{0\le t\le T} t\left(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}w_t\|_{L^2}^2\right) + \bar{\rho}^{\alpha}\int_0^T t\left(\|\nabla u_t\|_{L^2}^2 + \|\nabla w_t\|_{L^2}^2\right)\mathrm{d}t \le C\bar{\rho}^{\alpha}, \quad (3.34)$$

and

$$\sup_{0\le t\le T} t^2\left(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}w_t\|_{L^2}^2\right) + \bar{\rho}^{\alpha}\int_0^T t^2\left(\|\nabla u_t\|_{L^2}^2 + \|\nabla w_t\|_{L^2}^2\right)\mathrm{d}t \le C\bar{\rho}. \quad (3.35)$$

Provided $\bar{\rho} \ge M_3(\Omega, C_0, \bar{\mu}, \bar{\xi}, \bar{\eta}, \bar{\lambda}, \alpha, \beta, \|\nabla\rho_0\|_{L^q}, \|u_0\|_{H^2}, \|w_0\|_{H^2})$.

Proof. Taking the t-derivative of $(1.1)_2$–$(1.1)_3$ and multiply the resulting equation by $(tu_t, tw_t)^\top$, after integration by parts, we obtain

$$
\begin{aligned}
&\frac{t}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho(|u_t|^2+|w_t|^2)\mathrm{d}x + 2t\int \left(\bar{\mu}\rho^\alpha + \bar{\xi}\rho^\beta\right)|\mathcal{D}(u_t)|^2\mathrm{d}x \\
&+ 4\bar{\xi}t\int \rho^\beta|w_t|^2\mathrm{d}x + 2\bar{\eta}t\int \rho^\alpha|\mathcal{D}(w_t)|^2\mathrm{d}x + \bar{\lambda}t\int \rho^\alpha(\mathrm{div}w_t)^2\mathrm{d}x \\
=\ & t\int u\cdot\nabla\rho(|u_t|^2+|w_t|^2)\mathrm{d}x - t\int \rho u_t\cdot\nabla u\cdot u_t\mathrm{d}x - t\int \rho u_t\cdot\nabla w\cdot w_t\mathrm{d}x \\
&+ t\int (u\cdot\nabla\rho)(u\cdot\nabla u\cdot u_t)\mathrm{d}x + t\int (u\cdot\nabla\rho)(u\cdot\nabla w\cdot w_t)\mathrm{d}x + 4\beta\bar{\xi}t\int \rho^{\beta-1}(u\cdot\nabla\rho)w\cdot w_t\mathrm{d}x \\
&+ 2t\int \left(\alpha\bar{\mu}\rho^{\alpha-1} + \beta\bar{\xi}\rho^{\beta-1}\right)(u\cdot\nabla\rho)\mathcal{D}(u):\nabla u_t\mathrm{d}x \\
&+ 2\alpha\bar{\eta}t\int \rho^{\alpha-1}(u\cdot\nabla\rho)\mathcal{D}(w):\nabla w_t\mathrm{d}x + \alpha\bar{\lambda}t\int \rho^{\alpha-1}(u\cdot\nabla\rho)\mathrm{div}w\mathrm{div}w_t\mathrm{d}x \\
&- 2\beta\bar{\xi}t\int \rho^{\beta-1}(u\cdot\nabla\rho)\left(u_t\cdot(\nabla\times w) + w_t\cdot(\nabla\times u)\right)\mathrm{d}x + 2\bar{\xi}t\int \rho^\beta\left(u_t\cdot(\nabla\times w_t) + w_t\cdot(\nabla\times u_t)\right)\mathrm{d}x \\
=:\ & \sum_{i=1}^{11} J_i,
\end{aligned}
\tag{3.36}
$$

where the terms J_i $(i = 1, \cdots, 11)$ can be estimated as follows.

