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http://www.aimspress.com/journal/era

ERA, 33(12): 7570–7583.
DOI: 10.3934/era.2025334
Received: 21 October 2025
Revised: 07 December 2025
Accepted: 11 December 2025
Published: 17 December 2025

Research article

Characterizing skew n-derivations on triangular rings

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Abstract: Every ring containing a unity has a maximal right ring of quotients. In this paper, we investigate a skew n-derivation ϕ on a class of triangular rings and use the theory of maximal right ring of quotients to demonstrate that ϕ is an extremal skew n-derivation. This result not only significantly generalizes previous findings on derivations, but also forges a strong connection between the theory of derivations and that of rings of quotients, providing a tool for characterizing functional identities in various specific operator algebras.

Keywords: triangular ring; skew 3-derivation; skew n-derivation; maximal right ring of quotients; extremal skew n-derivation

1. Introduction

In 1956, Utumi [1] introduced the notion of the maximal left (or right) ring of quotients, also known as the Utumi ring of quotients, and demonstrated that every unital ring possesses a maximal left (or right) ring of quotients. For a comprehensive introduction to the maximal left (or right) ring of quotients, we refer the reader to [2, Chapter 2].

The investigation of various mappings on triangular rings and algebras has been a fruitful research direction. It was advanced by Cheung [3], who in 2001 characterized commuting mappings for a specific class of triangular algebras. A significant extension came from Wang et al. [4] in 2013, who studied n-derivations (with $n \geq 3$) on triangular algebras, building upon the work of Xu et al. [5] on semiprime rings. Around 2015, the theory of maximal rings of quotients became a key tool in this area. Eremita [6] and Wang [7] used it to investigate functional identities of degree 2. Subsequently, Eremita [8] described biderivations on triangular rings using this framework, improving upon Benkovič's results [9] for triangular algebras, while Wang [10] also examined biderivations from a different perspective. More recently, Liang et al. [11, 12] applied this theory to study bi-Lie n-derivations and n-Lie m-derivations.

The study of skew derivations holds a foundational position in noncommutative algebra and its applications. Skew derivations form the basis for constructing skew polynomial rings, which arise naturally in quantum groups and representation theory. Skew n-derivations extend this concept to the multilinear setting and are closely tied to classical problems in ring theory concerning commutativity and structural properties. This research direction can be traced back to Posner [13], who proved that the existence of a nonzero centralizing derivation on a prime ring $\mathcal{R}$ implies that $\mathcal{R}$ is commutative. A significant advancement was made by Brešar et al. [14], who systematically studied mappings satisfying specific identities on prime rings with involution. Their work established essential tools and a unified framework for analyzing multilinear mappings, including skew n-derivations, and laid the groundwork for characterizing such mappings across a range of algebraic structures, such as triangular algebras, matrix algebras, and prime rings with involution.

Notably, the first author and a colleague [15] utilized the maximal right ring of quotients to study skew biderivations. Despite this progress, the case of skew n-derivations (for $n \geq 3$) on triangular rings, which constitutes a natural and important extension, has remained entirely unexplored. Motivated by these developments, particularly the successful application of the maximal right ring of quotients, we systematically investigate skew n-derivations on triangular rings. The main goal of this paper is to show that, under certain natural conditions, every such skew n-derivation is extremal. This result provides a complete characterization of these mappings and embeds the study of skew n-derivations into the established theory of functional identities on triangular rings, highlighting the pivotal role of extremal mappings.

To prove these theorems, our strategy hinges on an inductive argument over n. The core of the induction lies in a meticulous analysis of the Peirce decomposition for the base case $n = 3$. Once the extremal form is confirmed for $n = 3$, the inductive step seamlessly propagates this structural description to all higher $n \geq 4$, ultimately showing that every skew n-derivation can be captured as an extremal skew n-derivation involving an element from $e\mathcal{T}f$.

The study of skew n-derivations is crucial for addressing problems in operator algebras and the general theory of functional identities, where such mappings are prevalent. Characterizing these derivations provides profound insights into the underlying algebraic structures. Our work not only generalizes prior results but also demonstrates the efficacy of using the maximal ring of quotients as a unifying tool. Future research may extend these findings to more general classes of rings or explore connections with other nonlinear mappings.

The structure of the paper is as follows. Section 2 covers the necessary preliminaries and tools. In Section 3, we first characterize the skew 3-derivation on the triangular ring $\mathcal{T}$ and demonstrate that it is extremal. Subsequently, an investigation of a skew n-derivation on $\mathcal{T}$ is conducted using induction.

2. Preliminaries

Let $\mathcal{T}$ be a ring with an automorphism σ. An additive mapping $d : \mathcal{T} \to \mathcal{T}$ is called a σ-derivation if $d(xy) = d(x)y + \sigma(x)d(y)$ holds for all $x, y \in \mathcal{T}$. A multiadditive mapping $\phi : \mathcal{T} \times \mathcal{T} \times \cdots \times \mathcal{T} \to \mathcal{T}$ is called a skew n-derivation with respect to σ if it is a σ-derivation of $\mathcal{T}$ for each argument ($n \geq 2$). Namely, when $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \in \mathcal{T}$ are fixed, then for all $x_i, y_i \in \mathcal{T}$, both

$$\phi(x_1, \ldots, x_i + y_i, \ldots, x_n) = \phi(x_1, \ldots, x_i, \ldots, x_n) + \phi(x_1, \ldots, y_i, \ldots, x_n)$$

and

$$\phi(x_1, \ldots, x_i y_i, \ldots, x_n) = \phi(x_1, \ldots, x_i, \ldots, x_n) y_i + \sigma(x_i)\phi(x_1, \ldots, y_i, \ldots, x_n)$$

always hold. Define

$$[x, y]_\sigma = \sigma(x)y - yx \tag{2.1}$$

for all $x, y \in \mathcal{T}$. We will call (2.1) the σ-commutator of $\mathcal{T}$. Note that

$$Z_\sigma(\mathcal{T}) = \{\lambda \in \mathcal{T} | [x, \lambda]_\sigma = 0 \text{ for all } x \in \mathcal{T}\}.$$

