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https://www.aimspress.com/journal/era

ERA, 33(12): 7509–7527.
DOI: 10.3934/era.2025331
Received: 13 October 2025
Revised: 29 November 2025
Accepted: 05 December 2025
Published: 15 December 2025

Research article

Boundedness and stability analysis of the two species predator-prey model with prey-taxis and nonlinear growth rate in predator

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Abstract: In this paper, we study initial-boundary value problem of a predator-prey model with taxis strategies and a nonlinear growth rate for the predator. We establish that, for any spatial dimension ($N \geq 1$), the model admits positive classical solutions that are globally existent and uniformly bounded. Our results demonstrate that the nonlinear growth rate can effectively restrain the aggregation of predators. Furthermore, by constructing a suitable energy functional and combining it with the previously established uniform boundedness result, we analyze the global asymptotic stability of the coexistence steady state.

Keywords: predator-prey model; nonlinear grow rate; existence; boundedness; dynamics behavior; global stability

1. Introduction

The predator-prey model is a crucial model for the interaction between biological population, which can be used to reveal the ecological complexity. To study predator-prey models, researchers typically formulate them as systems of partial differential equations (PDEs). This framework has yielded a wealth of research findings, including analyses of the boundedness of solutions, the long-time behavior of constant states, and the existence and non-existence of non-constant steady-state solutions and periodic solutions [1,2]. A typical PDE model is as follows:

$$\begin{cases} \frac{\partial u}{dt} = d_u \Delta u - au + eug(v), & x \in \Omega, t > 0, \\ \frac{\partial v}{dt} = d_v \Delta v + bv\left(1 - \frac{v}{K}\right) - ug(v), & x \in \Omega, t > 0, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ denotes a bounded domain with a smooth boundary $\partial\Omega$; $u(x,t)$ and $v(x,t)$ represent the densities of the predator and prey at time $t > 0$ and spatial position $x \in \Omega$, respectively; $g(v)$

is the so-called prey-dependent functional response; d_u and d_v are positive constants, the intensity of random dispersals of the species; $b > 0$ denotes the intrinsic growth rate of the prey; $K > 0$ represents the carrying capacity of the prey; $a > 0$ stands for the mortality rate of the predator; and $e > 0$ is the interaction strength between the predator and prey. For the homogeneous Dirichlet boundary condition, Zhou and Mu [3] have proven the existence of positive steady states for system (1.1) (with a Holling II-type functional response) by employing fixed-point index theory and bifurcation theory. Yi et al. [4] investigated the Hopf and Turing bifurcations of the system (1.1) (with Holling II-type functional response) under the homogeneous Neumann boundary condition.

In some existing literature, the growth function under consideration is linear, with the well-known logistic model serving as a typical example. However, in numerous biological phenomena, the growth rate is inherently dependent on population density. Recently, Yang et al. [5] considered the following diffusive predator-prey system, which incorporates a Holling II-type functional response and a nonlinear growth rate for the predator,

$$\begin{cases} u_t = d_u\Delta u + u\left(\frac{h}{1+ru} - a\right) + \frac{euv}{1+mv}, & x \in \Omega, t > 0, \\ v_t = d_v\Delta v + v(b - v) - \frac{uv}{1+mv}, & x \in \Omega, t > 0, \end{cases} \tag{1.2}$$

where r is the strength of density-dependence, and the predator population reproduces by the nonlinear function $\frac{h}{1+rv}$, which is called Beverton-Holt-like function [6,7]. The per capita reproduction rate (with a maximum value of h) decreases with density increasing. Under the homogeneous Dirichlet boundary condition, the existence, stability, and exact number of positive solutions for large values of m have been established in [6,7]. Chen and Yu [8] established the global attractivity of constant equilibria and proved the non-existence of non-constant positive steady states for system (1.2).

In 1987, Kareiva and Odell [9] first proposed a PDEs-based prey-taxis model to explain how area-restricted search gives rise to the phenomenon of predator aggregation:

$$\begin{cases} u_t = d_u\Delta u - \chi\nabla\cdot(u\nabla v) + uh(u) + eug(v), & x \in \Omega, t > 0, \\ v_t = d_v\Delta v + f(v) - ug(v), & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ (u, v)(x, 0) = (u, v)(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where $u(x, t)$ and $v(x, t)$ denote the densities of predators and preys, respectively. $\chi > 0$ represents the prey-taxis sensitivity coefficient, and $d_u, d_v > 0$ the diffusivity abilities of predator and prey, respectively. $e > 0$ denotes the intrinsic predation rate. The term $uh(u)$ describes the population kinetic of the predator, which is often used as the following forms of the function:

$$h(u) = a + bu, \quad a > 0, b \geq 0,$$

where a is the natural death rate and b denotes the rate of death resulting from the intra-specific competition. The term $f(v)$ is the growth function of prey, which is often used as logistic type and Allee effect type. The Allee effect refers to the phenomenon in ecology where population growth decreases at low population densities, often due to difficulties in mate finding, cooperative behaviors, or predator avoidance. Typical examples include reduced reproductive success in sparse insect populations and diminished survival in small social animal groups. This concept has been well documented and classified into weak and strong forms [10], and it plays an important role in mathematical models of population dynamics, particularly when low-density behavior influences long-term outcomes. The function $ug(v)$ represents the inter-specific interaction, and $g(v)$ is the functional response accounting for the intake rate of the predator as a function of prey density. Over the past decade, a wealth of research has emerged concerning two-species predator-prey models with prey-taxis. Researchers have increasingly focused on analyzing three key aspects of such models: the existence and boundedness of solutions, the long-time dynamical behavior of solutions [11–13], and pattern formation [14]. Recently, many researchers have focused on investigating three-population predator-prey models that incorporate prey-taxis. The interactions among these populations have yielded many mathematics and biology results [15–20].

Mathematical modeling offers a powerful way to analyze complex behaviors in ecological and epidemiological systems by turning real-world interactions into quantitative frameworks that allow for stability analysis, parameter sensitivity, and long-term simulation. For instance, Ahmed and Jawad [21] studied antibiotic effects on gut microbiota, Hakeem et al. [22] modeled desertification, Javaid et al. [23] explored predator-prey dynamics with Allee effects, and Ali et al. [24] developed an cholera model incorporating asymptomatic detection to evaluate control strategies. These models highlight how mathematical approaches can uncover hidden dynamics, guide interventions, and improve our understanding of system behavior.