$$
\begin{aligned}
J_1 \le\ & Ct\int |\nabla\rho\cdot u|(|u_t|^2+|w_t|^2)\mathrm{d}x \\
\le\ & Ct\|\nabla\rho\|_{L^q}\|u\|_{L^{\frac{2q}{q-2}}}(\|u_t\|_{L^4}^2 + \|w_t\|_{L^4}^2) \\
\le\ & Ct\bar{\rho}^{-\frac{1}{4}}\|\nabla\rho\|_{L^q}\|u\|_{L^2}^{\frac{q-3}{q}}\|\nabla u\|_{L^2}^{\frac{3}{q}}\left(\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}^{\frac{3}{2}} + \|\sqrt{\rho}w_t\|_{L^2}^{\frac{1}{2}}\|\nabla w_t\|_{L^2}^{\frac{3}{2}}\right) \\
\le\ & \frac{\bar{\mu}}{10}t\bar{\rho}^\alpha\|\nabla u_t\|_{L^2}^2 + \frac{\bar{\eta}}{10}t\bar{\rho}^\alpha\|\nabla w_t\|_{L^2}^2 + Ct\bar{\rho}^{-3\alpha-1}\|\nabla\rho\|_{L^q}^4\|\nabla u\|_{L^2}^{\frac{12}{q}}(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}w_t\|_{L^2}^2) \\
\le\ & \frac{\bar{\mu}}{10}t\bar{\rho}^\alpha\|\nabla u_t\|_{L^2}^2 + \frac{\bar{\eta}}{10}t\bar{\rho}^\alpha\|\nabla w_t\|_{L^2}^2 + Ct\bar{\rho}^{-3\alpha-1}\|\nabla u\|_{L^2}^{\frac{12}{q}}(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}w_t\|_{L^2}^2),
\end{aligned}
\tag{3.37}
$$

$$
\begin{aligned}
J_2 + J_3 \le\ & Ct\int (|\rho\nabla u||u_t|^2 + |\rho\nabla w||u_t||w_t|)\mathrm{d}x \\
\le\ & Ct\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^2}(\|\nabla u\|_{L^3}\|u_t\|_{L^6} + \|\nabla w\|_{L^3}\|w_t\|_{L^6}) \\
\le\ & Ct\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^2}(\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^6}^{\frac{1}{2}}\|\nabla u_t\|_{L^2} + \|\nabla w\|_{L^2}^{\frac{1}{2}}\|\nabla w\|_{L^6}^{\frac{1}{2}}\|\nabla w_t\|_{L^2}) \\
\le\ & \frac{\bar{\mu}}{10}t\bar{\rho}^\alpha\|\nabla u_t\|_{L^2}^2 + \frac{\bar{\eta}}{10}t\bar{\rho}^\alpha\|\nabla w_t\|_{L^2}^2 + Ct\bar{\rho}^{1-\alpha}\|\sqrt{\rho}u_t\|_{L^2}^2(\|\nabla u\|_{L^2}\|\nabla u\|_{H^1} + \|\nabla w\|_{L^2}\|\nabla w\|_{H^1}) \\
\le\ & \frac{\bar{\mu}}{10}t\bar{\rho}^\alpha\|\nabla u_t\|_{L^2}^2 + \frac{\bar{\eta}}{10}t\bar{\rho}^\alpha\|\nabla w_t\|_{L^2}^2 + Ct\bar{\rho}^{\frac{3}{2}-2\alpha}\|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}^3 + Ct\bar{\rho}^{3-3\alpha}\|\nabla u\|_{L^2}^4\|\sqrt{\rho}u_t\|_{L^2}^2 \\
&+ Ct\bar{\rho}^{1+\beta-2\alpha}\|\nabla u\|_{L^2}\|\nabla w\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}^2 + Ct\bar{\rho}^{\frac{3}{2}-2\alpha}\|\nabla w\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}^2\|\sqrt{\rho}w_t\|_{L^2} \\
&+ Ct\bar{\rho}^{3-3\alpha}\|\nabla u\|_{L^2}^2\|\nabla w\|_{L^2}^2\|\sqrt{\rho}u_t\|_{L^2}^2,
\end{aligned}
\tag{3.38}
$$