We first fix some notation. Let $\mathcal{T}$ be a ring, and let $Z(\mathcal{T})$ denote its center. The maximal right and left rings of quotients of $\mathcal{T}$ are denoted by $Q_r(\mathcal{T})$ and $Q_l(\mathcal{T})$, respectively. The center of $Q_r(\mathcal{T})$, which is the extended centroid of $\mathcal{T}$, is written as $C_r(\mathcal{T})$. Given two subsets $\mathcal{S}, \mathcal{P} \subseteq Q_r(\mathcal{T})$, we set

$$C_\sigma(\mathcal{S}, \mathcal{P}) = \{s \in \mathcal{S} | sx = \sigma(x)s \text{ for all } x \in \mathcal{P}\}.$$

Following [6], we recall that a right (resp. left) ideal I of $\mathcal{T}$ is dense if for every pair $0 \neq r_1 \in \mathcal{T}$, $r_2 \in \mathcal{T}$, there exists $r \in \mathcal{T}$ with $r_1 r \neq 0$ and $r_2 r \in I$ (resp. $rr_1 \neq 0$ and $rr_2 \in I$). The properties of $Q_l(\mathcal{T})$ given in [6] extend by symmetry to $Q_r(\mathcal{T})$.

Proposition 2.1. *Let $\mathcal{T}$ be a unital ring. The maximal right ring of quotients $Q_r(\mathcal{T})$ satisfies the following properties:*

(i) $\mathcal{T}$ is a subring of $Q_r(\mathcal{T})$ with the same 1,
(ii) for any $q \in Q_r(\mathcal{T})$ there exists a dense right ideal I of $\mathcal{T}$ such that $qI \subseteq \mathcal{T}$,
(iii) if $0 \neq q \in Q_r(\mathcal{T})$ and I is a dense right ideal of $\mathcal{T}$, then $qI \neq 0$,
(iv) for any dense right ideal I of $\mathcal{T}$ and a right $\mathcal{T}-$module homomorphism $f : I \to \mathcal{T}$, there exists $q \in Q_r(\mathcal{T})$ such that $f(x) = qx$ for all $x \in I$.

Let $\mathcal{T}$ be a unital ring with a nontrivial idempotent element e. The ring $\mathcal{T}$ is called a triangular ring if $e\mathcal{T}f$ is a left $e\mathcal{T}e$-module and also a right $f\mathcal{T}f$-module, and $f\mathcal{T}e = 0$, where $f = 1 - e$. Therefore, a triangular ring has the following Peirce decomposition:

$$\mathcal{T} = e\mathcal{T}e \oplus e\mathcal{T}f \oplus f\mathcal{T}f.$$

For any invertible element $a \in \mathcal{T}$, the mapping $\sigma_a(x) = a^{-1}xa$ is an automorphism of $\mathcal{T}$, called the inner automorphism defined by a. Cheung introduced the concept of a partible automorphism in [16]. An automorphism σ of $\mathcal{T}$ is partible with respect to $e\mathcal{T}e, e\mathcal{T}f, f\mathcal{T}f$ if it can be decomposed as $\sigma = \sigma_a \circ \bar{\sigma}$, where $\bar{\sigma}$ is an automorphism of $\mathcal{T}$ that preserves each component, meaning:

$$\bar{\sigma}(e\mathcal{T}e) = e\mathcal{T}e, \quad \bar{\sigma}(e\mathcal{T}f) = e\mathcal{T}f, \quad \bar{\sigma}(f\mathcal{T}f) = f\mathcal{T}f.$$

According to [16, Theorem 5.2.3], an automorphism σ is partible if and only if $\sigma(e\mathcal{T}e) \cap f\mathcal{T}f = \{0\}$ and $\sigma(f\mathcal{T}f) \cap e\mathcal{T}e = \{0\}$. Whenever these equivalent conditions hold, it also follows that $\sigma(e\mathcal{T}f) = e\mathcal{T}f$. For example, if subrings $e\mathcal{T}e$, $f\mathcal{T}f$ of a triangular ring $\mathcal{T}$ have only trivial idempotents, then any automorphism of $\mathcal{T}$ is a partible automorphism.

Next, we provide the definition of the extremal skew n-derivation through the following result.

Proposition 2.2. *Let $\mathcal{T}$ be a triangular ring. If there exists a fixed element $\lambda \in e\mathcal{T}f$ such that $[[x, y], \lambda]_\sigma = 0$, then the mapping*

$$\tau_n(x_1, x_2, \ldots, x_n) = [x_1, [x_2, \ldots, [x_n, \lambda]_\sigma \cdots]_\sigma]_\sigma$$

is a skew n-derivation, where $x, y, x_1, x_2, \ldots, x_n \in \mathcal{T}$.