Based on system (1.2), we consider the following diffusive predator-prey system with prey-taxis and a nonlinear growth rate for the predator,

$$\begin{cases} u_t = d_u\Delta u - \chi\nabla\cdot(u\nabla v) + u\left(\frac{h}{1+ru} - a\right) + eug(v), & x\in\Omega, t>0,\\ v_t = d_v\Delta v + v(b-v) - ug(v), & x\in\Omega, t>0,\\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega, t>0,\\ u(x,0)=u_0(x), v(x,0)=v_0(x), & x\in\Omega. \end{cases} \tag{1.4}$$

The initial data of system (1.4) satisfies

$$u_0, v_0 \in W^{1,p}(\Omega), v_0 \gneqq 0, (p>N), \tag{1.5}$$

and $g(s)$ satisfies the following hypotheses:
(H1) $g(s)\in C^1([0,\infty)), g(0)=0, g(s)>0$ in $(0,\infty)$, and $g'(s)>0$ on $[0,\infty)$.

The main results concerning the global existence and boundedness of solutions to system (1.4) are presented as follows.

Theorem 1.1. *Let $\Omega\subset\mathbb{R}^N (N\geq 1)$ be a bounded domain with smooth boundary. For any initial data satisfying condition* (1.5)*, there exists $\chi_1>0$ (depending on the initial data) such that if $\chi<\chi_1$, then problem* (1.4) *possesses a unique global nonnegative classical solution*

$$(u,v)\in[C^0(\overline{\Omega}\times[0,+\infty))\cap C^{2,1}(\overline{\Omega}\times(0,+\infty))]^2.$$

Moreover, the solution satisfies

$$\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{W^{1,\infty}} \leq C, \quad \textit{for all } t>0,$$

where $C>0$ is a constant independent of t.

Remark 1.2. *In order to better understand problem* (1.4)*, it is necessary to mention the general form of the prey-taxis model as follows:*

$$\begin{cases} u_t = d_u\Delta u - \chi\nabla\cdot(u\nabla v) + (\alpha - \beta u)u + \xi u f(v), & x \in \Omega, t > 0, \\ v_t = d_v\Delta v + (a - bv)v - uf(v), & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ (u, v)(x, 0) = (u_0, v_0)(x), & x \in \Omega. \end{cases} \tag{1.6}$$

Wang and Wang [25, Theorems 1.2 and 1.3] showed that, there exists a constant $\beta^* > 0$ *such that if*

$$\beta > \beta^*, \tag{1.7}$$

then (1.6) *has a unique nonnegative and global bounded solution in higher dimensions* ($N \geq 3$)*. We can distinguish the difference between* (1.4) *and* (1.6)*, that is, there is no interspecific competition of predators in* (1.4)*. In this paper, if the prey-taxis coefficient* χ *is small, then system* (1.4) *possesses bounded solutions.*

The second objective of this paper is to clarify the role of prey-taxis in regulating the stability of nonnegative spatially homogeneous equilibria. We consider $g(v) = v$, and let $a = b = e = 1$ then

$$\begin{cases} u_t = d_u\Delta u - \chi\nabla\cdot(u\nabla v) + u\left(\frac{h}{1+ru} - 1\right) + uv, & x \in \Omega, t > 0, \\ v_t = d_v\Delta v + v(1 - v) - uv, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ (u, v)(x, 0) = (u, v)(x), & x \in \Omega. \end{cases} \tag{1.8}$$

By direct computation, (1.8) have two semi-trivial equilibrium, $(0, 1)$ and $\left(\frac{h-1}{r}, 0\right)$, if $h > 1$. Furthermore, (1.8) has the unique positive constant solution

$$u^* = \frac{-1 + \sqrt{1 + 4hr}}{2r}, v^* = \frac{2r + 1 - \sqrt{1 + 4hr}}{2r}. \tag{1.9}$$

The positive constant solution (u^*, v^*) exists if $h < r + 1$.

Theorem 1.3. *Assume that* $h, r, \chi > 0$ *and* Ω *is a bounded domain with smooth boundary, then*

1) *If* $h < r + 1$*, then the co-existence steady state* (u^*, v^*) *is locally asymptotically stable;*
2) *If* $h > r + 1$*, then the semi-trivial steady state* $(u_*, 0)$ *is locally asymptotically stable;*
3) *The semi-trivial steady state* $(0, 1)$ *is unstable;*
4) *The extinction steady state* $(0, 0)$ *is unstable.*

We know from Theorem 1.3 that $(0, 1)$ and $(0, 0)$ are unstable, while (u^*, v^*) and $(u_*, 0)$ are linearly stable under suitable conditions. Hence, it is natural to study whether or not (u^*, v^*) and $(u_*, 0)$ are globally asymptotically stable. Then global stability results are stated in the following theorem.

Theorem 1.4. *Let* (u, v) *be a nonnegative global bounded classical solution of system* (1.8) *with initial data satisfying condition* (1.5).

(i) Suppose that $h < r + 1$ *and*

$$\chi^2 < \frac{4d_u d_v v^*}{u^* \|v\|_{L^\infty}^2},$$

then

$$\|u(\cdot, t) - u^*\|_{L^\infty} + \|v(\cdot, t) - v^*\|_{L^\infty} \to 0, as\ t \to \infty.$$

(ii) Suppose that $h \geq r + 1$ *and*

$$\chi^2 < \frac{4d_u d_v}{u_*}, \tag{1.10}$$

then

$$\lim_{t\to\infty} (\|u(\cdot, t) - u_*\|_{L^\infty} + \|v(\cdot, t)\|_{L^\infty}) = 0.$$

The arrangement of this article is as follows. In Section 2, we get some useful estimates of the solution and proof of Theorem 1.1. In Section 3, we consider the dynamics behavior of the solution. In Section 4, we give the simulations and discussion.

2. Existence and boundedness of global solutions

The existence of local solutions to system (1.4) is established via Amann's theorem [26]. First, we give the following lemma.

Lemma 2.1. *Let* $\Omega \subset \mathbb{R}^N (N \geq 1)$ *be a bounded domain. If the initial data satisfies condition* (1.5)*, then there exist a maximal existence time* $T_{\max} \in (0, \infty]$ *and a pair* (u, v) *of nonnegative functions*

$$(u, v) \in [C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(0, T_{\max})]^2,$$

which solves (1.5) *in the classical sense in* $\Omega \times (0, T_{\max})$*. Moreover, we have*

$$u > 0, 0 < v \leq B := \max\{\|v_0\|_{L^\infty(\Omega)}, b\}. \tag{2.1}$$

and either

$$T_{\max} = \infty \quad or \quad \limsup_{t \nearrow T_{\max}} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)}) = \infty. \tag{2.2}$$

Proof. The local-in-time existence and uniqueness of the classical solution to problem (1.4) follow from Amann's theorem [26, Theorems 14.4 and 14.6]. The estimate (2.1) can be derived via the maximum principle. Furthermore, the extensibility criterion (2.2) can be obtained directly from [27, Theorem 5.2].