where we have used (3.13) and (3.14). Similarly, we have

$$
\begin{aligned}
J_4+J_5 \leq\ & Ct\int|\nabla\rho||u|^2(|\nabla u||u_t|+|\nabla w||w_t|)\mathrm{d}x\\
\leq\ & Ct\|\nabla\rho\|_{L^q}\|u\|_{L^6}^2\left(\|\nabla u\|_{L^{\frac{2q}{q-2}}}\|u_t\|_{L^6}+\|\nabla w\|_{L^{\frac{2q}{q-2}}}\|w_t\|_{L^6}\right)\\
\leq\ & \frac{\bar\mu}{10}t\bar\rho^{\alpha}\|\nabla u_t\|_{L^2}^2+\frac{\bar\eta}{10}t\bar\rho^{\alpha}\|\nabla w_t\|_{L^2}^2+Ct\bar\rho^{-\alpha}\|\nabla\rho\|_{L^q}^2\|\nabla u\|_{L^2}^{\frac{6(q-1)}{q}}\|\nabla u\|_{H^1}^{\frac{6}{q}}\\
& +Ct\bar\rho^{-\alpha}\|\nabla\rho\|_{L^q}^2\|\nabla u\|_{L^2}^4\|\nabla w\|_{L^2}^{2-\frac{6}{q}}\|\nabla w\|_{H^1}^{\frac{6}{q}}\\
\leq\ & \frac{\bar\mu}{10}t\bar\rho^{\alpha}\|\nabla u_t\|_{L^2}^2+\frac{\bar\eta}{10}t\bar\rho^{\alpha}\|\nabla w_t\|_{L^2}^2+Ct\bar\rho^{-\alpha+\frac{6}{q}(\frac12-\alpha)}\|\nabla u\|_{L^2}^{\frac{6(q-1)}{q}}\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6}{q}}\\
& +Ct\rho^{-\alpha+2(1-\alpha)\frac{6}{q}}\|\nabla u\|_{L^2}^{6+\frac{12}{q}}+Ct\bar\rho^{-\alpha+\frac{6}{q}(\beta-\alpha)}\|\nabla u\|_{L^2}^{\frac{6(q-1)}{q}}\|\nabla w\|_{L^2}^{\frac{6}{q}}\\
& +Ct\bar\rho^{-\alpha+\frac{6}{q}(\frac12-\alpha)}\|\nabla u\|_{L^2}^4\|\nabla w\|_{L^2}^{2-\frac{6}{q}}\|\sqrt{\rho}w_t\|_{L^2}^{\frac{6}{q}}\\
& +Ct\rho^{-\alpha+2(1-\alpha)\frac{6}{q}}\|\nabla u\|_{L^2}^{4+\frac{12}{q}}\|\nabla w\|_{L^2}^2+Ct\bar\rho^{-\alpha+\frac{6}{q}(\beta-\alpha)}\|\nabla u\|_{L^2}^{4+\frac{6}{q}}\|\nabla w\|_{L^2}^{2-\frac{6}{q}}\\
\leq\ & \frac{\bar\mu}{10}t\bar\rho^{\alpha}\|\nabla u_t\|_{L^2}^2+\frac{\bar\eta}{10}t\bar\rho^{\alpha}\|\nabla w_t\|_{L^2}^2+Ct\left(\bar\rho^{-\alpha+2(1-\alpha)\frac{6}{q}}+\bar\rho^{-\alpha+\frac{6}{q}(\beta-\alpha)}\right)\|\nabla u\|_{L^2}^4\\
& +Ct\bar\rho^{-\alpha+\frac{6}{q}(\frac12-\alpha)}\left(\|\nabla u\|_{L^2}^2(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2)+\|\nabla u\|_{L^2}^{4+\frac{6}{q-3}}(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2)\right),
\end{aligned}
\tag{3.39}
$$

