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https://www.aimspress.com/journal/era

ERA, 33(12): 7551–7569.
DOI: 10.3934/era.2025333
Received: 24 October 2025
Revised: 27 November 2025
Accepted: 05 December 2025
Published: 17 December 2025

Theory article

Large-time behavior of solutions to the 2D generalized magneto-micropolar equations

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Abstract: This work provides rigorous verification of the super-smoothing effect of higher-order fractional Laplacian dissipation. Shang and Zhao (2017) have proved the global regularity of classical solutions of the 2D incompressible magneto-micropolar equations with linear velocity damping u, microrotational dissipation $(-\Delta)\omega$, and magnetic diffusion $(-\Delta)^{\beta}b, \beta > 1$. This paper is devoted to further investigating the large-time behavior of global smooth solutions of the system with $1 < \beta \leq \frac{3}{2}$. We apply the negative Sobolev space to overcome the difficulty caused by fractional-order dissipation and establish $\int_0^t \|\nabla b(\tau)\|_{L^2} d\tau \leq C$. Furthermore, by fully exploiting the special structure of the system and combining the properties of a heat operator with the generalized Fourier splitting methods, we obtain the decay rates of the solutions and their first-order derivatives for $1 \leq p \leq \frac{2}{\beta}$.

Keywords: magneto-micropolar equations; decay rates; fractional magnetic dissipation; linear velocity damping; Hardy-Littlewood-Sobolev inequality

1. Introduction

The magneto-micropolar equations illustrates the motion of electrically conductive micropolar fluids in the presence of a magnetic field. The 3D magneto-micropolar equations with fractional dissipation can be represented as

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + (\mu + \chi)\Lambda^{2\alpha} u = -\nabla \pi + (b \cdot \nabla)b + 2\chi \nabla \times \omega, \\ \partial_t \omega + (u \cdot \nabla)\omega - \lambda \nabla\nabla \cdot \omega + 4\chi\omega + \kappa\Lambda^{2\gamma}\omega = 2\chi \nabla \times u, \\ \partial_t b + (u \cdot \nabla)b + \nu\Lambda^{2\beta} b = (b \cdot \nabla)u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ (u, \omega, b)(x, 0) = (u_0(x), \omega_0(x), b_0(x), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^3$ and $t \geq 0$. The vectors $u(x,t)$, $b(x,t)$, and $\omega(x,t)$ denote the velocity of the fluid, the magnetic field, and the micro-rotational velocity, respectively. The scalars $\pi(x,t)$ denote the hydrostatic pressure. The positive parameters μ, χ, and $\frac{1}{\nu}$ are, respectively, the kinematic viscosity, vortex viscosity, and magnetic Reynolds number. α, κ and λ are angular viscosities. Here, $\Lambda = (-\Delta)^{\frac{1}{2}}$, for $\sigma \geq 0$, the fractional Laplacian operator Λ^σ is defined by the Fourier transform

$$\widehat{\Lambda^\sigma g}(\xi) = |\xi|^\sigma \hat{g}(\xi).$$

Standard Laplacian dissipation $\Lambda^2 = (-\Delta)$ describes a local, classical diffusion process. Higher-order fractional dissipation $\Lambda^\sigma = (-\Delta)^{\frac{\sigma}{2}}, \sigma > 2$ describes a non-local, long-range anomalous diffusion process [1]. The fractional Laplacian operator serves as a powerful mathematical model to describe or approximate certain complex physical processes, such as simulating plasmas with long-range interactions, flow in porous media, and representing hyperviscosity in turbulence modeling [2, 3]. In partial, if $\sigma = 0$, we define $\Lambda^\sigma(g) = g$. When $\alpha = \gamma = \beta = 1$, (1.1) becomes the classical magneto-micropolar equation with standard Laplacian operator dissipation.

Due to their rich phenomena, significant physical relevance, and mathematical complexity, magneto-micropolar equation (1.1) have attracted considerable attention from physicists and mathematicians. In 1974, Ahmadi and Shahinpoor [4] proposed the magneto-micropolar equations. In physics, the motion of aggregates of small, solid, ferromagnetic particles in viscous magnetic fluids can be described by the magneto-micropolar equations [5, 6]. In bioengineering, magneto-micropolar fluids can be employed to model the application of magnetic tracers in blood flow [7, 8]. On the one hand, for the blow-up criteria of smooth solutions to the Cauchy problem of the Eq (1.1), we refer to [9–11], and for regularity criteria of weak solutions of (1.1), we refer to [12, 13]; on the other hand, for large-time behavior of solutions to the Cauchy problem of (1.1), we refer to [14–16]. Recently, Zhai et al. [17] proved the stability for the 3D compressible magneto-micropolar equations with only velocity dissipation (i.e.$\mu, \chi >, \alpha = 1, \kappa = \nu = 0$) near a background magnetic field on $\mathbb{T}^3$.

When we set $u = (u_1, u_2, 0)$, $b = (b_1, b_2, 0)$, $\omega = (0, 0, \omega)$, (1.1) becomes the 2D magneto-micropolar equations

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + (\mu + \chi)\Lambda^{2\alpha} u = -\nabla\pi + (b \cdot \nabla)b + 2\chi\nabla \times \omega, \\ \partial_t \omega + (u \cdot \nabla)\omega + 4\chi\omega + \kappa\Lambda^{2\gamma}\omega = 2\chi\nabla \times u, \\ \partial_t b + (u \cdot \nabla)b + \nu\Lambda^{2\beta} b = (b \cdot \nabla)u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ (u, \omega, b)(x, 0) = (u_0(x), \omega_0(x), b_0(x). \end{cases} \tag{1.2}$$

If the magnetic field $b = 0$, then (1.2) reduces to the micropolar fluid system. Dong and Zhang [18] and Dong et al. [19] established the global existence and uniqueness of classical solutions to the 2D micropolar equation (1.2) with only velocity dissipation and with only angular velocity dissipation, respectively. Let $\omega, \chi = 0$; then (1.2) reduces to the incompressible magnetohydrodynamic (MHD) equations. Cao et al. [20] and Jiu and Zhao [21] independently presented the global regularity of classical solutions to the 2D MHD equation (1.2) with only magnetic diffusion $\mu, \chi = 0, \nu > 0, \beta > 1$. Dong et al. [22] proved that the 2D MHD equations with $(-\Delta)^\alpha u, \alpha > 0, \partial_{22} b_1, \partial_{11} b_2$ have a unique global smooth solution and obtained optimal large-time decay rates when the initial data is sufficiently smooth. Liu and Zhang [23] established the linear stability of the 2D MHD equation (1.2) with $\mu, \chi =$

$0, \nu > 0, \beta = 1$ when the magnetic field is close to the equilibrium state $e_2 = (0, 1)$ in the periodic domain $\mathbb{T}^2$. From the papers [20–23], we find that it is difficult to obtain the global smooth solution and the nonlinear stability of the 2D MHD equation (1.2) with only magnetic diffusion $\mu, \chi = 0, \nu > 0, \beta = 1$. Because of the similar structure of the 2D MHD equations and the 2D magneto-micropolar equations, it is difficult to prove the global regularity and the large-time behavior of solutions for (1.2) with $\mu, \chi = 0, \kappa, \nu > 0, \gamma = \beta = 1$.

This paper considers the well-posedness to the Cauchy problem of 2D magneto-micropolar equation (1.2). For the full dissipation case $\mu, \chi, \kappa, \nu > 0$ of the 2D magneto-micropolar equation (1.2), the literature [24, 25] proved the global existence and uniqueness of classical solutions. Later, the system (1.2) with partial dissipation and fractional dissipation has also made many important advancements (see, e.g., [26–28]). Especially, Shang and Wu [29] established the global smooth solutions for three types of 2D magneto-micropolar fluid equations with $\mu, \chi > 0, \kappa = \nu = 0, \alpha = 2$, and $\mu, \chi, \kappa, \nu > 0, \alpha > 0, \beta = 1, \gamma = 1$, and $\mu, \chi, \nu > 0, \kappa = 0, 1 < \alpha < 2, 0 < \beta < 1, \alpha + \beta \geq 2$, respectively. Shang-Zhao [26] established a regularity criterion for the 2D system without velocity dissipation for $\mu = \chi = 0, \kappa, \nu > 0, \alpha = 0, \beta = 1, \gamma = 1$ and the global regularity for $\mu = \chi = 0, \kappa, \nu > 0, \alpha = 0, \beta > 1, \gamma = 1$.