Lemma 2.2. *Let* (u, v) *be a solution of* (1.4)*. Then, u and v satisfy*

$$\int_\Omega u + e \int_\Omega v \leq C, \tag{2.3}$$

where positive constant C is independent of t.

Proof. From the equations of (1.4), a direct calculation indicates that

$$\begin{aligned}\frac{d}{dt}\left(\int_\Omega u+ev\right)=&\int_\Omega u\left(\frac{h}{1+ru}-a\right)+e\int_\Omega v(b-v)\\ \leq& h\int_\Omega\frac{u}{1+ru}-a\int_\Omega u+\frac{b^2e}{4}|\Omega|\end{aligned}$$

because of $x(b-x)\leq\frac{b^2}{4}$. Thus

$$\begin{aligned}\frac{d}{dt}\left(\int_\Omega u+e\int_\Omega v\right)\leq&-a\left(\int_\Omega u+e\int_\Omega v\right)+h\int_\Omega\frac{u}{1+ru}+aeB|\Omega|+\frac{b^2e}{4}|\Omega|\\ =&-a\left(\int_\Omega u+e\int_\Omega v\right)+h\int_\Omega\frac{[(1+ru)-1]}{r(1+ru)}+aeB|\Omega|+\frac{b^2e}{4}|\Omega|\\ \leq&-a\left(\int_\Omega u+e\int_\Omega v\right)+\frac{h}{r}|\Omega|+aeB|\Omega|+\frac{b^2e}{4}|\Omega|.\end{aligned}$$

Hence, we have

$$\frac{d}{dt}\left(\int_\Omega u+e\int_\Omega v\right)+a\left(\int_\Omega u+e\int_\Omega v\right)\leq C.$$

Accordingly, upon the ordinary differential equation comparison principle, we arrive at

$$\int_\Omega u+e\int_\Omega v\leq C,$$

for all $t\in(0,T_{\max})$, which directly yields (2.3).

In view of the facts $0\leq v\leq B$ and $g(s)\in C^1([0,\infty))$ (see Lemma 2.1 and (H1)), there is a constant $M_v>0$ independent of t such that

$$\|g(v(\cdot,t))\|_{L^\infty}\leq M_v,\qquad \text{for all } t\in(0,T_{\max}). \tag{2.4}$$

In this subsequent proof, we assume that

$$\sup_{0<t<T}\|u(\cdot,t)\|_{L^\infty}\leq M_u \tag{2.5}$$

with some $T\in(0,T_{\max})$.

Lemma 2.3. *Let $\Omega\subset\mathbb{R}^N(N\geq 1)$ be a bounded domain with smooth boundary. Then, for the classical solution (u,v) of (1.4), there exists a constant $M_2>0$ independent of t such that*

$$\|\nabla v(\cdot,t)\|_{L^\infty}\leq M_2\|v_0\|_{L^\infty}+M_2|\Omega|^{\frac{1}{p}}\left(\frac{b^2}{4}+M_uM_v\right),\quad \textit{for all } t\in(0,T). \tag{2.6}$$

Proof. Using the variation of constants formula to the second equation of (1.4), one has

$$v(\cdot,t)=e^{d_vt\Delta}v_0+\int_0^t e^{d_v(t-s)\Delta}\left[v(b-v)-ug(v)\right]ds.$$

Therefore,

$$\nabla v(\cdot,t) = \nabla e^{d_v t\Delta} v_0 + \int_0^t \nabla e^{d_v(t-s)\Delta}\left[v(b-v) - ug(v)\right] ds. \tag{2.7}$$

From Minkowski's inequality, (2.4), and (2.5), we get that there is $p > n$ such that

$$\|v(b-v) - ug(v)\|_{L^p} \le \frac{b^2}{4}|\Omega|^{\frac{1}{p}} + \|ug(v)\|_{L^p} \le \frac{b^2}{4}|\Omega|^{\frac{1}{p}} + M_u M_v |\Omega|^{\frac{1}{p}},$$

for all $t \in (0,T)$. Applying Lemma 1.3 in [28] to (2.7), we obtain

$$\begin{aligned}
&\|\nabla v(\cdot,t)\|_{L^\infty} \\
\le &\left\|\nabla e^{d_v t\Delta} v_0\right\|_{L^\infty} + \int_0^t \left\|\nabla e^{d_v(t-s)\Delta}\left[v(b-v) - ug(v)\right]\right\|_{L^\infty} ds \\
\le &Ce^{-d_v\lambda_1 t}\|v_0\|_{L^\infty} + C\int_0^t \left(1 + (t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right) e^{-d_v\lambda_1(t-s)}\|v(b-v) - ug(v)\|_{L^p} ds \\
\le &C\|v_0\|_{L^\infty} + C\left(\frac{b^2}{4}|\Omega|^{\frac{1}{p}} + M_u M_v|\Omega|^{\frac{1}{p}}\right)\int_0^\infty \left(1 + (t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right) e^{-d_v\lambda_1(t-s)} ds \\
:= &C\|v_0\|_{L^\infty} + C\left(\frac{b^2}{4}|\Omega|^{\frac{1}{p}} + M_u M_v|\Omega|^{\frac{1}{p}}\right) M_3, \qquad \text{for all } t \in (0,T),
\end{aligned}$$

where $\lambda_1 > 0$ denotes the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions, $M_3 = \int_0^\infty \left(1 + (t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right) e^{-d_v\lambda_1(t-s)} ds$, C is a positive constant. Let $M_2 = \max\{C, CM_3\}$, and we get (2.6).