Proof. In order to prove that τ_n is a skew n-derivation, we first show that τ_n is symmetric under transposition. It follows easily from (2.1) that

$$[x, [y, z]_\sigma]_\sigma = [[x, y], z]_\sigma + [y, [x, z]_\sigma]_\sigma \tag{2.2}$$

for all $x, y, z \in \mathcal{T}$. Since $[[x, y], \lambda]_\sigma = 0$, we derive

$$[[x, y], [x_1, \lambda]_\sigma]_\sigma = [[[x, y], x_1], \lambda]_\sigma + [x_1, [[x, y], \lambda]_\sigma]_\sigma = 0$$

for all $x, y, x_1 \in \mathcal{T}$. Using (2.2) and the above relation, we have

$$[[x, y], [x_1, [x_2, \lambda]_\sigma]_\sigma]_\sigma = [[[x, y], x_1], [x_2, \lambda]_\sigma]_\sigma + [x_1, [[x, y], [x_2, \lambda]_\sigma]_\sigma]_\sigma = 0.$$

That is, $[[x, y], \tau_2(x_1, x_2)]_\sigma = 0$. By repeatedly applying (2.2), we can continue this process to obtain

$$[[x, y], \tau_k(x_1, x_2, \ldots, x_k)]_\sigma = 0$$

for all $x, y, x_1, x_2, \ldots, x_k \in \mathcal{T}$, where $3 \leq k \leq n$. Therefore

$$\begin{aligned}
\tau_n(x_1, \ldots, x_{k-1}, x_k, \ldots, x_n) &= [x_1, \ldots, [x_{k-1}, [x_k, \tau_{n-k}(x_{k+1}, \ldots, x_n)]_\sigma]_\sigma \cdots]_\sigma \\
&= [x_1, \ldots, [[x_{k-1}, x_k], \tau_{n-k}(x_{k+1}, \ldots, x_n)]_\sigma \cdots]_\sigma \\
&\quad + [x_1, \ldots, [x_k, [x_{k-1}, \tau_{n-k}(x_{k+1}, \ldots, x_n)]_\sigma]_\sigma \cdots]_\sigma \\
&= [x_1, \ldots, [x_k, [x_{k-1}, \tau_{n-k}(x_{k+1}, \ldots, x_n)]_\sigma]_\sigma \cdots]_\sigma \\
&= \tau_n(x_1, \ldots, x_k, x_{k-1}, \ldots, x_n).
\end{aligned}$$

Since a permutation can be expressed as a product of adjacent transpositions, it follows that

$$\tau_n(x_1, x_2, \ldots, x_n) = \tau_n(x_{\rho(1)}, x_{\rho(2)}, \ldots, x_{\rho(n)}) \tag{2.3}$$

for all $x_1, x_2, \ldots, x_n \in \mathcal{T}$ and $\rho \in S_n$, where S_n denotes the symmetric group of degree n. Considering that $[xy, z]_\sigma = [x, z]_\sigma y + \sigma(x)[y, z]_\sigma$, we have

$$\begin{aligned}
\tau_n(x_1 y, x_2, \ldots, x_n) &= [x_1 y, [x_2, \ldots, [x_n, \lambda]_\sigma \cdots]_\sigma]_\sigma \\
&= [x_1, [x_2, \ldots, [x_n, \lambda]_\sigma \cdots]_\sigma]_\sigma y + \sigma(x_1)[y, [x_2, \ldots, [x_n, \lambda]_\sigma \cdots]_\sigma]_\sigma \\
&= \tau_n(x_1, x_2, \ldots, x_n) y + \sigma(x_1)\tau_n(y, x_2, \ldots, x_n)
\end{aligned}$$

for all $x_1, x_2, \ldots, x_n, y \in \mathcal{T}$. According to (2.3), we get

$$\tau_n(x_1, \ldots, x_i y, \ldots, x_n) = \tau_n(x_1, \ldots, x_i, \ldots, x_n) y + \sigma(x_i)\tau_n(x_1, \ldots, y, \ldots, x_n)$$

for all $x_1, \ldots, x_i, \ldots, x_n, y \in \mathcal{T}$. Hence, τ_n is a skew n-derivation.

We refer to the above skew n-derivation as the extremal skew n-derivation. The following conclusions are derived by reformulating theorems from the relevant literature; therefore, we will not present a proof here.

Lemma 2.1. *[17, Lemma 2.3] Let $\mathcal{T}$ be a ring and σ be an automorphism of $\mathcal{T}$. If ϕ is a skew biderivation of $\mathcal{T}$. Then*

$$\phi(x,y)[u,v] = [\sigma(x),\sigma(y)]\phi(u,v)$$

for all $x,y,u,v \in \mathcal{T}$.

Lemma 2.2. *[9, Lemma 4.2] Let $\phi : \mathcal{T}\times\mathcal{T}\times\mathcal{T} \to \mathcal{T}$ be a skew 3-derivation. Then*

(i) $\phi(x,y,1) = \phi(x,1,y) = \phi(1,x,y) = 0$ *for all $x,y \in \mathcal{T}$,*
(ii) $\phi(x,y,0) = \phi(x,0,y) = \phi(0,x,y) = 0$ *for all $x,y \in \mathcal{T}$.*

Lemma 2.3. *[18, Lemma 4.2] Suppose that $\phi : \mathcal{T}\times\mathcal{T} \to \mathcal{T}$ is a skew biderivation. Then*

$$(\phi(x,y)+\phi(y,x))[u,v] = 0 = [u,v](\phi(x,y)+\phi(y,x))$$

for all $x,y,u,v \in \mathcal{T}$.

3. Skew n-derivations

Since this paper will employ induction to study skew n-derivations, we now present some conclusions regarding skew 3-derivations.