For the large-time behavior of the 2D magneto-micropolar equation (1.2), there has been some research such as [30–32]. For the 2D and 3D magneto-micropolar equations with $\mu, \chi > 0$, $\alpha = 1$, and $\kappa = \nu = 0$, Wu and Zhang [33] proved that L^2-norms of the fluid velocity and micro-rotational velocity decay to zero, and L^2-norms of the magnetic field converge to a non-negative constant as $t \to \infty$ by applying the Fourier splitting method. Recently, Shang and Gu [34] established the decay estimates of small solutions to 2D magneto-micropolar equation (1.2) with $\mu, \chi, \kappa, \nu > 0, \alpha = 0, \gamma = 1, \beta = 1$. They also obtained the global existence of classical solution for 2D magneto-micropolar equation (1.2) with $\mu, \chi, \nu > 0, \alpha = 1, \kappa = 0, \partial_{yy} b_1, \partial_{xx} b_2$ for small initial data. Shang and Gu [35] improved the the decay rates of solutions to 2D magneto-micropolar equation (1.2) with $\mu, \chi, \kappa, \nu > 0, \alpha = 0, \gamma = 1, \beta = 1$ and derived the optimal decay estimate for $\|b\|_{L^\infty}$. Subsequently, Ye et al. [36] further improved the optimal time rates of weak solutions for the system in [35].

Motivated by [20, 21, 23, 26, 34–36], when higher-order fractional magnetic dissipation that takes the place of the standard Laplacian operator is introduced into the magneto-micropolar fluid model and the MHD equations, it describes a physical system in which magnetic field perturbations can be rapidly smoothed and dissipated in a non-local, long-range jumping manner. Moreover, it has a "super-smoothing" effect that enhances the overall stability of the system. This paper intends to present a rigorous mathematic theory to verify the physical smoothness effect of higher-order fractional magnetic dissipation in terms of the 2D magneto-micropolar equations. Shang and Zhao [26] have proved the global regularity of the magneto-micropolar equation (1.2) in the case $\mu, \chi, \kappa, \nu > 0, \alpha = 0, \gamma = 1, \beta > 1$; moreover, our goal is to further investigate the large time behavior of the magneto-micropolar equations

$$\begin{cases} \partial_t u + (\mu + \chi) u + (u \cdot \nabla) u + \nabla \pi = (b \cdot \nabla) b + 2\chi \nabla \times \omega, \\ \partial_t \omega - \kappa \Delta \omega + 4\chi \omega + (u \cdot \nabla) \omega = 2\chi \nabla \times u, \\ \partial_t b + \nu(-\Delta)^\beta b + (u \cdot \nabla) b = (b \cdot \nabla) u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ u(x, 0) = u_0(x), \quad \omega(x, 0) = \omega_0(x), \quad b(x, 0) = b_0(x). \end{cases} \tag{1.3}$$

2. Preliminaries

To prove the essential auxiliary estimates, we firstly recall the Hardy-Littlewood-Sobolev inequality for fractional integration (see [37]), where $\Lambda = (-\Delta)^{\frac{1}{2}}$ denotes the Zygmund operator.

Lemma 2.1. *Let* $0 < k < d$ *and* $1 < p < q < \infty$ *with* $\frac{k}{d} + \frac{1}{q} = \frac{1}{p}$. *Then*

$$\|\Lambda^{-k} f\|_{L^p(\mathbb{R}^d)} \leq C\|f\|_{L^q(\mathbb{R}^d)}.$$

Next, we recall the following classic L^p-L^q estimate for the heat operator.

Lemma 2.2. *(Schonbek [38]) Let* $\alpha > 0$, $1 \leq p \leq q \leq \infty$, *and* $m \geq 0$. *For any* $t > 0$, *we have*

$$\|\nabla^m e^{-\mu(-\Delta)^\alpha t} f\|_{L^q(\mathbb{R}^2)} \leq C t^{-\frac{m}{2\alpha} - \frac{1}{\alpha}(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^2)}.$$

3. Main result

The precise result is stated in the following theorem.

Theorem 3.1. *Let* $1 < \beta \leq \frac{3}{2}$, *and suppose* $(u_0, \omega_0) \in H^1(\mathbb{R}^2)$ *and* $b_0 \in L^p(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$ *for* $1 \leq p \leq \frac{2}{\beta}$, *such that* $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. *Let* (u, ω, b) *be a global solution to the system (1.3), and*

$$\kappa > \frac{4\chi^2}{\mu + \chi}.$$

Then for any $t > 0$, *we have the following decay upper bounds of decay rates:*

$$\|u\|_{L^2(\mathbb{R}^2)} + \|\omega\|_{L^2(\mathbb{R}^2)} \leq C(1+t)^{-\frac{1}{\beta p}(2-\epsilon)}, \quad 0 < \epsilon \leq \frac{3}{2} \tag{3.1}$$

$$\|b\|_{L^2(\mathbb{R}^2)} \leq C(1+t)^{-\frac{1}{\beta}(\frac{1}{p} - \frac{1}{2})}, \tag{3.2}$$

$$\|\nabla u\|_{L^2(\mathbb{R}^2)} + \|\nabla \omega\|_{L^2(\mathbb{R}^2)} \leq C(1+t)^{-\frac{1}{\beta p} - \frac{2\beta - p\beta + 2}{2(2\beta+1)p}} \tag{3.3}$$

$$\|\nabla b\|_{L^2(\mathbb{R}^2)} \leq C(1+t)^{-\frac{1}{\beta p}}. \tag{3.4}$$

Remark 3.1. *When we replace* $1 \leq p \leq \frac{2}{\beta}$ *with* $1 \leq p < 2$, *the results of Theorem 3.1 also hold to the system (1.3) with* $\beta = 1$. *And it is worth noting that it contains the results in [34, 35].*

Remark 3.2. *We require* $b_0 \in L^p(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$ *for* $1 \leq p \leq \frac{2}{\beta}$. *Mathematically, this implies that the system possesses finite total magnetic energy at the initial time, improves integrability properties, and provides the foundation for energy estimates. Physically, this formulation eliminates extreme situations where the initial magnetic field exhibits highly concentrated energy at singular points, prohibiting exceedingly singular "hot spots" or "current sheets" in the initial magnetic field. When combined with higher-order fractional dissipation* $-(\Delta)^\beta b, \beta > 1$ *in the magnetic field, we observe that a milder initial state coupled with a stronger dissipation mechanism together ensures the system's stability.*

Remark 3.3. *Because the velocity equations of the compressible magneto-micropolar fluid system are non-strictly hyperbolic in the absence of dissipation, their eigenvalues may coincide, leading to potential resonance of linear waves that can strengthen nonlinear effects. Furthermore, the pressure term is linked to the density through an equation of state, which increases the nonlinearity of the system. Mathematically, these factors make stability analysis considerably more challenging. This represents a worthwhile research problem, and we will continue to explore it in the future.*

4. The proof of Theorem 3.1

We remark that the decay rates on the system (1.3) are not trivial. Because of the momentum equation without kinematic viscosity dissipation and the magnetic field equation with fractional dissipation, the classic Schonbek's Fourier splitting methods [39], which rely on dissipation and Kato's methods [40], which are based on the $L^p - L^q$ estimates of a heat semigroup that cannot apply directly. In order to overcome the main difficulty, we first need to obtain the auxiliary decay rates of $\|(\nabla u, \nabla b, \nabla \omega)\|_{L^2}$. However, due to the index $\beta > 1$, the term $\int_0^t \|\nabla b(\tau)\|_{L^2} d\tau$ cannot be obtained by energy estimates. Fortunately, applying the negative Sobolev space, we can deduce that

$$\|\Lambda^{1-\beta} b(t)\|_{L^2}^2 + \nu \int_0^t \|\Lambda b(\tau)\|_{L^2}^2 d\tau \le C, \tag{4.1}$$

but it is necessary to restrict $1 < \beta \le \frac{3}{2}$ in order to make the above result hold. With the aid of the above estimates, we first derive

$$\|(\nabla u, \nabla b, \nabla \omega)\|_{L^2} \le C(1+t)^{-\frac{1}{2}}.$$

Furthermore, we also obtain auxiliary estimates

$$\|u\|_{L^2(\mathbb{R}^2)} + \|\omega\|_{L^2(\mathbb{R}^2)} \le C(1+t)^{-\frac{2}{3}}$$

and the sharp estimate

$$\|b\|_{L^2(\mathbb{R}^2)} \le C(1+t)^{-\frac{1}{\beta}(\frac{1}{p}-\frac{1}{2})}$$

with the aid of the properties of a heat operator.