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^N (N \ge 1)$ be a bounded domain with smooth boundary. Then, for the classical solution (u,v) of* (1.4)*, there exist constants $M_4, M_5 > 0$ and $j > \max\left\{\frac{n}{2}, 1\right\}$ such that*

$$\|u(\cdot,t)\|_{L^\infty} \le M_4 + \chi M_4 M_5 + M_4 M_u^{1-\frac{1}{j}}, \textit{ for all } t \in (0,T). \tag{2.8}$$

Proof. Applying (2.3)–(2.6) and Hölder's inequality, we obtain that there exist $q > n$ and $j > \max\left\{1, \frac{n}{2}\right\}$ such that

$$\|u\nabla v\|_{L^q} \le M_u|\Omega|^{\frac{1}{q}}\left(M_2\|v_0\|_{L^\infty} + M_2|\Omega|^{\frac{1}{p}}\left(\frac{b^2}{4} + M_u M_v\right)\right) := M_5,$$

$$\begin{aligned}
\left\|\frac{hu}{1+ru} + eug(v)\right\|_{L^j} &\le \frac{h}{r}|\Omega|^{\frac{1}{j}} + eM_v\|u\|_{L^j} \\
&\le \frac{h}{r}|\Omega|^{\frac{1}{j}} + eM_v\left(\|u\|_{L^\infty}^{1-\frac{1}{j}}\|u\|_{L^1}^{\frac{1}{j}}\right) \\
&\le \frac{h}{r}|\Omega|^{\frac{1}{j}} + eM_v M_1^{\frac{1}{j}} M_u^{1-\frac{1}{j}},
\end{aligned}$$

for all $t \in (0,T)$. The first equation of (1.4) can be rewritten as

$$u_t - d_u\Delta u + au = -\chi\nabla\cdot(u\nabla v) + \frac{hu}{1+ru} + eug(v).$$

In view of the variation of constants formula and Lemma 1.3 in [28], it yields

$$
\begin{aligned}
\|u(\cdot,t)\|_{L^\infty} \le{}& \left\|e^{t(d_u\Delta-a)}u_0\right\|_{L^\infty} + \chi\int_0^t \left\|e^{(t-s)(d_u\Delta-a)}\nabla\cdot(u\nabla v)\right\|_{L^\infty} ds \\
&+ \int_0^t \left\|e^{(t-s)(d_u\Delta-a)}\left(\frac{hu}{1+ru}+eug(v)\right)\right\|_{L^\infty} ds \\
\le{}& \|u_0\|_{L^\infty} + C\chi\int_0^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2q}}\right)e^{-(\lambda_1 d_u+a)(t-s)}\|u\nabla v\|_{L^q} ds \\
&+ C\int_0^t \left(1+(t-s)^{-\frac{n}{2j}}\right)e^{-a(t-s)}\left\|\frac{hu}{1+ru}+eug(v)\right\|_{L^j} ds \\
\le{}& \|u_0\|_{L^\infty} + C\chi M_5\int_0^\infty \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2q}}\right)e^{-(\lambda_1 d_u+a)(t-s)} ds \\
&+ C\left(\frac{h}{r}|\Omega|^{\frac{1}{j}} + eM_vM_1^{\frac{1}{j}}M_u^{1-\frac{1}{j}}\right)\int_0^\infty \left(1+(t-s)^{-\frac{n}{2j}}\right)e^{-a(t-s)} ds \\
\le{}& \|u_0\|_{L^\infty} + C\chi M_5M_6 + C\frac{h}{r}|\Omega|^{\frac{1}{j}}M_7 + CeM_vM_1^{\frac{1}{j}}M_u^{1-\frac{1}{j}}M_7,
\end{aligned}
$$

for all $t\in(0,T)$, where $\lambda_1>0$ denotes the first nonzero eigenvalue of $-\Delta$ in Ω, C is a positive constant, $M_6=\int_0^\infty \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2q}}\right)e^{-(\lambda_1 d_u+a)(t-s)}ds$, and $M_7=\int_0^\infty \left(1+(t-s)^{-\frac{n}{2j}}\right)e^{-a(t-s)}ds$. Let

$$
M_4=\max\left\{\|u_0\|_{L^\infty}+C\frac{h}{r}|\Omega|^{\frac{1}{j}}M_7, CM_6, CeM_vM_1^{\frac{1}{j}}M_7\right\},
$$

so we have (2.8).

Lemma 2.5. *Let $\Omega\subset\mathbb{R}^N(N\ge 1)$ be a bounded domain with smooth boundary. For the classical solution (u,v) of* (1.4)*, there exists $\chi_1>0$ (depending on the initial data) such that if $\chi<\chi_1$, then*

$$
\|\nabla v(\cdot,t)\|_{L^\infty}+\|u(\cdot,t)\|_{L^\infty}\le M_8,\quad \text{for all } t\in(0,T_{\max}), \tag{2.9}
$$

where $M_8>0$ is a constant independent of t.

Proof. We take M_u sufficiently large such that

$$
M_u>(8M_4)^j>8M_4>8\|u_0\|_{L^\infty}, \tag{2.10}
$$

where M_u, j and M_4 are defined in (2.5) and Lemma 2.4, respectively. Let

$$
\chi_1=\frac{M_u}{2M_4M_5}=\frac{1}{2M_4|\Omega|^{\frac{1}{q}}\left(M_2\|v_0\|_{L^\infty}+M_2|\Omega|^{\frac{1}{p}}\left(\frac{b^2}{4}+M_uM_v\right)\right)}
$$

and

$$
T:=\sup\left\{\hat{T}\in(0,T_{\max})\Big|\sup_{0<t<\hat{T}}\|u(\cdot,t)\|_{L^\infty(\Omega)}\le M_u\right\}.
$$

According to the continuity of u and (2.10), T is well-defined and positive. Next, we claim that $T = T_{\max}$ under the assumption $\chi < \chi_1$. From the facts (2.10) and Young's inequality, we obtain

$$\begin{aligned}\sup_{0<t<T} \|u(\cdot,t)\|_{L^\infty} &\leq M_4 + \chi M_4 M_5 + M_4 M_u^{1-\frac{1}{j}} \\ &\leq \frac{1}{8}M_u + \frac{1}{2}M_u + \frac{1}{8}M_u + 8^{j-1}M_4^j < M_u.\end{aligned}$$

Thus, using the continuity u again, one has $T = T_{\max}$ and

$$\sup_{0<t<T_{\max}} \|u(\cdot,t)\|_{L^\infty} \leq C.$$

By applying (2.6), we get

$$\sup_{0<t<T_{\max}} \|\nabla v(\cdot,t)\|_{L^\infty} \leq C.$$

This completes the proof of Lemma 2.5.

The proof of Theorem 1.1. From (2.1) and Lemma 2.5, we can find a constant C independent of t such that $\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{W^{1,\infty}} \leq C$, for all $t \in (0, T_{\max})$, which together with Lemma 2.1 implies Theorem 1.1.

3. Stability of constant steady state

Based on the previously established global boundedness result for the classical solutions of the system, our focus in this section shifts to analyzing the dynamics of the nonnegative constant steady states of system (1.8), which are (u^*, v^*), $\left(\frac{h-1}{r}, 0\right)$, and $(0, 1)$ of (1.8). Here, our discussion in this section is divided into two closely connected parts. In the first part, we analyze the local asymptotic stability of steady states; in the second part, we investigate the global asymptotic stability of the solutions.