Lemma 3.1. *Let $\mathcal{T}$ be a triangular ring and $\phi : \mathcal{T}\times\mathcal{T}\times\mathcal{T} \to \mathcal{T}$ be a skew 3-derivation. If $x,y \in \mathcal{T}$ satisfy $[x,y]=0$, then $\phi(x,y,z) \in e\mathcal{T}f$, $\phi(x,z,y) \in e\mathcal{T}f$, and $\phi(z,x,y) \in e\mathcal{T}f$ for all $z \in \mathcal{T}$.*

Proof. Fix $z \in \mathcal{T}$. Let $\phi_z(x,y) = \phi(x,y,z)$, then ϕ_z is a skew biderivation on $\mathcal{T}$. According to Lemma 2.1, we have

$$\phi_z(x,y)[u,v] = [\sigma(x),\sigma(y)]\phi_z(u,v)$$

for all $x,y,u,v \in \mathcal{T}$. It follows that

$$\phi(x,y,z)[u,v] = [\sigma(x),\sigma(y)]\phi(u,v,z) \tag{3.1}$$

for all $x,y,z,u,v \in \mathcal{T}$. Letting $u = e, v = eaf \in e\mathcal{T}f$ in (3.1), we obtain

$$\phi(x,y,z)[e,eaf] = [\sigma(x),\sigma(y)]\phi(e,eaf,z).$$

Hence,

$$\phi(x,y,z)eaf = \sigma([x,y])\phi(e,eaf,z) = 0. \tag{3.2}$$

On the other hand, by (3.1), we conclude

$$[\sigma(u),\sigma(v)]\phi(x,y,z) = \phi(u,v,z)[x,y].$$

It follows that

$$[e,eaf]\phi(x,y,z) = \phi(\sigma^{-1}(e),\sigma^{-1}(eaf),z)[x,y] = 0.$$

Thus

$$eaf\phi(x,y,z) = 0 \tag{3.3}$$

for all $x, y, z, a \in \mathcal{T}$. By [7, Lemma 2.1] and [6, Proposition 2.6], $\mathcal{T}f$ is a dense right ideal of $\mathcal{T}$ and $e\mathcal{T}$ is a dense left ideal of $\mathcal{T}$. Furthermore, according to (3.2), (3.3), and Proposition 2.1 (iii), we obtain

$$e\phi(x, y, z)e = f\phi(x, y, z)f = 0.$$

Therefore, when $[x, y] = 0$, it holds that $\phi(x, y, z) \in e\mathcal{T}f$ for all $z \in \mathcal{T}$.

Similarly, it can be proved that when $[x, y] = 0$, $\phi(x, z, y) \in e\mathcal{T}f$ and $\phi(z, x, y) \in e\mathcal{T}f$ hold .

Lemma 3.2. *Let $\mathcal{T}$ be a triangular ring and $\phi : \mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ be a skew 3-derivation with a partible automorphism σ. If $\phi(e, e, e) \neq 0$, then $\phi = \tau + \theta$, where $\tau(x, y, z) = [x, [y, [z, \phi(e, e, e)]_\sigma]_\sigma]_\sigma$ is an extremal skew 3-derivation, and θ is a skew 3-derivation such that $\theta(e, e, e) = 0$.*

Proof. Since $[e, e] = 0$, it follows from Lemma 3.1 that $\phi(e, e, e) = e\phi(e, e, e)f$. It is clear that $\phi(e, e, e) \notin Z_\sigma(\mathcal{T})$. Applying (3.1) yields that

$$\phi(e, e, e)[x, y] = [\sigma(e), \sigma(e)]\phi(x, y, e) = 0$$

and

$$[\sigma(x), \sigma(y)]\phi(e, e, e) = \phi(x, y, e)[e, e] = 0$$

for all $x, y \in \mathcal{T}$. The above two relations lead to

$$\sigma([x, y])\phi(e, e, e) - \phi(e, e, e)[x, y] = 0.$$

That is, $[[x, y], \phi(e, e, e)]_\sigma = 0$. According to Proposition 2.2, it is deduced that

$$\tau(x, y, z) = [x, [y, [z, \phi(e, e, e)]_\sigma]_\sigma]_\sigma$$

is an extremal skew 3-derivation on $\mathcal{T}$.

Let $\theta = \phi - \tau$. Then θ is a skew 3-derivation. Since σ is a partible automorphism, it follows that

$$\begin{aligned}
\theta(e, e, e) &= \phi(e, e, e) - \tau(e, e, e) \\
&= \phi(e, e, e) - [e, [e, [e, \phi(e, e, e)]_\sigma]_\sigma]_\sigma \\
&= \phi(e, e, e) - \sigma(e)\phi(e, e, e) \\
&= \sigma(f)\phi(e, e, e) \\
&= 0.
\end{aligned}$$

Theorem 3.1. *Let $\mathcal{T}$ be a 2-torsion-free triangular ring and σ be a partible automorphism. If the left annihilator of $[e\mathcal{T}e, e\mathcal{T}e]$ or $[f\mathcal{T}f, f\mathcal{T}f]$ in $Q_r(\mathcal{T})$ is zero. Then the skew 3-derivation on $\mathcal{T}$ is an extremal skew 3-derivation.*

Proof. Without loss of generality, we may assume that the left annihilator of $[e\mathcal{T}e, e\mathcal{T}e]$ in $Q_r(\mathcal{T})$ is zero. Let $\phi : \mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ be a skew 3-derivation. By Lemma 3.2, to prove that ϕ is an extremal skew 3-derivation, it suffices to show that $\phi(x, y, z) = 0$ for all $x, y, z \in \mathcal{T}$ under the condition that $\phi(e, e, e) = 0$.