By the generalized Fourier splitting methods, we derive the improved decay rates

$$\|\nabla u\|_{L^2(\mathbb{R}^2)} + \|\nabla \omega\|_{L^2(\mathbb{R}^2)} + \|\nabla b\|_{L^2(\mathbb{R}^2)} \le C(1+t)^{-\frac{1}{\beta p}}$$

and the optimal upper bounds

$$\|u\|_{L^2(\mathbb{R}^2)} + \|\omega\|_{L^2(\mathbb{R}^2)} \le C(1+t)^{-\frac{1}{\beta p}(2-\epsilon)}, \quad 0 < \epsilon \le \frac{3}{2}.$$

Ultimately, we apply the implicit decay estimate

$$\int_0^t (1+\tau)^n (\|\Omega\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 + \|\Lambda^\beta j\|_{L^2}^2) d\tau \le C(1+t)^{n-\frac{2}{\beta p}}$$

and obtain the decay rates

$$\|\nabla u\|_{L^2(\mathbb{R}^2)} + \|\nabla \omega\|_{L^2(\mathbb{R}^2)} \le C(1+t)^{-\frac{1}{\beta p}-\frac{2\beta - p\beta + 2}{2(2\beta+1)p}}.$$

4.1. The priori estimates

We now show the L^2-estimate for the solutions to the system (2.1). The result is stated in the following proposition.

Proposition 4.1. *Let* $(u_0, \omega_0, b_0) \in L^2(\mathbb{R}^2)$*. Then, any* $t > 0$, (u, ω, b) *of the system (1.3) satisfies*

$$\begin{aligned} &\|u\|_{L^2}^2 + \|\omega\|_{L^2}^2 + \|b\|_{L^2}^2 + 2\sigma \int_0^t \|u\|_{L^2}^2 d\tau + 8\chi \int_0^t \|\omega\|_{L^2}^2 d\tau \\ &+2(\kappa - \frac{4\chi^2}{\mu + \chi - \sigma}) \int_0^t \|\nabla\omega\|_{L^2}^2 d\tau + \nu \int_0^t \|\Lambda^\beta b\|_{L^2}^2 d\tau \le C. \end{aligned} \tag{4.2}$$

Proof: Taking the L^2-inner products of Eq (1.3) with u, ω and b, respectively, then adding the resulting equations together, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2 + \|\omega\|_{L^2}^2 + \|b\|_{L^2}^2) + (\mu + \chi)\|u\|_{L^2}^2 \\ &+4\chi\|\omega\|_{L^2}^2 + \kappa\|\nabla\omega\|_{L^2}^2 + \nu\|\Lambda^\beta b\|_{L^2}^2 \\ = \;& 2\chi \int_{\mathbb{R}^2} \nabla \times \omega \cdot u dxdy + 2\chi \int_{\mathbb{R}^2} \nabla \times u\omega dxdy. \end{aligned} \tag{4.3}$$

Applying Young's inequality, we have

$$\begin{aligned} &2\chi \int_{\mathbb{R}^2} \nabla \times \omega \cdot u dxdy + 2\chi \int_{\mathbb{R}^2} \nabla \times u\omega dxdy \\ = \;& 4\chi \int_{\mathbb{R}^2} \nabla \times \omega \cdot u dxdy \\ \le \;& (\mu + \chi - \sigma)\|u\|_{L^2}^2 + \frac{4\chi^2}{\mu + \chi - \sigma}\|\nabla\omega\|_{L^2}^2. \end{aligned}$$

Inserting the above estimate into (4.3) yields

$$\begin{aligned} &\frac{d}{dt}(\|u\|_{L^2}^2 + \|\omega\|_{L^2}^2 + \|b\|_{L^2}^2) + 2\sigma\|u\|_{L^2}^2 + 8\chi\|\omega\|_{L^2}^2 \\ &+2(\kappa - \frac{4\chi^2}{\mu + \chi - \sigma})\|\nabla\omega\|_{L^2}^2 + \nu\|\Lambda^\beta b\|_{L^2}^2 \le 0. \end{aligned} \tag{4.4}$$

Integrating (4.4) over $[0, t]$, we get the desired estimate (4.2), thus the proof of Proposition 4.1 is completed.

We now proceed to estimate the H^1-norm of (u, ω, b). Recalling the vorticity $\Omega = \nabla\times u = \partial_x u_2 - \partial_y u_1$ and the current density $j = \nabla \times b = \partial_x b_2 - \partial_y b_1$, we obtain

$$\begin{cases} \partial_t\Omega + u \cdot \nabla\Omega + (\mu + \chi)\Omega = b \cdot \nabla j - 2\chi\Delta\omega, \\ \partial_t\nabla\omega + \nabla(u \cdot \nabla\omega) - \kappa\Delta\nabla\omega + 4\chi\nabla\omega = 2\chi\nabla\Omega, \\ \partial_t j + u \cdot \nabla j + \nu(-\Delta)^\beta j = b \cdot \nabla\Omega + Q(\nabla u, \nabla b), \end{cases} \tag{4.5}$$

where $Q(\nabla u, \nabla b) = 2\partial_x b_1(\partial_x u_2 + \partial_y u_1) - 2\partial_x u_1(\partial_x b_2 + \partial_y b_1)$.

Proposition 4.2. *Let* $(u_0,\ \omega_0,\ b_0) \in H^1(\mathbb{R}^2)$. *Then, any* $t > 0$, $(\Omega,\ \nabla\omega,\ j)$ *obtains*

$$\begin{aligned} & \|\Omega(t)\|_{L^2}^2 + \|\nabla\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \sigma \int_0^t \|\Omega(\tau)\|_{L^2}^2 d\tau \\ & +2(\kappa - \frac{4\chi^2}{\mu+\chi-\sigma} - \sigma) \int_0^t \|\Delta\omega(\tau)\|_{L^2}^2 d\tau + \nu \int_0^t \|\Lambda^\beta j(\tau)\|_{L^2}^2 d\tau \\ \leq\ & (\|\Omega_0\|_{L^2}^2 + \|\nabla\omega_0\|_{L^2}^2 + \|j_0\|_{L^2}^2) e^{C\int_0^t (\|\nabla\omega(\tau)\|_{L^2}^2 + \|\Lambda^\beta b(\tau)\|_{L^2}^2) d\tau}, \end{aligned} \tag{4.6}$$

where $\sigma > 0$ *is chosen sufficiently small such that* $\kappa > \frac{4\chi^2}{\mu+\chi-\sigma} + \sigma$.