3.1. Local asymptotic behavior

In this subsection, we analyze the stability and instability of constant steady states. To begin, for the corresponding ODE, using eigenvalue analysis, we study the stability of these equilibrium points.

We first consider the following ordinary differential equation:

$$\begin{cases} u_t = u\left(\frac{h}{1+ru} - 1\right) + uv, & t > 0, \\ v_t = v(1-v) - uv, & t > 0, \\ u(0) = u_0, v(0) = v_0. \end{cases} \tag{3.1}$$

For this ODE, we have the following proposition.

Proposition 3.1. *Assume that $h, r > 0$.*

(i) If $h < r + 1$, then the co-existence equilibrium (u^*, v^*) of (3.1) is locally asymptotically stable;
(ii) If $h > r + 1$, then the semi-trivial equilibrium $(\frac{h-1}{r}, 0)$ is locally asymptotically stable.
(iii) The semi-trivial equilibrium $(0, 1)$ is a saddle point.

Proof. 1) It is easy to see that the stability of the equilibrium (u^*, v^*) can be determined by the following linearized matrix:

$$A_O = \begin{pmatrix} \frac{-hru^*}{(1+ru^*)^2} & u^* \\ -v^* & -v^* \end{pmatrix}.$$

The stability of (u^*, v^*) is reduced to consider the characteristic equation:

$$det(A_O - \mu I) = \mu^2 - T_O\mu + D_O$$

with

$$\begin{cases} T_O = \frac{-hru^*}{(1+ru^*)^2} - v^*, \\ D_O = \frac{hru^*v^*}{(1+ru^*)^2} + u^*v^*. \end{cases}$$

Then $T_O < 0$ and $D_O > 0$. Hence, (u^*, v^*) is locally asymptotically stable.

2) The stability of the equilibrium $(\frac{h-1}{r}, 0)$ can be determined by the following linearized matrix:

$$A_O^u = \begin{pmatrix} \frac{1-h}{h} & \frac{h-1}{r} \\ 0 & \frac{r-h+1}{r} \end{pmatrix}.$$

The stability of $(\frac{h-1}{r}, 0)$ is reduced to consider the characteristic equation:

$$det(A_O^u - \mu I) = \left(\mu + \frac{h-1}{h}\right)\left(\mu + \frac{h-r-1}{r}\right).$$

Thus, the semi-trivial equilibria $(\frac{h-1}{r}, 0)$ is locally asymptotically stable when $h > r + 1$.

3) The stability of the equilibrium $(0, 1)$ can be determined by the following linearized matrix:

$$A_O^v = \begin{pmatrix} h & 0 \\ -1 & -1 \end{pmatrix}.$$

The stability of $(0, 1)$ is reduced to consider the characteristic equation:

$$det(A_O^v - \mu I) = (\mu - h)(\mu + 1).$$

Hence, the characteristic equation must have a positive root h and a negative root -1. Then $(0, 1)$ is a saddle point. We complete the proof of Proposition 3.1.

Now, based on Proposition 3.1, we start to prove Theorem 1.3. Let $\{\lambda_i\}_{i=1}^{\infty}$ be the sequence of eigenvalues of operator $-\Delta$ over bounded domain Ω with homogeneous Neumann boundary condition satisfying monotonically increasing condition, i.e., $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \cdots$.

The proof of Theorem 1.3. 1) The linearization of system (1.8) around its constant steady state (u^*, v^*) can be written as

$$A_P = \begin{pmatrix} d_u\Delta u - \frac{hru^*}{(1+ru^*)^2} & -\chi u^*\Delta v + u^* \\ -v^* & d_v\Delta v - v^* \end{pmatrix}.$$

The stability is reduced to consider the characteristic equation:

$$det(\mathcal{A}_{\mathcal{P}} - \mu I) = \mu^2 - T_P\mu + D_P$$

with

$$\begin{cases} T_P = -(d_u + d_v)\lambda_i - \frac{hru^*}{(1+ru^*)^2} - v^*, \\ D_P = d_u d_v \lambda_i^2 + \left(d_u v^* + d_v \frac{hru^*}{(1+ru^*)^2} + \chi u^* v^*\right)\lambda_i + \frac{hru^* v^*}{(1+ru^*)^2} + u^* v^*. \end{cases}$$

One has $T_P < 0$ and $D_P > 0$. Hence, (u^*, v^*) is locally asymptotically stable.

2) The linearization of (1.8) at semi-trivial solution $(\frac{h-1}{r}, 0)$ can be expressed by

$$A_P^u = \begin{pmatrix} d_u\Delta u + \frac{1-h}{h} & -\chi u_* \Delta v + \frac{h-1}{r} \\ 0 & d_v \Delta v + \frac{r-h+1}{r} \end{pmatrix}.$$

So, the stability is reduced to consider the characteristic equation:

$$det(A_P^u - \mu I) = \left(\mu + d_u\lambda_i + \frac{h-1}{h}\right)\left(\mu + d_v\lambda_i + \frac{h-1-r}{r}\right).$$

Hence, $(\frac{h-1}{r}, 0)$ is locally asymptotically stable as $h > r + 1$.

3) The linearization of (1.8) at semi-trivial solution $(0, 1)$ can be expressed by

$$A_P^v = \begin{pmatrix} d_u\Delta u + h & 0 \\ -1 & d_v\Delta v - 1 \end{pmatrix}.$$

So the stability is reduced to consider the characteristic equation:

$$det(A_P^v - \mu I) = (\mu + d_u\lambda_i - h)(\mu + d_v\lambda_i + 1).$$

When $\lambda_i = 0$, the characteristic equation must have a positive root h. Hence, $(0, 1)$ is unstable.

3.2. Global asymptotic behavior

Next, we prove the global stability of (u^*, v^*) and $(\frac{h-1}{r}, 0)$ of (1.8) by constructing Lyapunov functionals.