$$
\begin{aligned}
& J_6+J_7+J_8+J_9+J_{10}\\
\leq\ & Ct\bar\rho^{\alpha-1}\int|u||\nabla\rho|\left(|w||w_t|+|\nabla w||\nabla w_t|+|\nabla u||\nabla u_t|+|\nabla w||u_t|+|\nabla u||w_t|\right)\mathrm{d}x\\
\leq\ & Ct\bar\rho^{\alpha-1}\|\nabla\rho\|_{L^q}\|u\|_{L^{\frac{3q}{q-3}}}(\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2}+\|\nabla w\|_{L^6}\|\nabla w_t\|_{L^2}+\|\nabla u\|_{L^2}\|w_t\|_{L^6}+\|\nabla w\|_{L^2}\|u_t\|_{L^6})\\
\leq\ & \frac{\bar\mu}{10}t\bar\rho^{\alpha}\|\nabla u_t\|_{L^2}^2+\frac{\bar\eta}{10}t\bar\rho^{\alpha}\|\nabla w_t\|_{L^2}^2+Ct\bar\rho^{\alpha-2}\|\nabla\rho\|_{L^q}^2\|\nabla u\|_{L^2}^{3-\frac{6}{q}}\|\nabla u\|_{H^1}^{\frac{6}{q}-1}(\|\nabla u\|_{H^1}^2+\|\nabla w\|_{H^1}^2)\\
\leq\ & \frac{\bar\mu}{10}t\bar\rho^{\alpha}\|\nabla u_t\|_{L^2}^2+\frac{\bar\eta}{10}t\bar\rho^{\alpha}\|\nabla w_t\|_{L^2}^2+Ct\bar\rho^{-\frac32+(\frac12-\alpha)\frac{6}{q}}\|\nabla u\|_{L^2}^{3-\frac{6}{q}}\|\sqrt{\rho}u_t\|_{L^2}^{1+\frac{6}{q}}+Ct\bar\rho^{-\alpha+2(1-\alpha)\frac{6}{q}}\|\nabla u\|_{L^2}^4\\
& +Ct\bar\rho^{\beta-2+(\beta-\alpha)\frac{6}{q}}\|\nabla u\|_{L^2}^{3-\frac{6}{q}}\|\nabla w\|_{L^2}^{1+\frac{6}{q}}+Ct\bar\rho^{-\frac32+(\frac12-\alpha)\frac{6}{q}}\|\nabla u\|_{L^2}^{3-\frac{6}{q}}\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6}{q}-1}\|\sqrt{\rho}w_t\|_{L^2}^2\\
& +Ct\bar\rho^{\beta-2+(\beta-\alpha)\frac{6}{q}}\|\nabla u\|_{L^2}^{5-\frac{6}{q}}\|\nabla w\|_{L^2}^{\frac{6}{q}-1}.
\end{aligned}
\tag{3.40}
$$

Next, taking M_3 large enough, such that

$$
M_3\geq M_2,\quad \text{and}\quad C\bar\rho^{\beta}\leq\frac{1}{10}\min\{\bar\mu,\bar\eta\}\bar\rho^{\alpha},
$$

we have

$$
\begin{aligned}
J_{11}\leq\ & Ct\bar\rho^{\beta}(\|u_t\|_{L^2}\|\nabla w_t\|_{L^2}+\|w_t\|_{L^2}\|\nabla u_t\|_{L^2})\\
\leq\ & Ct\bar\rho^{\beta}\|\nabla u_t\|_{L^2}\|\nabla w_t\|_{L^2}\\
\leq\ & Ct\bar\rho^{\beta}(\|\nabla u_t\|_{L^2}^2+\|\nabla w_t\|_{L^2}^2)\\
\leq\ & \frac{\bar\mu}{10}t\bar\rho^{\alpha}\|\nabla u_t\|_{L^2}^2+\frac{\bar\eta}{10}t\bar\rho^{\alpha}\|\nabla w_t\|_{L^2}^2.
\end{aligned}
\tag{3.41}
$$