Proof: Taking the L^2-inner product of the first, second, and third equations of (4.5) with Ω, $\nabla\omega$, and j respectively, and adding the resulting equations together, we have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}(\|\Omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 + \|j\|_{L^2}^2) + (\mu+\chi)\|\Omega\|_{L^2}^2 + 4\chi\|\nabla\omega\|_{L^2}^2 + \kappa\|\Delta\omega\|_{L^2}^2 + \nu\|\Lambda^\beta j\|_{L^2}^2 \\ =\ & -4\chi \int_{\mathbb{R}^2} \Omega\Delta\omega dxdy - \int_{\mathbb{R}^2} \nabla u \cdot \nabla\omega \cdot \nabla\omega dxdy + \int_{\mathbb{R}^2} Q(\nabla u, \nabla b) j dxdy. \end{aligned} \tag{4.7}$$

Applying Young's inequality yields

$$-4\chi \int_{\mathbb{R}^2} \Omega\Delta\omega dxdy \leq (\mu+\chi-\sigma)\|\Omega\|_{L^2}^2 + \frac{4\chi^2}{\mu+\chi-\sigma}\|\Delta\omega\|_{L^2}^2.$$

By Hölder's inequality, the Gagliardo-Nirenberg inequality, Young's inequality, and $\|\nabla u\|_{L^2} = \|\Omega\|_{L^2}$, we obtain

$$\begin{aligned} -\int_{\mathbb{R}^2} \nabla u \cdot \nabla\omega \cdot \nabla\omega dxdy &\leq \|\nabla u\|_{L^2}\|\nabla\omega\|_{L^4}^2 \\ &\leq C\|\Omega\|_{L^2}\|\nabla\omega\|_{L^2}\|\Delta\omega\|_{L^2} \\ &\leq \sigma\|\Delta\omega\|_{L^2}^2 + C\|\nabla\omega\|_{L^2}^2\|\Omega\|_{L^2}^2. \end{aligned}$$

Applying the Gagliardo-Nirenberg inequality again yields

$$\|j\|_{L^4} \leq C\|b\|_{L^2}^{\frac{\beta-1}{\beta}} \|\Lambda^\beta b\|_{L^2}^{\frac{2-\beta}{2\beta}} \|\Lambda^\beta j\|_{L^2}^{\frac{1}{2}};$$

this bound allows us to obtain

$$\begin{aligned} & \int_{\mathbb{R}^2} Q(\nabla u, \nabla b) j dxdy \\ =\ & 2\int_{\mathbb{R}^2} \partial_x b_1(\partial_x u_2 + \partial_y u_1) j dxdy - 2\int_{\mathbb{R}^2} \partial_x u_1(\partial_x b_2 + \partial_y b_1) j dxdy \\ \leq\ & C\|\Omega\|_{L^2}\|j\|_{L^4}^2 \\ \leq\ & C\|\Omega\|_{L^2}\|b\|_{L^2}^{\frac{2(\beta-1)}{\beta}} \|\Lambda^\beta b\|_{L^2}^{\frac{2-\beta}{\beta}} \|\Lambda^\beta j\|_{L^2} \\ \leq\ & \frac{\nu}{2}\|\Lambda^\beta j\|_{L^2}^2 + \frac{\sigma}{2}\|\Omega\|_{L^2}^2 + C\|b\|_{L^2}^{\frac{4(\beta-1)}{2-\beta}} \|\Lambda^\beta b\|_{L^2}^2\|\Omega\|_{L^2}^2. \end{aligned}$$

Inserting the above estimates to (4.7) yields

$$\begin{aligned}
&\frac{d}{dt}(\|\Omega\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2+\|j\|_{L^2}^2)+\sigma\|\Omega\|_{L^2}^2+2(\kappa-\frac{4\chi^2}{\mu+\chi-\sigma}-\sigma)\|\Delta\omega\|_{L^2}^2\\
&+8\chi\|\nabla\omega\|_{L^2}^2+\nu\|\Lambda^\beta j\|_{L^2}^2\\
\leq\ & C(\|\nabla\omega\|_{L^2}^2+\|b\|_{L^2}^{\frac{4(\beta-1)}{2-\beta}}\|\Lambda^\beta b\|_{L^2}^2)\|\Omega\|_{L^2}^2. \qquad (4.8)
\end{aligned}$$

Applying Grönwall's inequality gives (4.6). This completes the proof of Proposition 4.2.

In order to obtain the uniform bound of $\int_0^t\|\nabla b(\tau)\|_{L^2}^2d\tau$, we need to estimate the magnetic field b in negative Sobolev space $\dot{H}^{1-\beta}$; more concretely, we have the following result.

Proposition 4.3. *Under the same conditions of Theorem 3.1, then for any $t>0$, we have*

$$\|\Lambda^{1-\beta}b(t)\|_{L^2}^2+\nu\int_0^t\|\nabla b(\tau)\|_{L^2}^2d\tau\leq C. \qquad (4.9)$$

Proof: Applying $\Lambda^{1-\beta}$ to both sides of the Eq $(1.3)_3$ and taking the L^2-inner product of the resulting equation with $\Lambda^{1-\beta}$ yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Lambda^{1-\beta}b\|_{L^2}^2+\nu\|\nabla b\|_{L^2}^2\\
=\ &-\int_{\mathbb{R}^2}\Lambda^{1-\beta}(u\cdot\nabla b)\cdot\Lambda^{1-\beta}bdxdy+\int_{\mathbb{R}^2}\Lambda^{1-\beta}(b\cdot\nabla u)\cdot\Lambda^{1-\beta}bdxdy. \qquad (4.10)
\end{aligned}$$

Applying Lemma 2.1, Hölder's inequality, the Gagliardo-Nirenberg inequality, and Young's inequality, we have

$$\begin{aligned}
&-\int_{\mathbb{R}^2}\Lambda^{1-\beta}(u\cdot\nabla b)\cdot\Lambda^{1-\beta}bdxdy+\int_{\mathbb{R}^2}\Lambda^{1-\beta}(b\cdot\nabla u)\cdot\Lambda^{1-\beta}bdxdy\\
=\ &-\int_{\mathbb{R}^2}\Lambda^{1-2\beta}(u\cdot\nabla b+b\cdot\nabla u)\cdot\Lambda bdxdy\\
\leq\ &(\|\Lambda^{2-2\beta}(u\otimes b)\|_{L^2}+\|\Lambda^{2-2\beta}(b\otimes u)\|_{L^2})\|\Lambda b\|_{L^2}\\
\leq\ &C(\|u\otimes b\|_{L^{\frac{2}{2\beta-1}}}+\|b\otimes u\|_{L^{\frac{2}{2\beta-1}}})\|\Lambda b\|_{L^2}\\
\leq\ &C\|b\|_{L^{\frac{1}{\beta-1}}}\|u\|_{L^2}\|\Lambda b\|_{L^2}\\
\leq\ &C\|b\|_{L^2}^{2(\beta-1)}\|\Lambda b\|_{L^2}^{3-2\beta}\|u\|_{L^2}\|\Lambda b\|_{L^2}\\
\leq\ &\frac{\nu}{2}\|\Lambda b\|_{L^2}^2+C(\|b\|_{L^2}^2+\|j\|_{L^2}^2)\|u\|_{L^2}^2.
\end{aligned}$$

Inserting the above result into (4.10) and integrating time, we obtain

$$\begin{aligned}
&\|\Lambda^{1-\beta}b(t)\|_{L^2}^2+\nu\int_0^t\|\nabla b(\tau)\|_{L^2}^2d\tau\\
\leq\ &\|\Lambda^{1-\beta}b_0\|_{L^2}^2+C\int_0^t(\|b(\tau)\|_{L^2}^2+\|j(\tau)\|_{L^2}^2)\|u(\tau)\|_{L^2}^2d\tau
\end{aligned}$$

$$
\begin{aligned}
&\leq C\|b_0\|_{L^p}^{\frac{2(\beta-1)p}{2-p}}\|b_0\|_{L^2}^{\frac{2(2-\beta p)}{2-p}}+C\sup_{0\leq\tau\leq t}(\|b(\tau)\|_{L^2}^2+\|j(\tau)\|_{L^2}^2)\int_0^t\|u(\tau)\|_{L^2}^2d\tau\\
&\leq C,
\end{aligned}
$$

where $1\leq p\leq\frac{2}{\beta}$. Thus the proof of Proposition 4.3 is completed.

4.2. The proof of Theorem 3.1

This section is devoted to the proof of the large-time behavior of the solutions and their first-order derivatives to system (1.3).