Lemma 3.2. *Let* (u, v) *be a nonnegative global bounded classical solution to system* (1.8) *and* (u^*, v^*) *be defined by* (1.9)*. Suppose that* $h < r + 1$ *and*

$$\chi^2 < \frac{4 d_u d_v v^*}{u^* \|v\|^2_{L^\infty(\Omega)}}, \tag{3.2}$$

so, we have

$$\lim_{t\to\infty} \left(\|u(\cdot, t) - u^*\|_{L^\infty(\Omega)} + \|v(\cdot, t) - v^*\|_{L^\infty(\Omega)}\right) = 0.$$

Proof. Let

$$E_1(t) = \int_\Omega \left(u - u^* - u^* \ln \frac{u}{u^*}\right) + \int_\Omega \left(v - v^* - v^* \ln \frac{v}{v^*}\right).$$

Clearly, $E_1(t)$ is a nonnegative function, and $E_1(t) = 0$ if and only if $(u, v) = (u^*, v^*)$. Differentiating $E_1(t)$ with respect t and using (1.8), we have

$$\begin{aligned}
\frac{d}{dt}E_1(t) =& \int_\Omega \frac{u-u^*}{u}u_t + \int_\Omega \frac{v-v^*}{v}v_t \\
=& - d_u u^* \int_\Omega \frac{|\nabla u|^2}{u^2} + \chi u^* \int_\Omega \frac{\nabla u \cdot \nabla v}{u} + \int_\Omega (u-u^*)\left(\frac{h}{1+ru} - 1 + v\right) \\
& - d_v v^* \int_\Omega \frac{|\nabla v|^2}{v^2} + \int_\Omega (v-v^*)(1-v-u) \\
=& - d_u u^* \int_\Omega \frac{|\nabla u|^2}{u^2} + \chi u^* \int_\Omega \frac{\nabla u \cdot \nabla v}{u} - d_v v^* \int_\Omega \frac{|\nabla v|^2}{v^2} \\
& - hr \int_\Omega \frac{(u-u^*)^2}{(1+ru^*)(1+ru)} - \int_\Omega (v-v^*)^2 \\
:=& - \int_\Omega X_1 A_1 X_1^T - hr \int_\Omega \frac{(u-u^*)^2}{(1+ru^*)(1+ru)} - \int_\Omega (v-v^*)^2,
\end{aligned} \tag{3.3}$$

where $X_1 = \left(\frac{\nabla u}{u}, \frac{\nabla v}{v}\right)$ and A_1 is a matrix denoted by

$$A_1 = \begin{pmatrix} d_u u^* & -\frac{\chi u^* v}{2} \\ -\frac{\chi u^* v}{2} & d_v v^* \end{pmatrix}.$$

One can verify that if (3.2) holds, the matrix A_1 is positive definite, and hence there exists a positive constant α_1 such that

$$- \int_\Omega X_1 A_1 X_1^T \le -\alpha_1 \int_\Omega \left(\frac{|\nabla u|^2}{u^2} + \frac{|\nabla v|^2}{v^2}\right). \tag{3.4}$$

Substituting (3.4) into (3.3), we have

$$\frac{d}{dt}E_1(t) \le -\alpha_1 \int_\Omega \left(\frac{|\nabla u|^2}{u^2} + \frac{|\nabla v|^2}{v^2}\right) - hr \int_\Omega \frac{(u-u^*)^2}{(1+ru^*)(1+ru)} - \int_\Omega (v-v^*)^2 \le 0,$$

where "=" holds if and only if $(u, v) = (u^*, v^*)$. Then using the LaSalle invariance principle [29, Theorem 3], we see that the solution (u, v) converges to (u^*, v^*) as $t \to \infty$ in L^∞.

Now, we shall show the global asymptotical stability of constant steady state $(\frac{h-1}{r}, 0) := (u_*, 0)$ of (1.8).

Lemma 3.3. *Let* (u, v) *be a nonnegative global bounded classical solution to system* (1.8). *Suppose that* $h \ge r + 1$ *and*

$$\chi^2 < \frac{4 d_u d_v}{u_*}, \tag{3.5}$$

so, we have

$$\lim_{t\to\infty} (\|u(\cdot, t) - u_*\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}) = 0.$$

Proof. Define

$$E_2(t) = \int_\Omega \left(u - u_* - u_* \ln \frac{u}{u_*}\right) + \frac{1}{2}\int_\Omega v^2 + \int_\Omega v.$$

Note that $E_2(t)$ is a nonnegative function and $E_2(t) = 0$ if and only if $(u, v) = (u_*, 0)$. Differentiating $E_2(t)$ with respect t, we have

$$\begin{aligned}
\frac{d}{dt}E_2(t) =& - d_u u_* \int_\Omega \frac{|\nabla u|^2}{u^2} + \chi u_* \int_\Omega \frac{\nabla u \cdot \nabla v}{u} + \int_\Omega (u - u_*)\left(\frac{h}{1 + ru} - 1 + v\right) \\
& - d_v \int_\Omega |\nabla v|^2 + \int_\Omega v^2(1 - v - u) + \int_\Omega v(1 - v - u) \\
=& - d_u u_* \int_\Omega \frac{|\nabla u|^2}{u^2} + \chi u_* \int_\Omega \frac{\nabla u \cdot \nabla v}{u} - d_v \int_\Omega |\nabla v|^2 \\
& - hr \int_\Omega \frac{(u - u_*)^2}{(1 + ru)(1 + ru_*)} + \int_\Omega v(u - u_*) + \int_\Omega v^2 - \int_\Omega v^3 - \int_\Omega uv^2 \\
& - \int_\Omega v(u - u_*) - (u_* - 1)\int_\Omega v - \int_\Omega v^2 \\
=& - d_u u_* \int_\Omega \frac{|\nabla u|^2}{u^2} + \chi u_* \int_\Omega \frac{\nabla u \cdot \nabla v}{u} - d_v \int_\Omega |\nabla v|^2 \\
& - hr \int_\Omega \frac{(u - u_*)^2}{(1 + ru)(1 + ru_*)} - \int_\Omega v^3 - \int_\Omega uv^2 - (u_* - 1)\int_\Omega v \\
:=& - \int_\Omega X_2 A_2 X_2^T - hr \int_\Omega \frac{(u - u_*)^2}{(1 + ru_*)(1 + ru)} - (u_* - 1)\int_\Omega v \\
& - \int_\Omega v^3 - \int_\Omega uv^2,
\end{aligned} \tag{3.6}$$

where $X_2 = \left(\frac{\nabla u}{u}, \nabla v\right)$ and matrix A_2 is expressed as

$$A_2 = \begin{pmatrix} d_u u_* & -\frac{\chi u_*}{2} \\ -\frac{\chi u_*}{2} & d_v \end{pmatrix}.$$

We can verify that if (3.5) holds, the matrix A_2 is positive definite, and hence there is a constant $\alpha_2 > 0$ such that

$$-\int_\Omega X_2 A_2 X_2^T \le -\alpha_2 \int_\Omega \left(\frac{|\nabla u|^2}{u^2} + |\nabla v|^2\right). \tag{3.7}$$