Substituting (3.37)–(3.41) into (3.36), then using (3.4) and (3.8), we deduce

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(t\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2\right)\right)+t\bar{\rho}^{\alpha}\left(\bar{\mu}\|\nabla u_t\|_{L^2}^2+\bar{\eta}\|\nabla w_t\|_{L^2}^2\right)\\
\leq\ &\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2+Ct\left(\bar{\rho}^{\frac{3}{2}-2\alpha}+\bar{\rho}^{-\frac{3}{2}+(\frac{1}{2}-\alpha)\frac{6}{q}}\right)(\|\nabla u\|_{L^2}+\|\nabla w\|_{L^2})\left(\|\sqrt{\rho}u_t\|_{L^2}^3+\|\sqrt{\rho}w_t\|_{L^2}^3\right)\\
&+Ct\left(\bar{\rho}^{3-3\alpha}+\bar{\rho}^{1+\beta-2\alpha}+\bar{\rho}^{-\alpha+\frac{6}{q}(\frac{1}{2}-\alpha)}\right)\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right)\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2\right)\\
&+Ct\left(\bar{\rho}^{-\alpha+2(1-\alpha)\frac{6}{q}}+\bar{\rho}^{-\alpha+\frac{6}{q}(\beta-\alpha)}+\bar{\rho}^{-\alpha+\frac{6}{q}(\frac{1}{2}-\alpha)}+\bar{\rho}^{\beta-2+(\beta-\alpha)\frac{6}{q}}\right)\left(\|\nabla u\|_{L^2}^4+\|\nabla w\|_{L^2}^4\right),
\end{aligned}
\tag{3.42}
$$

due to $q\in(3,6)$ and (3.4). Thus, Gronwall's inequality yields

$$
\begin{aligned}
&\sup_{0\leq t\leq T}t\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2\right)+\bar{\rho}^{\alpha}\int_0^T t\left(\|\nabla u_t\|_{L^2}^2+\|\nabla w_t\|_{L^2}^2\right)\mathrm{d}t\\
\leq\ &C\int_0^T\left(t\left(\bar{\rho}^{-\alpha+2(1-\alpha)\frac{6}{q}}+\bar{\rho}^{-\alpha+\frac{6}{q}(\beta-\alpha)}+\bar{\rho}^{-\alpha+\frac{6}{q}(\frac{1}{2}-\alpha)}+\bar{\rho}^{\beta-2+(\beta-\alpha)\frac{6}{q}}\right)\left(\|\nabla u\|_{L^2}^4+\|\nabla w\|_{L^2}^4\right)\right.\\
&\left.+\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2\right)\mathrm{d}t\\
&\cdot\exp\left\{\left(\bar{\rho}^{\frac{3}{2}-2\alpha}+\bar{\rho}^{-\frac{3}{2}+(\frac{1}{2}-\alpha)\frac{6}{q}}\right)\int_0^T(\|\nabla u\|_{L^2}+\|\nabla w\|_{L^2})(\|\sqrt{\rho}u_t\|_{L^2}+\|\sqrt{\rho}w_t\|_{L^2})\,\mathrm{d}t\right\}\\
&\cdot\exp\left\{\left(\bar{\rho}^{3-3\alpha}+\bar{\rho}^{1+\beta-2\alpha}+\bar{\rho}^{-\alpha+\frac{6}{q}(\frac{1}{2}-\alpha)}\right)\int_0^T\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right)\mathrm{d}t\right\}.
\end{aligned}
\tag{3.43}
$$

Now we estimate the terms on the right-hand side of (3.43). By (3.4), (3.8), and (3.18), we have

$$
\int_0^T t\left(\|\nabla u\|_{L^2}^4+\|\nabla w\|_{L^2}^4\right)\mathrm{d}t\leq\sup_{t\in[0,T]}t(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2)\int_0^T(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2)\mathrm{d}t\leq C\bar{\rho}^{2(1-\alpha)}. \tag{3.44}
$$

Next, Hölder's inequality yields

$$
\begin{aligned}
&\int_0^T(\|\nabla u\|_{L^2}+\|\nabla w\|_{L^2})(\|\sqrt{\rho}u_t\|_{L^2}+\|\sqrt{\rho}w_t\|_{L^2})\,\mathrm{d}t\\
\leq\ &\left(\int_0^T(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2)\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^T\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2\right)\mathrm{d}t\right)^{\frac{1}{2}}\leq C\bar{\rho}^{\frac{1}{2}}.
\end{aligned}
\tag{3.45}
$$