We employ the Peirce decomposition to express $\phi(x, y, z)$ as a sum of components. A series of claims will then be established to analyze each component individually. This process ultimately leads to the conclusion that $\phi(x, y, z) = 0$ for all $x, y, z \in \mathcal{T}$ when $\phi(e, e, e) = 0$.

Claim 1. *For any $x, y, z \in \mathcal{T}$, we have*

$$\phi(exe, fyf, eze) = \phi(exe, fyf, fzf) = 0,$$
$$\phi(fxf, eye, eze) = \phi(fxf, eye, fzf) = 0,$$
$$\phi(exe, eye, fzf) = \phi(fxf, fyf, eze) = 0.$$

By Lemma 3.1, it follows from $[exe, fyf] = 0$ that

$$\phi(exe, fyf, z) \in e\mathcal{T}f$$

for all $x, y, z \in \mathcal{T}$. Given that σ is a partible automorphism and that $\phi(e, f, e) = \phi(e, 1, e) - \phi(e, e, e) = 0$, we obtain

$$\begin{aligned} \phi(exe, fyf, e) &= \phi(exe, f, e)fyf + \sigma(f)\phi(exe, fyf, e) \\ &= \phi(exe, f, e)fyf \\ &= \phi(exe, f, e)efyf + \sigma(exe)\phi(e, f, e)fyf \\ &= 0. \end{aligned} \tag{3.4}$$

Therefore

$$\begin{aligned} \phi(exe, fyf, eze) &= \phi(exe, f, eze)fyf + \sigma(f)\phi(exe, fyf, eze) \\ &= \phi(exe, f, eze)efyf + \sigma(eze)\phi(exe, f, e)fyf \\ &= 0 \end{aligned}$$

for all $x, y, z \in \mathcal{T}$. In view of (3.4) and Lemma 2.2, we get $\phi(exe, fyf, f) = 0$. Consequently

$$\phi(exe, fyf, fzf) = \phi(exe, fyf, f)fzf + \sigma(f)\phi(exe, fyf, fzf) = 0.$$

Analogously,

$$\phi(fxf, eye, eze) = \phi(fxf, eye, fzf) = \phi(exe, eye, fzf) = \phi(fxf, fyf, eze) = 0.$$

Claim 2. *For any $x, y, z \in \mathcal{T}$, we have*

$$\phi(exe, fyf, ezf) = \phi(fxf, eye, ezf) = 0;$$
$$\phi(exe, eyf, fzf) = \phi(fxf, eyf, eze) = 0;$$
$$\phi(exf, eye, fzf) = \phi(exf, fyf, eze) = 0.$$

Since $\phi(e, f, exf) \in e\mathcal{T}f$, we can define a mapping $\psi : \mathcal{T}f \to e\mathcal{T}f$ as

$$\psi(x) = \phi(e, f, exf).$$

For any $r \in \mathcal{T}$, $x \in \mathcal{T}f$, we get

$$\begin{aligned} \psi(xr) &= \phi(e, f, exrf) = \phi(e, f, exfrf) \\ &= \phi(e, f, exf)frf + \sigma(exf)\phi(e, f, frf) \\ &= \psi(x)r. \end{aligned}$$

Therefore, according to Proposition 2.1 (iv), there exists $q \in Q_r(\mathcal{T})$ such that $\psi(x) = qx$, $x \in \mathcal{T}f$. In particular, $\psi(f) = qf = 0$, which implies that $q = qe$, and thus $\psi(x) = eqex$, $x \in \mathcal{T}f$. By Lemma 3.1, we arrive at

$$\begin{aligned}\psi(rx) &= \phi(e, f, erxf) = \phi(e, f, rexf)\\ &= \sigma(r)\phi(e, f, exf) + \phi(e, f, r)exf\\ &= \sigma(r)\psi(x)\end{aligned}$$

for all $r \in e\mathcal{T}e$, $x \in \mathcal{T}f$. It follows that $eqerx = \sigma(r)eqex$, which means that $(eqer - \sigma(r)eqe)x = 0$, $x \in \mathcal{T}f$. Since $\mathcal{T}f$ is a dense right ideal, we have $eqer = \sigma(r)eqe$, $r \in e\mathcal{T}e$. Thus, $eqe \in C_\sigma(eQ_r(\mathcal{T})e, e\mathcal{T}e)$. Let $\alpha = eqe$. Then $\phi(e, f, exf) = \alpha exf$ for all $x \in \mathcal{T}$. Similarly, there exist $\beta, \gamma, \varepsilon \in C_\sigma(eQ_r(\mathcal{T})e, e\mathcal{T}e)$ such that

$$\phi(f, e, exf) = \beta exf; \quad \phi(f, exf, e) = \gamma exf; \quad \phi(exf, e, f) = \varepsilon exf$$

for all $x \in \mathcal{T}$. By $\phi(f, e, exf) = -\phi(e, e, exf) = \phi(e, f, exf)$, we have

$$\phi(f, e, exf) = \phi(e, f, exf) = \alpha exf. \tag{3.5}$$

According to Lemma 2.3, we get $[u, v](\phi(f, e, exf) + \phi(f, exf, e)) = 0$. It follows that

$$\begin{aligned}0 &= [\sigma(ese), \sigma(ete)](\phi(f, e, exf) + \phi(f, exf, e))\\ &= \sigma([ese, ete])(\alpha exf + \gamma exf)\\ &= \sigma([ese, ete])(\alpha + \gamma)exf\\ &= (\alpha + \gamma)[ese, ete]exf\end{aligned}$$