4.2.1. Decay estimates for $\|(\nabla u,\nabla b,\nabla\omega)\|_{L^2}$ and $\|(u,b,\omega)\|_{L^2}$

With Propositions 4.1, 4.2, and 4.3 at our disposal, we investigate the auxiliary decay rate of $\|(\nabla u,\ \nabla\omega,\ \nabla b)\|_{L^2}$.

Proposition 4.4. *Assume the initial values* $(u_0,\ \omega_0,\ b_0)\in H^1(\mathbb{R}^2)$*; then we have*

$$
\|\nabla u(t)\|_{L^2}+\|\nabla\omega(t)\|_{L^2}+\|\nabla b(t)\|_{L^2}\leq C(1+t)^{-\frac{1}{2}}. \tag{4.11}
$$

Proof: When $t\geq 1$, by Proposition 4.1, 4.2, and 4.3, we have

$$
\int_0^\infty\|\nabla\omega(\tau)\|_{L^2}^2d\tau\leq C(\|u_0\|_{L^2}^2+\|\omega_0\|_{L^2}^2+\|b_0\|_{L^2}^2), \tag{4.12}
$$

$$
\begin{aligned}
&\int_0^\infty\|\nabla u(\tau)\|_{L^2}^2d\tau=\int_0^\infty\|\Omega(\tau)\|_{L^2}^2d\tau\\
\leq\ & C(\|\Omega_0\|_{L^2}^2+\|\nabla\omega_0\|_{L^2}^2+\|j_0\|_{L^2}^2)e^{C(\|u_0\|_{L^2}^2+\|\omega_0\|_{L^2}^2+\|b_0\|_{L^2}^2)},
\end{aligned} \tag{4.13}
$$

and

$$
\int_0^t\|\nabla b(\tau)\|_{L^2}^2d\tau\leq C. \tag{4.14}
$$

Combining (4.2) with (4.8), we obtain

$$
\begin{aligned}
&\|\Omega(t)\|_{L^2}^2+\|\nabla\omega(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2\\
\leq\ & C(\|\Omega(s)\|_{L^2}^2+\|\nabla\omega(s)\|_{L^2}^2+\|j(s)\|_{L^2}^2)e^{C\int_s^t(\|\nabla\omega(\tau)\|_{L^2}^2+\|\Lambda^\beta b(\tau)\|_{L^2}^2)d\tau}\\
\leq\ & C(\|\Omega(s)\|_{L^2}^2+\|\nabla\omega(s)\|_{L^2}^2+\|j(s)\|_{L^2}^2)e^{C(\|u_0\|_{L^2}^2+\|\omega_0\|_{L^2}^2+\|b_0\|_{L^2}^2)}.
\end{aligned} \tag{4.15}
$$

Integrating (4.15) in $(\frac{t}{2},\ t)$ with respect to s and applying (4.12)–(4.14), we derive

$$
\begin{aligned}
&t(\|\Omega(t)\|_{L^2}^2+\|\nabla\omega(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2)\\
\leq\ & 2Ce^{C(\|u_0\|_{L^2}^2+\|\omega_0\|_{L^2}^2+\|b_0\|_{L^2}^2)}\int_{\frac{t}{2}}^t(\|\Omega(s)\|_{L^2}^2+\|\nabla\omega(s)\|_{L^2}^2+\|j(s)\|_{L^2}^2)ds
\end{aligned}
$$

$$\leq \quad C.$$

It implies

$$\|\Omega(t)\|_{L^2}^2 + \|\nabla\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 \leq C(1+t)^{-1}.$$

For $0 < t < 1$, applying Proposition 4.2, we obtain

$$\|\Omega(t)\|_{L^2}^2 + \|\nabla\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 \leq C \leq C(1+t)^{-1}.$$

In view of $\|\Omega\|_{L^2}^2 = \|\nabla u\|_{L^2}^2$, and $\|j\|_{L^2}^2 = \|\nabla b\|_{L^2}^2$,we complete the proof of Proposition 4.4.

Now, we apply the properties of a heat operator to investigate the decay rate of $\|(u, \omega, b)\|_{L^2}$.

Proposition 4.5. *Assume the initial values* $(u_0,\ \omega_0,\ b_0)$ *satisfy the assumptions stated in Theorem 3.1; then we have*

$$\|u(t)\|_{L^2} + \|\omega(t)\|_{L^2} \leq C(1+t)^{-\frac{2}{3}}, \tag{4.16}$$

$$\|b(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{\beta}(\frac{1}{p}-\frac{1}{2})}. \tag{4.17}$$

Proof: Taking the L^2-inner products to the first and second equations of (1.3) with u and ω, then adding the resulting equations together, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|u(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2) + (\mu+\chi)\|u\|_{L^2}^2 + 4\chi\|\omega\|_{L^2}^2 + \kappa\|\nabla\omega\|_{L^2}^2\\
= &\int_{\mathbb{R}^2} b\cdot\nabla b\cdot u dxdy + 4\chi\int_{\mathbb{R}^2} \nabla\times\omega\cdot u dxdy\\
\leq &\|b\|_{L^4}\|\nabla b\|_{L^2}\|u\|_{L^4} + (\mu+\chi-\sigma)\|u\|_{L^2}^2 + \frac{4\chi^2}{\mu+\chi-\sigma}\|\nabla\omega\|_{L^2}^2\\
\leq &C\|b\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^2}^{\frac{3}{2}}\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}} + (\mu+\chi-\sigma)\|u\|_{L^2}^2 + \frac{4\chi^2}{\mu+\chi-\sigma}\|\nabla\omega\|_{L^2}^2;
\end{aligned} \tag{4.18}$$

then integrating (4.18) in time, we have

$$\begin{aligned}
&\|u(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2\\
\leq &e^{-\varrho t}(\|u_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2) + C\int_0^t e^{-\varrho(t-\tau)}\|b(\tau)\|_{L^2}^{\frac{1}{2}}\|\nabla b(\tau)\|_{L^2}^{\frac{3}{2}}\|u(\tau)\|_{L^2}^{\frac{1}{2}}\|\nabla u(\tau)\|_{L^2}^{\frac{1}{2}}d\tau,
\end{aligned} \tag{4.19}$$

where $\varrho = \min\{2\sigma,\ 8\chi\}$.

By (4.2), (4.13), and (4.14), we easily obtain

$$\begin{aligned}
&\int_0^{\frac{t}{2}} e^{-\varrho(t-\tau)}\|b(\tau)\|_{L^2}^{\frac{1}{2}}\|\nabla b(\tau)\|_{L^2}^{\frac{3}{2}}\|u(\tau)\|_{L^2}^{\frac{1}{2}}\|\nabla u(\tau)\|_{L^2}^{\frac{1}{2}}d\tau\\
\leq &Ce^{-\frac{\varrho t}{2}}\int_0^{\frac{t}{2}} \|\nabla b(\tau)\|_{L^2}^{\frac{3}{2}}\|\nabla u(\tau)\|_{L^2}^{\frac{1}{2}}d\tau
\end{aligned}$$

$$\begin{aligned}
&\leq \quad Ce^{-\frac{\varrho t}{2}}\int_0^{\frac{t}{2}}(\|\nabla b(\tau)\|_{L^2}^2+\|\nabla u(\tau)\|_{L^2}^2)d\tau \\
&\leq \quad Ce^{-\frac{\varrho t}{2}}. \qquad (4.20)
\end{aligned}$$