Inserting (3.44)–(3.45) into (3.43), one gets

$$
\sup_{0\leq t\leq T}t\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2\right)+\bar{\rho}^{\alpha}\int_0^T t\left(\|\nabla u_t\|_{L^2}^2+\|\nabla w_t\|_{L^2}^2\right)\mathrm{d}t\leq C\bar{\rho}^{\alpha}\exp\{C\bar{\rho}^{-A}\}\leq C\bar{\rho}^{\alpha}. \tag{3.46}
$$

For some constant $A>0$. This completes the proof of (3.34).

On the other hand, multiplying (3.42) by t, one has

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(t^2\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2\right)\right)+t^2\bar{\rho}^{\alpha}\left(\bar{\mu}\|\nabla u_t\|_{L^2}^2+\bar{\eta}\|\nabla w_t\|_{L^2}^2\right)\\
\leq\ &Ct(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2)+Ct^2\left(\bar{\rho}^{\frac{3}{2}-2\alpha}+\bar{\rho}^{-\frac{3}{2}+(\frac{1}{2}-\alpha)\frac{6}{q}}\right)(\|\nabla u\|_{L^2}+\|\nabla w\|_{L^2})\left(\|\sqrt{\rho}u_t\|_{L^2}^3+\|\sqrt{\rho}w_t\|_{L^2}^3\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ Ct^2\left(\bar{\rho}^{3-3\alpha} + \bar{\rho}^{1+\beta-2\alpha} + \bar{\rho}^{-\alpha+\frac{6}{q}(\frac{1}{2}-\alpha)}\right)\left(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right)\left(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}w_t\|_{L^2}^2\right)\\
&+ Ct^2\left(\bar{\rho}^{-\alpha+2(1-\alpha)\frac{6}{q}} + \bar{\rho}^{-\alpha+\frac{6}{q}(\beta-\alpha)} + \bar{\rho}^{-\alpha+\frac{6}{q}(\frac{1}{2}-\alpha)} + \bar{\rho}^{\beta-2+(\beta-\alpha)\frac{6}{q}}\right)\left(\|\nabla u\|_{L^2}^4 + \|\nabla w\|_{L^2}^4\right). \qquad (3.47)
\end{aligned}
$$

Applying Gronwall's inequality yields

$$
\begin{aligned}
&\sup_{0\le t\le T} t^2\left(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}w_t\|_{L^2}^2\right) + \bar{\rho}^{\alpha}\int_0^T t^2\left(\|\nabla u_t\|_{L^2}^2 + \|\nabla w_t\|_{L^2}^2\right)\mathrm{d}t\\
\le\ & C\int_0^T \Big(t^2\left(\bar{\rho}^{-\alpha+2(1-\alpha)\frac{6}{q}} + \bar{\rho}^{-\alpha+\frac{6}{q}(\beta-\alpha)} + \bar{\rho}^{-\alpha+\frac{6}{q}(\frac{1}{2}-\alpha)} + \bar{\rho}^{\beta-2+(\beta-\alpha)\frac{6}{q}}\right)\left(\|\nabla u\|_{L^2}^4 + \|\nabla w\|_{L^2}^4\right)\\
&+ t(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}w_t\|_{L^2}^2)\Big)\mathrm{d}t\cdot\exp\{\bar{\rho}^{-A}\}\\
\le\ & C\bar{\rho}\cdot\exp\{\bar{\rho}^{-A}\}\\
\le\ & C\bar{\rho}, \qquad (3.48)
\end{aligned}
$$

which completes the proof of (3.35). □

Finally, we are in a position to close the a priori assumption of $\mathcal{E}_\rho$, the key observation is that $\|\nabla u\|_{L_t^1 L_x^\infty}$ is uniformly bounded with respect to time T.