for all $s, t, x \in \mathcal{T}$. Since $\mathcal{T}f$ is a dense right ideal and the left annihilator of $[e\mathcal{T}e, e\mathcal{T}e]$ is zero, we obtain $\alpha = -\gamma$, which implies that

$$\phi(f, exf, e) = -\alpha exf.$$

Therefore

$$\phi(e, exf, f) = \phi(f, exf, e) = -\alpha exf \tag{3.6}$$

for all $x \in \mathcal{T}$. In view of $[u, v](\phi(f, e, exf) + \phi(exf, e, f)) = 0$, we get

$$\phi(exf, e, f) = -\phi(f, e, exf) = -\alpha exf. \tag{3.7}$$

On the other hand, using $[u, v](\phi(e, exf, f) + \phi(exf, e, f)) = 0$, we obtain

$$\phi(exf, e, f) = -\phi(e, exf, f) = \alpha exf \tag{3.8}$$

for all $x \in \mathcal{T}$. Comparing (3.7) and (3.8), we have $\alpha = 0$. Therefore, according to (3.5)–(3.7), we arrive at

$$\phi(f, e, exf) = \phi(e, f, exf) = \phi(e, exf, f) = \phi(f, exf, e) = \phi(exf, e, f) = \phi(exf, f, e) = 0. \tag{3.9}$$

Consequently

$$\begin{aligned}
\phi(exe, fyf, ezf) &= \phi(exe, fyf, ezf)e + \sigma(exe)\phi(e, fyf, ezf) \\
&= \sigma(exe)\sigma(f)\phi(e, fyf, ezf) + \sigma(exe)\phi(e, f, ezf)fyf \\
&= 0, \\
\phi(fxf, eye, ezf) &= \phi(fxf, eye, ezf)e + \sigma(eye)\phi(fxf, e, ezf) \\
&= \sigma(eye)\sigma(f)\phi(fxf, e, ezf) + \sigma(eye)\phi(f, e, ezf)fxf \\
&= 0, \\
\phi(exe, eyf, fzf) &= \phi(exe, eyf, f)fzf + \sigma(f)\phi(exe, eyf, fzf) \\
&= \sigma(exe)\phi(e, eyf, f)fzf + \phi(exe, eyf, f)efzf \\
&= 0, \\
\phi(fxf, eyf, eze) &= \phi(f, eyf, eze)fxf + \sigma(f)\phi(fxf, eyf, eze) \\
&= \sigma(eze)\phi(f, eyf, e)fxf + \phi(f, eyf, eze)efxf \\
&= 0, \\
\phi(exf, eye, fzf) &= \phi(exf, eye, f)fzf + \sigma(f)\phi(exf, eye, fzf) \\
&= \sigma(eye)\phi(exf, e, f)fzf + \phi(exf, eye, f)efzf \\
&= 0, \\
\phi(exf, fyf, eze) &= \phi(exf, f, eze)fyf + \sigma(f)\phi(exf, fyf, eze) \\
&= \sigma(eze)\phi(exf, f, e)fyf + \phi(exf, f, eze)efyf \\
&= 0
\end{aligned}$$

for all $x, y, z \in \mathcal{T}$.

Claim 3. *For any $x, y, z \in \mathcal{T}$, we have*

$$\begin{aligned}
\phi(exe, eye, ezf) = \phi(fxf, fyf, ezf) = 0; \\
\phi(exe, eyf, eze) = \phi(fxf, eyf, fzf) = 0; \\
\phi(exf, eye, eze) = \phi(exf, fyf, fzf) = 0; \\
\phi(exe, eye, eze) = \phi(fxf, fyf, fzf) = 0.
\end{aligned}$$

By $\phi(exe, e, e) \in e\mathcal{T}f$ and $\phi(e, e, e) = 0$, we have

$$\phi(exe, e, e) = \phi(exe, e, e)e + \sigma(exe)\phi(e, e, e) = 0.$$

Therefore

$$\phi(exe, eye, e) = \phi(exe, eye, e)e + \sigma(eye)\phi(exe, e, e) = 0 \tag{3.10}$$

for all $x, y \in \mathcal{T}$. Using (3.10) and the fact that $\phi(exe, eye, 1) = 0$, we derive $\phi(exe, eye, f) = 0$. It follows that

$$\begin{aligned}
\phi(exe, eye, ezf) &= \phi(exe, eye, e)ezf + \sigma(e)\phi(exe, eye, ezf) \\
&= \sigma(e)\sigma(ezf)\phi(exe, eye, f) + \sigma(e)\phi(exe, eye, ezf)f \\
&= \sigma(e)\phi(exe, eye, ezf)f
\end{aligned}$$

for all $x, y, z \in \mathcal{T}$. Since σ is a partible automorphism, we get $\phi(exe, eye, ezf) \in e\mathcal{T}f$. Applying (3.9) yields that

$$\phi(e, e, ezf) = \phi(f, e, ezf) = 0.$$

Therefore

$$\begin{aligned}\phi(exe, eye, ezf) &= \phi(exe, eye, ezf)e + \sigma(exe)\phi(e, eye, ezf)\\ &= \sigma(exe)\sigma(eye)\phi(e, e, ezf) + \sigma(exe)\phi(e, eye, ezf)e \\ &= 0\end{aligned} \tag{3.11}$$

for all $x, y, z \in \mathcal{T}$.

In the same way, we conclude

$$\phi(fxf, fyf, ezf) = \phi(exe, eyf, eze) = \phi(fxf, eyf, fzf) = \phi(exf, eye, eze) = \phi(exf, fyf, fzf) = 0$$

for all $x, y, z \in \mathcal{T}$.