Set

$$\mathcal{M}(t)=\sup_{0\leq\tau\leq t}\{(1+\tau)^{\frac{1}{2}}(\|\nabla u(\tau)\|_{L^2}+\|\nabla b(\tau)\|_{L^2})$$

and

$$\mathcal{N}(t)=\sup_{0\leq\tau\leq t}\{(1+\tau)^{\frac{2}{3}}(\|u(\tau)\|_{L^2}+\|\omega\|_{L^2})\};$$

then applying Proposition 4.1 and Proposition 4.4 can yield

$$\begin{aligned}
&\int_{\frac{t}{2}}^t e^{-\varrho(t-\tau)}\|b(\tau)\|_{L^2}^{\frac{1}{2}}\|\nabla b(\tau)\|_{L^2}^{\frac{3}{2}}\|u(\tau)\|_{L^2}^{\frac{1}{2}}\|\nabla u(\tau)\|_{L^2}^{\frac{1}{2}}d\tau \\
&\leq \quad C\mathcal{M}^2(t)\int_{\frac{t}{2}}^t e^{-\varrho(t-\tau)}(1+\tau)^{-\frac{4}{3}}[(1+\tau)^{\frac{2}{3}}\|u(\tau)\|_{L^2}]^{\frac{1}{2}}d\tau \\
&\leq \quad C\mathcal{M}^2(t)\mathcal{N}^{\frac{1}{2}}(t)\int_{\frac{t}{2}}^t e^{-\varrho(t-\tau)}(1+\tau)^{-\frac{4}{3}}d\tau. \qquad (4.21)
\end{aligned}$$

Inserting (4.20) and (4.21) into (4.19), due to $\mathcal{M}(t)\leq C$, we have

$$\mathcal{N}^2\leq C(1+t)^{\frac{4}{3}}e^{-\frac{\varrho t}{2}}+C\mathcal{N}^{\frac{1}{2}}(t).$$

By Young's inequality, we get

$$\|u(t)\|_{L^2}+\|\omega(t)\|_{L^2}\leq C(1+t)^{-\frac{2}{3}}. \qquad (4.22)$$

Therefore, we obtain the first decay estimate of Proposition 4.5.

To obtain the decay estimate of b, we write the equation of $(1.3)_3$ into integral form

$$b(x,y,t) \quad = \quad e^{-\nu\Lambda^{2\beta}t}b_0+\int_0^t\nabla e^{-\nu\Lambda^{2\beta}(t-\tau)}(b\otimes u-u\otimes b)(\tau)d\tau. \qquad (4.23)$$

For $t\geq 1$, appying Lemma 2.2 gets

$$\|e^{-\nu\Lambda^{2\beta}t}b_0\|_{L^2}\leq C(1+t)^{-\frac{1}{\beta}(\frac{1}{p}-\frac{1}{2})}.$$

Using Lemma 2.2 and (4.22), we have for $t\geq 1$,

$$\begin{aligned}
&\|\int_0^t\nabla e^{-\nu\Lambda^{2\beta}(t-\tau)}(b\otimes u-u\otimes b)(\tau)d\tau\|_{L^2} \\
&\leq \quad C\int_0^t(t-\tau)^{-\frac{1}{\beta}}\|(b\otimes u-u\otimes b)(\tau)\|_{L^1}d\tau \\
&\leq \quad C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{\beta}}\|u\|_{L^2}\|b\|_{L^2}d\tau+C\int_{\frac{t}{2}}^t(t-\tau)^{-\frac{1}{\beta}}\|u\|_{L^2}\|b\|_{L^2}d\tau
\end{aligned}$$

$$
\begin{aligned}
&\le C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{\beta}}(1+\tau)^{-\frac{2}{3}}d\tau + C\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1}{\beta}}(1+\tau)^{-\frac{2}{3}}d\tau\\
&\le C(1+t)^{-\frac{1}{\beta}+\frac{1}{3}}.
\end{aligned}
$$

Therefore, for $1<\beta\le\frac{3}{2}$, we obtain

$$
\begin{aligned}
\|b\|_{L^2} &\le C(1+t)^{-\frac{1}{\beta}(\frac{1}{p}-\frac{1}{2})} + C(1+t)^{-\frac{1}{\beta}+\frac{1}{3}}\\
&\le C(1+t)^{-\frac{1}{\beta}(\frac{1}{p}-\frac{1}{2})}.
\end{aligned}
$$

For $0<t<1$, applying Proposition 4.1 yields

$$
\|b\|_{L^2} \le C \le C(1+t)^{-\frac{1}{\beta}(\frac{1}{p}-\frac{1}{2})}.
$$

Thus we complete the proof of Proposition 4.5.

4.2.2. Faster decay estimates for $\|(\nabla u,\nabla b,\nabla\omega)\|_{L^2}$ and $\|(u,\,\omega)\|_{L^2}$

In this subsection, we are devoted to improving the decay rates of $\|(\nabla u,\nabla b,\nabla\omega)\|_{L^2}$ and $\|(u,\,\omega)\|_{L^2}$. Applying generalized Fourier splitting methods, we begin by obtaining the improved decay estimates $\|\nabla u(t)\|_{L^2}+\|\nabla b(t)\|_{L^2}+\|\nabla\omega(t)\|_{L^2}\le C(1+t)^{-\frac{1}{\beta p}}$. Then, by refined calculations, we establish the optimal decay rates $\|u(t)\|_{L^2}+\|\omega(t)\|_{L^2}\le C(1+t)^{-\frac{1}{\beta p}(2-\epsilon)}$.

Proposition 4.6. *Assume the initial values* $(u_0,\,\omega_0,\,b_0)$ *satisfy the assumptions stated in Theorem 3.1; then we have*

$$
\|\nabla u(t)\|_{L^2}+\|\nabla b(t)\|_{L^2}+\|\nabla\omega(t)\|_{L^2}\le C(1+t)^{-\frac{1}{\beta p}}, \tag{4.24}
$$

$$
\|u(t)\|_{L^2}+\|\omega(t)\|_{L^2}\le C(1+t)^{-\frac{1}{\beta p}(2-\epsilon)}, \tag{4.25}
$$

where $0<\epsilon\le\frac{2}{3}$.

Proof: Let $\varpi=\min\{\sigma,8\chi,\nu\}$; then we can obtain by (4.8) and Proposition 4.1 that

$$
\begin{aligned}
&\frac{d}{dt}(\|\Omega\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2+\|j\|_{L^2}^2)+\varpi(\|\Omega\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2+\|\Lambda^\beta j\|_{L^2}^2)\\
\le\ & C(\|\nabla\omega\|_{L^2}^2+\|\Lambda^\beta b\|_{L^2}^2)\|\Omega\|_{L^2}^2.
\end{aligned}\tag{4.26}
$$

Taking

$$
B(t)=\{\xi\in\mathbb{R}^2\,\big|\,|\xi|^{2\beta}\le\frac{n}{1+t}\},
$$

and dividing the domain $\mathbb{R}^2$ into $B(t)$ and $B(t)^c$, obey

$$
\begin{aligned}
\|\Lambda^\beta j\|_{L^2}^2 &= \int_{\mathbb{R}^2}|\xi|^{2\beta}|\widehat{j}(\xi,t)|^2d\xi\\
&\ge \frac{n}{1+t}\int_{B(t)^c}|\widehat{j}(\xi,t)|^2d\xi
\end{aligned}
$$

$$\geq \frac{n}{1+t}\|j\|_{L^2}^2 - \frac{n}{1+t}\int_{B(t)} |\widehat{j}(\xi,t)|^2 d\xi.$$

Then for $t \geq t_0 > n$, we have

$$\begin{aligned}
&\frac{d}{dt}(\|\Omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 + \|j\|_{L^2}^2) + \frac{\varpi n}{1+t}(\|\Omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 + \|j\|_{L^2}^2)\\
\leq\ & \frac{\varpi n}{1+t}\int_{B(t)} |\widehat{j}(\xi,t)|^2 d\xi + C(\|\nabla\omega\|_{L^2}^2 + \|\Lambda^\beta b\|_{L^2}^2)\|\Omega\|_{L^2}^2\\
\leq\ & \frac{\varpi n}{1+t}\int_{B(t)} |\xi|^2|\widehat{b}(\xi,t)|^2 d\xi + C(\|\nabla\omega\|_{L^2}^2 + \|\Lambda^\beta b\|_{L^2}^2)\|\Omega\|_{L^2}^2\\
\leq\ & C(1+t)^{-1-\frac{1}{\beta}}\|b\|_{L^2}^2 + C(\|\nabla\omega\|_{L^2}^2 + \|\Lambda^\beta b\|_{L^2}^2)\|\Omega\|_{L^2}^2\\
\leq\ & C(1+t)^{-1-\frac{2}{\beta p}} + C(\|\nabla\omega\|_{L^2}^2 + \|\Lambda^\beta b\|_{L^2}^2)\|\Omega\|_{L^2}^2.
\end{aligned}$$