Lemma 3.6. *Under the conditions of Proposition 3.1, there exists a positive constant M_4 such that*

$$\mathcal{E}_\rho(T) \le 2\mathcal{E}_\rho(0), \qquad (3.49)$$

provided $\bar{\rho} \ge M_4(\Omega, C_0, \bar{\mu}, \bar{\xi}, \bar{\eta}, \bar{\lambda}, \alpha, \beta, \|\nabla\rho_0\|_{L^q}, \|u_0\|_{H^2}, \|w_0\|_{H^2})$.

Proof. Applying the operator ∇ to $(1.1)_1$, we get

$$\nabla\rho_t + u\cdot\nabla^2\rho + \nabla u\cdot\nabla\rho = 0. \qquad (3.50)$$

Making the L^2-inner product of (3.50) with $|\nabla\rho|^{q-2}\nabla\rho$ and then integrating by parts, we have

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho\|_{L^q}^q = -\int \nabla\rho\cdot\nabla u\cdot\nabla\rho|\nabla\rho|^{q-2}\mathrm{d}x \le C\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^q}^q. \qquad (3.51)$$

It follows from Gronwall's inequality that

$$\|\nabla\rho\|_{L^q} \le \|\nabla\rho_0\|_{L^q}\cdot\exp\Big\{C\int_0^T \|\nabla u\|_{L^\infty}\mathrm{d}t\Big\}. \qquad (3.52)$$

Therefore, if we can prove the estimate

$$\int_0^T \|\nabla u\|_{L^\infty}\mathrm{d}t \le C\bar{\rho}^{-B}, \qquad (3.53)$$

for some constant $B > 0$, then by taking M_4 sufficiently large such that $M_4 \ge M_3$ and $C\bar{\rho}^{-B} \le \ln 2$, we complete the proof of Lemma 3.6.

Adding (3.15) and (3.16) together, we get

$$\|u\|_{W^{2,q}}+\|w\|_{W^{2,q}}\le C\left(\bar\rho^{-\alpha}(\|\rho u_t\|_{L^q}+\|\rho w_t\|_{L^q})+\bar\rho^{(1-\alpha)\frac{5q-6}{q}}\|\nabla u\|_{L^2}^{\frac{5q-6}{q}}(\|\nabla u\|_{L^2}+\|\nabla w\|_{L^2})\right). \tag{3.54}$$

It follows from (3.8), (3.17), and (3.54) that

$$\begin{aligned}
&\int_0^T\|\nabla u\|_{L^\infty}\mathrm{d}t\le C\int_0^T\|\nabla u\|_{W^{1,q}}\mathrm{d}t\\
\le\ & C\bar\rho^{-\alpha}\int_0^T(\|\rho u_t\|_{L^q}+\|\rho w_t\|_{L^q})\,\mathrm{d}t+C\bar\rho^{(1-\alpha)\frac{5q-6}{q}}\int_0^T\|\nabla u\|_{L^2}^{\frac{5q-6}{q}}(\|\nabla u\|_{L^2}+\|\nabla w\|_{L^2})\mathrm{d}t\\
\le\ & C\bar\rho^{-\alpha}\int_0^T(\|\rho u_t\|_{L^q}+\|\rho w_t\|_{L^q})\,\mathrm{d}t+C\bar\rho^{(1-\alpha)\frac{5q-6}{q}}\int_0^T\|\nabla u\|_{L^2}^{2}\mathrm{d}t\\
\le\ & C\bar\rho^{-\alpha}\int_0^T(\|\rho u_t\|_{L^q}+\|\rho w_t\|_{L^q})\,\mathrm{d}t+C\bar\rho^{\frac{6(q-1)}{q}(1-\alpha)}.
\end{aligned}\tag{3.55}$$