Since $\phi(exe, eye, eze) = \phi(exe, eye, 1) - \phi(exe, eye, ezf) - \phi(exe, eye, fzf)$, it follows from (3.11) and Claim 1 that $\phi(exe, eye, eze) = 0$. Similarly, we have $\phi(fxf, fyf, fzf) = 0$ for all $x, y, z \in \mathcal{T}$.

Claim 4. *For any $x, y, z \in \mathcal{T}$, we have*

$$\begin{aligned}&\phi(exf, eyf, eze) = \phi(exf, eyf, fzf) = 0,\\ &\phi(exe, eyf, ezf) = \phi(fxf, eyf, ezf) = 0,\\ &\phi(exf, eye, ezf) = \phi(exf, fyf, ezf) = 0,\\ &\phi(exf, eyf, ezf) = 0.\end{aligned}$$

Fix $x \in \mathcal{T}$ and define a mapping $\psi_x : \mathcal{T}f \to e\mathcal{T}f$ as $\psi_x(y) = \phi(exf, eyf, e)$. According to Claims 2 and 3, we have

$$\phi(exf, r, e) = \phi(exf, ere, e) + \phi(exf, frf, e) + \phi(exf, erf, e) \in e\mathcal{T}f$$

for all $r \in \mathcal{T}$. It follows that

$$\begin{aligned}\psi_x(yr) &= \phi(exf, eyrf, e) = \phi(exf, eyfr, e)\\ &= \sigma(eyf)\phi(exf, r, e) + \phi(exf, eyf, e)r\\ &= \psi_x(y)r\end{aligned}$$

for all $y \in \mathcal{T}f$, $r \in \mathcal{T}$. Thus, there exists $q_x \in Q_r(\mathcal{T})$ such that $\psi_x(y) = q_x y$, $y \in \mathcal{T}f$. In particular, $\psi_x(f) = q_x f = 0$. That is, $q_x = q_x e$. This implies that $\psi_x(y) = eq_x ey$. In addition, we have

$$\begin{aligned}\psi_x(ry) &= \phi(exf, eryf, e) = \phi(exf, reyf, e)\\ &= \sigma(r)\phi(exf, eyf, e) + \phi(exf, r, e)eyf\\ &= \sigma(r)\psi_x(y)\end{aligned}$$

for all $y \in \mathcal{T}f$, $r \in e\mathcal{T}e$. Therefore, $eq_x ery = \sigma(r)eq_x ey$, which implies that $(eq_x er - \sigma(r)eq_x e)\mathcal{T}f = 0$. This leads us to conclude that $eq_x er = \sigma(r)eq_x e$. That is $eq_x e \in C_\sigma(eQ_r(\mathcal{T})e, e\mathcal{T}e)$. Since

$$\begin{aligned}0 &= \phi(exe, eye, e)[ex'f, ey'f] = [\sigma(exe), \sigma(eye)]\phi(ex'f, ey'f, e)\\ &= [\sigma(exe), \sigma(eye)]eq_{x'}ey'f\end{aligned}$$

for all $x, y, x', y' \in \mathcal{T}$, we have $\sigma([exe, eye])eq_{x'}e = 0$. It follows from $eq_{x'}e \in C_\sigma(eQ_r(\mathcal{T})e, e\mathcal{T}e)$ that $eq_{x'}e[exe, eye] = 0$. By the assumption, we conclude $eq_{x'}e = 0$, which implies $\phi(exf, eyf, e) = 0$ for all $x, y \in \mathcal{T}$. Thus,

$$\phi(exf, eyf, eze) = \phi(exf, eyf, eze)e + \sigma(eze)\phi(exf, eyf, e) = 0$$

for all $x, y, z \in \mathcal{T}$. Since $\phi(exf, eyf, f) = -\phi(exf, eyf, e) = 0$, we get

$$\phi(exf, eyf, fzf) = \sigma(f)\phi(exf, eyf, fzf) + \phi(exf, eyf, f)fzf = 0.$$

Consequently

$$\phi(exf, eyf, ezf) = \phi(exf, eyf, 1) - \phi(exf, eyf, eze) - \phi(exf, eyf, fzf) = 0$$

for all $x, y, z \in \mathcal{T}$. Likewise, we can derive that

$$\begin{aligned}\phi(exe, eyf, ezf) &= \phi(fxf, eyf, ezf) = 0,\\ \phi(exf, eye, ezf) &= \phi(exf, fyf, ezf) = 0\end{aligned}$$

for all $x, y, z \in \mathcal{T}$.

Due to

$$\begin{aligned}\phi(x, y, z) &= \phi(exe + exf + fxf, eye + eyf + fyf, eze + ezf + fzf)\\ &= \phi(exe, eye, eze) + \phi(exe, eye, ezf) + \phi(exe, eye, fzf)\\ &+ \phi(exe, eyf, eze) + \phi(exe, eyf, ezf) + \phi(exe, eyf, fzf)\\ &+ \phi(exe, fyf, eze) + \phi(exe, fyf, ezf) + \phi(exe, fyf, fzf)\\ &+ \phi(exf, eye, eze) + \phi(exf, eye, ezf) + \phi(exf, eye, fzf)\\ &+ \phi(exf, eyf, eze) + \phi(exf, eyf, ezf) + \phi(exf, eyf, fzf)\\ &+ \phi(exf, fyf, eze) + \phi(exf, fyf, ezf) + \phi(exf, fyf, fzf)\\ &+ \phi(fxf, eye, eze) + \phi(fxf, eye, ezf) + \phi(fxf, eye, fzf)\\ &+ \phi(fxf, eyf, eze) + \phi(fxf, eyf, ezf) + \phi(fxf, eyf, fzf)\\ &+ \phi(fxf, fyf, eze) + \phi(fxf, fyf, ezf) + \phi(fxf, fyf, fzf)\end{aligned}$$

for all $x, y, z \in \mathcal{T}$, based on Claims 1–4, the skew 3-derivation ϕ satisfying $\phi(e, e, e) = 0$ is zero. Thus, in view of Lemma 3.2, Theorem 3.1 holds.