Multiplying the above inequality by $(1+t)^{\varpi n}$ ($\varpi n > 5$) can deduce

$$\begin{aligned}
&\frac{d}{dt}[(1+t)^{\varpi n}(\|\Omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 + \|j\|_{L^2}^2)]\\
\leq\ & C(1+t)^{\varpi n-1-\frac{2}{\beta p}} + C(\|\nabla\omega\|_{L^2}^2 + \|\Lambda^\beta b\|_{L^2}^2)[(1+t)^{\varpi n}\|\Omega\|_{L^2}^2].
\end{aligned}$$

By Grönwall's inequality, we have

$$(1+t)^{\varpi n}(\|\Omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 + \|j\|_{L^2}^2) \leq C(1+t)^{\varpi n-\frac{2}{\beta p}};$$

this implies for $t \geq t_0$

$$\|\Omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 + \|j\|_{L^2}^2 \leq C(1+t)^{-\frac{2}{\beta p}}.$$

For $0 < t < t_0$, (4.6) implies

$$\|\Omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 + \|j\|_{L^2}^2 \leq C \leq C(1+t)^{-\frac{2}{\beta p}}.$$

Therefore, we have

$$\|\Omega\|_{L^2} + \|\nabla\omega\|_{L^2} + \|j\|_{L^2} \leq C(1+t)^{-\frac{1}{\beta p}}.$$

In addition, multiplying both sides of (4.26) by $(1+t)^n$ for $n > 5$ yields

$$\begin{aligned}
&\frac{d}{dt}[(1+t)^n(\|\Omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 + \|j\|_{L^2}^2)] + \varpi(1+t)^n(\|\Omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 + \|\Lambda^\beta j\|_{L^2}^2)\\
\leq\ & n(1+t)^{n-1}(\|\Omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 + \|j\|_{L^2}^2) + C(1+t)^n(\|\nabla\omega\|_{L^2}^2 + \|\Lambda^\beta b\|_{L^2}^2)\|\Omega\|_{L^2}^2\\
\leq\ & n(1+t)^{n-1-\frac{2}{\beta p}} + C(1+t)^{n-\frac{2}{\beta p}}(\|\nabla\omega\|_{L^2}^2 + \|\Lambda^\beta b\|_{L^2}^2).
\end{aligned} \tag{4.27}$$

Integrating (4.27) in time, we can obtain

$$\int_0^t (1+\tau)^n(\|\Omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 + \|\Lambda^\beta j\|_{L^2}^2)d\tau \leq C(1+t)^{n-\frac{2}{\beta p}}. \tag{4.28}$$

Next, our goal is to improve the decay rates of $\|u(t)\|_{L^2}$ and $\|\omega(t)\|_{L^2}$. We re-estimate the terms on the right-hand side of (4.18), for $r > 2$, which yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|u(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2) + (\mu+\chi)\|u\|_{L^2}^2 + 4\chi\|\omega\|_{L^2}^2 + \kappa\|\nabla\omega\|_{L^2}^2 \\
= &\int_{\mathbb{R}^2} b\cdot\nabla b\cdot u dxdy + 4\chi\int_{\mathbb{R}^2}\nabla\times\omega\cdot u dxdy \\
\leq &\|b\|_{L^r}\|\nabla b\|_{L^2}\|u\|_{L^{\frac{2r}{r-2}}} + (\mu+\chi-\sigma)\|u\|_{L^2}^2 + \frac{4\chi^2}{\mu+\chi-\sigma}\|\nabla\omega\|_{L^2}^2 \\
\leq &C\|b\|_{L^2}^{\frac{2}{r}}\|\nabla b\|_{L^2}^{2-\frac{2}{r}}\|u\|_{L^2}^{1-\frac{2}{r}}\|\nabla u\|_{L^2}^{\frac{2}{r}} + (\mu+\chi-\sigma)\|u\|_{L^2}^2 + \frac{4\chi^2}{\mu+\chi-\sigma}\|\nabla\omega\|_{L^2}^2.
\end{aligned}
\tag{4.29}
$$

Integrating (4.29) in time, we obtain

$$
\begin{aligned}
&\|u(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2 \\
\leq\ &e^{-\varrho t}(\|u_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2) + C\int_0^{\frac{t}{2}} e^{-\varrho(t-\tau)}\|b\|_{L^2}^{\frac{2}{r}}\|\nabla b\|_{L^2}^{2-\frac{2}{r}}\|u\|_{L^2}^{1-\frac{2}{r}}\|\nabla u\|_{L^2}^{\frac{2}{r}}d\tau \\
&+C\int_{\frac{t}{2}}^{t} e^{-\varrho(t-\tau)}\|b\|_{L^2}^{\frac{2}{r}}\|\nabla b\|_{L^2}^{2-\frac{2}{r}}\|u\|_{L^2}^{1-\frac{2}{r}}\|\nabla u\|_{L^2}^{\frac{2}{r}}d\tau.
\end{aligned}
\tag{4.30}
$$

For $t \geq 1$, using Proposition 4.1 and Proposition 4.2, we get

$$
\begin{aligned}
&\int_0^{\frac{t}{2}} e^{-\varrho(t-\tau)}\|b\|_{L^2}^{\frac{2}{r}}\|\nabla b\|_{L^2}^{2-\frac{2}{r}}\|u\|_{L^2}^{1-\frac{2}{r}}\|\nabla u\|_{L^2}^{\frac{2}{r}}d\tau \\
\leq\ &Ce^{-\frac{\varrho t}{2}}\int_0^{\frac{t}{2}}\|\nabla b\|_{L^2}^{2-\frac{2}{r}}\|\nabla u\|_{L^2}^{\frac{2}{r}}d\tau \\
\leq\ &Ce^{-\frac{\varrho t}{2}}\Big(\int_0^{\frac{t}{2}}\|\nabla b\|_{L^2}^2 d\tau\Big)^{1-\frac{1}{r}}\Big(\int_0^{\frac{t}{2}}\|\nabla u\|_{L^2}^2 d\tau\Big)^{\frac{1}{r}} \\
\leq\ &Ce^{-\frac{\varrho t}{2}}.
\end{aligned}
\tag{4.31}
$$

Applying Proposition 4.4 and Proposition 4.5, we obtain

$$
\begin{aligned}
&\int_{\frac{t}{2}}^{t} e^{-\varrho(t-\tau)}\|b\|_{L^2}^{\frac{2}{r}}\|\nabla b\|_{L^2}^{2-\frac{2}{r}}\|u\|_{L^2}^{1-\frac{2}{r}}\|\nabla u\|_{L^2}^{\frac{2}{r}}d\tau \\
\leq\ &CQ^{1-\frac{2}{r}}(t)\int_{\frac{t}{2}}^{t} e^{-\varrho(t-\tau)}(1+\tau)^{-\frac{2(2r+2-p)}{\beta p(r+2)}}d\tau \\
\leq\ &CQ^{1-\frac{2}{r}}(t),
\end{aligned}
\tag{4.32}
$$

where

$$
Q(t) = \sup_{0\leq s\leq t}\{(1+s)^{\frac{2r+2-p}{\beta p(r+2)}}(\|u(s)\|_{L^2} + \|\omega(s)\|_{L^2})\}.
$$

Inserting the estimates (4.31) and (4.32) into (4.30) gets

$$
Q^2(t) \leq C + CQ^{1-\frac{2}{r}}(t),
$$

which implies, for all $t > 0$,

$$\|u(t)\|_{L^2} + \|\omega(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{\beta p}(2-\epsilon)},$$

where r is sufficiently large such that $0 < \epsilon = \frac{2+p}{r+2} \leq \frac{2}{3}$. Thus we complete the proof of Proposition 4.6.