Using the Gagliardo-Nirenberg inequality, (3.34), and (3.35), we have

$$\begin{aligned}
\int_0^T(\|\rho u_t\|_{L^q}+\|\rho w_t\|_{L^q})\,\mathrm{d}t\le\ & C\bar\rho^{\frac{5q-6}{4q}}\int_0^T\left(\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{2q}}\|\nabla u_t\|_{L^2}^{\frac{3(q-2)}{2q}}+\|\sqrt{\rho}w_t\|_{L^2}^{\frac{6-q}{2q}}\|\nabla w_t\|_{L^2}^{\frac{3(q-2)}{2q}}\right)\mathrm{d}t\\
\le\ & C\bar\rho^{\frac{5q-6}{4q}}\left(\sup_{0\le t\le\min\{1,T\}}t\|\sqrt{\rho}u_t\|_{L^2}^2\mathrm{d}t\right)^{\frac{6-q}{4q}}\\
&\cdot\left(\int_0^{\min\{1,T\}}t\|\nabla u_t\|_{L^2}^2\mathrm{d}t\right)^{\frac{3(q-2)}{4q}}\left(\int_0^{\min\{1,T\}}t^{-\frac{2q}{q+6}}\mathrm{d}t\right)^{\frac{q+6}{4q}}\\
&+C\bar\rho^{\frac{5q-6}{4q}}\left(\sup_{\min\{1,T\}\le t\le T}t^2\|\sqrt{\rho}u_t\|_{L^2}^2\mathrm{d}t\right)^{\frac{6-q}{4q}}\\
&\cdot\left(\int_{\min\{1,T\}}^T t^2\|\nabla u_t\|_{L^2}^2\mathrm{d}t\right)^{\frac{3(q-2)}{4q}}\left(\int_{\min\{1,T\}}^T t^{-\frac{4q}{q+6}}\mathrm{d}t\right)^{\frac{q+6}{4q}}\\
&+C\bar\rho^{\frac{5q-6}{4q}}\left(\sup_{0\le t\le\min\{1,T\}}t\|\sqrt{\rho}w_t\|_{L^2}^2\mathrm{d}t\right)^{\frac{6-q}{4q}}\\
&\cdot\left(\int_0^{\min\{1,T\}}t\|\nabla w_t\|_{L^2}^2\mathrm{d}t\right)^{\frac{3(q-2)}{4q}}\left(\int_0^{\min\{1,T\}}t^{-\frac{2q}{q+6}}\mathrm{d}t\right)^{\frac{q+6}{4q}}\\
&+C\bar\rho^{\frac{5q-6}{4q}}\left(\sup_{\min\{1,T\}\le t\le T}t^2\|\sqrt{\rho}w_t\|_{L^2}^2\mathrm{d}t\right)^{\frac{6-q}{4q}}\\
&\cdot\left(\int_{\min\{1,T\}}^T t^2\|\nabla w_t\|_{L^2}^2\mathrm{d}t\right)^{\frac{3(q-2)}{4q}}\left(\int_{\min\{1,T\}}^T t^{-\frac{4q}{q+6}}\mathrm{d}t\right)^{\frac{q+6}{4q}}\\
\le\ & C\bar\rho^{\frac{5q-6}{4q}+\alpha\frac{6-q}{4q}}+C\bar\rho^{\frac{5q-6}{4q}+\frac{6-q}{4q}+(1-\alpha)\frac{3(q-2)}{4q}}\\
\le\ & C\bar\rho^{\frac{5q-6}{4q}+\alpha\frac{6-q}{4q}}.
\end{aligned}\tag{3.56}$$

Combining (3.55) and (3.56), we obtain

$$\int_0^T \|\nabla u\|_{L^\infty} \mathrm{d}t \le C\bar{\rho}^{\frac{5q-6}{4q}+\alpha\frac{6-q}{4q}-\alpha} + C\bar{\rho}^{\frac{6(q-1)}{q}(1-\alpha)} \le C\bar{\rho}^{-D}, \tag{3.57}$$

where

$$D = \min\left\{\frac{5q-6}{4q}(\alpha-1), \frac{6(q-1)}{q}(\alpha-1)\right\}.$$

Thus, we complete the proof. □

At last, combining Lemmata 3.4 and 3.6 then taking $M = M_4$, we have proved Proposition 3.1. With the help of Lemmata 3.1–3.6, Theorem 1.1 can be established through a standard bootstrap argument (see [29]). We omit it here.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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