Theorem 3.2. *Let $\mathcal{T}$ be a 2-torsion-free triangular ring and σ be a partible automorphism. If the left annihilator of $[e\mathcal{T}e, e\mathcal{T}e]$ or $[f\mathcal{T}f, f\mathcal{T}f]$ in $Q_r(\mathcal{T})$ is zero. Then the skew n-derivation $\phi : \mathcal{T} \times \mathcal{T} \times \cdots \times \mathcal{T} \to \mathcal{T}$ is an extremal skew n-derivation ($n \geq 3$).*

Proof. The theorem is proved by induction on n. If $n = 3$, then the conclusion is derived from Theorem 3.1. Now, we consider the case $n \geq 4$. Let $x_4, x_5, \ldots, x_n \in \mathcal{T}$ be fixed and define

$$\phi'(x_1, x_2, x_3) = \phi(x_1, x_2, \ldots, x_n)$$

for all $x_1, x_2, x_3 \in \mathcal{T}$. Then $\phi' : \mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is a skew 3-derivation. According to Theorem 3.1, we have

$$\phi'(x_1, x_2, x_3) = [x_1, [x_2, [x_3, \lambda]_\sigma]_\sigma]_\sigma,$$

where λ is determined by $x_4, \ldots, x_n$ and satisfies $[[x, y], \lambda]_\sigma = 0$. Since $\lambda \in e\mathcal{T}f$ and σ is a partible automorphism, it follows that

$$\phi'(\sigma^{-1}(e), \sigma^{-1}(e), \sigma^{-1}(e)) = [\sigma^{-1}(e), [\sigma^{-1}(e), [\sigma^{-1}(e), \lambda]_\sigma]_\sigma]_\sigma = \lambda.$$

Therefore

$$\phi(\sigma^{-1}(e), \sigma^{-1}(e), \sigma^{-1}(e), x_4, \ldots, x_n) = \lambda.$$

This implies that

$$\phi(x_1, x_2, \ldots, x_n) = [x_1, [x_2, [x_3, \phi(\sigma^{-1}(e), \sigma^{-1}(e), \sigma^{-1}(e), x_4, \ldots, x_n)]_\sigma]_\sigma]_\sigma. \quad (3.12)$$

Since $\phi(\sigma^{-1}(e), x_2, \ldots, x_n)$ acts as a skew $(n-1)$-derivation on $\mathcal{T}$, we apply the induction hypothesis to obtain

$$\phi(\sigma^{-1}(e), x_2, \ldots, x_n) = [x_2, [x_3, \ldots, [x_n, \lambda']_\sigma \cdots]_\sigma]_\sigma, \quad (3.13)$$

where $\lambda' \in e\mathcal{T}f$ satisfies $[[x, y], \lambda']_\sigma = 0$. In particular, applying (3.13) yields that

$$\phi(\sigma^{-1}(e), \sigma^{-1}(e), \sigma^{-1}(e), x_4, \ldots, x_n) = [x_4, [x_5, \ldots, [x_n, \lambda']_\sigma \cdots]_\sigma]_\sigma. \quad (3.14)$$

Applying (3.14) to (3.12), we arrive at

$$\phi(x_1, x_2, \ldots, x_n) = [x_1, [x_2, [x_3, [x_4, \ldots, [x_n, \lambda']_\sigma \cdots]_\sigma]_\sigma]_\sigma]_\sigma$$

for all $x_1, x_2, \ldots, x_n \in \mathcal{T}$. By Proposition 2.2, the conclusion holds.

4. Conclusions

In this paper, we have systematically investigated skew n-derivations on triangular rings and established that, under certain natural conditions, every such skew n-derivation is extremal. Our main results, Theorems 3.1 and 3.2, show that if $\mathcal{T}$ is a 2-torsion-free triangular ring with a partible automorphism σ and if the left annihilator of $[e\mathcal{T}e, e\mathcal{T}e]$ or $[f\mathcal{T}f, f\mathcal{T}f]$ in $Q_r(\mathcal{T})$ is zero, then every skew n-derivation for $n \geq 3$ is an extremal skew n-derivation.

The proof proceeds by induction on n, with the base case $n = 3$ treated in detail via a series of technical claims that analyze the behavior of the derivation on the Peirce components of the triangular ring. The induction step then extends this structure to arbitrary ($n \geq 4$), revealing that the skew n-derivation can always be expressed as an iterated σ-commutator with a fixed element in $e\mathcal{T}f$.

Extremal skew n-derivations represent a generalization of inner skew derivations. Future research could explore their classification within broader algebraic structures, such as Hopf algebras and color algebras; establish connections with homological algebra and noncommutative geometry; and apply them to characterize multilinear identities on operator algebras. Such work holds the potential to build a bridge between classical derivation theory and modern multilinear structures.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The research was supported by the project of Jilin Education Department (No. JJKH20250926KJ).

Conflict of interest

The authors declare there are no conflicts of interest.

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