4.2.3. Faster decay estimates for $\|(\nabla u, \nabla\omega)\|_{L^2}$

In this subsection, we are devoted to investigating the decay estimate for $\|(\nabla u, \nabla\omega)\|_{L^2}$. We are now in the position to state one of the main results.

Proposition 4.7. *Assume the initial values* (u_0, ω_0, b_0) *satisfy the assumptions stated in Theorem 3.1. Then we have*

$$\|\nabla u(t)\|_{L^2} + \|\nabla\omega(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{\beta p}-\frac{2\beta-\beta p+2}{2(2\beta+1)p}}.$$

Proof: Taking the L^2-inner products of the first and second equations of (4.5) with Ω and $\nabla\omega$, respectively, and adding the resulting equations together, we gain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2) + (\mu+\chi)\|\Omega\|_{L^2}^2 + 4\chi\|\nabla\omega\|_{L^2}^2 + \kappa\|\Delta\omega\|_{L^2}^2 \\
= &\int_{\mathbb{R}^2}(b\cdot\nabla j)\Omega dxdy - 4\chi\int_{\mathbb{R}^2}\Omega\Delta\omega dxdy - \int_{\mathbb{R}^2}\nabla\omega\cdot\nabla u\cdot\nabla\omega dxdy. \qquad (4.33)
\end{aligned}$$

Applying Hölder's inequality and Young's inequality yields

$$\begin{aligned}
&\int_{\mathbb{R}^2}(b\cdot\nabla j)\Omega dxdy \\
\leq\ & \|b\|_{L^\infty}\|\nabla j\|_{L^2}\|\Omega\|_{L^2} \\
\leq\ & C\|b\|_{L^2}^{\frac{\beta}{1+\beta}}\|\Lambda^\beta j\|_{L^2}^{\frac{1}{1+\beta}}\|j\|_{L^2}^{\frac{\beta-1}{\beta}}\|\Lambda^\beta j\|_{L^2}^{\frac{1}{\beta}}\|\Omega\|_{L^2} \\
\leq\ & \frac{\sigma}{2}\|\Omega\|_{L^2}^2 + C\|b\|_{L^2}^{\frac{2\beta^2}{2\beta+1}}\|j\|_{L^2}^{\frac{2(\beta^2-1)}{2\beta+1}}\|\Omega\|_{L^2}^{\frac{2(\beta+1-\beta^2)}{2\beta+1}}\|\Lambda^\beta j\|_{L^2}^2 \\
\leq\ & \frac{\sigma}{2}\|\Omega\|_{L^2}^2 + C(1+t)^{-\frac{2\beta-\beta p+2}{(2\beta+1)p}}\|\Lambda^\beta j\|_{L^2}^2,
\end{aligned}$$

$$-4\chi\int_{\mathbb{R}^2}\Omega\Delta\omega dxdy \leq (\mu+\chi-\sigma)\|\Omega\|_{L^2}^2 + \frac{4\chi^2}{\mu+\chi-\sigma}\|\Delta\omega\|_{L^2},$$

and

$$\begin{aligned}
-\int_{\mathbb{R}^2}\nabla\omega\cdot\nabla u\cdot\nabla\omega dxdy &\leq \frac{\sigma}{2}\|\Delta\omega\|_{L^2}^2 + C\|\nabla\omega\|_{L^2}^2\|\Omega\|_{L^2}^2 \\
&\leq \frac{\sigma}{2}\|\Delta\omega\|_{L^2}^2 + C(1+t)^{-\frac{2}{\beta p}}\|\Omega\|_{L^2}^2.
\end{aligned}$$

Inserting the above estimates into (4.33) and multiplying both sides of the resulting equation by $(1+t)^n$, we can obtain

$$\frac{d}{dt}\left((1+t)^n(\|\Omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2)\right)$$

$$\leq \quad C(1+t)^n((1+t)^{-\frac{2\beta-p\beta+2}{(2\beta+1)p}}\|\Lambda^\beta j\|_{L^2}^2+(1+t)^{-\frac{2}{\beta p}}\|\Omega\|_{L^2}^2),$$

where we have assumed that there exists a $t_0 > 1$ such that $5 < n < \varpi(1+t)$ for $t > t_0$ and $\varpi = \min\{\sigma, 8\chi\}$.

Integrating in time (t_0, t) for the above inequality and (4.28), we can obtain

$$\begin{aligned}
& (1+t)^n(\|\Omega\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2) \\
\leq\ & C(1+t_0)^n(\|\Omega(t_0)\|_{L^2}^2+\|\nabla\omega(t_0)\|_{L^2}^2) \\
& +C\int_{t_0}^t(1+\tau)^{n-\frac{2\beta-p\beta+2}{(2\beta+1)p}}\|\Lambda^\beta j\|_{L^2}^2 d\tau+C\int_{t_0}^t(1+\tau)^{n-\frac{2}{\beta p}}\|\Omega\|_{L^2}^2 d\tau \\
\leq\ & C+C(1+t)^{n-\frac{2\beta-p\beta+2}{(2\beta+1)p}-\frac{2}{\beta p}}.
\end{aligned}$$

It implies for $t > t_0$

$$\|\Omega\|_{L^2}+\|\nabla\omega\|_{L^2}\leq C(1+t)^{-\frac{1}{\beta p}-\frac{2\beta-p\beta+2}{2(2\beta+1)p}}.$$

For fixed t_0, when $0 < t \leq t_0$, using Proposition 4.2 can deduce

$$\|\nabla u\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2\leq C\leq C(1+t)^{-\frac{1}{\beta p}-\frac{2\beta-p\beta+2}{2(2\beta+1)p}}.$$

Therefore, the Proposition 4.7 is obtained. To summarize, the proof of Theorem 3.1 is completed.

5. Conclusions

This paper establishes the large-time behavior of solutions to the 2D magneto-micropolar equations with linear velocity damping and fractional magnetic diffusion. To overcome the difficulty caused by fractional-order dissipation, we apply the negative Sobolev space and prove $\int_0^t \|\nabla b(\tau)\|_{L^2} d\tau \leq C$. However, it is necessary to restrict $1 < \beta \leq \frac{3}{2}$ for this result to hold. Furthermore, for the initial values $(u_0, \omega_0, b_0) \in H^1(\mathbb{R}^2)$, we establish that $\|\nabla u(t)\|_{L^2} + \|\nabla\omega(t)\|_{L^2} + \|\nabla b(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}}$ and $\|u(t)\|_{L^2} + \|\omega(t)\|_{L^2} \leq C(1+t)^{-\frac{2}{3}}$. In addition, under the additional assumption that $b_0 \in L^p(\mathbb{R}^2)$ $(1 \leq p \leq \frac{2}{\beta})$, we apply the properties of a heat operator, generalized Fourier splitting methods, and iterative methods and obtain the desired decay rates of the solutions and their first-order derivatives presented in Theorem 3.1. It is worth noting that in order to improve the decay estimate from $\|u(t)\|_{L^2} + \|\omega(t)\|_{L^2} \leq C(1+t)^{-\frac{2}{3}}$ to $\|u(t)\|_{L^2} + \|\omega(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{\beta p}(2-\epsilon)}$, we require $0 < \epsilon \leq \frac{2}{3}$. In our approach, it is optimal to have the positive number ϵ as small as possible. Therefore this restriction is reasonable. This study not only demonstrates the stabilizing effect of the magnetic field on the fluid flow, but also provides a framework for investigating the decay properties of other fluid models with fractional dissipation.

Acknowledgments

Guo was supported by the Key Research Project Plan for Higher Education Institutions of Henan Province (No. 24B110012), the Young Scientists Fund of the Natural Science Foundation of Henan (No. 242300420922), and the Scientific Research Funds from the Shangqiu Normal University (No. 7001120). Li was supported by the National Natural Science Foundation of China (No. 12201074), the Natural Science Foundation of Jiangsu Province (No. BK20220614), and the Natural Science Fund Project of Colleges in Jiangsu Province (No. 22KJB110008).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

No conflict of interest exits in the submission of this manuscript.

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