In view of $u_* = \frac{h-1}{r}$ and $h \ge r + 1$, we have

$$-(u_* - 1)\int_\Omega v \le 0. \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.6), we have

$$\frac{d}{dt}E_2(t) \le -\alpha_2 \int_\Omega \left(\frac{|\nabla u|^2}{u^2} + |\nabla v|^2\right) - hr \int_\Omega \frac{(u - u_*)^2}{(1 + ru)(1 + ru_*)} - \int_\Omega v^3 \le 0,$$

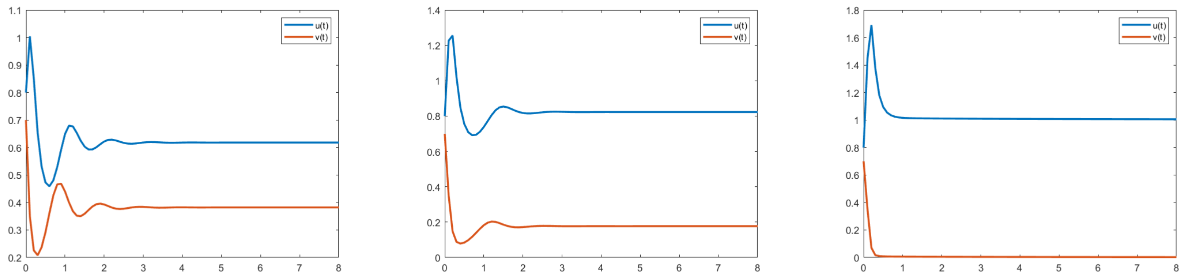
where "=" holds if and only if $(u, v) = (u_*, 0)$. Then using the LaSalle invariance principle [29, Theorem 3], we obtain that the solution (u, v) converges to $(u_*, 0)$ as $t \to \infty$ in L^∞. This completes the proof.

The proof of Theorem 1.4. Theorem 1.4 is a consequence of Lemmas 3.2 and 3.3.

4. Simulation and discussion

4.1. Simulations

First, we consider corresponding ODE system (3.1). First, let $h = 1, r = 1$, and initial function $(u_0, v_0) = (0.80, 0.70)$. Second, there exist only the constant equilibria $(0, 0), (0, 1)$, and $(0.62, 0.38)$ for the system (3.1). We observe that as time goes by, the numerical solution of (u, v) converges to $(0.62, 0.38)$ (see Figure 1(a)). Then, we take $h = 1.5, r = 1$. The initial function remain the same. Then, there exist the constant equilibria $(0, 0), (0, 1), (0.5, 0)$, and $(0.82, 0.18)$ for the system (3.1). We observe that as time evolves, the numerical solution of (u, v) converges to positive constant equilibrium $(0.82, 0.18)$ (see Figure 1(b)). Third, we choose $h = 2, r = 1$ and the same initial value. Then, there exist the constant equilibria $(0, 0), (0, 1)$, and $(1, 0)$ for the system (3.1). We observe that the numerical solution of (u, v) converges to semi-equilibrium $(1, 0)$ (see Figure 1(c)).



(a) (b) (c)

Figure 1. (a) $h = 1, r = 1$; (b) $h = 1.5, r = 1$; (c) $h = 2, r = 1$. The initial data is $(u_0, v_0) = (0.80, 0.70)$.

Next, we carry out some numerical analysis about the stability of the constant solution for system (1.8). We take $d_u = 1, d_v = 0.1, h = 1, r = 1$, and $\Omega = (0, 3\pi)$. Then, there exist only the constant equilibria $(0, 0), (0, 1)$, and $(0.62, 0.38)$ for the system (1.8). According to Theorem 1.4, when $\chi < 0.5$, $(0.62, 0.38)$ is globally asymptotically stable. Let $\chi = 0.4, 1$, and 100, respectively. We set initial function $(u_0, v_0) = (0.62 + 0.05 \sin x, 0.38 + 0.05 \cos x)$, which is a small disturbance near the constant equilibrium point $(0.62, 0.38)$. We also choose initial functions $(u_0, v_0) = (0.62 \pm 0.5 \sin x, 0.38 \pm 0.5 \cos x)$. We observe that the numerical solution of (u, v) converges to $(0.62, 0.38)$ in all cases. Here, we only present numerical simulation pictures when $\chi = 1$ and $(u_0, v_0) = (0.62 + 0.05 \sin x, 0.38 + 0.05 \cos x)$ (see Figure 2).

Then, let $d_u = 1, d_v = 0.1, h = 1.5, r = 1$, and $\Omega = (0, 3\pi)$. There are the constant steady states $(0, 0), (0, 1), (0.5, 0)$, and $(0.82, 0.18)$ for the system (1.8). From Theorem 1.4, we know that $(0.82, 0.18)$ is globally asymptotically stable when $\chi < 0.3$. We take $\chi = 0.2, 1$, and 100, and initial function $(u_0, v_0) = (0.82 \pm 0.05 \sin x \pm 0.5 \sin x, 0.18 \pm 0.05 \cos x \pm 0.5 \cos x)$, respectively. We observe that as time evolves, the numerical solution converges to positive constant steady state $(0.82, 0.18)$ in all cases. Here, we only present numerical simulation pictures when $\chi = 1$ and $(u_0, v_0) = (0.82+0.05 \sin x, 0.18+$

0.05 cos x), as shown in Figure 3(a).

Finally, we choose $d_u = 1, d_v = 0.1, h = 2, r = 1, \chi = 1$, and $\Omega = (0, 3\pi)$. Therefore, there exist the constant equilibria $(0, 0), (0, 1)$, and $(1, 0)$ for the system (1.8). Let initial function $(u_0, v_0) = (0.80 + 0.05 \sin x, 0.70 + 0.05 \cos x)$. We observe that the numerical solution of (u, v) converges to semi-equilibrium $(1, 0)$ (see Figure 4).

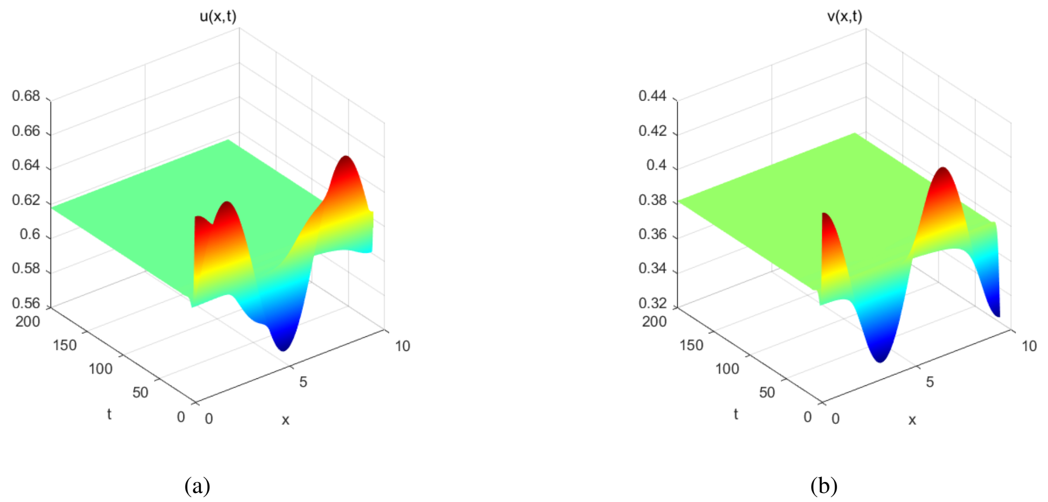


Figure 2. $d_u = 1, d_v = 0.1, h = 1, r = 1, \chi = 1$, and $\Omega = (0, 3\pi)$ and initial function $(u_0, v_0) = (0.62 + 0.05 \sin x, 0.38 + 0.05 \cos x)$.

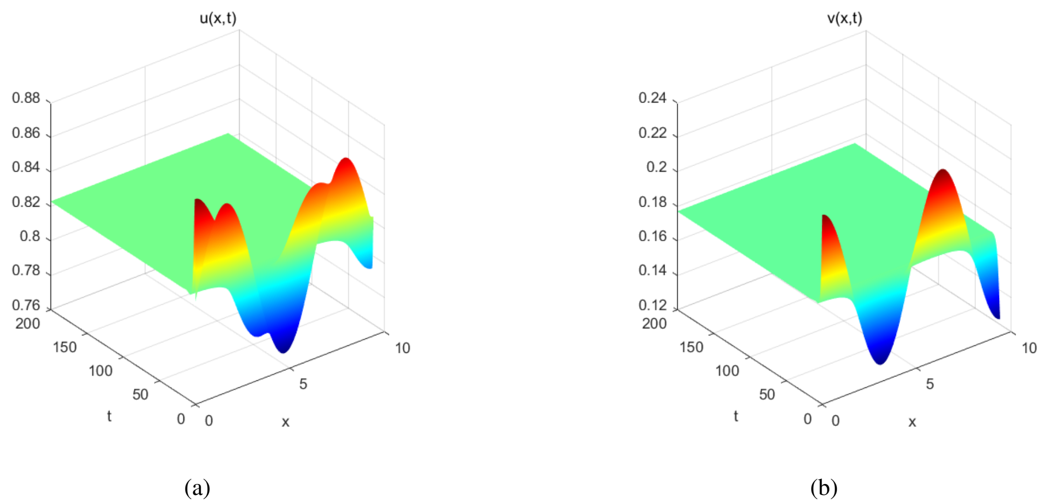


Figure 3. $d_u = 1, d_v = 0.1, h = 1.5, r = 1, \chi = 1$, and $\Omega = (0, 3\pi)$ and initial function $(u_0, v_0) = (0.82 + 0.05 \sin x, 0.18 + 0.05 \cos x)$.

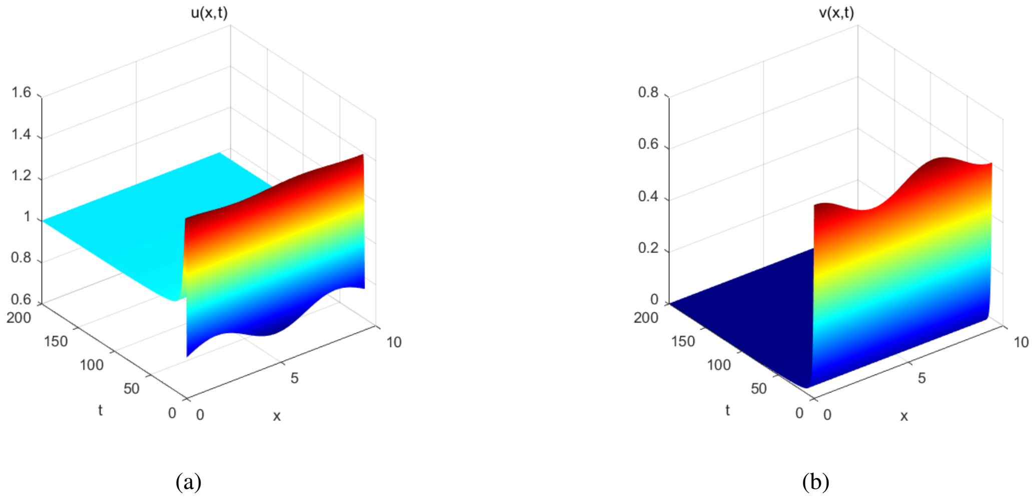


Figure 4. $d_u = 1, d_v = 0.1, h = 2, r = 1, \chi = 1$, and $\Omega = (0, 3\pi)$ and initial function $(u_0, v_0) = (0.80 + 0.05 \sin x, 0.70 + 0.05 \cos x)$

4.2. Discussion

We investigate the dynamics of a two-component predator-prey model that incorporates taxis mechanisms and nonlinear growth for the predator population. The main contributions of this paper are composed of two parts. The first one is the global boundedness of the classical solution in any dimensions. The second is the influence of the taxis strategies on the pattern formation. Through rigorous mathematical analysis, we find that the prey-taxis does not lead to pattern formation, and the results of numerical simulations support our theoretical results. Based on the previous analysis, we have provided the results on the local and global stability of positive constant equilibria and semi-equilibria for system (1.8) and its corresponding ODE system (3.1). In this section, we present some examples to numerically illustrate the stability of constant steady state. On the other hand, we note that the proof of global boundedness of solutions relies on the validity of Condition (H1), and the conditions derived for global stability are sufficient but not necessary. These constraints restrict the applicability of our results to cases where such conditions hold, leaving room for improvement. Future work could focus on relaxing these conditions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

Jianhua Li was supported by the Innovation Ability Improvement Project of College and Universities of Gansu Province (No. 2020B-215) and the Young Doctor Fund Project of Gansu Province (No. 2022QB151). Yingyuan Mi was supported by the Foundamental Research Funds for the Central Universities (No. 31920250078) and Research Projects for Talent Introduction at

Northwest Minzu University (no.xbmuyjrc202518).

Conflict of interest

The authors declare there are no conflicts of interest.

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